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# On the two-dimensional Pauli operator with a finite number of Aharonov-Bohm solenoids

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# On the two-dimensional Pauli operator with a finite number of Aharonov-Bohm solenoids

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**Abstract**

This licentiate thesis consists of two papers comparing different two-dimensional self-adjoint Pauli operators corresponding to a singular magnetic field with a finite number of Aharonov-Bohm solenoids.

In the first paper the Pauli operator is defined via a quadratic form. This Pauli operator has the good property of being spin-flip invariant. We prove an Aharonov-Casher type formula for the operator.

In the second paper we study the Pauli operators that are obtained as the square of a self-adjoint Dirac operator. Among these operators there is essentially only one that is spin-flip invariant. Again, we prove Aharonov-Casher type formulas for the different operators. There are only two of them that admits zero-modes, one of them according to the original Aharonov-Casher formula. However these two extensions are very asymmetric, not being spin-flip invariant.

We conclude that none of the studied operators satisfy all good properties one would expect from a self-adjoint Pauli operator. The question of which of the Pauli operators describe the real physical situation is still open.

**Keywords:** Pauli operator, Dirac operator, self-adjoint extensions, Aharonov-Bohm effect, Aharonov-Casher formula.

**AMS 2000 Subject Classification:** 35P15, 35Q40, 81Q10

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# Introduction

A charged spin-$\frac{1}{2}$ quantum particle, moving in a plane, affected by a magnetic field $\mathbf{B}$ orthogonal to the plane, is described by the self-adjoint Pauli operator.

If we introduce coordinates $(x_1, x_2, x_3)$ for $\mathbb{R}^3$ in such a way that the particle is living in the $x_1x_2$-plane we can write the magnetic field $\mathbf{B}$ as

$$\mathbf{B} = (0, 0, B(x_1, x_2))\,.$$

A magnetic vector potential is a vector valued function $\mathbf{A}$ satisfying $\mathbf{B} = \operatorname{curl}\mathbf{A}$. The vector potential is not uniquely determined. If $\tilde{\mathbf{A}} = \mathbf{A} + \nabla f$ for some sufficient regular function $f$, then $\operatorname{curl}\tilde{\mathbf{A}} = \operatorname{curl}\mathbf{A}$. In the case with an orthogonal magnetic field $\mathbf{B}$, the magnetic vector potential $\mathbf{A}$ can be given by

$$\mathbf{A} = (A_1(x_1, x_2), A_2(x_1, x_2), 0)\,, \quad \text{where } B(x_1, x_2) = \frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2}.$$

Since the magnetic field and the magnetic vector potential can be expressed using the coordinates $x_1$ and $x_2$ only, we omit the third coordinate $x_3$ and talk about the magnetic field $B(x_1, x_2)$ and the magnetic vector potential $\mathbf{A} = (A_1, A_2)$. Moreover it is convenient to identify a point $(x_1, x_2)$ in $\mathbb{R}^2$ with the complex number $z = x_1 + ix_2$.

Using the notations

$$\Pi_k = -i\frac{\partial}{\partial x_k} + A_k, \quad k = 1, 2$$

and

$$Q_\pm = \Pi_1 \pm i\Pi_2,$$

the Pauli operator can be defined via the quadratic form

$$p[\psi] = ||Q_+\psi_+||^2 + ||Q_-\psi_-||^2, \quad \psi = \begin{pmatrix}\psi_+\\ \psi_-\end{pmatrix} \in L_2(\mathbb{R}^2)^2 \tag{1}$$

on $L_2(\mathbb{R}^2)^2$, the domain being the closure in the sense of the metrics $p[\psi]$ of the core consisting of smooth compactly supported functions. With this notation the Pauli operator can be written as

$$P = \begin{pmatrix}P_+ & 0\\ 0 & P_-\end{pmatrix} = \begin{pmatrix}Q_+^*Q_+ & 0\\ 0 & Q_-^*Q_-\end{pmatrix}.$$

However, it is proved in [Sob96] that the definition of the quadratic form $p[\psi]$ requires that $\mathbf{A}$ belongs to $L_{2,\mathrm{loc}}(\mathbb{R}^2)^2$.

We are interested in the situation where the magnetic field is more singular. We let the magnetic field $B$ consist of a singular part with a finite number of Aharonov-Bohm (AB) solenoids (see [AB59]) and a regular part with compact support, that is, we let

$$B(z) = B_0(z) + \sum_{j=1}^{n} 2\pi\alpha_j\delta_{z_j}. \tag{2}$$

Here $B_0 \in C_0^1(\mathbb{R}^2)$, $\Lambda = \{z_j\}_1^n$ denotes the distinct points where the AB solenoids are located and $\alpha_j$ is the intensity (flux divided by $2\pi$) at the solenoid located at the point $z_j$.

In the case where $B$ has only one AB solenoid (located at the origin) this magnetic field is given by $2\pi\alpha\delta_0$, and in this simple case a magnetic vector potential $\mathbf{A}$ is given by

$$\mathbf{A}(x_1, x_2) = \frac{\alpha}{x_1^2 + x_2^2} \, (-x_2, x_1)$$

which is not in $L_{2,\mathrm{loc}}(\mathbb{R}^2)^2$. Hence there is a problem in defining the self-adjoint Pauli operator corresponding to this kind of singular magnetic fields.

The Pauli operator, originally defined on smooth functions with compact support not touching the singular points $\Lambda$, is not essentially self-adjoint (see [AT98, DŠ98, EŠV02, GŠ04a, GŠ04b]). Hence there are several self-adjoint extensions, and the question of which of them describes the physics in the best way arises.

All self-adjoint extensions can be defined by a method of Krein and von Neumann (see [AG93]). But this method requires that one can describe the deficiency spaces explicitly, and this is not the case when the magnetic field has several AB solenoids.

However, there are methods to define some of the self-adjoint extensions. These methods are described in the next subsections. To be able to compare different self-adjoint extensions we must have some properties that are natural to expect from a self-adjoint Pauli operator. Let us list the properties we are looking at:

(P1) The Pauli operator should be gauge invariant, that is, changing the AB intensity by an integer multiple shall result in an unitarily equivalent operator.

(P2) Since the particle moves in a plane and the magnetic field is orthogonal to the plane, the situation should be the similar if the magnetic field change sign and the spin-up and spin-down components also change places. We say that the operator is spin-flip invariant if it transforms in a (anti)-unitarily way when these transformations are applied.

(P3a) It should be possible to approximate the Pauli operator in some sense with operators corresponding to more regular magnetic fields. Most of all one would like to be able to approximate the Pauli operator as a Pauli Hamiltonian, and then we say that it satisfies property (P3a).

(P3b) If it is not possible to approximate the Pauli operator as a Pauli Hamiltonian, but each component with probably different approximation parameters, we say that it satisfies property (P3b).

(P4) The Pauli operator is classically the square of the Dirac operator. If this is the case we say that it satisfies property (P4).

(P5) There is a classical result about the dimension of the kernel of the Pauli operator, usually called the Aharonov-Casher formula (see [AC79]). It says that if $\Phi = \frac{1}{2\pi} \int B d\lambda$ is the total flux divided by $2\pi$ (here $d\lambda$ denotes the Lebesgue measure in the plane), then the dimension of the kernel of the Pauli operator is $\lfloor |\Phi| \rfloor - 1$ or $\lfloor |\Phi| \rfloor$ depending on if $\Phi$ is an integer or not. It is not that important that the Pauli operator satisfies the original Aharonov-Casher formula, but physicists are interested in the number of zero-modes (see for example the discussion in [BRF$^+$02] and the references there). We use it merely as a tool to easily distinguish between different extensions.

(P6) The Pauli operator should satisfy a locality property. The asymptotics of functions in the domain at one AB solenoid should not depend on the asymptotics at the other AB solenoids.

## Paper I

Denoting by $h$ a potential satisfying $\Delta h = B$ in the sense of distributions, the quadratic form

$$\pi[\psi] = \left\| 2\frac{\partial}{\partial \bar{z}} \left(e^{-h}\psi_+\right) e^h \right\|^2 + \left\| 2\frac{\partial}{\partial z} \left(e^{h}\psi_-\right) e^{-h} \right\|^2, \quad \psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} \in L_2(\mathbb{R}^2)^2 \qquad (3)$$

is considered in [EV02] (the magnetic field considered in that article is more general than the one we consider here). The derivatives are taken in the space $\mathcal{D}'(\mathbb{R}^2)$. It is proved that if the AB intensities belong to $(-1, 1)$ then (3) is well-defined. Moreover it agrees with the quadratic form (1) for less singular magnetic fields. The Pauli operator is then defined to be the self-adjoint operator corresponding to this quadratic form. We show that this operator lacks the property of being spin-flip invariant. This motivates Paper I, where the setting is slightly different. We define the Pauli operator via the same quadratic form (3) but with the crucial difference that we consider the differentiations in the expression of the quadratic form to take place in the distribution space $\mathcal{D}'(\mathbb{R}^2 \setminus \Lambda)$ instead of $\mathcal{D}'(\mathbb{R}^2)$. This enables us to define the Pauli operator via the quadratic form for arbitrary AB intensities, and what is more important the Pauli operator defined in this way turns out to be spin-flip invariant.

We show that the Pauli operator we defined, called the Maximal Pauli operator in the paper, satisfies the properties (P1), (P2), (P3b) and (P6). We check in Paper II that it is not the square of a self-adjoint Dirac operator. Moreover we show that the operator may have zero-modes, but the number of zero-modes are not given by the original Aharonov-Casher formula.

We also check that the Pauli extension defined in [EV02] satisfies (P1), (P3a), (P4), (P5) and (P6). It does not satisfy (P2), and we see in Paper II that it is the square of a Dirac operator that may describe different physics at different AB solenoids.

## Paper II

In the second paper we take a closer look at the Pauli operators that can be realized as the square of a self-adjoint Dirac operator, see (P4). To be able to define these Pauli operators we first have to define the Dirac operators corresponding to the magnetic field (2).

These Dirac operator $D$ is formally defined as

$$D\psi = 2i \begin{pmatrix} 0 & \frac{\partial}{\partial z}\left(e^{h}\psi_-\right)e^{-h} \\ \frac{\partial}{\partial \bar{z}}\left(e^{-h}\psi_+\right)e^{h} & 0 \end{pmatrix}, \quad \psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} \in L_2(\mathbb{R}^2)^2,$$

but one has to specify the domain of the operator to make it self-adjoint.

We start by studying the Dirac operator corresponding to one AB solenoid, first defined on compactly supported functions not touching the singular point. This operator is not essentially self-adjoint, but there is a one-parameter family of self-adjoint extensions, described in [dSG89, Tam03]. The different extensions describe different physics near the AB solenoid. Mathematically this is realized through different asymptotics near the AB solenoid of the functions in the domains of the different extensions.

Having the self-adjoint Dirac extensions defined for one AB solenoid, inspired by property (6), we define them for several AB solenoids by choosing one of the extensions at each solenoid and gluing them together as in [AR04]. Then we prove that this procedure gives, in fact, a self-adjoint operator. It is natural to choose the same extension at all solenoids so the final operator describes the same physics everywhere.

The Pauli operators are then defined as the square of one of these Dirac operators. All these Pauli operators satisfy the properties (P1), (P3b), (P4) and (P6). Only two of them satisfy (P2) and these two extensions do not have any zero-modes.

There are two extensions satisfying (P3a). These are the only extensions of this type where the spin-up and spin-down components are not coupled, which implies that they admit zero-modes. One of them satisfies the original Aharonov-Casher formula.

### Summary

Among the self-adjoint Pauli operators defined in these two papers none of them satisfies all the wanted properties listed above. We leave it for the physicists to decide which of them is describing the real physical situation in the best way.

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# Paper I

# On the Aharonov-Casher formula for different self-adjoint extensions of the Pauli operator with singular magnetic field

Mikael Persson

**Abstract**

Two different self-adjoint Pauli extensions describing a spin-1/2 two-dimensional quantum system with singular magnetic field are studied. An Aharonov-Casher type formula is proved for the maximal Pauli extension and the possibility of approximation of the two different self-adjoint extensions by operators with regular magnetic fields is investigated.

## 1 Introduction

Two-dimensional spin-1/2 quantum systems involving magnetic fields are described by the self-adjoint Pauli operator. One interesting question about such systems is the appearance of zero modes (eigenfunctions with eigenvalue zero). Aharonov and Casher proved in [AC79] that if the magnetic field is bounded and compactly supported, then zero modes can arise, and the number of zero modes is simply connected to the total flux of the magnetic field. Since then, Aharonov-Casher type formulas have been proved for more and more singular magnetic fields in different settings, see [CFKS87, GG02, LL58, Mil82]. Recently they were proved for measure-valued magnetic fields in [EV02] by Erdős and Vougalter.

We are interested in the Pauli operator when the magnetic field consists of a regular part with compact support and a singular part with a finite number of Aharonov-Bohm (AB) solenoids [AB59]. The Pauli operator for such singular magnetic fields, defined initially on smooth functions with support not touching the singularities, is not essentially self-adjoint. Thus there are several ways of defining the self-adjoint Pauli extension, depending on what boundary conditions one sets at the AB solenoids, see [AT98, DŠ98, EŠV02, GŠ04a, GŠ04b]. Different extensions describe different physics, and there is a discussion going on about which extensions describe the real physical situation.

There are two possible approaches to making the choice of the extension: trying to describe boundary conditions at the singularities by means of modeling actual interaction of the particle with an AB solenoid, or considering approximations of singular fields by regular ones, see [BP03, Tam03]. We are going to study the maximal extension introduced in [GG02], called the Maximal Pauli operator, and compare it with the extension defined in [EV02], that we will call the EV Pauli operator. These two extensions were recently studied in [RS05] in the presence of infinite number of AB solenoids, and it was proved that a magnetic field with infinite flux gives an infinite-dimensional space of zero modes for both extensions.

When studying the Pauli operator in the presence of AB solenoids one must always keep in mind the possibility to reduce the intensities of solenoids by arbitrary integers by means of singular gauge transformations. In Section 2 we define both extensions via quadratic forms. The Maximal Pauli operator can be defined directly for arbitrary strength of the AB fluxes, while the EV Pauli operator is defined via gauge transformations if the AB intensities do not belong to the interval $[-1/2, 1/2)$.

The EV Pauli operator has the advantage that the Aharonov-Casher type formula in its original form holds even for singular AB magnetic fields. However, as we show, it does not satisfy another natural requirement, that the number of zero modes is invariant under the change of sign of the magnetic field. This absence of invariance exhibits itself only if both singular and regular parts of the field are present. This justifies our attempt to study the Maximal Pauli operator which lacks the latter disadvantage. The price we have to pay for this is that our Aharonov-Casher type formula has certain extra terms.

For the Dirac operators with strongly singular magnetic field the question on the number of zero modes was considered in [HO01]. The definition of the self-adjoint operator considered there is close to the one in Erdős-Vougalter, however it is not gauge invariant, therefore the Aharonov Casher-type formula obtained in [HO01] depends on intensity of each AB solenoid separately.

In Section 3 we establish that the Maximal Pauli operator is gauge invariant and that changing the sign of the magnetic field leads to anti-unitarily equivalence. Our main result is the Aharonov-Casher type formula for the Maximal Pauli operator. An interesting fact is that this operator can have both spin-up and spin-down zero modes, in contrary to the EV Pauli operator and the Pauli operator for less singular magnetic fields, which have either spin-up or spin-down zero modes, but not both. In [GG02] a setting with an infinite lattice of AB solenoids with equal AB flux at each solenoid is studied, having both spin-up and spin-down zero modes, both with infinite multiplicity.

In Section 4 we discuss the approximation by more regular fields in the sense of Borg and Pulé, see [BP03]. It turns out that both the Maximal Pauli operator and the EV Pauli operator can be approximated in this way. However, the EV Pauli operator can be approximated as a Pauli Hamiltonian, while the Maximal Pauli operator can only be approximated one component at a time. Since different ways of approximating the magnetic field may lead to different results, see [BV93, Tam03], we leave the question if the Maximal Pauli operator can be approximated as Pauli Hamiltonian open.

## 2 Definition of the Pauli operators

The Pauli operator is formally defined as

$$P = (\sigma \cdot (-i\nabla + \mathbf{A}))^2 = (-i\nabla + \mathbf{A})^2 + \sigma_3 B$$

on $L_2(\mathbb{R}^2) \otimes \mathbb{C}^2$. Here $\sigma = (\sigma_1, \sigma_2)$, where $\sigma_1$, $\sigma_2$ and $\sigma_3$ are the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \text{ and } \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$\mathbf{A}$ is the real magnetic vector potential and $B = \mathrm{curl}(\mathbf{A})$ is the magnetic field. This definition does not work if the magnetic field $B$ is too singular, see the discussion in [EV02, Sob96]. If $\mathbf{A} \in L_{2,\mathrm{loc}}(\mathbb{R}^2)$, using the notations $\Pi_k = -i\partial_k + A_k$, for $k = 1, 2$, $Q_\pm = \Pi_1 \pm i\Pi_2$ and $\lambda$ for the Lebesgue measure, the Pauli operator can be defined via the quadratic form

$$p[\psi] = \|Q_+\psi_+\|^2 + \|Q_-\psi_-\|^2 = \int |\sigma \cdot (-i\nabla + \mathbf{A})\psi|^2 d\lambda(x), \tag{1}$$

the domain being the closure in the sense of the metrics $p[\psi]$ of the core consisting of smooth compactly supported functions. With this notation, we can write the Pauli operator $P$ as

$$P = \begin{pmatrix} P_+ & 0 \\ 0 & P_- \end{pmatrix} = \begin{pmatrix} Q_+^* Q_+ & 0 \\ 0 & Q_-^* Q_- \end{pmatrix}. \tag{2}$$

However, defining the Pauli operator via the quadratic form $p[\psi]$ in (1) requires that the vector potential $\mathbf{A}$ belongs to $L_{2,\mathrm{loc}}(\mathbb{R}^2)$, otherwise $p[\psi]$ can be infinite for nice functions $\psi$, see [Sob96]. If the magnetic field consists of only one AB solenoid located at the origin with intensity (flux divided by $2\pi$) $\alpha$, then the magnetic vector potential $\mathbf{A}$ is given by $\mathbf{A}(x_1, x_2) = \frac{\alpha}{x_1^2 + x_2^2}(-x_2, x_1)$ which is not in $L_{2,\mathrm{loc}}(\mathbb{R}^2)$. Here, and elsewhere we identify a point $(x_1, x_2)$ in the two-dimensional space $\mathbb{R}^2$ with $z = x_1 + ix_2$ in the complex plane $\mathbb{C}$.

Following [EV02], we will define the Pauli operator via another quadratic form, that agrees with $p[\psi]$ for less singular magnetic fields. We start by describing the magnetic field.

Even though the Pauli operator can be defined for more general magnetic fields, in order to demonstrate the main features of the study, without extra technicalities, we restrict ourself to a magnetic field consisting of a sum of two parts, the first being a smooth function with compact support, the second consisting of finitely many AB solenoids. Let $\Lambda = \{z_j\}_{j=1}^n$ be a set of distinct points in $\mathbb{C}$ and let $\alpha_j \in \mathbb{R} \setminus \mathbb{Z}$. The magnetic field we will study in this paper has the form

$$B(z) = B_0(z) + \sum_{j=1}^{n} 2\pi\alpha_j \delta_{z_j}, \tag{3}$$

where $B_0 \in C_0^1(\mathbb{R}^2)$. In [EV02] the magnetic field is given by a signed real regular Borel measure $\mu$ on $\mathbb{R}^2$ with locally finite total variation. It is clear that $\mu = B_0(z)d\lambda(z) + \sum_{j=1}^n 2\pi\alpha_j\delta_{z_j}$ is such a measure.

