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PREPRINT 2006:15

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Göteborg, August 2006

Preprint 2006:15
ISSN 1652-9715

Matematiska vetenskaper
Göteborg 2006

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ABSTRACT. In this paper we study bilinear Hankel forms of higher weights on Hardy spaces in several dimensions (see [Su1] and [Su2] for Hankel forms of higher weights on weighted Bergman spaces). For the case of weight zero we get a full characterization of $\mathcal{S}_p$ class Hankel forms, $1 \leq p < \infty$, in terms of the membership for the symbols to be in certain Besov spaces. Also, in this case, if a Hankel form is bounded, then the symbol satisfies a certain Carleson measure criterion. For the case of higher weights, we find sufficient criteria for Hankel forms to be in class $\mathcal{S}_p$, $1 \leq p \leq 2$.

1. INTRODUCTION

Schatten-von Neumann class Hankel forms of higher weights on Bergman spaces are characterized in [Su1] and [Su2]. In the same way, as for the case of Bergman spaces, Hankel forms of higher weights on a Hardy space are explicit characterizations of irreducible components in the tensor product of Hardy spaces under the Möbius group, see [PZ].

In this paper we use the same notations as in [Su1] and [Su2]. Now, let $\partial\mathbb{B}$ be the boundary of the unit ball $\mathbb{B}$ in $\mathbb{C}^d$. We denote by H_F^s the *bilinear Hankel forms of weight* s on the Hardy space $H^2(\partial\mathbb{B})$ if

$$(1)\quad H_F^s(f,g) = \int_{\mathbb{B}} \left\langle \otimes^m B^t(z,z)\mathcal{T}_s(f,g)(z), F(z)\right\rangle (1-|z|^2)^{d-1}\, dm(z)\,,$$

where $\mathcal{T}_s$ is the transvectant given by

$$\mathcal{T}_s(f,g)(z) = \sum_{k=0}^{s} \binom{s}{k} (-1)^{s-k} \frac{\partial^k f(z) \odot \partial^{s-k} g(z)}{(d)_k (d)_{s-k}}\,,$$

Date: August 8, 2006.
2000 *Mathematics Subject Classification.* 32A25, 32A35, 32A37, 47B35.

and $(a)_k = a(a+1)\cdots(a+k-1)$ is the Pochammer symbol. The tensor-valued function F is called the *symbol* corresponding to the Hankel form H_F^s. In fact, this is the limiting case $\nu = d$ of (7) in [Su1].

In Section 2 we establish the Schatten-von Neumann class criteria for bilinear Hankel forms of weight zero. In this case we get a full characterization of $\mathcal{S}_p$ class Hankel forms, $1 \leq p < \infty$, in terms of the membership for the symbols in certain Besov spaces. Also a sufficient criterion for boundedness, in terms of Carleson measures, is presented there. The main theorems in Section 2 are Theorem 2.5 and Theorem 2.16. In section 3 we study the case of higher weight. Here a new difficulty appears. The transvectant does not behave in the same way as for the case of Bergman spaces, see Example 3.5. Therefore we cannot generalize the techniques used in [Su1] to find boundedness and compactness criteria, but we establish sufficient criteria for Hankel forms of nonzero weight to be of class $\mathcal{S}_p$, $1 \leq p \leq 2$, see Theorem 3.9.

Notation. If $\|\cdot\|_1$ and $\|\cdot\|_2$ are two equivalent norms on a vector space X, then we write $\|x\|_1 \simeq \|x\|_2$, $x \in X$. Also, for two functions f and g we write $f \lesssim g$ if there is a constant $C > 0$, independent of the variables in questions, such that $Cf(z) \leq g(z)$.

Acknowledgements. I thank Edgar Tchoundja for many fruitful discussions on the relationship between symbols and on Carleson measures.

2. Hankel forms of weight zero

To find the Schatten-von Neumann class Hankel forms of weight zero on Hardy spaces we shall rewrite H_F^0 in terms of the small Hankel operators studied in [Z]. The problem then boils down to finding the relationship between the corresponding symbols.

The Hankel form H_G in [Z] is given by

$$H_G(f,g) = \int_{\partial\mathbb{B}} \overline{G(w)} f(w) g(w)\, d\sigma(w)\,, \tag{2}$$

where $d\sigma$ is the normalized area measure on $\partial\mathbb{B}$. By the reproducing property on the Hardy space $H^2(\partial\mathbb{B})$ we have the following relationship between F and G.

Lemma 2.1. *Let H^0_F be given by (1) and H_G by (2). Then $H^0_F = H_G$, if and only if*

$$(R+d)_d G(w) = c_d (d)_d F(w)\,, \tag{3}$$

where c_d is a normalization constant for dm on $\mathbb{B}$.

Proof. Let $f, g \in H^2(\partial\mathbb{B})$. Using the reproducing property of fg and Fubini-Tonelli's theorem,

$$\int_{\mathbb{B}} f(z)g(z)\overline{F(z)}(1-|z|^2)^{d-1}\,dm(z) \\ = \int_{\partial\mathbb{B}} f(w)g(w) \int_{\mathbb{B}} \frac{\overline{F(z)}(1-|z|^2)^{d-1}}{(1-\langle z,w\rangle)^d}\,dm(z)\,d\sigma(w)\,.$$

Hence $H^0_F = H_G$ if and only if

$$G(w) = \int_{\mathbb{B}} \frac{F(z)(1-|z|^2)^{d-1}}{(1-\langle w,z\rangle)^d}\,dm(z)\,. \tag{4}$$

Apply the radial differentiation R,

$$\begin{aligned} & (RG)(w) \\ = \; & \sum_{i=1}^{d} w_i \frac{\partial G}{\partial w_i}(w) = d\int_{\mathbb{B}} \frac{\langle w,z\rangle F(z)(1-|z|^2)^{d-1}}{(1-\langle w,z\rangle)^{d+1}}\,dm(z) \\ = \; & -d\int_{\mathbb{B}} \frac{F(z)(1-|z|^2)^{d-1}}{(1-\langle w,z\rangle)^{d}}\,dm(z) + d\int_{\mathbb{B}} \frac{F(z)(1-|z|^2)^{d-1}}{(1-\langle w,z\rangle)^{d+1}}\,dm(z)\,, \end{aligned}$$

so that

$$(R+d)G(w) = d\int_{\mathbb{B}} \frac{F(z)(1-|z|^2)^{d-1}}{(1-\langle w,z\rangle)^{d+1}}\,dm(z)\,.$$

Repeating this procedure,

$$(R+d)_d G(w) = (d)_d \int_{\mathbb{B}} \frac{F(z)(1-|z|^2)^{d-1}}{(1-\langle w,z\rangle)^{2d}}\,dm(z)\,.$$

Hence, by the reproducing property on the Bergman space $L^2_a(dm)$, equation (4) can be reformulated into

$$(R+d)_d G(w) = c_d (d)_d F(w)\,, \tag{5}$$

where c_d is a normalization constant for dm on $\mathbb{B}$. On the other hand, if $F(w) = (R+d)_d G(w)/c_d(d)_d$ then equation (4) holds by symmetry of $(R+a)$, $a>0$, w.r.t. the inner product

$$\int_{\mathbb{B}} h_1(z)\overline{h_2(z)}(1-|z|^2)^{d-1}\,dm(z)\,.$$

□

Remark 2.2. For convenience we denote $D = R+1$. Then D is symmetric w.r.t. the inner product

$$\langle h_1, h_2\rangle_{\alpha'} = \int_{\mathbb{B}} h_1(z)\overline{h_2(z)}(1-|z|^2)^{\alpha'}\,dm(z)$$

where $\alpha' > -1$ and $h_1, h_2 : \mathbb{B} \to \mathbb{C}$ are holomorphic.

2.1. **Schatten-von Neumann class $\mathcal{S}_p$ Hankel forms.** In this subsection we present sufficient and necessary conditions for Hankel forms of weight zero to be in Schatten-von Neumann class $\mathcal{S}_p$, $1 \le p < \infty$, see Theorem 2.5, and the following lemmas are useful in the proof this theorem.

Lemma 2.3. *Let $a_1, \cdots a_k, b_1, \cdots, b_k > 0$, $\alpha > -1$ and $1 < p < \infty$. Then*

$$\|(R+a_k)\cdots(R+a_1)f\|_\alpha \simeq \|(R+b_k)\cdots(R+b_1)f\|_\alpha$$

for all holomorphic $f : \mathbb{B} \to \mathbb{C}$, where $\|f\|_\alpha = \|f\|_{L^p((1-|z|^2)^\alpha dm(z))}$.