The function $h_0$ given by

$$h_0(z) = \frac{1}{2\pi}\int \log|z-z'|B_0(z')d\lambda(z')$$

satisfies $\Delta h_0 = B_0$ since $B_0 \in C_0^1(\mathbb{R}^2)$ and $\Delta \log|z-z_j| = 2\pi\delta_{z_j}$ in the sense of distributions. The function

$$h(z) = h_0(z) + \sum_{j=1}^n \alpha_j \log|z-z_j|$$

satisfies $\Delta h = B$. It is easily seen that $h_0(z) \sim \Phi_0 \log|z|$ as $|z| \to \infty$, and thus the asymptotics of $e^{h(z)}$ is

$$e^{\pm h(z)} \sim \begin{cases} |z|^{\pm\Phi}, & |z| \to \infty \\ |z-z_j|^{\pm\alpha_j}, & z \to z_j, \end{cases}$$

where $\Phi_0 = \frac{1}{2\pi}\int B_0(z)d\lambda(z)$ and $\Phi = \frac{1}{2\pi}\int B(z)d\lambda(z) = \Phi_0 + \sum_{j=1}^n \alpha_j$.

We are now ready to define the two self-adjoint Pauli operators. The decisive difference between them is the sense in which we are taking derivatives. This leads to different domains, and, as we will see in later sections, to different properties of the operators. Let us introduce notations for taking derivatives on the different spaces of distributions. Remember that $\Lambda = \{z_j\}_{j=1}^n$ is a finite set of distinct points in $\mathbb{C}$. We let the derivatives in $\mathcal{D}'(\mathbb{R}^2)$ be denoted by $\partial$ and the derivatives in $\mathcal{D}'(\mathbb{R}^2 \setminus \Lambda)$ be denoted by $\partial$ with a tilde over it, that is $\tilde{\partial}$. Thus, for example, by $\partial_z$ we mean $\frac{\partial}{\partial z}$ in the space $\mathcal{D}'(\mathbb{R}^2)$ and by $\tilde{\partial}_z$ we mean $\frac{\partial}{\partial z}$ in the space $\mathcal{D}'(\mathbb{R}^2 \setminus \Lambda)$.

## 2.1 The EV Pauli operator

We follow [EV02] and define the sesqui-linear forms $\pi_+$ and $\pi_-$ by

$$\pi_+^h(\psi_+,\xi_+) = 4\int \overline{\partial_{\bar z}\left(e^{-h}\psi_+\right)}\partial_{\bar z}\left(e^{-h}\xi_+\right)e^{2h}d\lambda(z),$$
$$\mathcal{D}(\pi_+^h) = \left\{\psi_+ \in L_2(\mathbb{R}^2) \;:\; \pi_+^h(\psi_+,\psi_+) < \infty\right\},$$

and

$$\pi_-^h(\psi_-,\xi_-) = 4\int \overline{\partial_{z}\left(e^{h}\psi_-\right)}\partial_{z}\left(e^{h}\xi_-\right)e^{-2h}d\lambda(z),$$
$$\mathcal{D}(\pi_-^h) = \left\{\psi_- \in L_2(\mathbb{R}^2) \;:\; \pi_-^h(\psi_-,\psi_-) < \infty\right\}.$$

Set

$$\pi^h(\psi,\xi) = \pi_+^h(\psi_+,\xi_+) + \pi_-^h(\psi_-,\xi_-),$$
$$\mathcal{D}(\pi^h) = \mathcal{D}(\pi_+^h) \oplus \mathcal{D}(\pi_-^h) = \left\{\psi = \begin{pmatrix}\psi_+ \\ \psi_-\end{pmatrix} \in L_2(\mathbb{R}^2)\otimes\mathbb{C}^2 \;:\; \pi^h(\psi,\psi) < \infty\right\}.$$

Let us make more accurate the description of the domains of the forms $\pi^h_\pm$ and $\pi^h$. For example, what is required of a function $\psi_+$ to be in $\mathcal{D}(\pi^h_+)$? It should belong to $L_2(\mathbb{R}^2)$, and the expression

$$\pi^h_+(\psi_+, \psi_+) = 4 \int \left|\partial_{\bar{z}} \left(e^{-h}\psi_+\right)\right|^2 e^{2h} d\lambda(z)$$

should have a meaning and be finite. This means that the distribution $\partial_{\bar{z}} \left(e^{-h}\psi_+\right)$ actually must be a function and its modulus multiplied with $e^h$ must belong to $L_2(\mathbb{R}^2)$, that is $|\partial_{\bar{z}} \left(e^{-h}\psi_+\right)| e^h \in L_2(\mathbb{R}^2)$. This forces all the intensities $\alpha_j$ to be in the interval $(-1, 1)$, see [EV02].

Next we define the norm by

$$|||\psi|||^2_{\pi^h} = |||\psi_+|||^2_{\pi^h_+} + |||\psi_-|||^2_{\pi^h_-} ,$$

where

$$|||\psi_+|||^2_{\pi^h_+} = \|\psi_+\|^2 + \left\|\partial_{\bar{z}} \left(e^{-h}\psi_+\right) e^h\right\|^2$$

and

$$|||\psi_-|||^2_{\pi^h_-} = \|\psi_-\|^2 + \left\|\partial_{z} \left(e^{h}\psi_-\right) e^{-h}\right\|^2 .$$

This form $\pi^h$ is symmetric, nonnegative and closed with respect to $\|\cdot\|$, again see [EV02], and hence it defines a unique self-adjoint operator $\mathcal{P}_h$ via

$$\mathcal{D}(\mathcal{P}_h) = \{\psi \in \mathcal{D}(\pi^h) \ : \ \pi^h(\psi, \cdot) \in \left(L_2(\mathbb{R}^2) \otimes \mathbb{C}^2\right)\} \tag{4}$$

and

$$(\mathcal{P}_h\psi, \xi) = \pi^h(\psi, \xi), \quad \psi \in \mathcal{D}(\mathcal{P}_h), \xi \in \mathcal{D}(\pi^h). \tag{5}$$

We call this operator $\mathcal{P}_h$ the *non-reduced EV Pauli operator*.

If some intensities $\alpha_j$ belongs to $\mathbb{R}\setminus[-1/2, 1/2)$, we let $\alpha^*_j$ be the unique real number in $[-1/2, 1/2)$ such that $\alpha_j$ and $\alpha^*_j$ differ only by an integer, that is $\alpha^*_j - \alpha_j = m_j \in \mathbb{Z}$. We define the *reduced EV Pauli operator* (or just the *EV Pauli operator*), $P_h$, to be

$$P_h = \exp(i\phi)\mathcal{P}_h \exp(-i\phi) \tag{6}$$

where $\phi(z) = \sum_{j=1}^n m_j \arg(z - z_j)$. Hence, if there are some $\alpha_j$ outside the interval $(-1, 1)$ only the reduced EV Pauli operator is well-defined. If all the intensities $\alpha_j$ belong to the interval $[-1/2, 1/2)$ then we do not have to perform the reduction and hence there is only one definition. However, if there are intensities $\alpha_j$ inside the interval $(-1, 1)$ but outside the interval $[-1/2, 1/2)$ then we have two different definitions of the EV Pauli operator, the direct one and the one obtained by reduction. In the next section we will show that these two operators are not the same.

## 2.2 The Maximal Pauli operator

Let $\alpha_j \in \mathbb{R} \setminus \mathbb{Z}$. We define the forms

$$\mathfrak{p}_+^h(\psi_+, \xi_+) = 4 \int \overline{\tilde{\partial}_{\bar{z}} \left(e^{-h}\psi_+\right)} \tilde{\partial}_{\bar{z}} \left(e^{-h}\xi_+\right) e^{2h} d\lambda(z),$$
$$\mathcal{D}(\mathfrak{p}_+^h) = \left\{\psi_+ \in L_2(\mathbb{R}^2) \ : \ \mathfrak{p}_+^h(\psi_+, \psi_+) < \infty\right\},$$

and

$$\mathfrak{p}_-^h(\psi_-, \xi_-) = 4 \int \overline{\tilde{\partial}_{z} \left(e^{h}\psi_-\right)} \tilde{\partial}_{z} \left(e^{h}\xi_-\right) e^{-2h} d\lambda(z),$$
$$\mathcal{D}(\mathfrak{p}_-^h) = \left\{\psi_- \in L_2(\mathbb{R}^2) \ : \ \mathfrak{p}_-^h(\psi_-, \psi_-) < \infty\right\}.$$

Set

$$\mathfrak{p}^h(\psi, \xi) = \mathfrak{p}_+^h(\psi_+, \xi_+) + \mathfrak{p}_-^h(\psi_-, \xi_-),$$
$$\mathcal{D}(\mathfrak{p}^h) = \mathcal{D}(\mathfrak{p}_+^h) \oplus \mathcal{D}(\mathfrak{p}_-^h) = \left\{\psi = \begin{pmatrix}\psi_+ \\ \psi_-\end{pmatrix} \in L_2(\mathbb{R}^2) \otimes \mathbb{C}^2 \ : \ \mathfrak{p}^h(\psi, \psi) < \infty\right\}.$$

Again, let us make clear about the domains of the forms. For a function $\psi_+$ to be in $\mathcal{D}(\mathfrak{p}_+^h)$ it is required that $\psi_+ \in L_2(\mathbb{R}^2)$ and that $\tilde{\partial}_z(e^{-h}\psi_+)$ is a function. After taking the modulus of this derivative and multiplying by $e^h$ we should get into $L_2(\mathbb{R}^2 \setminus \Lambda)$, that is $|\tilde{\partial}_{\bar{z}}(e^{-h}\psi_+)|e^h \in L_2(\mathbb{R}^2 \setminus \Lambda)$. Note that the form $\mathfrak{p}^h$ does not feel the AB fluxes at $\Lambda$ since the derivatives are taken in the space $\mathcal{D}'(\mathbb{R}^2 \setminus \Lambda)$, and integration does not feel $\Lambda$ either since $\Lambda$ has Lebesgue measure zero. This enable the AB solenoids to have intensities that lie outside $(-1, 1)$.

Also, define the norm

$$|||\psi_h|||_{\mathfrak{p}^h}^2 = |||\psi_+|||_{\mathfrak{p}_+^h}^2 + |||\psi_-|||_{\mathfrak{p}_-^h}^2 ,$$

where

$$|||\psi_+|||_{\mathfrak{p}_+^h}^2 = \|\psi_+\|^2 + \left\|\tilde{\partial}_{\bar{z}} \left(e^{-h}\psi_+\right) e^h\right\|^2$$

and

$$|||\psi_-|||_{\mathfrak{p}_-^h}^2 = \|\psi_-\|^2 + \left\|\tilde{\partial}_{z} \left(e^{h}\psi_-\right) e^{-h}\right\|^2.$$

**Proposition 2.1.** *The form $\mathfrak{p}^h$ defined above is symmetric, nonnegative and closed with respect to $\|\cdot\|$.*

*Proof.* It is clear that $\mathfrak{p}^h$ is symmetric and nonnegative. Let $\psi_n = (\psi_{n,+}, \psi_{n,-})$ be a Cauchy sequence in the norm $|||\cdot|||_{\mathfrak{p}^h}$. This implies that $\psi_{n,\pm} \to \psi_\pm$ in $L_2(d\lambda(z))$, $\tilde{\partial}_{\bar{z}}\left(e^{-h}\psi_{n,+}\right) \to u_+$ in $L_2(e^{2h}d\lambda(z))$ and $\tilde{\partial}_z(e^h\psi_{n,-}) \to u_-$ in $L_2(e^{-2h}d\lambda(z))$. We have to show that $\tilde{\partial}_{\bar{z}}\left(e^{-h}\psi_+\right) = u_+$ and $\tilde{\partial}_z(e^h\psi_-) = u_-$. For any test-function $\phi \in C_0^\infty(\mathbb{R}^2 \setminus \Lambda)$,

$$\left|\int \bar\phi\left(u_+ - \tilde\partial_{\bar z}\left(e^{-h}\psi_+\right)\right) d\lambda(z)\right| \le \left|\int \bar\phi\left(u_+ - \tilde\partial_{\bar z}\left(e^{-h}\psi_{n,+}\right)\right)\right| + \left|\int \tilde\partial_{\bar z}(\bar\phi) e^{-h}\left(\psi_+ - \psi_{n,+}\right)\right|$$
$$\le \|\bar\phi e^{-h}\| \cdot \left\|u_+ - \tilde\partial_{\bar z}\left(e^{-h}\psi_{n,+}\right)\right\|_{L_2(e^{2h})} + \left\|\tilde\partial_{\bar z}(\bar\phi)e^{-h}\right\| \cdot \|\psi_+ - \psi_{n,+}\|.$$

The last expression tends to zero as $n \to \infty$, since the first terms in each sum is bounded (thanks to $\phi$) and the other one tends to zero. The proof is the same for the spin down component. This shows that $\mathfrak{p}^h$ is closed. □

Hence $\mathfrak{p}^h$ defines a unique self-adjoint operator $\mathfrak{P}_h$ via

$$\mathcal{D}(\mathfrak{P}_h) = \{\psi \in \mathcal{D}(\mathfrak{p}^h) \ : \ \mathfrak{p}^h(\psi,\cdot) \in \left(L_2(\mathbb{R}^2)\otimes\mathbb{C}^2\right)\} \tag{7}$$

and

$$(\mathfrak{P}_h\psi,\xi) = \mathfrak{p}^h(\psi,\xi), \quad \psi \in \mathcal{D}(\mathfrak{P}_h), \xi \in \mathcal{D}(\mathfrak{p}^h). \tag{8}$$

We call this operator $\mathfrak{P}_h$ the *Maximal Pauli operator*.

# 3 Properties of the Pauli operators

In this section we will compare some properties of the two Pauli operators $P_h$ and $\mathfrak{P}_h$ defined in the previous section. We start by showing that $\mathfrak{P}_h$ is gauge invariant while the non-reduced EV Pauli operator $\mathcal{P}_h$ is not.

## 3.1 Gauge transformations

Let $B(z) = B_0(z) + \sum_{j=1}^n 2\pi\alpha_j\delta_{z_j}$ be the same magnetic field as before and let $\hat B(z)$ be another magnetic field that differs from $B(z)$ only by some multiples of the delta functions, that is $\hat B(z) - B(z) = \sum_{j=1}^n 2\pi m_j \delta_{z_j}$, where $m_j$ are integers, not all zero. Then the corresponding scalar potentials $\hat h(z)$ and $h(z)$ differ only by the corresponding logarithms $\hat h(z) - h(z) = \sum_{j=1}^n m_j \log|z-z_j|$. With $\phi(z) = \sum_{j=1}^n m_j \arg(z-z_j)$ we get $\hat h(z) + i\phi(z) = h(z) + \sum_{j=1}^n m_j \log(z-z_j)$. This function is multivalued, however, since $m_j$ are integers, we have

$$\partial_{\bar z}\left(\hat h(z) + i\phi(z)\right) = \partial_{\bar z} h(z) + \sum_{j=1}^n m_j \partial_{\bar z}\log(z-z_j), \tag{9}$$

$$\tilde\partial_{\bar z}\left(\hat h(z) + i\phi(z)\right) = \tilde\partial_{\bar z} h(z), \text{ and} \tag{10}$$

$$e^{\hat h + i\phi} = e^h \prod_{j=1}^m (z-z_j)^{m_j}. \tag{11}$$

Let us check what happens with $\mathfrak{p}^h$ when we do gauge transforms. Let $\psi = (\psi_+, \psi_-)^t \in \mathcal{D}(\mathfrak{p}^h)$. We should check that $e^{-i\phi}\psi$ belongs to $\mathcal{D}(\mathfrak{p}^{\hat{h}})$, where $\phi(z) = \sum_{j=1}^n m_j \arg(z - z_j)$ is the harmonic conjugate to $\hat{h}(z) - h(z)$. We do this for $\mathfrak{p}_+^{\hat{h}}$. It is similar for $\mathfrak{p}_-^{\hat{h}}$. Since $\psi_+ \in \mathcal{D}(\mathfrak{p}_+^h)$ we know that $\tilde{\partial}_{\bar{z}}(\psi_+ e^{-h}) \in L_{1,\text{loc}}(\mathbb{R}^2 \setminus \Lambda)$. Let us check that $\tilde{\partial}_{\bar{z}}(\hat{\psi}_+ e^{-\hat{h}}) \in L_{1,\text{loc}}(\mathbb{R}^2 \setminus \Lambda)$. Again, by (11) we have

$$\begin{aligned}\tilde{\partial}_{\bar{z}}(\hat{\psi}_+ e^{-\hat{h}}) &= \tilde{\partial}_{\bar{z}}\left(\psi_+ e^{-h}\prod_{j=1}^n (z - z_j)^{-m_j}\right)\\ &= \tilde{\partial}_{\bar{z}}\left(\psi_+ e^{-h}\right)\prod_{j=1}^n (z - z_j)^{-m_j} + \psi_+ e^{-h}\tilde{\partial}_{\bar{z}}\left(\prod_{j=1}^n (z - z_j)^{-m_j}\right),\end{aligned}$$

which clearly belongs to $L_{1,\text{loc}}(\mathbb{R}^2 \setminus \Lambda)$.

Next we should check that $|\tilde{\partial}_{\bar{z}}(\hat{\psi}_+ e^{-\hat{h}})|e^{\hat{h}}$ belongs to $L_2(\mathbb{R}^2 \setminus \Lambda)$ under the assumption that $|\tilde{\partial}_{\bar{z}}(\psi_+ e^{-h})|e^h$ belongs to $L_2(\mathbb{R}^2 \setminus \Lambda)$. A calculation using (10) and (11) gives

$$\begin{aligned}\left|\tilde{\partial}_{\bar{z}}\left(e^{-\hat{h}}\hat{\psi}_+\right)\right| e^{\hat{h}} &= \left|\tilde{\partial}_{\bar{z}}\left(e^{-\hat{h}-i\phi}\psi_+(z)\right)\right| e^{\hat{h}}\\ &= \left|\left(\tilde{\partial}_{\bar{z}}(-h(z))\psi_+ + \tilde{\partial}_{\bar{z}}\psi_+(z)\right) e^{-h}\prod_{j=1}^n (z - z_j)^{-m_j}\right| e^h \prod_{j=1}^n |z - z_j|^{m_j} \qquad (12)\\ &= \left|\tilde{\partial}_{\bar{z}}\left(e^{-h}\psi_+\right)\right| e^h.\end{aligned}$$

Hence $\psi_+ \in \mathcal{D}(\mathfrak{p}_+^h)$ implies $\hat{\psi}_+ = e^{-i\phi}\psi_+ \in \mathcal{D}(\mathfrak{p}_+^{\hat{h}})$. In the same way it follows that $\psi_- \in \mathcal{D}(\mathfrak{p}_-^h)$ implies that $\hat{\psi}_- = e^{-i\phi}\psi_- \in \mathcal{D}(\mathfrak{p}_-^{\hat{h}})$. Thus $e^{-i\phi}\mathcal{D}(\mathfrak{p}^h) \subset \mathcal{D}(\mathfrak{p}^{\hat{h}})$. In the same way we can show that $e^{i\phi}\mathcal{D}(\mathfrak{p}^{\hat{h}}) \subset \mathcal{D}(\mathfrak{p}^h)$, and thus we can conclude that $e^{-i\phi}\mathcal{D}(\mathfrak{p}^h) = \mathcal{D}(\mathfrak{p}^{\hat{h}})$. From the calculation in (12) and a similar calculation for the spin-down component $\psi_-$ it also follows that

$$\begin{aligned}\mathfrak{p}^{\hat{h}}\left(e^{-i\phi}\psi, e^{-i\phi}\psi\right) &= 4\int \left|\tilde{\partial}_{\bar{z}}\left(e^{-\hat{h}-i\phi}\psi_+\right)\right|^2 e^{2\hat{h}} + \left|\tilde{\partial}_z\left(e^{\hat{h}-i\phi}\psi_-\right)\right|^2 e^{-2\hat{h}} d\lambda(z)\\ &= 4\int \left|\tilde{\partial}_{\bar{z}}\left(e^{-h}\psi_+\right)\right|^2 e^{2h} + \left|\tilde{\partial}_z\left(e^{h}\psi_-\right)\right|^2 e^{-2h} d\lambda(z)\\ &= \mathfrak{p}^h(\psi, \psi).\end{aligned}$$

Hence we can conclude that if $\psi \in \mathcal{D}(\mathfrak{P}_h)$ and $\xi \in \mathcal{D}(\mathfrak{p}^h)$ then $e^{-i\phi}\psi \in \mathcal{D}(\mathfrak{P}_{\hat{h}})$ and $e^{-i\phi}\xi \in \mathcal{D}(\mathfrak{p}^{\hat{h}})$. If we denote by $U_\phi$ the unitary operator of multiplication by $e^{i\phi}$, then we get

$$(\mathfrak{P}_h\psi, \xi) = \mathfrak{p}^h(\psi, \xi) = \mathfrak{p}^{\hat{h}}(U_\phi^*\psi, U_\phi^*\xi) = (\mathfrak{P}_{\hat{h}}U_\phi^*\psi, U_\phi^*\xi) = (U_\phi\mathfrak{P}_{\hat{h}}U_\phi^*\psi, \xi),$$

and hence $\mathfrak{P}_h$ and $\mathfrak{P}_{\hat{h}}$ are unitarily equivalent. We have proved the following proposition.

**Proposition 3.1.** *Let $B$ and $\hat{B}$ be two singular magnetic fields as in* (3)*, with difference $\hat{B} - B = \sum_{j=1}^{n} 2\pi m_j \delta_{z_j}$, where $m_j$ are integers, not all equal to zero. Then their corresponding Maximal Pauli operators defined by* (7) *and* (8) *are unitarily equivalent.*

To see that $\mathcal{P}_h$ is not gauge invariant it is enough to look at an example. Note that this operator is defined only for intensities belonging to the interval $(-1, 1)$. Let $n = 1$, $z_1 = 0$, $\alpha_1 = -1/2$ and $m_1 = 1$, so the two magnetic fields are $B(z) = B_0(z) - \pi\delta_0$ and $\hat{B}(z) = B_0(z) + \pi\delta_0$. The scalar potentials are given by $h(z) = h_0(z) - \frac{1}{2}\log|z|$ and $\hat{h}(z) = h_0(z) + \frac{1}{2}\log|z|$ respectively, where $h_0(z)$ is a smooth function with asymptotics $\Phi_0 \log|z|$ as $|z| \to \infty$. We should show that $\mathcal{D}(\pi^{\hat{h}})$ is not given by $e^{-i\phi}\mathcal{D}(\pi^h)$, where $\phi(z) = \arg(z)$. Then it follows that $\pi^h$ and $\pi^{\hat{h}}$ do not define unitarily equivalent operators.

Let $\psi_+ \in \mathcal{D}(\pi_+^h)$. This means, in particular, that $\partial_{\bar{z}}(\psi_+ e^{-h})$ belongs to $L_{1,\mathrm{loc}}(\mathbb{R}^2)$. Now let $\hat{\psi}_+ = e^{-i\phi}\psi_+$. Then, according to (11) we get

$$\partial_{\bar{z}}(\hat{\psi}_+ e^{-\hat{h}}) = \partial_{\bar{z}}(\psi_+ e^{-\hat{h}-i\phi}) = \partial_{\bar{z}}\left(\frac{\psi_+ e^{-h}}{z}\right) = \partial_{\bar{z}}(\psi_+ e^{-h})\frac{1}{z} + \psi_+ e^{-h}\pi\delta_0$$

which is not in $L_{1,\mathrm{loc}}(\mathbb{R}^2)$ since it is a distribution involving $\delta_0$ (for non-smooth $\psi_+$ it is not even well-defined). Thus we have $\mathcal{D}(\pi_+^{\hat{h}}) \neq e^{-i\phi}\mathcal{D}(\pi_+^h)$ and hence $\mathcal{D}(\pi^{\hat{h}}) \neq e^{-i\phi}\mathcal{D}(\pi^h)$ so $\pi^h$ and $\pi^{\hat{h}}$ are not defining unitarily equivalent operators $\mathcal{P}_h$ and $\mathcal{P}_{\hat{h}}$.

## 3.2 Zero modes

When studying spectral properties of the operator $\mathfrak{P}_h$ it is sufficient to consider AB intensities $\alpha_j$ that belong to the interval $(0, 1)$, since the operator is gauge invariant. See the discussion after the proof of Theorem 3.3 for more details about what happens when we do gauge transformations.

**Lemma 3.2.** *Let $c_j \in \mathbb{C}$ and $z_j \in \mathbb{C}$, $j = 1, \ldots, n$, where $z_j \neq z_i$ if $j \neq i$ and not all $c_j$ are equal to zero. Then*

$$\sum_{j=1}^{n} \frac{c_j}{z - z_j} \sim |z|^{-l-1}, \quad |z| \to \infty, \tag{13}$$

*where $l$ is the smallest nonnegative integer such that $\sum_{j=1}^{n} c_j z_j^l \neq 0$.*

*Proof.* If $|z|$ is large in comparison with all $|z_j|$ we have

$$\begin{aligned}\sum_{j=1}^{n} \frac{c_j}{z-z_j} &= \frac{1}{z}\sum_{j=1}^{n} \frac{c_j}{1-z_j/z} \\ &= \sum_{k=0}^{\infty}\left(\sum_{j=1}^{n} c_j z_j^k\right)\frac{1}{z^{k+1}} \\ &= \left(\sum_{j=1}^{n} c_j z_j^l\right)\frac{1}{z^{l+1}} + O(|z|^{-l-2})\end{aligned}$$

and thus $\sum_{j=1}^{n} \frac{c_j}{z-z_j} \sim |z|^{-l-1}$ as $|z| \to \infty$. □

**Remark.** We note that $l$ in Lemma 3.2 may never be greater than $n-1$. Indeed, if $l \geq n$ then we would have the linear system of equations $\{\sum_{j=1}^{n} c_j z_j^k = 0\}_{k=0}^{n-1}$. But the determinant of this system is $\prod_{i>j}(z_i - z_j) \neq 0$, and this would force all $c_j$ to be zero.

Note also that for $l < n$ we have a linear system of $l$ equations $\{\sum_{j=1}^{n} c_j z_j^k = 0\}_{k=0}^{l-1}$ with $n$ unknowns $c_j$, and that the $l \times n$ matrix $\{z_j^k\}$ has rank $l$. ■

**Theorem 3.3.** *Let $B(z)$ be the magnetic field* (3) *with all $\alpha_j \in (0,1)$, and let $\mathfrak{P}_h$ be the Pauli operator defined by* (7) *and* (8) *in Section 2 corresponding to $B(z)$. Then*

$$\dim\ker \mathfrak{P}_h = \{n - \Phi\} + \{\Phi\}\,, \tag{14}$$

*where $\Phi = \frac{1}{2\pi}\int B(z)d\lambda(z)$, and $\{x\}$ denotes the largest integer strictly less than $x$ if $x > 1$ and $0$ if $x \leq 1$. Using the notations $Q_\pm$ introduced in Section 2, we also have*

$$\dim\ker Q_+ = \{n - \Phi\} \quad \textit{and} \quad \dim\ker Q_- = \{\Phi\}\,. \tag{15}$$

*Proof.* We follow the reasoning originating in [AC79], with necessary modifications. First we note that $(\psi_+, \psi_-)^t$ belongs to $\ker \mathfrak{P}_h$ if and only if $\psi_+$ belongs to $\ker Q_+$ and $\psi_-$ belongs to $\ker Q_-$, which is equivalent to

$$\tilde{\partial}_{\bar z}\left(e^{-h}\psi_+\right) = 0 \quad \text{and} \quad \tilde{\partial}_z\left(e^{h}\psi_-\right) = 0. \tag{16}$$

This means exactly that $f_+(z) = e^{-h}\psi_+(z)$ is holomorphic and $f_-(z) = e^{h}\psi_-(z)$ is anti-holomorphic in $z \in \mathbb{R}^2 \setminus \Lambda$. It is the change in the domain where the functions are holomorphic that influences the result.

Let us start with the spin-up component $\psi_+$. The function $f_+$ is allowed to have poles of order at most one at $z_j$, $j = 1, \ldots, n$, and no others, since $e^h \sim |z - z_j|^{\alpha_j}$ as $z \to z_j$ and $\psi_+ = f_+ e^h$ should belong to $L_2(\mathbb{R}^2)$. Hence there exist constants $c_j$ such that the function $f_+(z) - \sum_{j=1}^{n} \frac{c_j}{z-z_j}$ is entire. From the asymptotics $e^h \sim |z|^{\Phi}$, $|z| \to \infty$, it follows that $f_+ - \sum_{j=1}^{n} \frac{c_j}{z-z_j}$ may only be a polynomial of degree at most $N = -\Phi - 2$. Hence

$$f_+(z) = \sum_{j=1}^{n} \frac{c_j}{z-z_j} + a_0 + a_1 z + \ldots a_N z^N,$$

where we let the polynomial part disappear if $N < 0$. Now, the asymptotics for $\psi_+$ is

$$\psi_+(z) \sim |z|^{-l-1+\Phi} + |z|^{N+\Phi}, \quad |z| \to \infty,$$

where $l$ is the smallest nonnegative integer such that $\sum_{j=1}^{n} c_j z_j^l \neq 0$. To have $\psi_+$ in $L_2(\mathbb{R}^2)$ we take $l$ to be the smallest nonnegative integer strictly greater than $\Phi$. Remember also from the remark after Lemma 3.2 that $l \leq n-1$. We get three cases. If $\Phi < -1$, then all complex numbers $c_j$ can be chosen freely, and a polynomial of degree $\{-\Phi\} - 1$ may be added which results $\{n - \Phi\}$ degrees of freedom. If $-1 \leq \Phi < n-1$ we have no contribution from the polynomial, and we have to choose the coefficients $c_j$ such that $\sum_{j=1}^{n} c_j z_j^k = 0$ for $k = 0, 1, \ldots, l-1$. The dimension of the null-space of the matrix $\{z_j^k\}$ is $n - l = \{n - \Phi\}$. If $\Phi \geq n-1$ then we must have all coefficients $c_j$ equal to zero and we get no contribution from the polynomial. Hence, in all three cases we have $\{n - \Phi\}$ spin-up zero modes.

Let us now focus on the spin-down component $\psi_-$. The function $f_-$ may not have any singularities, since the asymptotics of $e^{-h}$ is $|z - z_j|^{-\alpha_j}$ as $z \to z_j$. Hence $f_-$ must be entire. Moreover, $f_-$ may grow no faster than a polynomial of degree $\Phi - 1$ for $\psi_-$ to be in $L_2(\mathbb{R}^2)$. Thus $f_-$ has to be a polynomial of degree at most $\{\Phi\} - 1$, which gives us $\{\Phi\}$ spin-down zero modes. □

The number of zero modes for $\mathfrak{P}_h$ and $P_h$ are not the same. The Aharonov-Casher theorem for the EV Pauli operator (Theorem 3.1 in [EV02]) states for the field under consideration:

**Theorem 3.4.** *Let $B(z)$ be as in* (3) *and let $\hat{B}(z)$ be the unique magnetic field where all AB intensities $\alpha_j$ are reduced to the interval $[-1/2, 1/2)$, that is $\hat{B}(z) = B(z) + \sum_{j=1}^{n} 2\pi m_j \delta_{z_j}$, where $\alpha_j + m_j \in [-1/2, 1/2)$. Let $\Phi = \frac{1}{2\pi}\int \hat{B}(z) d\lambda(z)$. Then the dimension of the kernel of the EV Pauli operator $P_h$ is given by $\{|\Phi|\}$. All zero modes belong only to the spin-up or only to the spin-down component (depending on the sign of $\Phi$).*

Below we explain by some concrete examples how the spectral properties of the two Pauli operators $\mathfrak{P}_h$ and $P_h$ differ.

**Example 3.5.** Since $P_h$ is not gauge invariant we must not expect that the number of zero modes of $P_h$ is invariant under gauge transforms. To see that this property in fact can fail, let us look at the Pauli operators $P_{h_1}$ and $P_{h_2}$ induced by the magnetic fields

$$\begin{aligned} B_1(z) &= B_0(z) + \pi\delta_0, \text{ and} \\ B_2(z) &= B_0(z) - \pi\delta_0 \end{aligned}$$

respectively, where $B_0$ has compact support and $\Phi_0 = \frac{1}{2\pi}\int B_0(z) d\lambda(z) = \frac{3}{4}$. Then $B_2$ is reduced (that is, its AB intensity belong to $[-1/2, 1/2)$) but $B_1$ has to be reduced. Due to Theorem 3.4, the EV Pauli operators $P_{h_1}$ and $P_{h_2}$ corresponding to $B_1$ and $B_2$ have no zero modes. However, a direct computation for the non-reduced

EV Pauli operator $\mathcal{P}_{h_1}$ corresponding to $B_1$ shows that it actually has one zero mode. The situation is getting more interesting when we look at the operator that should correspond to $B_3 = B_0(z) + 3\pi\delta_0$. The AB intensity for $B_3$ is too strong so we have to make a reduction. In [EV02] the reduction is made to the interval $[-1/2, 1/2)$, and we have followed this convention, but physically there is nothing that says that this is the natural choice. Reducing the AB intensity of $B_3$ to $-1/2$ gives an operator with no zero modes and reducing it to $1/2$ gives an operator with one zero mode.

The Maximal Pauli operators $\mathfrak{P}_{h_1}$, $\mathfrak{P}_{h_2}$ and $\mathfrak{P}_{h_3}$ for these three magnetic fields all have one zero mode. This is easily seen by applying Theorem 3.3 to $\mathfrak{P}_{h_1}$ and then using the fact that the operators are unitarily equivalent.

However, more understanding is achieved when looking more closely at how the eigenfunctions for these three Maximal Pauli operators look like. Let $h_k$ be the scalar potential for $B_k$, $k = 1, 2, 3$. Then, as we have seen before $h_1(z) = h_0(z) + \frac{1}{2}\log|z|$, $h_2(z) = h_0(z) - \frac{1}{2}\log|z|$ and $h_3(z) = h_0(z) + \frac{3}{2}\log|z|$ where $h_0(z)$ corresponds to $B_0(z)$. Following the reasoning from the proof of Theorem 3.3 we see that the solution space to $\mathfrak{P}_{h_1}\psi = 0$ is spanned by $\psi = (0, e^{-h_1})^t$.

Next, we see what the solutions to $\mathfrak{P}_{h_2}\psi = 0$ look like. Now we have $\Phi_2 = \frac{1}{2\pi}\int B_2(z)d\lambda(z) = 1/4 > 0$. Let us begin with the spin-up component $\psi_+$. This time, the holomorphic $f_+ = e^{-h_2}\psi_+$ may not have any poles since then $\psi_+$ would not belong to $L_2(\mathbb{R}^2)$, and $f_+(z) = e^{-h_2}\psi_+(z) \to 0$ as $|z| \to \infty$, so we must have $f_+ \equiv 0$, and thus $\psi_+ \equiv 0$. For $\psi_-(z)$ to be in $L_2(\mathbb{R}^2)$ it is possible for $f_-$ to have a pole of order 1 at the origin. Hence there exist a constant $c$ such that $f_-(z) - c/\bar{z}$ is anti-holomorphic in the whole plane. The function $f_-(z) \to 0$ as $|z| \to \infty$ since the total intensity $\Phi_2 > 0$. This implies, by Liouville's theorem, that $f_-(z) \equiv c/\bar{z}$, so the solution space to $\mathfrak{P}_{h_2}\psi = 0$ is spanned by $\psi(z) = (0, e^{-h_2}/\bar{z})$.

Finally, let us determine the solutions to $\mathfrak{P}_{h_3}\psi = 0$. Now $\Phi_3 = \frac{1}{2\pi}\int B_3(z)d\lambda(z) = 9/4$. Consider the spin-up part $\psi_+$. For $\psi_+$ to be in $L_2(\mathbb{R}^2)$ our function $f_+$ may have a pole of order no more than two at the origin. As before, there exist constants $c_1$ and $c_2$ such that $f_+(z) - c_1/z - c_2/z^2$ is entire and its limit is zero as $|z| \to \infty$, and thus $f_+(z) \equiv c_1/z + c_2/z^2$. Again, both $c_1$ and $c_2$ must vanish for $\psi_+$ to be in $L_2(\mathbb{R}^2)$ (otherwise we would not stay in $L_2$ at infinity). Thus $\psi_+ \equiv 0$. On the other hand, the function $f_-$ may not have any poles (these poles would push $\psi_-$ out of $L_2(\mathbb{R}^2)$), so it is anti-holomorphic in the whole plane. It also may grow no faster than $|z|^{5/4}$ as $|z| \to \infty$, and thus $f_-$ has to be a first order polynomial in $\bar{z}$, that is $f_-(z) = c_0 + c_1\bar{z}$. Moreover for $\psi_-$ to be in $L_2(\mathbb{R}^2)$ it must have a zero of order 1 at the origin, and thus $f_-(z) = c_1\bar{z}$. We conclude that the solutions to $\mathfrak{P}_{h_3}\psi = 0$ are spanned by $(0, \bar{z}e^{-h_3})^t$.
■

A natural property one should expect of a reasonably defined Pauli operator is that its spectral properties are invariant under the reversing the direction of the magnetic field: $B \mapsto -B$. The corresponding operators are formally anti-unitary equivalent under the transformation $\psi \mapsto \bar{\psi}$ and interchanging of $\psi_+$ and $\psi_-$.

**Example 3.6.** The number of zero modes for $P_h$ is not invariant under $B(z) \mapsto -B(z)$, which we should not expect since the interval $[-1/2, 1/2)$ is not symmetric. We check this by showing that the number of zero modes are not the same. To see this, let

$B(z) = B_0(z) + \pi\delta_0$, where $B_0$ has compact support and $\Phi_0 = \frac{1}{2\pi}\int B_0(z)d\lambda(z) = \frac{3}{4}$. Then $B$ has to be reduced since the AB intensity at zero is $1/2 \notin [-1/2, 1/2)$. After reduction we get the magnetic field $\hat{B}(z) = B_0(z) - \pi\delta_0$, and we can apply Theorem 3.4. Let $\hat{\Phi} = \frac{1}{2\pi}\int \hat{B}d\lambda(z) = \frac{1}{4}$. Thus the number of zero modes for $P_h$ is 0. Now look at the Pauli operator $P_{-h}$ defined by the magnetic field $B_-(z) = -B(z) = -B_0(z) - \pi\delta_0$. This magnetic field is reduced and thus we can apply Theorem 3.4 directly. The total intensity is $\Phi_- = \frac{1}{2\pi}\int -B(z)d\lambda(z) = -\frac{5}{4}$, so the number of zero modes for $P_{-h}$ is 1. If $B$ has several AB fluxes then the difference in the number of zero modes of $P_h$ and $P_{-h}$ can be made arbitrarily large.