Proof. This result follows using the same arguments as in the proof of Theorem 5.3 in [BB]. □

Lemma 2.4. *If $\alpha > -1$, then*

$$\left\|\big((R+\alpha+d+1)f\big)(\cdot)(1-|\cdot|^2)\right\|_\alpha \simeq \|f\|_\alpha\,,$$

for all holomorphic $f : \mathbb{B} \to \mathbb{C}$.

Proof. If $\beta > 0$, then

$$\frac{1}{\beta}\left((R+\beta)\left(\frac{1}{(1-\langle\cdot, w\rangle)^\beta}\right)\right)(z) = \frac{1}{(1-\langle z, w\rangle)^{\beta+1}}\,,$$

and hence the result follows by Theorem 2.19 in [Zhu1]. □

Theorem 2.5. *The Hankel form H_F^0 is of Schatten-von Neumann class $\mathcal{S}_p$, for $1 < p < \infty$, if and only*

$$\|F(\cdot)(1-|\cdot|^2)^d\|_{L^p(d\iota)} = \left(\int_{\mathbb{B}} \left|F(z)(1-|z|^2)^d\right|^p\,d\iota(z)\right)^{1/p} < \infty\,.$$

Also, H_F^0 is of trace class $\mathcal{S}_1$ if and only if $DF \in L^1(dm)$.

Remark 2.6. The measure $d\iota(z) = (1-|z|^2)^{-(d+1)}\, dm(z)$ is a Möbius invariant measure on $\mathbb{B}$.

To prove the theorem we use Theorem 1 in [Z] (see also Theorem C in [FeldR]).

Theorem 2.7. *Let $\alpha > -1$ and $1 \le p < \infty$. Then the Hankel form H_G, defined by (2), is of Schatten-von Neumann class $\mathcal{S}_p$ if and only if*

$$\sum_{|\alpha|=d+1} \left\| (\partial^\alpha G(\cdot))\, (1-|\cdot|^2)^{d+1} \right\|_{L^p(d\iota)} < \infty.$$

Proof of Theorem 2.5. We shall make use of the fact that $H_F^0 = H_G$ if and only if F and G satisfies equation (3) which follows by Lemma 2.1. Then $DF(z) = c_d D(R+d)_d G(z)$. In view of Theorem 2.7, it is enough to prove that, for $1 < p < \infty$,

$$\left\| F(\cdot)(1-|\cdot|^2)^d \right\|_{L^p(d\iota)} \simeq \left\| (DF(\cdot))\, (1-|\cdot|^2)^{d+1} \right\|_{L^p(d\iota)} \tag{6}$$

and that, for $1 \le p < \infty$,

$$\begin{aligned} &\left\| \left(D^{d+1}G(\cdot)\right) (1-|\cdot|^2)^{d+1} \right\|_{L^p(d\iota)} \\ &\quad \simeq \sum_{|\alpha|=d+1} \left\| (\partial^\alpha G(\cdot))\, (1-|\cdot|^2)^{d+1} \right\|_{L^p(d\iota)} + \sum_{|\alpha|\le d} \left| (\partial^\alpha G)\,(0) \right|, \end{aligned} \tag{7}$$

since then it will follow by Lemma 2.3 that

$$\begin{aligned} &\left\| F(\cdot)(1-|\cdot|^2)^d \right\|_{L^p(d\iota)} \\ &\quad \simeq \left\| (DF(\cdot))\, (1-|\cdot|^2)^{d+1} \right\|_{L^p(d\iota)} \\ &\quad \simeq \left\| \left(D^{d+1}G(\cdot)\right) (1-|\cdot|^2)^{d+1} \right\|_{L^p(d\iota)} \\ &\quad \simeq \sum_{|\alpha|=d+1} \left\| (\partial^\alpha G(\cdot))\, (1-|\cdot|^2)^{d+1} \right\|_{L^p(d\iota)} + \sum_{|\alpha|\le d} \left| (\partial^\alpha G)\,(0) \right|. \end{aligned}$$

Actually, (6) is a direct consequence of Lemma 2.3 and Lemma 2.4, and (7) is a consequence of Theorem 5.3 in [BB]. □

2.2. **Bounded Hankel forms.** In this subsection we present a necessary condition for Hankel forms of weight zero to be bounded, see Theorem 2.16. First we need some preliminaries, which basically can be found in [Zhu1] but is presented here just to make it easier for the reader. We remark also that equivalence in Lemma 2.12 holds in the one dimensional case, due to Corollary 15 in [Zhu2].