**Remark.** If there are only AB solenoids then the EV Pauli operator preserves the number of zero modes under $B \mapsto -B$, so the absence of spin-flip invariance can be noticed only in the presence of both AB and nice part.

**Example 3.7.** The number of zero modes for $\mathfrak{P}_h$ is invariant under $B(z) \mapsto -B(z)$. Since it is clear that the number of zero modes is invariant under $z \mapsto \bar{z}$ we look instead at how the Pauli operators change when we do $B(z) \mapsto \hat{B}(z) = -B(\bar{z})$. If we set $\zeta = \bar{z}$ we get $\hat{B}(\zeta) = -B(z)$ and the scalar potentials satisfy $\hat{h}(\zeta) = -h(z)$. Assume that $\psi = (\psi_+(z), \psi_-(z))^t \in \mathcal{D}(\mathfrak{p}^h)$. Then

$$\begin{aligned}
&\mathfrak{p}^{h(z)}\left(\begin{pmatrix}\psi_+(z)\\ \psi_-(z)\end{pmatrix}, \begin{pmatrix}\psi_+(z)\\ \psi_-(z)\end{pmatrix}\right) = \\
&= 4\int \left|\tilde{\partial}_{\bar{z}}(\psi_+(z)e^{-h(z)})\right|^2 e^{2h(z)} + \left|\tilde{\partial}_z(\psi_-(z)e^{h(z)}\right|^2 e^{-2h(z)} d\lambda(z) \\
&= 4\int \left|\tilde{\partial}_\zeta(\psi_+(\bar{\zeta})e^{\hat{h}(\zeta)}\right|^2 e^{-2\hat{h}(\zeta)} + \left|\tilde{\partial}_{\bar{\zeta}}(\psi_-(\bar{\zeta})e^{-\hat{h}(\zeta)}\right|^2 e^{2\hat{h}(\zeta)} d\lambda(\zeta) \\
&= \mathfrak{p}^{\hat{h}(\bar{z})}\left(\begin{pmatrix}\psi_-(z)\\ \psi_+(z)\end{pmatrix}, \begin{pmatrix}\psi_-(z)\\ \psi_+(z)\end{pmatrix}\right)
\end{aligned}$$

Hence we see that $(\psi_+, \psi_-)^t$ belongs to $\mathcal{D}(\mathfrak{P}_{h(z)})$ if and only if $(\psi_-, \psi_+)^t$ belongs to $\mathcal{D}(\mathfrak{P}_{\hat{h}(\bar{z})})$ and then $\mathfrak{P}_{\hat{h}(\bar{z})} = \mathfrak{P}_{h(z)}V$ where $V : L_2(\mathbb{R}^2) \otimes \mathbb{C}^2 \to L_2(\mathbb{R}^2) \otimes \mathbb{C}^2$ is the isometric operator given by $V((\psi_+, \psi_-)^t) = (\psi_-, \psi_+)^t$. Hence it is clear that $\mathfrak{P}_{\hat{h}(\bar{z})}$ and $\mathfrak{P}_{h(z)}$ have the same number of zero modes. ∎

**Example 3.8.** In the previous example we saw that changing the sign of the magnetic field results in unitarily equivalent Maximal Pauli operators. This implies that the number of zero modes for the Maximal Pauli operators corresponding to $B$ and $-B$ are the same. This, however, can be seen directly from the Aharonov-Casher formula in Theorem 3.3. To be able to apply the theorem to $-B = -B_0 - \sum_{j=1}^n 2\pi\alpha_j\delta_j$ we have to do gauge transformations, adding 1 to all the AB intensities, resulting in $\hat{B} = -B_0 + \sum_{j=1}^n 2\pi(1-\alpha_j)\delta_j$. Now according to Theorem 3.3 the number of zero modes of $\mathfrak{P}_{-h}$ is equal to

$$\dim\ker\mathfrak{P}_{-h} = \{\hat{\Phi}\} + \{n - \hat{\Phi}\} = \{n - \Phi\} + \{\Phi\} = \dim\ker\mathfrak{P}_h,$$

where we have used that $\hat{\Phi} = \frac{1}{2\pi}\int \hat{B}d\lambda(z) = n - \Phi$. ∎

# 4 Approximation by regular fields

We have mentioned that the different Pauli extensions depend on which boundary conditions are induced at the AB fluxes. Let us now make this more precise. Since the self-adjoint extension only depends on the boundary condition at the AB solenoids it is enough to study the case of one such solenoid and no smooth field. For simplicity, let the solenoid be located at the origin, with intensity $\alpha \in (0,1)$, that is, let the magnetic field be given by $B = 2\pi\alpha\delta_0$. We consider self-adjoint extensions of the Pauli operator $P$ that can be written in the form

$$P = \begin{pmatrix} P_+ & 0 \\ 0 & P_- \end{pmatrix} = \begin{pmatrix} Q_+^* Q_+ & 0 \\ 0 & Q_-^* Q_-, \end{pmatrix}$$

with some explicitly chosen closed operators $Q_\pm$. It is exactly such extensions $P$ that can be defined by the quadratic form (1). A function $\psi_+$ belongs to $\mathcal{D}(P_+)$ if and only if $\psi_+$ belongs to $\mathcal{D}(Q_+)$ and $Q_+\psi_+$ belongs to $\mathcal{D}(Q_+^*)$, and similarly for $P_-$.

With each self-adjoint extension $P_\pm = Q_\pm^* Q_\pm$ one can associate (see [DŠ98, EŠV02, GŠ04a, Tam03]) functionals $c_{-\alpha}^\pm$, $c_\alpha^\pm$, $c_{\alpha-1}^\pm$ and $c_{1-\alpha}^\pm$, by

$$\begin{aligned}
c_{-\alpha}^\pm(\psi_\pm) &= \lim_{r\to 0} r^\alpha \frac{1}{2\pi}\int_0^{2\pi} \psi_\pm d\theta, \\
c_{\alpha}^\pm(\psi_\pm) &= \lim_{r\to 0} r^{-\alpha} \left(\frac{1}{2\pi}\int_0^{2\pi} \psi_\pm d\theta - r^{-\alpha} c_{\alpha}^\pm(\psi_\pm)\right), \\
c_{\alpha-1}^\pm(\psi_\pm) &= \lim_{r\to 0} r^{1-\alpha} \frac{1}{2\pi}\int_0^{2\pi} \psi_\pm e^{i\theta} d\theta, \text{ and} \\
c_{1-\alpha}^\pm(\psi_\pm) &= \lim_{r\to 0} r^{\alpha-1} \left(\frac{1}{2\pi}\int_0^{2\pi} \psi_\pm e^{i\theta} d\theta - r^{\alpha-1} c_{1-\alpha}^\pm(\psi_\pm)\right).
\end{aligned}$$

such that $\psi_\pm \in \mathcal{D}(P_\pm)$ if and only if

$$\psi_\pm \sim c_{-\alpha}^\pm r^{-\alpha} + c_\alpha^\pm r^\alpha + c_{\alpha-1}^\pm r^{\alpha-1} e^{-i\theta} + c_{1-\alpha}^\pm r^{1-\alpha} e^{-i\theta} + O(r^\gamma) \tag{17}$$

as $r \to 0$, where $\gamma = \min(1+\alpha, 2-\alpha)$ and $z = re^{i\theta}$.

Any two nontrivial independent linear relations between these functionals determine a self-adjoint extension. In order that the operator be rotation-invariant, none of these relations may involve both $\alpha$ and $1-\alpha$ terms simultaneously. Accordingly, the parameters $\nu_0^\pm = c_\alpha^\pm / c_{-\alpha}^\pm$ and $\nu_1^\pm = c_{1-\alpha}^\pm / c_{\alpha-1}^\pm$, with possible values in $(-\infty, \infty]$, are introduced in [BP03], and it is proved that the operators $P_\pm$ can be approximated by operators with regularized magnetic fields in the norm resolvent sense if and only if $\nu_0^\pm = \infty$ and $\nu_1^\pm \in (-\infty, \infty]$ or if $\nu_0^\pm \in (-\infty, \infty]$ and $\nu_1^\pm = \infty$.

Before we check what parameters the Maximal and EV Pauli operators correspond to, let us in a few words discuss how the approximation in [BP03] works.

The vector magnetic potential $\mathbf{A}$ is approximated with the vector potential

$$\mathbf{A}_R(z) = \begin{cases} \mathbf{A}(z) & |z| > R \\ 0 & |z| < R \end{cases}$$

avoiding the singularity in the origin. The corresponding Hamiltonian $H_R$, formally defined as

$$H_R = (i\nabla + \mathbf{A}_R)^2 + \frac{\beta}{R}\delta(r - R),$$

where $\beta = \beta(\alpha, R)$, is studied. It is decomposed into angular momentum operators $h_{m,R}$. Only the operators $h_{m,R}$ where $m = 0$ or $m = 1$ have nontrivial deficiency space. Let $h^{\beta}_{m,R}$ be self-adjoint extensions of $h_{m,R}$ and let $H^{\beta}_R = \bigoplus_{m=-\infty}^{\infty} h^{\beta}_{m,R}$. Theorem 1 in [BP03] says (here we use the notation $\nu_0$ and $\nu_1$ for what could be $\nu_0^{\pm}$ and $\nu_1^{\pm}$ respectively):

(I) If

$$\frac{\beta(\alpha, R) + \alpha}{R^{2\alpha}} \to 2\alpha\nu_0$$

then $H^{\beta}_R$ converges in the norm resolvent sense to one component of the Pauli Hamiltonian corresponding to $\nu_1 = \infty$.

(II) If

$$\frac{\beta(\alpha, R) - \alpha + 2}{R^{2(1-\alpha)}} \to 2(1 - \alpha)\nu_1$$

then $H^{\beta}_R$ converges in the norm resolvent sense to one component of the Pauli Hamiltonian corresponding to $\nu_0 = \infty$.

We are now going to check what parameters the Maximal and EV Pauli operators corresponds to. Generally, for the function $\psi_+$ to be in $\mathcal{D}(P_+)$, it must belong to $\mathcal{D}(Q_+)$ and $Q_+\psi_+$ must belong to $\mathcal{D}(Q_+^*)$. We will find out what is required for a function $g$ to be in $\mathcal{D}(Q_+^*)$. Take any $\phi_+ \in \mathcal{D}(Q_+)$, then the integration by parts on the domain $\varepsilon < |z|$ gives

$$\begin{aligned}
\langle g, Q_+\phi_+\rangle &= \lim_{\varepsilon\to 0}\int_{|z|>\varepsilon} g(z)\overline{-2i\frac{\partial}{\partial \bar{z}}(e^{-h}\phi_+(z))e^{h}d\lambda(z)} \\
&= -\lim_{\varepsilon\to 0}\int_{|z|>\varepsilon} -2i\frac{\partial}{\partial z}(g(z)e^{h})e^{-h}\overline{\phi_+(z)}d\lambda(z) \\
&\quad \lim_{\varepsilon\to 0}\varepsilon\int_0^{2\pi} g(\varepsilon e^{i\theta})\overline{\phi_+(\varepsilon e^{i\theta})}e^{-i\theta}d\theta \\
&= \langle Q_-g, \phi_+\rangle + \lim_{\varepsilon\to 0}\frac{\varepsilon}{2}\int_0^{2\pi} g(\varepsilon e^{i\theta})\overline{\phi_+(\varepsilon e^{i\theta})}e^{-i\theta}d\theta
\end{aligned}$$

Hence, for $g$ to belong to $\mathcal{D}(Q_+^*)$ it is necessary and sufficient that

$$\lim_{\varepsilon\to 0}\varepsilon\int_0^{2\pi} g(\varepsilon e^{i\theta})\overline{\phi_+(\varepsilon e^{i\theta})}e^{-i\theta}d\theta = 0$$

for all $\phi_+ \in \mathcal{D}(p_+)$, and thus for $Q_+\psi_+$ to belong to $\mathcal{D}(Q_+^*)$ it is necessary and sufficient that

$$\lim_{\varepsilon\to 0}\varepsilon\int_0^{2\pi}\left(\frac{\partial}{\partial\bar z}(e^{-h}\psi_+)e^h\right)\Big|_{z=\varepsilon e^{i\theta}}\overline{\phi_+(\varepsilon e^{i\theta})}e^{-i\theta}d\theta = 0$$

for all $\phi_+ \in \mathcal{D}(p_+)$. We know that $\psi_+$ has asymptotics $\psi_+ \sim c^+_{-\alpha}r^{-\alpha} + c^+_\alpha r^\alpha + c^+_{\alpha-1}r^{\alpha-1}e^{-i\theta} + c^+_{1-\alpha}r^{1-\alpha}e^{-i\theta} + O(r^\gamma)$ and that $\frac{\partial}{\partial\bar z} = \frac{e^{i\theta}}{2}\left(\frac{\partial}{\partial r} + \frac{i}{r}\frac{\partial}{\partial\theta}\right)$ in polar coordinates. A calculation gives

$$\varepsilon\frac{\partial}{\partial\bar z}(e^{-h}\psi_+)e^h e^{-i\theta}\Big|_{z=\varepsilon e^{i\theta}} \sim -2\alpha c^+_{-\alpha}\varepsilon^{-\alpha} + 2(1-\alpha)c^+_{1-\alpha}\varepsilon^{1-\alpha}e^{-i\theta} + O(r^\gamma),$$

hence we must have

$$\lim_{\varepsilon\to 0}\int_0^{2\pi}\left(-2\alpha c^+_{-\alpha}\varepsilon^{-\alpha} + 2(1-\alpha)c^+_{1-\alpha}\varepsilon^{1-\alpha}e^{-i\theta}\right)\overline{\phi_+(\varepsilon e^{i\theta})}d\theta = 0 \tag{18}$$

for all $\phi_+ \in \mathcal{D}(p_+)$. A similar calculation for the spin-down component yields

$$\lim_{\varepsilon\to 0}\int_0^{2\pi}\left(2\alpha c^-_{\alpha}\varepsilon^{\alpha} + 2(\alpha-1)c^-_{\alpha-1}\varepsilon^{\alpha-1}e^{i\theta}\right)\overline{\phi_-(\varepsilon e^{i\theta})}d\theta = 0. \tag{19}$$

To calculate what parameters $\nu_0^\pm$ and $\nu_1^\pm$ the Maximal and EV Pauli extensions correspond to, it is enough to study the asymptotics of the functions in the form core.

Let us first consider the Maximal Pauli extension. Functions on the form $(\phi_0^+ c/z)e^h$ constitute a form core for $\mathfrak{p}_+^h$, where $\phi_0$ is smooth. Hence there are elements in $\mathcal{D}(\mathfrak{p}_+^h)$ that asymptotically behave as $r^\alpha$ and also elements with asymptotics $r^{\alpha-1}e^{-i\theta}$. According to (18) this means that $c^+_{-\alpha}$ and $c^+_{1-\alpha}$ must be zero. Similarly, the elements that behave like $r^{-\alpha}$ and elements that behave like $r^{1-\alpha}e^{i\theta}$ constitute a form core for $\mathfrak{p}_-^h$, which by (19) forces $c^-_\alpha$ and $c^-_{\alpha-1}$ to be zero. The parameters $\nu_0^\pm$ and $\nu_1^\pm$ are given by $\nu_0^+ = c^+_\alpha/c^+_{-\alpha} = \infty$, $\nu_1^+ = c^+_{1-\alpha}/c^+_{\alpha-1} = 0$, $\nu_0^- = c^-_\alpha/c^-_{-\alpha} = 0$ and $\nu_1^- = c^-_{1-\alpha}/c^-_{\alpha-1} = \infty$. We see that the spin-up component can be approximated as in (II), while the spin-down component can be approximated as in (I).

Let us now consider the EV Pauli extension, and study the case when $\alpha \in (0, 1/2)$. The case $\alpha < 0$ follows in a a similar way. A form core for $\pi_+^h$ is given by $e^h\phi_0$ where $\phi_0$ is smooth, see [EV02]. These functions have asymptotic behavior $r^\alpha$. From (18) follows that $c^+_{-\alpha}$ must vanish. However, $\psi_+$ belonging to $\mathcal{D}(Q_+)$ must also belong to $\mathcal{D}(\pi_+^h)$ and since the functions in the form core for $\pi_+^h$ behave as $r^\alpha$ or nicer, we see that the term $c^+_{\alpha-1}r^{\alpha-1}e^{-i\theta}$ gets too singular to be in $\mathcal{D}(Q_+)$ if $c^+_{\alpha-1} \neq 0$, and hence $c^+_{\alpha-1}$ must be zero.

Similarly, a form core for $\pi_-^h$ is given by $e^{-h}\phi_0$, with $\phi_0$ smooth. Functions in this form core have asymptotic behavior $r^{-\alpha}$ or $r^{-\alpha+1}e^{i\theta}$ which forces $c_\alpha^-$ and $c_{\alpha-1}^-$ to be zero.

Hence the parameters $\nu_0^\pm$ and $\nu_1^\pm$ are given by $\nu_0^+ = c_\alpha^+/c_{-\alpha}^+ = \infty$, $\nu_1^+ = c_{1-\alpha}^+/c_{\alpha-1}^+ = \infty$, $\nu_0^- = c_\alpha^-/c_{-\alpha}^- = 0$ and $\nu_1^- = c_{1-\alpha}^-/c_{\alpha-1}^- = \infty$.

We conclude that the spin-up part of the EV Pauli operator can be approximated in either of the ways (I) or (II), while the spin-down part can be approximated in way (I).

**Remark.** From the calculations above it follows that the EV Pauli operator can be approximated as a Pauli Hamiltonian in the sense of [BP03], while the Maximal Pauli operator cannot be approximated as a Pauli Hamiltonian, since the spin-up and spin-down components are approximated in different ways.

Since AB is defined up to a singular gauge transformation and regular fields can not be transformed in this way it is unclear which additional physical requirements or principles can decide on which way of approximation is the most physically reasonable.

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# Paper II

# On the Dirac and Pauli operators with several Aharonov-Bohm solenoids

Mikael Persson

**Abstract**

We study the self-adjoint Pauli operators that can be realized as the square of a self-adjoint Dirac operator and correspond to a magnetic field consisting of a finite number of Aharonov-Bohm solenoids and a regular part, and prove an Aharonov-Casher type formula for the number of zero-modes for these operators. We also see that essentially only one of the Pauli operators are spin-flip invariant, and this operator does not have any zero-modes.

## 1 Introduction

Two-dimensional spin-$\frac{1}{2}$ non-relativistic quantum systems with magnetic fields are described by the Pauli operator. For non-singular magnetic fields the Pauli operator is usually defined as the square of the Dirac operator. However, for more singular magnetic fields, such as the delta field an Aharonov-Bohm (AB) solenoid generates (see [AB59]), the situation is more complicated, because then there are many self-adjoint extensions of both the Dirac and the Pauli operator, originally defined on smooth functions with compact support not touching the singular points. The different extensions describe different physics, and it is not clear which extensions describe the real physical situation.

One of the ways to control the properties of self-adjoint extensions consists in the analysis of the dimension of the null-space of the operator. The dimension of the kernel of the Pauli operator is usually given by the Aharonov-Casher formula (see [AC79]). It has been proved in different settings, see [CFKS87, GG02, Mil82]. Recently it was also proved for one of the extensions for a very singular magnetic field (containing the case with AB solenoids) in [EV02]. Another extension was introduced in [GG02], and in [Per05] an Aharonov-Casher type formula was proved for that extension.