Definition 2.8 (See [Zhu1])**.** Let $\zeta \in \partial\mathbb{B}$ and $r > 0$ and let

$$Q_r(\zeta) = \{z \in \mathbb{B} : d(z, \zeta) < r\}$$

where $d(z, \zeta) = |1 - \langle z, \zeta\rangle|^{1/2}$ is the non-isotropic metric on $\partial\mathbb{B}$. A positive Borel measure μ in $\mathbb{B}$ is called a *Carleson* measure if there exists a constant $C > 0$ such that

$$\mu((Q_r(\zeta))) \leq Cr^{2d}$$

for all $\zeta \in \partial\mathbb{B}$ and $r > 0$.

Lemma 2.9 (Theorem 5.4 in [Zhu1])**.** *A positive Borel measure μ in $\mathbb{B}$ is Carleson if and only if*

$$\sup_{z\in\mathbb{B}} \int_{\mathbb{B}} P(z, w)\, d\mu(w) < \infty\,,$$

where

$$P(z, w) = \frac{(1 - |z|^2)^d}{|1 - \langle z, w\rangle|^{2d}}\,; \quad z, w \in \mathbb{B}\,.$$

Lemma 2.10 (Theorem 5.9 in [Zhu1])**.** *A positive Borel measure μ in $\mathbb{B}$ is Carleson if and only if there exists a constant $C > 0$ such that*

$$\int_{\mathbb{B}} |f(z)|^2\, d\mu(z) \leq C\|f\|^2_{H^2(\partial\mathbb{B})}$$

for all $f \in H^2(\partial\mathbb{B})$.

Lemma 2.11 (Theorem 50 in [ZhZh])**.** *Let μ be a positive Borel measure in $\mathbb{B}$. Then the following conditions are equivalent*

(a) *There is a constant $C > 0$ such that*

$$\int_{\mathbb{B}} |(Rf)(z)|^2 d\mu(z) \leq C\|f\|^2_{H^2(\partial\mathbb{B})}$$

for all $f \in H^2(\partial\mathbb{B})$.

(b) *There is a constant $C > 0$ such that*

$$\mu(Q_r(\zeta)) \leq Cr^{2(d+2)}$$

for all $r > 0$ and $\zeta \in \partial\mathbb{B}$.

Lemma 2.12. *Let $\alpha > -1$. For any holomorphic function $g : \mathbb{B} \to \mathbb{C}$, if $d\mu_1(z) = |g(z)|^2(1 - |z|^2)^\alpha\, dm(z)$ is a Carleson measure then so is $d\mu_2(z) = |Rg(z)|^2(1 - |z|^2)^{\alpha+2}\, dm(z)$.*

Proof. If $d\mu_1$ is a Carleson measure then there is a constant $C > 0$ such that

$$\int_{Q_r(\zeta)} (1-|z|^2)^2\, d\mu_1(z) \le 4r^4 \int_{Q_r(\zeta)} d\mu_1(z) \le 4Cr^{2(d+2)}\,,$$

for all $r > 0$ and $\zeta \in \mathbb{B}$, so that $(1-|z|^2)^2 d\mu_1(z)$ satisfies the condition (b) in Lemma 2.11. Hence there is a constant $C_1 > 0$ such that

$$\int_{\mathbb{B}} |(Rf)(z)|^2 (1-|z|^2)^2\, d\mu_1(z) \le C_1 \|f\|_{H^2(\partial\mathbb{B})} \tag{8}$$

for all $f \in H^2(\partial\mathbb{B})$. By Theorem 2.16 in [Zhu1] (used on fg, assuming, without loss of generality, that $f(0) = 0$) and by inequality (8),

$$\begin{aligned}
&\left(\int_{\mathbb{B}} |f(z)|^2\, d\mu_2(z)\right)^{1/2} \\
&\le \left(\int_{\mathbb{B}} |(R(fg))(z)|^2 (1-|z|^2)^{\alpha+2}\, dm(z)\right)^{1/2} + \\
&\qquad\qquad \left(\int_{\mathbb{B}} |(Rf)(z)|^2 (1-|z|^2)^2\, d\mu_1(z)\right)^{1/2} \\
&\le C_2 \left(\int_{\mathbb{B}} |f(z)|^2\, d\mu_1(z)\right)^{1/2} + C_1 \|f\|_{H^2(\partial\mathbb{B})} \le C_3 \|f\|_{H^2(\partial\mathbb{B})}
\end{aligned}$$

for all $f \in H^2(\partial\mathbb{B})$ so that $d\mu_2$ is Carleson by Lemma 2.10. □