The Pauli operators we are going to study in this article are the ones defined as the square of some self-adjoint Dirac operator. The goal of this article is to study this family of self-adjoint Pauli operators corresponding to a magnetic field consisting of finitely many AB solenoids and a smooth field with compact support, and to find an Aharonov-Casher type formula for these Pauli operators.

A natural property to expect from the Pauli operator is that it transforms in an (anti)-unitarily way when the sign of the magnetic field is changed and the spin-up and spin-down components are switched. This property is usually called spin-flip invariance, and we want to answer the question of which Pauli operators defined satisfy it. To be able to answer these questions we have to do some work with the self-adjoint Dirac operators corresponding to the same magnetic field.

The Dirac operator with a strongly singular magnetic field has been studied before in [dSG89, Ara93, HO01, AH05, Tam03]. In [HO01] a formula for the dimension of the kernel of the Dirac operator was proved for two different asymmetric self-adjoint extensions, and it was demonstrated that, in fact, this dimension may differ for self-adjoint realizations, each of them seeming to be quite natural. These extensions are closely related to the ones introduced in [Ara93]. In both these articles the magnetic field is the same as the one we consider (the one in [Ara93] does not have the regular part), with the addition of even more singular terms containing derivatives of the delta distributions. Using gauge transformations one can disregard these derivatives.

In [Tam03] it was proved that one of these asymmetric Dirac extensions can be approximated by operators corresponding to more regular magnetic fields. However, this extension, as we show, lacks the property of being spin-flip invariant.

In Section 2 we study the Dirac operator. In order to be able to treat the general case, we need first to repeat in details the description of all self-adjoint extensions corresponding to only one AB solenoid. After that we define the Dirac operator corresponding to the magnetic field consisting of several AB solenoids and a regular part with compact support. We use the same method of gluing together different self-adjoint extensions as in [AR04], and prove that the obtained operator is self-adjoint. After that we check which extensions are spin-flip invariant and finally we prove a formula for the dimension of the kernel of the Dirac extensions.

In Section 3 we look at the Pauli operators that are the square of some self-adjoint Dirac operator defined in Section 2. We show exactly which Pauli extensions are obtained in this way, in the terms of the asymptotics of the functions in the domain of the Pauli operator at the points where the singular AB solenoids are located. We also find an Aharonov-Casher type formula for these Pauli operators. It turns out that there are only two of them that have zero-modes. These two extensions are very asymmetric though, admitting singularities in one component only, which looks rather non-physical. All the other extensions have singularities in both the spin-up and spin-down components, and they are coupled.

It turns out that the Pauli operator studied in [EV02] is a mixture of these two asymmetric extensions, admitting different physical situations at different AB solenoids. This has to do with the normalization of the AB intensities. In that article the AB intensities were chosen to be normalized to the interval $[-1/2, 1/2)$. If any of the intervals $(0, 1)$ or $(-1, 0)$ would have been chosen instead, then the Pauli operator would have been one of the asymmetric ones studied in this article. Both these asymmetric operators have the advantage that they describe the same physics at all AB solenoids. In the end of the article we present a discussion of the properties of the self-adjoint Pauli extensions with respect to different ways of normalization of AB intensities.

## 2 The Dirac operator

The goal in this section is to describe the self-adjoint Dirac operators corresponding to a magnetic field consisting of several (but finitely many) AB solenoids together with a smooth field, and to find an Aharonov-Casher type formula for the dimension of the kernel of these self-adjoint operators. Let us introduce some notations that will be used throughout the article. We identify a point $(x_1, x_2)$ in $\mathbb{R}^2$ with the complex number $z = x_1 + ix_2$, and we will often write $z$ in polar coordinates, $z = re^{i\theta}$. Sometimes it will be convenient to use the polar coordinates $r_j e^{i\theta_j}$ with $z_j$ as the origin. The magnetic field will consist of a regular part $B_0 \in C_0^1(\mathbb{R}^2)$ and a singular part consisting of $n$ AB solenoids located at the points $\Lambda = \{z_j\}_1^n$, so that the magnetic field $B$ will have the form

$$B(z) = B_0(z) + \sum_{j=1}^{n} 2\pi\alpha_j \delta_{z_j}. \tag{2.1}$$

Owing to gauge equivalence (see [Tam03]) we can assume that all the AB intensities $\alpha_j$ (fluxes divided by $2\pi$) belong to the interval $(0, 1)$. All derivatives will be considered in the distribution space $\mathcal{D}'(\mathbb{C} \setminus \Lambda)$. We will denote by $h$ a magnetic scalar potential satisfying $\Delta h = B$. The magnetic scalar potential is uniquely defined modulo addition of a harmonic function. We will use the scalar potential

$$h(z) = \frac{1}{2\pi} \int B_0(\zeta) \log|z - \zeta| d\lambda(\zeta) + \sum_{j=1}^{n} \alpha_j \log|z - z_j| = h_0(z) + \sum_{j=1}^{n} h_j(z), \tag{2.2}$$

where $d\lambda$ is the Lebesgue measure. The *actions* $q_\pm^h$, which will be used to describe how the Dirac operator acts, are defined by

$$q_+^h u = -2i \tfrac{\partial}{\partial \bar{z}} \left(ue^{-h}\right) e^h$$

and

$$q_-^h u = -2i \tfrac{\partial}{\partial z} \left(ue^{h}\right) e^{-h}.$$

These actions $q_+^h$ and $q_-^h$ are usually called the spin-up and spin-down actions, respectively. The Dirac action is given by

$$\mathfrak{d}^h = \begin{pmatrix} 0 & q_-^h \\ q_+^h & 0 \end{pmatrix}.$$

To be able to describe the self-adjoint Dirac operators with several AB solenoids we first study the self-adjoint extensions of the Dirac operator with one AB solenoid, originally defined on smooth functions with compact support not touching the singular point.

### 2.1 The Dirac operator with one AB solenoid

The case of one AB solenoid has been studied before (see [dSG89, Tam03]), and we just sketch the way it was done since we need the detailed information about these extensions for our further analysis. We let the AB solenoid have intensity $\alpha = \alpha_1 \in (0,1)$ and be located at the origin. We will describe all self-adjoint extensions of the Dirac operator originally defined on $(C_0^\infty(\mathbb{R}^2 \setminus \{0\}))^2$. Let us drop the superscript $h$ of $q_\pm^h$ and $\mathfrak{d}^h$ in this subsection. The actions $q_\pm$ can in this case be written as

$$q_+ = -2i\left(\frac{\partial}{\partial \bar{z}} - \frac{\alpha}{2\bar{z}}\right) = -ie^{i\theta}\left(\frac{\partial}{\partial r} + \frac{i}{r}\frac{\partial}{\partial \theta} - \frac{\alpha}{r}\right) \tag{2.3}$$

and

$$q_- = -2i\left(\frac{\partial}{\partial z} + \frac{\alpha}{2z}\right) = -ie^{-i\theta}\left(\frac{\partial}{\partial r} - \frac{i}{r}\frac{\partial}{\partial \theta} + \frac{\alpha}{r}\right). \tag{2.4}$$

First we define the minimal Dirac operator on $(C_0^\infty(\mathbb{R}^2 \setminus \{0\}))^2$. This operator will not be essentially self-adjoint, that is, its graph norm closure with respect to the graph norm $\left(||\psi||^2 + ||\mathfrak{d}\psi||^2\right)^{1/2}$ will not be self-adjoint. However, the closure can be used to describe all self-adjoint extensions by the method of Krein and von Neumann described in [AG93].

The minimal Dirac operator $\mathfrak{D}_{\min}$, obviously symmetric, is defined by

$$\begin{aligned} \mathcal{D}(\mathfrak{D}_{\min}) &= (C_0^\infty(\mathbb{R}^2 \setminus \{0\}))^2, \\ \mathfrak{D}_{\min}\psi &= \mathfrak{d}\psi, \quad \text{for } \psi \in \mathcal{D}(\mathfrak{D}_{\min}). \end{aligned}$$

Then $\overline{\mathfrak{D}}_{\min}$ is given by

$$\mathcal{D}(\overline{\mathfrak{D}}_{\min}) = \left\{\psi \in (H^1)^2 : \lim_{z\to 0} \psi_\pm(z) = 0\right\}, \tag{2.5}$$

$$\overline{\mathfrak{D}}_{\min}\psi = \mathfrak{d}\psi, \quad \text{for } \psi \in \mathcal{D}(\overline{\mathfrak{D}}_{\min}). \tag{2.6}$$

To be able to use the method of Krein–von Neumann, we have to describe the adjoint $\overline{\mathfrak{D}}^*_{\min}$ of the operator $\overline{\mathfrak{D}}_{\min}$. This is done by an integration by parts, and the result is

$$\begin{aligned} \mathcal{D}(\overline{\mathfrak{D}}^*_{\min}) &= \left\{\psi \in (L_2(\mathbb{R}^2))^2 : ||\mathfrak{d}\psi|| < \infty\right\}, \\ \overline{\mathfrak{D}}^*_{\min}\psi &= \mathfrak{d}\psi, \quad \text{for } \psi \in \mathcal{D}(\overline{\mathfrak{D}}^*_{\min}). \end{aligned}$$

Next we want to describe the deficiency spaces $\chi_\pm$ defined by $\chi_\pm = \ker(\overline{\mathfrak{D}}^*_{\min} \mp i)$. This reduces to find solutions in $L_2$ to the system of equations

$$\begin{cases} q_-q_+\psi_+ + \psi_+ = 0, \\ \qquad\qquad \psi_- = \mp i q_+\psi_+. \end{cases} \tag{2.7}$$

Here, the $\mp$-sign refer to the different deficiency spaces $\chi_\pm$. Let us start by solving the first equation in (2.7). Since $\frac{e^{im\theta}}{\sqrt{2\pi}}$, $m \in \mathbb{Z}$ is complete in $L_2(S^1)$ we look for solutions $\psi_+$ on the form

$$\psi_+(re^{i\theta}) = \sum_{m=-\infty}^{\infty} u_m(r)\frac{e^{im\theta}}{\sqrt{2\pi}}.$$

Then the equation $q_-q_+\psi_+ + \psi_+ = 0$ is equivalent to the infinite system of ordinary differential equations:

$$-r^2u_m''(r) - ru_m'(r) + (r^2 + (m+\alpha)^2)u_m(r) = 0, \quad m \in \mathbb{Z}.$$

Thus $u_m(r)$ is given by

$$u_m(r) = A_m I_{|m+\alpha|}(r) + B_m K_{|m+\alpha|}(r)$$

where $I_\nu$ and $K_\nu$ are the usual modified Bessel functions of order $\nu$. The asymptotics of these Bessel functions at zero and at infinity (see [Wat95]) forces $A_m$ to be zero for all $m$ and $B_m$ to be zero if $m \notin \{-1, 0\}$. If $m$ is $-1$ or $0$ then $K_{|m+\alpha|}(r)$ belongs to $L_2(rdr)$. Thus we have solutions on the form

$$\psi_+(z) = B_{-1}K_{1-\alpha}(r)e^{-i\theta} + B_0K_\alpha(r).$$

We can now calculate $\psi_-$ from the second equation in (2.7) to get

$$\begin{aligned}\psi_-(z) &= \mp iq_+\psi_+(z) \qquad (2.8)\\ &= \mp e^{i\theta}\left(\frac{\partial}{\partial r} + \frac{i}{r}\frac{\partial}{\partial\theta} - \frac{\alpha}{r}\right)\left(B_{-1}K_{1-\alpha}(r)e^{-i\theta} + B_0K_\alpha(r)\right)\\ &= \pm B_{-1}K_\alpha(r) \pm B_0e^{-i\theta}K_{1+\alpha}(r),\end{aligned}$$

and thus $B_0 = 0$ since $K_{1+\alpha}(r)$ does not belong to $L_2(rdr)$.

We see that $\overline{\mathfrak{D}}_{\min}$ has deficiency index $(1,1)$, and the deficiency spaces $\chi_\pm$ are one-dimensional and are spanned by

$$\xi_\pm(re^{i\theta}) = \begin{pmatrix} K_{1-\alpha}(r)e^{-i\theta} \\ \pm K_\alpha(r)\end{pmatrix}. \qquad (2.9)$$

Denote by $U$ any unitary operator from $\chi_+$ to $\chi_-$. Then $U$ takes $\xi_+$ to $e^{i\tau}\xi_-$ for some $\tau \in [0, 2\pi)$. All self-adjoint extensions can be parameterized by $\tau$ as

$$\mathcal{D}(\mathfrak{D}^\tau) = \left\{\psi \in (L_2(\mathbb{R}^2))^2 : \ \psi = \psi_0 + c(\xi_+ + e^{i\tau}\xi_-), \psi_0 \in \mathcal{D}(\overline{\mathfrak{D}}_{\min}), \ c \in \mathbb{C}\right\} \qquad (2.10)$$

$$\mathfrak{D}^\tau\psi = \mathfrak{d}\psi_0 + ic(\xi_+ - e^{i\tau}\xi_-), \quad \text{for } \psi \in \mathcal{D}(\mathfrak{D}^\tau). \qquad (2.11)$$

It is also possible to describe the self-adjoint extensions by studying the asymptotic behavior of the functions in the domain at the origin. To see this, let us define the linear functionals $c^\pm_{-\alpha}$ and $c^\pm_{\alpha-1}$ on $\mathcal{D}(\mathfrak{D}^\tau)$ as

$$c^\pm_{-\alpha}(\psi_\pm) = \lim_{r\to 0} r^\alpha \frac{1}{2\pi}\int_0^{2\pi} \psi_\pm(re^{i\theta})d\theta, \text{ and} \qquad (2.12)$$

$$c^\pm_{\alpha-1}(\psi_\pm) = \lim_{r\to 0} r^{1-\alpha}\frac{1}{2\pi}\int_0^{2\pi}\psi_\pm(re^{i\theta})e^{i\theta}d\theta. \qquad (2.13)$$

For $\psi = \psi_0 + c(\xi_+ + e^{i\tau}\xi_-)$ in $\mathcal{D}(\mathfrak{D}^\tau)$, where $\psi_0 \in \mathcal{D}(\overline{\mathfrak{D}}_{\min})$, applying these functionals gives no contribution from $\psi_0$ since the limit of functions in $\mathcal{D}(\overline{\mathfrak{D}}_{\min})$ tends to zero at the origin. Let us introduce the notation $\sigma(\alpha) = \Gamma(\alpha)2^\alpha$. Using the asymptotics for the Bessel functions we get

$$\begin{aligned} c^+_{-\alpha}(\psi_+) &= 0 \\ c^-_{-\alpha}(\psi_-) &= \frac{c}{2}(1 - e^{i\tau})\sigma(\alpha) \\ c^+_{\alpha-1}(\psi_+) &= \frac{c}{2}(1 + e^{i\tau})\sigma(1 - \alpha) \\ c^-_{\alpha-1}(\psi_-) &= 0 \end{aligned} \tag{2.14}$$

for such functions $\psi \in \mathcal{D}(\mathfrak{D}^\tau)$. Here $c$ is the same constant as in (2.10). An equivalent description of all self-adjoint Dirac extensions is

$$\mathcal{D}(\mathfrak{D}^\tau) = \Big\{ \psi \in (L_2(\mathbb{R}^2))^2 : \ \mathfrak{d}\psi \in (L_2(\mathbb{R}^2))^2; \tag{2.15}$$

$$\frac{c^+_{\alpha-1}(\psi_+)}{c^-_{-\alpha}(\psi_-)} = i\cot(\tau/2)\frac{\sigma(1-\alpha)}{\sigma(\alpha)}, \text{ and}$$

$$c^+_{-\alpha}(\psi_+) = c^-_{\alpha-1}(\psi_-) = 0 \Big\},$$

$$\mathfrak{D}^\tau \psi = \mathfrak{d}\psi, \quad \text{for } \psi \in \mathcal{D}(\mathfrak{D}^\tau). \tag{2.16}$$

## 2.2 The Dirac operator with several AB solenoids together with a regular field

In this subsection we are going to study the Dirac operator for a magnetic field consisting of a finite number of AB solenoids together with a regular background field. We will use the same method as in [AR04] to glue together the different self-adjoint Dirac operators corresponding to only one AB solenoid and the self-adjoint Dirac operator corresponding to the regular magnetic field.

We start by defining the Dirac operator with two AB solenoids together with a smooth field. The general case does not give any extra difficulties. Let the magnetic field $B$ consist of a smooth field $B_0$ with compact support and two AB solenoids located at $z_1$ and $z_2$ with intensities $\alpha_1$ and $\alpha_2$,

$$B(z) = B_0(z) + 2\pi\alpha_1\delta_{z_1} + 2\pi\alpha_2\delta_{z_2}. \tag{2.17}$$

In this case our scalar potential $h$ can be written as

$$h(z) = h_0(z) + h_1(z) + h_2(z) = \frac{1}{2\pi}(\log|\cdot| * B_0)(z) + \alpha_1 \log|z - z_1| + \alpha_2 \log|z - z_2|.$$

From the previous section we have self-adjoint Dirac operators $\mathfrak{D}^{\tau_1,h_1}$ and $\mathfrak{D}^{\tau_2,h_2}$ corresponding to each of the AB solenoids separately. Let us drop the parameters $\tau_1$ and $\tau_2$ from the superscripts. So, for example, when we write $\mathfrak{D}^{h_1}$ we mean some self-adjoint extension with one AB solenoid located at $z_1$.

Let $\varphi_j \in C_0^\infty(\mathbb{R}^2)$, $j = 1, 2$, be equal to 1 in a neighborhood of $z_j$ and have small support not touching a neighborhood of $z_k$, $k \neq j$ and $0 \leq \varphi_j \leq 1$. Let $\varphi_0 = 1 - \varphi_1 - \varphi_2$. We denote by $E_{jk}$ the set $\operatorname{supp}\varphi_j \cap \operatorname{supp}\varphi_k$.

Let us introduce the multiplication operators $V^{h_j}$ as

$$V^{h_j} = 2i \begin{pmatrix} 0 & -\frac{\partial}{\partial z} h_j \\ \frac{\partial}{\partial \bar{z}} h_j & 0 \end{pmatrix}.$$

Note that $V^{h_0}$ is bounded from $(L_2)^2$ to $(L_2)^2$. For $j \neq 0$ we will be sure to apply the operators $V^{h_j}$ only on functions being zero in a neighborhood of the singular points $z_j$.