Definition 2.13 (See [Zhu1])**.** Let BMOA denote the space of functions $f \in H^2(\partial\mathbb{B})$ such that

$$\|f\|^2_{BMO} = |f(0)|^2 + \sup_{Q(\zeta,r)} \frac{1}{Q(\zeta,r)} \int_{Q(\zeta,r)} |f(\xi) - f_{Q(\zeta,r)}|^2\, d\sigma(\xi) < \infty\,,$$

where, for any $\zeta \in \partial\mathbb{B}$ and $r > 0$,

$$Q(\zeta,r) = \left\{\xi \in \partial\mathbb{B} : |1 - \langle\zeta,\xi\rangle|^{1/2} < r\right\},$$

and

$$f_{Q(\zeta,r)} = \frac{1}{Q(\zeta,r)} \int_{Q(\zeta,r)} f(\xi)\, d\sigma(\xi)\,.$$

Lemma 2.14 (Theorem 5.3 in [Zhu1])**.** *A function $f \in H^2(\partial\mathbb{B})$ belongs to BMOA if and only if*

$$\sup_{z\in\mathbb{B}} \int_{\partial\mathbb{B}} |f(\varphi_z(\zeta)) - f(z)|^2\, d\sigma(\zeta) < \infty\,,$$

where φ_z is the linear fractional map given by (8) in [Su1].

Lemma 2.15. *If f is in BMOA, then $|f(z)|^2\, dm(z)$ is a Carleson measure on $\mathbb{B}$.*

Proof. If f is in BMOA, then there is a constant $C > 0$ such that

$$\begin{aligned} &\sup_{z\in\mathbb{B}} \int_{\mathbb{B}} P(z,w)|f(w) - f(0)|^2\, dm(w) \\ &\quad\leq\ C \cdot \sup_{z\in\mathbb{B}} \int_{\partial\mathbb{B}} P(z,\zeta)|f(\zeta) - f(0)|^2\, d\sigma(\zeta) \\ &\quad=\ C \cdot \sup_{z\in\mathbb{B}} \int_{\partial\mathbb{B}} |f(\varphi_z(\zeta)) - f(z)|^2\, d\sigma(\zeta) < \infty\,, \end{aligned}$$

by Lemma 2.14. Then $|f(w)-f(0)|^2\, dm(w)$ is Carleson by Lemma 2.9, so that $|f(w)|^2\, dm(w)$ is Carleson. □

Theorem 2.16. *If the Hankel form H_F^0 is bounded then*

$$|F(z)|^2(1 - |z|^2)^{2d-1}\, dm(z)$$

is a Carleson measure on $\mathbb{B}$.

Proof. The classical Hankel form (small Hankel operator) H_G on the Hardy space $H^2(\partial\mathbb{B})$, as in [Z], is bounded if and only if $G \in BMOA$ and by Theorem 5.14 in [Zhu1],

$$G \in BMOA \quad \Longleftrightarrow \quad |(RG)(z)|^2(1 - |z|^2)\, dm(z) \text{ is Carleson}\,.$$

Now, $H_F^0 = H_G$ if and only if the equation (3) holds. Hence, if H_F^0 is bounded, then $|(RG)(z)|^2(1 - |z|^2)\, dm(z)$ is a Carleson measure and, since G is in BMOA, then $|((R+d)G)(z)|^2(1-|z|^2)\, dm(z)$ is a Carleson measure by Lemma 2.15. Using Lemma 2.12,

$$|(R(R + d)G)(z)|^2(1 - |z|^2)^{2\cdot 2-1} \quad \text{is Carleson,}$$

and hence $|((R+d+1)(R+d)G)(z)|^2(1 - |z|^2)^{2\cdot 2-1}\, dm(z)$ is Carleson. Repeating this procedure we get that

$$|((R + d)_d G)(z)|^2(1 - |z|^2)^{2d-1}\, dm(z) \quad \text{is Carleson.}$$

By equation (3), the proof is complete. □

3. The case $s = 1, 2, 3, \cdots$

In this section we study the class $\mathcal{S}_p$ properties, $1 \le p \le 2$, for the case $s \ge 1$. Denote by $\mathcal{H}^p_{d,s}$ the space of holomorphic functions $F : \mathbb{B} \to \odot^s V'$ such that $\|F\|_{d,s,p} < \infty$ where

$$\|F\|_{d,s,p} = \left(\int_{\mathbb{B}} \left\langle \otimes^s B^t(z,z) F(z), F(z) \right\rangle^{p/2} (1-|z|^2)^{pd} \frac{dm(z)}{(1-|z|^2)^{d+1}} \right)^{1/p} .$$