**Definition 2.1.** The Dirac operator $\mathfrak{D}^h$ corresponding to the magnetic field $B$ in (2.17) is defined as

$$\mathcal{D}(\mathfrak{D}^h) = \{\psi \in (L_2(\mathbb{R}^2))^2 : \ \varphi_j \psi \in \mathcal{D}(\mathfrak{D}^{h_j}), \ j = 0, 1, 2\}$$

and

$$\begin{aligned} \mathfrak{D}^h \psi =& (\mathfrak{D}^{h_0} + V^{h_1} + V^{h_2})(\varphi_0 \psi) \\ &+ (\mathfrak{D}^{h_1} + V^{h_0} + V^{h_2})(\varphi_1 \psi) \\ &+ (\mathfrak{D}^{h_2} + V^{h_0} + V^{h_1})(\varphi_2 \psi) \end{aligned}$$

for $\psi \in \mathcal{D}(\mathfrak{D}^h)$. □

The definition is independent of the choice of the partition of unity $1 = \varphi_0 + \varphi_1 + \varphi_2$. Indeed, let $1 = \tilde{\varphi}_0 + \tilde{\varphi}_1 + \tilde{\varphi}_2$ be another choice. For $\psi \in \mathcal{D}(\mathfrak{D}^h)$ we have

$$\begin{aligned} \mathfrak{D}^h \psi &= (\mathfrak{D}^{h_0} + V^{h_1} + V^{h_2})(\varphi_0 \psi) + (\mathfrak{D}^{h_1} + V^{h_0} + V^{h_2})(\varphi_1 \psi) + (\mathfrak{D}^{h_2} + V^{h_0} + V^{h_1})(\varphi_2 \psi) \\ &= (\mathfrak{D}^{h_0} + V^{h_1} + V^{h_2})(\varphi_0(\tilde{\varphi}_0 + \tilde{\varphi}_1 + \tilde{\varphi}_2)\psi) \\ &\quad + (\mathfrak{D}^{h_1} + V^{h_0} + V^{h_2})(\varphi_1(\tilde{\varphi}_0 + \tilde{\varphi}_1 + \tilde{\varphi}_2)\psi) \\ &\quad + (\mathfrak{D}^{h_2} + V^{h_0} + V^{h_1})(\varphi_2(\tilde{\varphi}_0 + \tilde{\varphi}_1 + \tilde{\varphi}_2)\psi) \\ &= (\mathfrak{D}^{h_0} + V^{h_1} + V^{h_2})(\tilde{\varphi}_0(\varphi_0 + \varphi_1 + \varphi_2)\psi) \\ &\quad + (\mathfrak{D}^{h_1} + V^{h_0} + V^{h_2})(\tilde{\varphi}_1(\varphi_0 + \varphi_1 + \varphi_2)\psi) \\ &\quad + (\mathfrak{D}^{h_2} + V^{h_0} + V^{h_1})(\tilde{\varphi}_2(\varphi_0 + \varphi_1 + \varphi_2)\psi) \\ &= (\mathfrak{D}^{h_0} + V^{h_1} + V^{h_2})(\tilde{\varphi}_0 \psi) + (\mathfrak{D}^{h_1} + V^{h_0} + V^{h_2})(\tilde{\varphi}_1 \psi) + (\mathfrak{D}^{h_2} + V^{h_0} + V^{h_1})(\tilde{\varphi}_2 \psi) \end{aligned}$$

Here we have used that $\varphi_j \tilde{\varphi}_k \psi$ belongs to the domain of both $\mathfrak{D}^{h_j}$ and $\mathfrak{D}^{h_k}$, so we are allowed to rearrange the terms as we want. Hence the definition of $\mathfrak{D}^h$ is independent of the partition of unity.

**Theorem 2.2.** *The Dirac operator $\mathfrak{D}^h$ is self-adjoint.*

For the proof, we need some lemmas.

**Lemma 2.3.** *The Dirac operator $\mathfrak{D} : (L_2)^2 \to (L_2)^2$ without any magnetic field is a self-adjoint operator with the Sobolev space $(H^1)^2$ as domain.*

**Proof.** See [Tha92]. □

**Lemma 2.4.** *The Dirac operator $\mathfrak{D}^{h_0}$ corresponding to the magnetic field $B_0$ is self-adjoint with domain $(H^1)^2$.*

**Proof.** The operator $\mathfrak{D}^{h_0}$ can be written as $\mathfrak{D}^{h_0} = \mathfrak{D} + V^{h_0}$ and the multiplication operator $V^{h_0}$ is relatively bounded with respect to $\mathfrak{D}$ with relative bound zero, so the lemma follows from the Kato-Rellich theorem. □

**Lemma 2.5.** *Let $T$ be a bounded operator from $(L_2)^2$ to $(H^1)^2$ and let $V$ be a function, $V(z) \to 0$ as $|z| \to \infty$. Then the composition $VT$ is compact in $(L_2)^2$.*

**Proof.** For $n = 1, 2, \ldots$ we write $V$ as $V = V_n + \tilde{V}_n$, where

$$V_n(z) = \begin{cases} V(z) & |V(z)| > \frac{1}{n} \\ 0 & |V(z)| \le \frac{1}{n}. \end{cases}$$

The functions $V_n$ all have compact support, so the operators $V_nT$ are compact. But $||V_nT - VT|| \le \frac{1}{n}||T||$ for all $n = 1, 2, \ldots$, so $VT$ is also compact. □

**Remark.** Note that Lemma 2.5 is also true for $2 \times 2$ matrix valued functions $V$ where all components tend to zero at infinity. It also holds if $T$ is bounded from $L_2$ to $H^1$.

**Lemma 2.6.** *Let $0 \ne s \in \mathbb{R}$ and let $\varphi \in C_0^\infty(\mathbb{R}^2)$ with zero in its support. Then the operator $\varphi R$ is compact, where $R = (\mathfrak{D}^\tau + is)^{-1}$ and $\mathfrak{D}^\tau$ is any self-adjoint extension of the Dirac operator corresponding to one AB solenoid (which is assumed to be located at the origin).*

**Proof.** First, $\varphi R$ is compact if and only if $\varphi R(\varphi R)^* = \varphi RR^*\varphi$ is compact. To show that $\varphi RR^*\varphi$ is compact, it is sufficient to show that $\varphi RR^*$ is compact.

The operator $RR^*$ is equal to $\left((\mathfrak{D}^\tau)^2 + s^2\right)^{-1}$. Note that $(\mathfrak{D}^\tau)^2$ is a self-adjoint Pauli operator corresponding to the same magnetic field (see Section 3.2 for a discussion of the Pauli operators that are the square of some Dirac operator). If we denote by $\mathfrak{P}$ any other self-adjoint Pauli operator corresponding to this magnetic field, then by the Krein resolvent formula (see [AG93]) the resolvents of $(\mathfrak{D}^\tau)^2$ and $\mathfrak{P}$ differ by a finite rank operator. Thus, it is enough to show that $\varphi(\mathfrak{P}+s^2)^{-1}$ is compact for a convenient choice of self-adjoint Pauli extension $\mathfrak{P}$. Let us choose $\mathfrak{P}$ to be the Friedrichs extension. The functions in the domain of this extension $\mathfrak{P}$ are zero at the origin so

$$\mathfrak{P} = \begin{pmatrix} H & 0 \\ 0 & H \end{pmatrix},$$

where $H$ is the Friedrichs extension of the Schrödinger operator corresponding to the same magnetic field (see [GŠ04] for a discussion of this). Hence it is enough to show that $\varphi(H + s^2)^{-1}$ is compact.

Let $H_0 = -\Delta$ be the operator corresponding to no magnetic field. Then, by the diamagnetic inequality (see [MOR04]) it follows that $|\varphi(H+s^2)^{-1}u| \le \varphi(H_0+s^2)^{-1}|u|$ (pointwise) for all $u \in L_2$. This inequality implies that $\varphi(H + s^2)^{-1}$ is compact if $\varphi(H_0 + s^2)^{-1}$ is compact (see [DF79] and [Pit79]).

The compactness of $\varphi(H_0 + s^2)^{-1}$ follows from Lemma 2.5 since $T = (H_0 + s^2)^{-1}$ is bounded from $L_2$ to $H^1$. □

**Lemma 2.7.** *$\mathfrak{D}^h$ is a symmetric operator.*

**Proof.** Let $\psi$ and $\tilde{\psi}$ belong to $\mathcal{D}(\mathfrak{D}^h)$. Then

$$\begin{aligned}\langle \mathfrak{D}^h \psi, \tilde{\psi} \rangle &= \int (\mathfrak{D}^h \psi) \cdot \bar{\tilde{\psi}} d\lambda(z) \\ &= \int ((\varphi_0 + \varphi_1 + \varphi_2)\mathfrak{D}^h \psi) \cdot \bar{\tilde{\psi}} d\lambda(z).\end{aligned}$$

The symmetry now follows from integration by parts, noticing that (for example)

$$\int \varphi_0 (\mathfrak{D}^{h_0} + V^{h_1} + V^{h_2})(\varphi_0 \psi) \cdot \bar{\tilde{\psi}} d\lambda(z) = \int \varphi_0 \psi \cdot \overline{(\mathfrak{D}^{h_0} + V^{h_1} + V^{h_2})(\varphi_0 \tilde{\psi})} d\lambda(z)$$

and

$$\begin{aligned}\int \varphi_1 (\mathfrak{D}^{h_0} + V^{h_1} + V^{h_2})(\varphi_0 \psi) \cdot \bar{\tilde{\psi}} d\lambda(z) &= \int_{E_{01}} (\mathfrak{D}^{h_0} + V^{h_1} + V^{h_2})(\varphi_0 \psi) \cdot \overline{\varphi_1 \tilde{\psi}} d\lambda(z) \\ &= \int \varphi_0 \psi \cdot \overline{(\mathfrak{D}^{h_1} + V^{h_0} + V^{h_2})(\varphi_1 \tilde{\psi})} d\lambda(z)\end{aligned}$$

and similar for the other terms. Adding all terms we see that

$$\langle \mathfrak{D}^h \psi, \tilde{\psi} \rangle = \langle \psi, \mathfrak{D}^h \tilde{\psi} \rangle$$

so $\mathfrak{D}^h$ is symmetric. □

In the following lemma we look at our operator as acting from its domain $\mathcal{D}(\mathfrak{D}^h)$ considered as a Hilbert space equipped with graph norm

$$\begin{aligned}||\psi||^2_{\mathfrak{D}^h} &= ||(\mathfrak{D}^{h_0} + V^{h_1} + V^{h_2})(\varphi_0 \psi)||^2 \\ &+ ||(\mathfrak{D}^{h_1} + V^{h_0} + V^{h_2})(\varphi_1 \psi)||^2 \\ &+ ||(\mathfrak{D}^{h_2} + V^{h_0} + V^{h_1})(\varphi_2 \psi)||^2 \\ &+ ||\psi||^2,\end{aligned}$$

to $(L_2)^2$. We will also need the graph norms of the other self-adjoint operators $\mathfrak{D}^{h_j}$, $j = 0, 1, 2$,

$$||\psi||^2_{\mathfrak{D}^{h_j}} = ||\mathfrak{D}^{h_j} \psi||^2 + ||\psi||^2, \quad \text{for } j = 0, 1, 2.$$

**Lemma 2.8.** *Let $0 \neq s \in \mathbb{R}$ be fixed. The operator $\mathfrak{D}^h + is : (\mathcal{D}(\mathfrak{D}^h), || \cdot ||_{\mathfrak{D}^h}) \to (L_2(\mathbb{R}^2))^2$ is a bounded Fredholm operator with index zero.*

**Proof.** First, it is clear that $\mathfrak{D}^h + is$ is bounded from the domain space with graph norm. To show that $\mathfrak{D}^h + is$ is a Fredholm operator, it is enough to find a left and a right parametrix (see [Agr90]). We start by finding a right parametrix. Let $R_j$ denote the resolvent $R_j = (\mathfrak{D}^{h_j} + is)^{-1}$, $j = 0, 1, 2$, and define the operator $R : (L_2)^2 \to (L_2)^2$ as

$$Ru = \varphi_0 R_0 u + \varphi_1 R_1 u + \varphi_2 R_2 u, \quad \text{for } u \in (L_2)^2.$$

For $u \in (L_2)^2$ we have $\varphi_j \varphi_k R_j u \in (H^1)^2$ and being zero in a neighborhood of the singular point(s) if $j \neq k$. Thus

$$(\mathfrak{D}^{h_k} + V^{h_j})(\varphi_j \varphi_k R_j u) = (\mathfrak{D}^{h_j} + V^{h_k})(\varphi_j \varphi_k R_j u), \quad j \neq k.$$

From this it follows that, if $u \in (L_2)^2$ and $j, k, l \in \{0, 1, 2,\ j \neq k \neq l\}$, then

$$\begin{aligned}
(\mathfrak{D}^h + is)(\varphi_j R_j u) &= (\mathfrak{D}^{h_j} + V^{h_k} + V^{h_l} + is)(\varphi_j^2 R_j u) \\
&\quad + (\mathfrak{D}^{h_k} + V^{h_j} + V^{h_l} + is)(\varphi_k \varphi_j R_j u) \\
&\quad + (\mathfrak{D}^{h_l} + V^{h_j} + V^{h_k} + is)(\varphi_l \varphi_j R_j u) \\
&= (\mathfrak{D}^{h_j} + V^{h_k} + V^{h_l} + is)(\varphi_j R_j u) \\
&= \varphi_j (\mathfrak{D}^{h_j} + V^{h_k} + V^{h_l} + is)(R_j u) + \mathfrak{D}(\varphi_j) R_j u \\
&= \varphi_j f + (V^{h_k} + V^{h_l} + \mathfrak{D}(\varphi_j)) R_j u.
\end{aligned}$$

Thus

$$\begin{aligned}
(\mathfrak{D}^h + is)Ru &= \mathfrak{D}^h(\varphi_0 R_0 u) + \mathfrak{D}^h(\varphi_1 R_1 u) + \mathfrak{D}^h(\varphi_2 R_2 u) \\
&= \varphi_0 u + \left((V^{h_1} + V^{h_2})\varphi_0 + \mathfrak{D}(\varphi_0)\right) R_0 u \\
&\quad + \varphi_1 u + \left((V^{h_0} + V^{h_2})\varphi_1 + \mathfrak{D}(\varphi_1)\right) R_1 u \\
&\quad + \varphi_2 u + \left((V^{h_0} + V^{h_1})\varphi_2 + \mathfrak{D}(\varphi_2)\right) R_2 u \\
&= u + Ku
\end{aligned}$$

where $K : (L_2)^2 \to (L_2)^2$ is the operator

$$\begin{aligned}
Ku &= \left((V^{h_1} + V^{h_2})\varphi_0 + \mathfrak{D}(\varphi_0)\right) R_0 u \\
&\quad + \left((V^{h_0} + V^{h_2})\varphi_1 + \mathfrak{D}(\varphi_1)\right) R_1 u \\
&\quad + \left((V^{h_0} + V^{h_1})\varphi_2 + \mathfrak{D}(\varphi_2)\right) R_2 u.
\end{aligned}$$

$K$ is compact. Indeed, the first part is compact according to Lemma 2.5 since $R_0$ is bounded from $(L_2)^2$ to $(H^1)^2$ and the matrix-valued function $(V^{h_1} + V^{h_2})\varphi_0 + \mathfrak{D}(\varphi_0)$ tends to zero at infinity. The other two parts are compact by Lemma 2.6. Hence $K$ is compact, so $R$ is a right parametrix.

To see that $R$ is also a left parametrix, it is enough to show that the a priori estimate

$$||\psi||^2_{\mathfrak{D}^h} \lesssim ||(\mathfrak{D}^h + is)\psi||^2 + ||\psi||^2 \tag{2.18}$$

holds for all $\psi \in \mathcal{D}(\mathfrak{D}^h)$. We use the symbol $\lesssim$ to express that the left-hand side is less than or equal to a constant times the right-hand side.

Since $\mathfrak{D}^{h_j}$, $j = 0, 1, 2$, are self-adjoint operators, we have the a priori estimates

$$||\varphi_j \psi||_{\mathfrak{D}^{h_j}} \lesssim ||(\mathfrak{D}^{h_j} + V^{h_k} + V^{h_l} + is)(\varphi_j \psi)||, \quad j, k, l \in \{0, 1, 2\}\ j \neq k \neq l.$$

We get

$$\begin{aligned}||\psi||^2_{\mathfrak{D}^h} &\lesssim ||\varphi_0\psi||^2_{\mathfrak{D}^{h_0}} + ||\varphi_1\psi||^2_{\mathfrak{D}^{h_1}} + ||\varphi_2\psi||^2_{\mathfrak{D}^{h_2}} \\ &\lesssim ||(\mathfrak{D}^{h_0} + V^{h_1} + V^{h_2} + is)(\varphi_0\psi)||^2 \\ &\quad + ||(\mathfrak{D}^{h_1} + V^{h_0} + V^{h_2} + is)(\varphi_1\psi)||^2 \\ &\quad + ||(\mathfrak{D}^{h_2} + V^{h_0} + V^{h_1} + is)(\varphi_2\psi)||^2\end{aligned}$$

Next we express the square of the norms as $||(\mathfrak{D}^h + is)\psi||^2$ minus a sum of inner product terms

$$\big\langle(\mathfrak{D}^{h_j} + V^{h_k} + V^{h_l} + is)(\varphi_j\psi), (\mathfrak{D}^{h_k} + V^{h_j} + V^{h_l} + is)(\varphi_k\psi)\big\rangle$$

as $j, k$ and $l$ runs over $\{0, 1, 2\}$ but are all different. Clearly, for each such term we have the estimate

$$\begin{aligned}&\big|\big\langle(\mathfrak{D}^{h_j} + V^{h_k} + V^{h_l} + is)(\varphi_j\psi), (\mathfrak{D}^{h_k} + V^{h_j} + V^{h_l} + is)(\varphi_k\psi)\big\rangle\big| \\ &\quad\lesssim ||\chi_{E_{jk}}(\mathfrak{D}^{h_j} + V^{h_k} + V^{h_l} + is)(\varphi_j\psi)|| \times ||\chi_{E_{jk}}(\mathfrak{D}^{h_k} + V^{h_j} + V^{h_l} + is)(\varphi_k\psi)|| \\ &\quad\lesssim ||\psi||^2.\end{aligned}$$

We see that (2.18) follows, so $R$ is also a left parametrix. Thus $\mathfrak{D}^h + is$ is a Fredholm operator.

To see that $\mathfrak{D}^h + is$ has index zero, we note that since $\mathfrak{D}$ with domain $(H^1)^2$ is self-adjoint, it has index zero and $R_s := (\mathfrak{D} + is)^{-1}$ is a parametrix for $\mathfrak{D} + is$. The operator

$$R - R_s = \varphi_0 R_0 + \varphi_1 R_1 + \varphi_2 R_2 - R_s$$

is compact. To see this, write $R - R_s$ as

$$R - R_s = (\varphi_0 - 1)R_0 + \varphi_1 R_1 + \varphi_2 R_2 + (R_0 - R_s).$$

The first term is compact according to Lemma 2.5, the second and third according to Lemma 2.6. For the last term we note that $R_0 - R_s = -R_0 V^{h_0} R_s$. The compactness of $V^{h_0} R_s$ follows from Lemma 2.5. Composition with the bounded operator $R_0$ preserves compactness. Thus $R - R_s$ is compact. It follows that

$$\operatorname{ind}(R) = \operatorname{ind}(R_s).$$

Since $R$ and $R_s$ are parametrices for $\mathfrak{D}^h + is$ and $\mathfrak{D} + is$ respectively, it holds that

$$\operatorname{ind}(R) + \operatorname{ind}(\mathfrak{D}^h + is) = \operatorname{ind}(R(\mathfrak{D}^h + is)) = 0,$$

and

$$\operatorname{ind}(R_s) + \operatorname{ind}(\mathfrak{D} + is) = \operatorname{ind}(R_s(\mathfrak{D} + is)) = 0.$$

Hence

$$\operatorname{ind}(\mathfrak{D}^h + is) = -\operatorname{ind}(R) = -\operatorname{ind}(R_s) = \operatorname{ind}(\mathfrak{D} + is) = 0.$$

□

**Proof (of Theorem 2.2).** We know from Lemma 2.7 that $\mathfrak{D}^h$ is symmetric, so for $0 \neq s \in \mathbb{R}$ we have

$$||(\mathfrak{D}^h + is)\psi||^2 = ||\mathfrak{D}^h\psi||^2 + s^2||\psi||^2 \geq s^2||\psi||^2.$$

It follows that $\dim\ker(\mathfrak{D}^h + is) = 0$. From Lemma 2.8 we have that $\mathfrak{D}^h + is$ has index zero, so $\dim\operatorname{Ker}((\mathfrak{D}^h)^* - is) = 0$. Choosing $s$ positive and negative respectively gives that the deficiency indices for $\mathfrak{D}^h$ is $(0,0)$, so $\mathfrak{D}^h$ is self-adjoint. □

## 2.3 Spin flip invariance

Since the particle we are studying moves only in a plane, and the magnetic field is orthogonal to this plane, physically it should be no difference if the sign of the magnetic field is changed. This transformation has to come together with a flip of the spin-up and spin-down components and a renormalization of the AB intensities. We say that a self-adjoint extension is spin flip invariant if, after applying these transformations, we end up with a (anti)-unitarily equivalent operator. We will show that there are only two values of $\tau$ for which the Dirac operator $\mathfrak{D}^{\tau,h}$ is spin flip invariant. Let $\boldsymbol{\tau} = (\tau_1, \dots, \tau_n)$ and denote the Dirac operator by $\mathfrak{D}^{\boldsymbol{\tau},h}$. We will use the linear functionals

$$c^{\pm}_{-\alpha_j}(\psi_\pm) = \lim_{r_j\to 0} r_j^{\alpha_j}\frac{1}{2\pi}\int_0^{2\pi}\psi_\pm(r_je^{i\theta_j})d\theta_j, \text{ and} \tag{2.19}$$

$$c^{\pm}_{\alpha_j-1}(\psi_\pm) = \lim_{r_j\to 0} r_j^{1-\alpha_j}\frac{1}{2\pi}\int_0^{2\pi}\psi_\pm(r_je^{i\theta_j})e^{i\theta_j}d\theta_j. \tag{2.20}$$

We define anti-unitarily operator $V : (L_2(\mathbb{R}^2))^2 \to (L_2(\mathbb{R}^2))^2$ as the spin-flip operator that takes $(\psi_+, \psi_-)^t$ to $(\overline{\psi_-}, \overline{\psi_+})^t$.

**Proposition 2.9.** *The operators $\mathfrak{D}^{\boldsymbol{\tau}',-h}$ and $\mathfrak{D}^{\boldsymbol{\tau},h}$ are anti-unitarily equivalent via the operator $V$ if and only if for all $j = 1, \dots, n$ we have $\tau_j' + \tau_j = \pi$ or $\tau_j' + \tau_j = 3\pi$.*

**Proof.** Let $\beta_j = 1 - \alpha_j$ be the normalized AB intensities for the magnetic field $-B$ that corresponds to $\mathfrak{D}^{\boldsymbol{\tau}',-h}$. A function $\psi$ in the domain of $\mathfrak{D}^{\boldsymbol{\tau}',-h}$ has the asymptotics

$$\psi \sim \frac{c_j}{2}\begin{pmatrix}(1+e^{i\tau_j'})\sigma(1-\beta_j)r_j^{\beta_j-1} + O(r_j^{1-\beta_j})\\ (1-e^{i\tau_j'})\sigma(\beta_j)r^{-\beta_j}e^{i\theta_j} + O(r_j^{\beta_j})\end{pmatrix}$$

as $z \to z_j$ for some constant $c_j \in \mathbb{C}$. We see that $V\psi$ has the asymptotics

$$\begin{aligned} V\psi &\sim \frac{\bar{c_j}}{2}\begin{pmatrix}(1-e^{-i\tau_j'})\sigma(\beta_j)r_j^{-\beta_j}e^{-i\theta_j} + O(r_j^{\beta_j})\\ (1+e^{-i\tau_j'})\sigma(1-\beta_j)r_j^{\beta_j-1} + O(r_j^{1-\beta_j})\end{pmatrix}\\ &\sim \frac{\bar{c_j}}{2}\begin{pmatrix}(1-e^{-i\tau_j'})\sigma(1-\alpha_j)r_j^{\alpha_j-1}e^{-i\theta_j} + O(r_j^{1-\alpha_j})\\ (1+e^{-i\tau_j'})\sigma(\alpha_j)r_j^{-\alpha_j} + O(r_j^{\alpha_j})\end{pmatrix}\end{aligned}$$

Applying the functionals (2.19) and (2.20) we see that $V\psi$ satisfies

$$\frac{c^{+}_{\alpha_j-1}((V\psi)_+)}{c^{-}_{-\alpha_j}((V\psi)_-)} = i\tan(\tau_j'/2)\frac{\sigma(1-\alpha_j)}{\sigma(\alpha_j)}$$

and $c^{-}_{\alpha_j-1}((V\psi_-)_-) = c^{+}_{-\alpha_j}((V\psi_+)_+) = 0$, so the requirements that the domain change properly is that

$$\tan(\tau_j'/2) = \cot(\tau_j/2), \quad \text{for } j = 1, \dots, n.$$

We see that $\tau_j/2$ and $\pi/2 - \tau_j'/2$ must differ by a integer multiple of $\pi$. Both $\tau_j$ and $\tau_j'$ belong to the interval $[0, 2\pi)$, so the only possibilities are $\tau_j' + \tau_j = \pi$ or $\tau_j' + \tau_j = 3\pi$.

**Corollary 2.10.** *The operators $\mathfrak{D}^{\boldsymbol{\tau},-h}$ and $\mathfrak{D}^{\boldsymbol{\tau},h}$ are anti-unitarily equivalent via the operator $V$ if and only if for all $j = 1, \dots, n$ we have $\tau_j = \pi/2$ or $\tau_j = 3\pi/2$.*

**Proof.** Take $\tau_j' = \tau_j$ in the previous Proposition.

If we let $W : (L_2(\mathbb{R}^2))^2 \to (L_2(\mathbb{R}^2))^2$ be the operator that takes $(\psi_+, \psi_-)^t$ to $(\psi_-, \psi_+)^t$ we get some other symmetries if we compose it with the gauge transform that only act on the spin-up component.

**Proposition 2.11.** *The operators $\mathfrak{D}^{\boldsymbol{\tau}',-h}$ and $\mathfrak{D}^{\boldsymbol{\tau},h}$ are unitarily equivalent via the operator $W$ composed with a gauge multiplication of $e^{-2i\sum\theta_j}$ of the spin-up component if and only if $|\tau_j' - \tau_j| = \pi$ for all $j = 1, \dots, n$.*

**Proof.** The proof goes on as in the proof of Proposition 2.9. This time the requirement on $\tau_j$ and $\tau_j'$ becomes

$$-\tan(\tau_j'/2) = \cot(\tau_j/2), \quad \text{for } j = 1, \dots, n$$

which gives $|\tau_j' - \tau_j| = \pi$ for all $j = 1, \dots, n$.

## 2.4 Zero-modes

Let us calculate the dimension of the kernel of $\mathfrak{D}^h$ under the assumption that $\tau_j = \tau$ for all $j = 1, \dots, n$, which means that we assume that we have the same physical conditions of the behavior of the particle close to all solenoids. Denote by $\Phi$ the total flux of $B$ divided by $2\pi$, that is

$$\Phi = \frac{1}{2\pi}\int B(z)d\lambda(z) = \frac{1}{2\pi}\int B_0(z)d\lambda(z) + \sum_{j=1}^{n}\alpha_j.$$

As usual, the definition of the total flux is a matter of agreement, due to the arbitrariness in the choice of normalization for AB intensities. The asymptotics of $e^h$ at infinity and at the singular points $\Lambda$ are given by

$$e^h \sim \begin{cases} |z|^{\Phi} & |z| \to \infty \\ |z - z_j|^{\alpha_j} & z \to z_j. \end{cases} \tag{2.21}$$

Remember also that the functions in the domain of $\mathfrak{D}^h$ must satisfy

$$\frac{c^+_{\alpha_j-1}(\psi_+)}{c^-_{-\alpha_j}(\psi_-)} = i\cot(\tau_j/2)\frac{\sigma(1-\alpha_j)}{\sigma(\alpha_j)}, \quad j=1,\ldots,n. \tag{2.22}$$

Let $\{x\}$ denote the lower integer part, that is

$$\{x\} = \begin{cases} \lfloor x \rfloor & x>1 \text{ and } x \notin \mathbb{N} \\ x-1 & x>1 \text{ and } x \in \mathbb{N} \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 2.12.** *If $\tau_j = \tau$, $j=1,\ldots,n$ then the dimension of the kernel of $\mathfrak{D}^h$ is given by*

$$\dim\ker\mathfrak{D}^h = \begin{cases} \{|n-\Phi|\} & \text{if } \tau = 0, \\ \{|\Phi|\} & \text{if } \tau=\pi, \\ 0 & \text{otherwise.} \end{cases}$$

The proof follows the same idea as the original proof by Aharonov-Casher with the same changes as in [Per05] and using the fact that the spin-up and spin-down components are coupled if $\tau \notin \{0,\pi\}$.

**Proof.** We start by calculating the zero-modes as if the spin-up and spin-down components were not coupled. Then we can study the spin-up and spin-down components separately.

Let us start with the spin-up component, that is, we start by studying the solutions to $q_+\psi_+ = 0$. This is equivalent to $\frac{\partial}{\partial\bar{z}}(e^{-h}\psi_+) = 0$, and thus the function $f_+ = e^{-h}\psi_+$ must be analytic in $\mathbb{C}\setminus\Lambda$. The behavior of $f_+$ at the singular points $\Lambda$ is different for different values of the parameter $\tau$, but a pole of order at most $\{-\Phi\}-1$ at infinity is allowed independently of the value of $\tau$.

**Case I**, $\tau=\pi$: For $\psi_+$ to belong to $L_2$, we see from (2.21) that the function $f_+$ is not allowed to have any poles at the singular points $\Lambda$. Thus, if $\tau=\pi$ then $f_+$ may be a polynomial of order at most $\{-\Phi\}-1$. There are $\{-\Phi\}$ many linearly independent such polynomials.

**Case II**, $\tau\neq\pi$: From (2.22) we see that a pole of order at most one is allowed at each $z_j\in\Lambda$. The calculation in [Per05] then yields that the dimension is $\{n-\Phi\}$.

Let us now turn to the spin-down component. We look for solutions to the equation $q_-\psi_- = 0$, which is equivalent to finding solutions to $\frac{\partial}{\partial z}(e^h\psi_-) = 0$. If we now let $f_- = e^h\psi_-$, then $f_-$ must be anti-analytic in $\mathbb{C}\setminus\Lambda$, and from the asymptotics (2.21) we see that $f_-$ may have a polynomial part of degree at most $\{\Phi\}-1$ independent of the value of the parameter $\tau$. Again we get two different cases for the behavior of the functions at the singular points $\Lambda$.

**Case I**, $\tau=0$: In this case we see from (2.22) that no singular parts for $\psi_-$ are allowed at $\Lambda$, and hence $f_-$ must have a zero of order at least 1 at each point in $\Lambda$. That is we have a polynomial in $\bar{z}$ of degree $\{\Phi\}-1$ with $n$ predicted zeroes. There are $\{\Phi-n\}$ linearly independent polynomials of this type.

**Case II**, $\tau \neq 0$: Now $f_-$ must be a polynomial in $\bar{z}$ of degree at most $\{\Phi\} - 1$, but without any forced zeroes. Thus the dimension of the kernel is $\{\Phi\}$.

Since the spin-up and spin-down components are not coupled in the cases $\tau = 0$ and $\tau = \pi$ these calculations above yield

$$\dim \ker \mathfrak{D}^h = \begin{cases} \{|n - \Phi|\} & \text{if } \tau = 0, \\ \{|\Phi|\} & \text{if } \tau = \pi. \end{cases}$$

Let us now assume that $\tau \notin \{0, \pi\}$. We should evaluate how the spin-up zero-modes match the spin-down zero-modes to satisfy the conditions at the singularities. First we note that to be able to have zero-modes both $\{n - \Phi\}$ and $\{\Phi\}$ must be positive. From the calculations in the last two paragraphs of the proof of Theorem 3.3 in [Per05] it follows that $f_+$ must be of the form

$$f_+(z) = \sum_{j=1}^{n} \frac{c_j}{z - z_j}$$

where

$$\sum_{j=1}^{n} c_j z_j^k = 0, \quad \text{for } k = 0, 1, \ldots, n - \{n - \Phi\} - 1 \tag{2.23}$$

and $f_-(z)$ must be a polynomial in $\bar{z}$ of degree at most $\{\Phi\} - 1$. Actually, we will show that even if the degree of the polynomial $f_-$ is $\{\Phi\}$ or in some cases $\{\Phi\} + 1$ all coefficients of the polynomial must be zero. Let us define the natural number $m$ as $m = n - \{n - \Phi\} - 1$ and note that $m = \lfloor \Phi \rfloor$. Let

$$f_-(z) = \sum_{j=0}^{m} a_j \bar{z}^j. \tag{2.24}$$

From the asymptotics (2.22) we see that

$$\frac{c_j}{f_-(z_j)} = e^{-2h_0(z_j)} \prod_{l \neq j} \left(|z_j - z_l|^{-2\alpha_l}\right) i \cot(\tau/2) \frac{\sigma(1 - \alpha_j)}{\sigma(\alpha_j)}, \quad j = 1, \ldots, n.$$

From the requirements (2.23) of the coefficients $c_j$ we get

$$0 = \sum_{j=1}^{n} c_j z_j^k = i \cot(\tau/2) \sum_{j=1}^{n} b_j f_-(z_j) z_j^k, \quad k = 0, 1, \ldots, m, \tag{2.25}$$

where

$$b_j = e^{-2h_0(z_j)} \prod_{l \neq j} \left(|z_j - z_l|^{-2\alpha_l}\right) \frac{\sigma(1 - \alpha_j)}{\sigma(\alpha_j)}.$$

We introduce the vector $\mathbf{a} = (a_0, \ldots, a_m)^t$ where $a_j$ are the coefficients in (2.24). Let us also introduce the matrix $V$ as

$$V = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ z_1 & z_2 & \cdots & z_n \\ \vdots & \vdots & \ddots & \vdots \\ z_1^m & z_2^m & \cdots & z_n^m \end{pmatrix},$$

and the diagonal matrix $B$ having the positive number $b_j$ at the $j$th diagonal position. Then (2.25) can be written as

$$i \cot(\tau/2) V B V^* \mathbf{a} = 0.$$

The matrix $VBV^*$ is clearly Hermitian and since $B$ is positive, we can write $VBV^*$ as $(V\sqrt{B})(V\sqrt{B})^*$. Hence the null space of $VBV^*$ is the same as that of the matrix $(V\sqrt{B})^* = \sqrt{B}V^*$. Since $V^*$ is (a part of) a Vandermonde matrix it has full rang, so the dimension of the null space of $\sqrt{B}V^*$ is zero. Hence the polynomial $f_-$, and thus also $\psi_-$, must be zero. Since the spin-up and spin-down components are coupled, it follows that $\psi_+$ is also zero. Consequently, $\dim\ker\mathfrak{D}^h = 0$, and the proof is complete. □

# 3 The Pauli operator

In this section we will study the Pauli operator corresponding to the magnetic field (2.1), obtained as the square of a self-adjoint Dirac operator. We look first at all Pauli operators corresponding to one AB solenoid. This has already been done in [GŠ04] so we only sketch the way it was done.

## 3.1 The Pauli operator with one AB solenoid

Let the AB solenoid with intensity $\alpha \in (0,1)$ be located at the origin. We start by defining the Pauli operator on $(C_0^\infty(\mathbb{R}^2 \setminus \{0\}))^2$. Let

$$\mathfrak{P}_{\min} = \begin{pmatrix} P_{+,\min} & 0 \\ 0 & P_{-,\min} \end{pmatrix},$$

where $P_{+,\min} f = q_- q_+ f$ and $P_{-,\min} f = q_+ q_- f$ are the same actions, both with domain $C_0^\infty(\mathbb{R}^2 \setminus \{0\})$. They are acting as

$$q_+ q_- = q_- q_+ = -\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\left(i\frac{\partial}{\partial \theta} - \alpha\right)^2$$

Following [AT98] we get the closure of these operators and then the adjoint of them. It turns out that the adjoint is given by

$$\mathcal{D}(\overline{\mathfrak{P}}_{\min}^*) = \left\{\psi \in (L_2)^2 : \ ||\mathfrak{d}^2\psi|| < \infty\right\}, \quad \text{and}$$
$$\overline{\mathfrak{P}}_{\min}^*\psi = \mathfrak{d}^2\psi, \quad \text{for } \psi \in \mathcal{D}(\overline{\mathfrak{P}}_{\min}^*).$$

We want to describe the deficiency spaces to be able to use the Krein-von Neumann theory. We look for functions belonging to $\chi_\pm = \ker(\overline{\mathfrak{P}}^*_{\min} \mp i)$. The differential equations are the same for the spin-up and spin-down components, so it is enough to study the spin-up one.

The function $\psi_+$ should satisfy $q_+q_-\psi_+ = \pm i\psi_+$. If we write $\psi_+$ as $\psi_+(re^{i\theta}) = \sum_{m=-\infty}^{\infty} u_m(r)\frac{e^{im\theta}}{\sqrt{2\pi}}$ we end up solving equations

$$r^2u_m''(r) + ru_m'(r) - (\mp ir^2 + (\alpha+m)^2)u_m(r) = 0$$

If we let $\lambda_\pm = e^{\mp i\pi/4}$ then the only solutions belonging to $L_2(rdr)$ are the ones for $m=-1$ and $m=0$. The solutions are

$$u_m(r) = A_m^\pm K_{|m+\alpha|}(\lambda_\pm r)e^{im\theta}, \quad m=-1,0.$$

Hence, the deficiency spaces look like

$$\begin{aligned}\chi_\pm = \operatorname{span}\Bigg\{ &\begin{pmatrix} K_\alpha(\lambda_\pm r) \\ K_\alpha(\lambda_\pm r)\end{pmatrix}, \begin{pmatrix} K_\alpha(\lambda_\pm r) \\ K_{1-\alpha}(\lambda_\pm r)e^{-i\theta}\end{pmatrix}, \\ &\begin{pmatrix} K_{1-\alpha}(\lambda_\pm r)e^{-i\theta} \\ K_\alpha(\lambda_\pm r)\end{pmatrix}, \begin{pmatrix} K_{1-\alpha}(\lambda_\pm r)e^{-i\theta} \\ K_{1-\alpha}(\lambda_\pm r)e^{-i\theta}\end{pmatrix}\Bigg\}.\end{aligned}$$

So the deficiency indices are $(4,4)$. To each self-adjoint Pauli extension $\mathfrak{P}$ one can associate functionals $c^\pm_{-\alpha}$, $c^\pm_\alpha$, $c^\pm_{\alpha-1}$ and $c^\pm_{1-\alpha}$ (see [GŠ04]), by

$$\begin{aligned}
c^\pm_{-\alpha}(\psi_\pm) &= \lim_{r\to 0} r^\alpha \frac{1}{2\pi}\int_0^{2\pi} \psi_\pm(re^{i\theta})d\theta, \\
c^\pm_{\alpha}(\psi_\pm) &= \lim_{r\to 0} r^{-\alpha} \left(\frac{1}{2\pi}\int_0^{2\pi} \psi_\pm(re^{i\theta})d\theta - r^{-\alpha}c^\pm_{-\alpha}(\psi_\pm)\right), \\
c^\pm_{\alpha-1}(\psi_\pm) &= \lim_{r\to 0} r^{1-\alpha} \frac{1}{2\pi}\int_0^{2\pi} \psi_\pm(re^{i\theta})e^{i\theta}d\theta, \text{ and} \\
c^\pm_{1-\alpha}(\psi_\pm) &= \lim_{r\to 0} r^{\alpha-1} \left(\frac{1}{2\pi}\int_0^{2\pi} \psi_\pm(re^{i\theta})e^{i\theta}d\theta - r^{\alpha-1}c^\pm_{\alpha-1}(\psi_\pm)\right).
\end{aligned}$$

such that $\psi = (\psi_+,\psi_-)^t \in \mathcal{D}(\mathfrak{P})$ if and only if

$$\psi(re^{i\theta}) \sim \begin{pmatrix} c^+_{\alpha-1}r^{\alpha-1}e^{-i\theta} + c^+_{1-\alpha}r^{1-\alpha}e^{-i\theta} + c^+_{-\alpha}r^{-\alpha} + c^+_\alpha r^\alpha + O(r^\gamma) \\ c^-_{\alpha-1}r^{\alpha-1}e^{-i\theta} + c^-_{1-\alpha}r^{1-\alpha}e^{-i\theta} + c^-_{-\alpha}r^{-\alpha} + c^-_\alpha r^\alpha + O(r^\gamma)\end{pmatrix} \tag{3.1}$$

as $r\to 0$, where $\gamma = \min(1+\alpha, 2-\alpha)$.

## 3.2 The Pauli operators with several AB solenoids

Since there are more self-adjoint Pauli extensions than Dirac extensions corresponding to our singular magnetic field it is clear that not all Pauli operators can be obtained as the square of a self-adjoint Dirac operator. Here we will study the Pauli operators that can be obtained in this way.

**Definition 3.1.** We define the Pauli operator $\mathfrak{P}^h$ as $(\mathfrak{D}^h)^2$ where $\mathfrak{D}^h$ is a self-adjoint Dirac operator defined in Definition 2.1. This means that

$$\mathcal{D}(\mathfrak{P}^h) = \{\psi \in (L_2)^2 : \ \mathfrak{d}^h\psi \in \mathcal{D}(\mathfrak{D}^h)\},$$
$$\mathfrak{P}^h\psi = (\mathfrak{d}^h)^2\psi, \quad \text{for } \psi \in \mathcal{D}(\mathfrak{P}^h).$$

□

Let us again introduce the boundary value linear functionals acting on $\mathcal{D}(\mathfrak{P}^h)$, but this time for all singular points $\Lambda$. For $j = 1, \ldots, n$, let

$$\begin{aligned}
c^{\pm}_{-\alpha_j}(\psi_\pm) &= \lim_{r_j\to 0} r^{\alpha_j}\frac{1}{2\pi}\int_0^{2\pi}\psi_\pm(r_je^{i\theta_j})d\theta_j,\\
c^{\pm}_{\alpha_j}(\psi_\pm) &= \lim_{r_j\to 0} r^{-\alpha_j}\left(\frac{1}{2\pi}\int_0^{2\pi}\psi_\pm(r_je^{i\theta_j})d\theta_j - r^{-\alpha_j}c^{\pm}_{-\alpha_j}(\psi_\pm)\right),\\
c^{\pm}_{\alpha_j-1}(\psi_\pm) &= \lim_{r_j\to 0} r^{1-\alpha_j}\frac{1}{2\pi}\int_0^{2\pi}\psi_\pm(r_je^{i\theta_j})e^{i\theta_j}d\theta_j, \text{ and}\\
c^{\pm}_{1-\alpha_j}(\psi_\pm) &= \lim_{r_j\to 0} r_j^{\alpha_j-1}\left(\frac{1}{2\pi}\int_0^{2\pi}\psi_\pm(r_je^{i\theta_j})e^{i\theta_j}d\theta_j - r_j^{\alpha_j-1}c^{\pm}_{\alpha_j-1}(\psi_\pm)\right).
\end{aligned}$$

These functionals catches the singular behavior of the functions $\psi_\pm$ at the AB solenoids.

**Proposition 3.2.** *For an arbitrary self-adjoint Pauli extension $\mathfrak{P}$, it is the square of some self-adjoint Dirac extension $\mathfrak{D}^h$ if and only if the following equations are satisfied for all $\psi \in \mathcal{D}(\mathfrak{P})$*

$$\frac{c^{+}_{\alpha_j-1}(\psi_+)}{c^{-}_{-\alpha_j}(\psi_-)} = i\cot(\tau_j/2)\frac{\sigma(1-\alpha_j)}{\sigma(\alpha_j)}, \tag{3.2}$$

$$\frac{c^{-}_{\alpha_j}(\psi_-)}{c^{+}_{1-\alpha_j}(\psi_+)} = i\cot(\tau_j/2)\frac{\sigma(-\alpha_j)}{\sigma(\alpha_j-1)}, \tag{3.3}$$

$$c^{+}_{-\alpha_j}(\psi_+) = 0, \ \textit{and} \tag{3.4}$$

$$c^{-}_{\alpha_j-1}(\psi_-) = 0. \tag{3.5}$$

**Proof.** Let $\mathfrak{D}^h$ be a given self-adjoint Dirac extension, and let $\psi$ belong to $\mathcal{D}(\mathfrak{D}^h)$. Then for some constants $c_j$ we have

$$\psi \sim \frac{c_j}{2}\begin{pmatrix}(1+e^{i\tau_j})\sigma(1-\alpha_j)r_j^{\alpha_j-1}e^{-i\theta_j} + O(r_j^{\alpha_j})\\ (1-e^{i\tau_j})\sigma(\alpha_j)r_j^{-\alpha_j} + O(r_j^{1-\alpha_j})\end{pmatrix}, \quad \text{for } j = 1, \ldots, n$$

We want to find the next part of the asymptotical expansion for $\psi$ such that

$$\mathfrak{D}^h(\psi) \sim \frac{a_j}{2}\begin{pmatrix}(1+e^{i\tau_j})\sigma(1-\alpha_j)r_j^{\alpha_j-1}e^{-i\theta_j}\\ (1-e^{i\tau_j})\sigma(\alpha_j)r_j^{-\alpha_j}\end{pmatrix}, \quad j = 1, \ldots, n$$

for some constants $a_j$. We get that $\psi$ must have the asymptotics

$$\psi \sim \begin{pmatrix} \frac{c_j}{2}(1+e^{i\tau_j})\sigma(1-\alpha_j)r_j^{\alpha_j-1}e^{-i\theta_j} - \frac{ia_j}{2}(1-e^{i\tau_j})\sigma(\alpha_j-1)r_j^{1-\alpha_j}e^{-i\theta_j} + O(r_j^{2-\alpha_j}) \\ \frac{c_j}{2}(1-e^{i\tau_j})\sigma(\alpha_j)r_j^{-\alpha_j} - \frac{ia_j}{2}(1+e^{i\tau_j})\sigma(-\alpha_j)r_j^{\alpha} + O(r_j^{1+\alpha_j}) \end{pmatrix}$$

as $z$ tends to $z_j$, for $j = 1, \ldots, n$. From this it follows that

$$c^+_{\alpha_j-1}(\psi_+) = \frac{c_j}{2}(1+e^{i\tau_j})\sigma(1-\alpha_j), \tag{3.6}$$

$$c^+_{1-\alpha_j}(\psi_+) = -\frac{ia_j}{2}(1-e^{i\tau_j})\sigma(\alpha_j-1), \tag{3.7}$$

$$c^-_{-\alpha_j}(\psi_-) = \frac{c_j}{2}(1-e^{i\tau_j})\sigma(\alpha_j), \text{ and} \tag{3.8}$$

$$c^-_{\alpha_j}(\psi_-) = -\frac{ia_j}{2}(1+e^{i\tau_j})\sigma(-\alpha_j). \tag{3.9}$$

Moreover

$$c^+_{-\alpha_j}(\psi_+) = 0, \text{ and} \tag{3.10}$$

$$c^-_{\alpha_j-1}(\psi_-) = 0 \tag{3.11}$$

since no such singular functions belong to $\mathcal{D}(\mathfrak{D}^h)$. The coefficients

$$c^+_{\alpha_j}(\psi_+) = \text{arbitrary, and} \tag{3.12}$$

$$c^-_{1-\alpha_j}(\psi_-) = \text{arbitrary} \tag{3.13}$$

since such terms disappear (near the singular point) when applying $\mathfrak{D}^h$. □

**Remark.** The definition of $\mathfrak{P}^h$ can be written as

$$\mathcal{D}(\mathfrak{P}^h) = \Big\{\psi \in (L_2(\mathbb{R}^2))^2 : \ \mathfrak{d}^2\psi \in (L_2(\mathbb{R}^2))^2; \\ \text{equations (3.2)–(3.5) hold for all } \psi\Big\}.$$

We see also that $\mathcal{D}(\mathfrak{P}^h)$ is exactly the subset of $\mathcal{D}(\mathfrak{D}^h)$ for which also the condition (3.3) holds. □

## 3.3 Spin-flip invariance and Zero-modes

**Proposition 3.3.** *The only self-adjoint Pauli extensions $\mathfrak{P}^{\boldsymbol{\tau},h} = (\mathfrak{D}^{\boldsymbol{\tau},h})^2$ that are spin-flip invariant under the transform $V$ are these where for all $j = 1, \ldots, n$ we have $\tau_j = \pi/2$ or $\tau_j = 3\pi/2$.*

The proof is more or less the same as for the Dirac operators.

**Proof.** Let $\beta_j = 1 - \alpha_j$. We have seen that a typical element $\psi$ in $\mathcal{D}(\mathfrak{P}^{\tau,-h})$ has the asymptotics

$$\psi \sim \begin{pmatrix} \frac{c_j}{2}(1+e^{i\tau_j})\sigma(1-\beta_j)r_j^{\beta_j-1} - \frac{ia_j}{2}(1-e^{i\tau_j})\sigma(\beta_j-1)r_j^{1-\beta_j} + O(r_j^{2-\beta_j}) \\ \frac{c_j}{2}(1-e^{i\tau_j})\sigma(\beta_j)r_j^{-\beta_j}e^{i\theta_j} - \frac{ia_j}{2}(1+e^{i\tau_j})\sigma(-\beta_j)r_j^{\beta_j}e^{i\theta_j} + O(r_j^{1+\beta_j}) \end{pmatrix},$$

and thus

$$V\psi \sim \begin{pmatrix} \frac{\overline{c_j}}{2}(1+e^{-i\tau_j})\sigma(1-\alpha_j)r^{\alpha_j-1}e^{-i\theta_j} - \frac{\overline{ia_j}}{2}(1-e^{-i\tau_j})\sigma(\alpha_j-1)r_j^{1-\alpha_j}e^{-i\theta_j} + O(r_j^{2-\alpha_j}) \\ \frac{\overline{c_j}}{2}(1-e^{-i\tau_j})\sigma(\alpha_j)r_j^{-\alpha_j} - \frac{\overline{ia_j}}{2}(1+e^{-i\tau_j})\sigma(-\alpha_j)r_j^{\alpha_j} + O(r_j^{1+\alpha_j}) \end{pmatrix}.$$

Since

$$\frac{c^+_{\alpha_j-1}((V\psi)_+)}{c^-_{-\alpha_j}((V\psi)_-)} = i\tan(\tau_j/2)\frac{\sigma(1-\alpha_j)}{\sigma(\alpha_j)}$$

and

$$\frac{c^-_{\alpha_j}((V\psi)_-)}{c^+_{1-\alpha_j}((V\psi)_+)} = i\tan(\tau_j/2)\frac{\sigma(-\alpha_j)}{\sigma(\alpha_j-1)}$$

it follows from (3.2) and (3.3) that $\tau_j$ must satisfy $\cot(\tau_j/2) = \tan(\tau_j/2)$, which, again, gives $\tau_j = \pi/2$ or $\tau_j = 3\pi/2$.

**Theorem 3.4.** *If $\tau_j = \tau$, $j = 1, \ldots, n$ then the dimension of the kernel of $\mathfrak{P}^h$ is given by*

$$\dim\ker\mathfrak{P}^h = \begin{cases} \{|n-\Phi|\} & \text{if } \tau = 0, \\ \{|\Phi|\} & \text{if } \tau = \pi, \\ 0 & \text{otherwise.} \end{cases}$$

**Proof.** This is a direct consequence of Theorem 2.12 since $\ker\mathfrak{P}^h = \ker\mathfrak{D}^h$. □

## 3.4 Discussion

Let us compare the different self-adjoint Pauli operators from [EV02] (which we will denote by $\mathfrak{P}_{\mathrm{EV}}$) and [Per05] (which we will denote by $\mathfrak{P}_{\max}$) with the ones obtained above as the square of a self-adjoint Dirac operator. It is easier to do this comparison if we have the same AB flux normalization for all operators. Thus, we let all AB intensities $\alpha_j$ belong to the interval $(0,1)$. In the case of the Pauli operator $\mathfrak{P}_{\mathrm{EV}}$, where the AB intensities were normalized to $[-1/2, 1/2)$, we have to do a gauge transformation if there are intensities $\alpha_j$ belonging to $[-1/2, 0)$. This is not a problem, since $\mathfrak{P}_{\mathrm{EV}}$ is gauge invariant.

Before we do the general comparison, let us look at a concrete example, showing that in general, neither of $\mathfrak{P}_{\max}$ nor $\mathfrak{P}_{\mathrm{EV}}$ is the square of a Dirac operator where we have the same physical situation at all singular points $\Lambda$. We do this, by giving a magnetic field that gives different number of zero modes for all operators.

**Example 3.5.** Let the magnetic field have nine AB solenoids, six of them with AB intensity 0.2 and three of them with AB intensity 0.9. Also, let the regular field have total flux $\Phi_0 = 2.0$. Then the total flux is $\Phi = 5.9$. If we normalize to $[-1/2, 1/2)$, the total flux is $\tilde{\Phi} = 2.9$. Thus, the number of zero-modes for the different Pauli operators is given by Table 1. We see that they are all different. It turns out though, that

Table 1: Number of zero-modes for the different Pauli extensions in a concrete example.

| Pauli operator | Number of zero-modes |
|---|---:|
| $\mathfrak{P}_{\max}$ | $\{\Phi\} + \{n - \Phi\} = 8$ |
| $\mathfrak{P}_{\mathrm{EV}}$ | $\{\lvert\tilde{\Phi}\rvert\} = 2$ |
| $\mathfrak{D}^2$, $\tau_j = 0$ for all $j = 1, \dots, 9$ | $\{\lvert n - \Phi\rvert\} = 3$ |
| $\mathfrak{D}^2$, $\tau_j = \pi$ for all $j = 1, \dots, 9$ | $\{\lvert\Phi\rvert\} = 5$ |
| $\mathfrak{D}^2$, $\tau_j = \tau \notin \{0, \pi\}$ for all $j = 1, \dots, 9$ | 0 |

$\mathfrak{P}_{\mathrm{EV}}$ is the square of a Dirac operator, where $\tau_j = \pi$ at the singular points $z_j$ where $\alpha_j \in (0, 1/2)$ and $\tau_j = 0$ at the points $z_j$ where $\alpha_j \in [1/2, 1)$, as we will see below. ∎

In Table 2 we see a comparison of the boundary conditions of the Pauli operators obtained above that are the square of a Dirac operator and the Maximal and EV Pauli operators (see [Per05, EV02]). We see that $\mathfrak{P}_{\max}$ is not the square of a Dirac operator. However, if we let

$$\tau_j = \begin{cases} \pi, & \text{if} \quad 0 < \alpha_j < 1/2 \\ 0, & \text{if} \quad 1/2 \leq \alpha_j < 1 \end{cases}, \quad j = 1, \dots, n,$$

and $\boldsymbol{\tau} = (\tau_1, \dots, \tau_n)$, then $\mathfrak{P}_{\mathrm{EV}}$ is the square of the self-adjoint Dirac operator corresponding to $\boldsymbol{\tau}$. Note that it is possible to have different physical situations at the singular points $\Lambda$. Indeed, if not all intensities $\alpha_j$ belong to either $(0, 1/2)$ or $[1/2, 1)$ then this is the case.

**Remark.** If the AB intensities in [EV02] would have been normalized to $(0, 1)$ instead of $[-1/2, 1/2)$, then the operator $\mathfrak{P}_{\mathrm{EV}}$ would have become the square of the Dirac operator where $\tau_j = \pi$ for all $j = 1, \dots, n$. If the AB intensities would have been normalized to $(-1, 0)$ then $\mathfrak{P}_{\mathrm{EV}}$ would have been the square of the Dirac operator where $\tau_j = 0$ for all $j = 1, \dots, n$. □

Among the Pauli operators studied in this article, the ones for $\tau = \pi/2$ (which is (anti)-unitarily equivalent to the one for $\tau = 3\pi/2$), $\tau = 0$ and $\tau = \pi$ seems to be the most interesting ones. For $\tau = \pi/2$ we get a very symmetric domain of the operator, which implies that the operator is spin-flip invariant. Lacking zero-modes, it does not satisfy the original Aharonov-Casher formula, but it can be approximated (at least component-wise) according to Table 2 and the result in [BP03] (see the end of [Per05] for a discussion of this).

Table 2: The boundary value conditions for the squared Dirac operators compared with the ones for the Maximal and EV Pauli operators.

| | $\mathfrak{P}^h = (\mathfrak{D}^h)^2$ | $\mathfrak{P}_{\max}$ | $\mathfrak{P}_{\mathrm{EV}}$ |
|---|---|---|---|
| $\frac{c^+_{\alpha_j}}{c^+_{-\alpha_j}}$ | $\infty$ | $\infty$ | $\infty$ |
| $\frac{c^+_{1-\alpha_j}}{c^+_{\alpha_j-1}}$ | $-\frac{a_j}{c_j}\frac{\sigma(\alpha_j-1)}{\sigma(1-\alpha_j)}\tan(\tau_j/2)$ | $0$ | $\begin{cases}\infty, & \text{if } 0<\alpha_j<1/2\\ 0, & \text{if } 1/2\le\alpha_j<1\end{cases}$ |
| $\frac{c^-_{\alpha_j}}{c^-_{-\alpha_j}}$ | $\frac{a_j}{c_j}\frac{\sigma(-\alpha_j)}{\sigma(\alpha_j)}\cot(\tau_j/2)$ | $0$ | $\begin{cases}0, & \text{if } 0<\alpha_j<1/2\\ \infty, & \text{if } 1/2\le\alpha_j<1\end{cases}$ |
| $\frac{c^-_{1-\alpha_j}}{c^-_{\alpha_j-1}}$ | $\infty$ | $\infty$ | $\infty$ |

The Pauli operators corresponding to $\tau = 0$ and $\tau = \pi$ have very asymmetric domains. Only one of the components contain singular terms at the points $\Lambda$. This lack of symmetry implies that these extensions are not spin-flip invariant. On the other hand, the Pauli operator corresponding to $\tau = \pi$ does satisfy the original Aharonov-Casher formula and there is no doubt that both of these Pauli operators can be approximated as in [BP03], even as Pauli Hamiltonians.

The Maximal Pauli operator studied [Per05] is spin-flip invariant and has zero-modes, even more than is present in the original Aharonov-Casher formula. At least component-wise it can be approximated as in [BP03]. However, it is not the square of a self-adjoint Dirac operator.

It is still not clear which Pauli extension that describes the physics in the best way.

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