This space is a well-defined Banach space if $1 \le p < \infty$. Also, denote by $\mathcal{H}^\infty_{d,s}$ the space of holomorphic functions such that

$$\|F\|_{d,s,\infty} = \sup_{z \in \mathbb{B}} \left\| (1-|z|^2)^d \otimes^s B^t(z,z)^{1/2} F(z) \right\| < \infty .$$

3.1. **Results about $\mathcal{H}^p_{\nu,s}$ for $\nu \ge d$.** In [Su1] and in [Su2] there are several results about $\mathcal{H}^p_{\nu,s}$ for $s = 0, 1, 2, \cdots$ and $\nu > d$. Now, if we consider $s = 1, 2, \cdots$, i.e., $s \ne 0$, then we can use the same arguments as in [Su1] and [Su2] to generalize results about $\mathcal{H}^p_{\nu,s}$ for $\nu > d$ to $\nu \ge d$, where $1 \le p \le \infty$. Hence, the results below will be stated without proofs. The reader is referred to [Su1] and [Su2] for more details.

Lemma 3.1. *Let $\nu \ge d$ and let s be a positive integer. Then the reproducing kernel of $\mathcal{H}^2_{\nu,s}$ is, up to a nonzero constant c, given by*

$$K_{\nu,s}(w,z) = (1 - \langle w, z \rangle)^{-2\nu} \otimes^s B^t(w,z)^{-1} .$$

Namely, for any $v \in \odot^s V'$ and any $F \in \mathcal{H}^2_{\nu,s}$,

$$\begin{aligned} \langle F(z), v \rangle &= c \langle F, K_{\nu,s}(\cdot, z) v \rangle_{\nu,s,2} \\ &= c \int_{\mathbb{B}} \left\langle \otimes^s B^t(w,w) F(w), K_{\nu,s}(w,z) v \right\rangle (1-|w|^2)^{2\nu} \, d\iota(w) . \end{aligned}$$

Let $\mathcal{H}'_{\nu,s}$ be the space of holomorphic functions $F : \mathbb{B} \to \odot^s V'$ such that the corresponding bilinear Hankel form on $H^2(\partial\mathbb{B}) \otimes H^2(\partial\mathbb{B})$, defined by (1), is of Hilbert-Schmidt class $\mathcal{S}_2$. The norm on $\mathcal{H}'_{\nu,s}$ is given by $\|F\|'_{\nu,s} = \|H^s_F\|_{\mathcal{S}_2}$.

Theorem 3.2. *Let $\nu \ge d$ and let s be a nonnegative integer. Then there is a constant $C_{\nu,s}$ such that*

$$\|F\|'_{\nu,s} = C_{\nu,s} \|F\|_{\nu,s,2}$$

Theorem 3.3. *Let $\nu \geq d$ and let s be a positive integer. If $1 < p < 2$, then*

$$\mathcal{H}^p_{\nu,s} = (\mathcal{H}^1_{\nu,s}, \mathcal{H}^2_{\nu,s})_{[2(1-1/p)]} \,.$$

Theorem 3.4. *Let $\nu \geq d$ and let s be a positive integer. Then $F \in \mathcal{H}^1_{d,s}$ if and only if there is a function $a \in l^1(\mathbb{B}, \odot^s V')$ with support in $\{z_j\}_{j=1}^{\infty} \subset \mathbb{B}$, $a_j = a(z_j)$, with $\sum_{j=0}^{\infty} \|a_j\| < \infty$ such that*

$$F(w) = \sum_{j=1}^{\infty} (1 - |z_j|^2)^d K_{d,s}(w, z_j) \otimes^s B^t(z_j, z_j)^{1/2} a_j \,. \tag{9}$$

3.2. **The transvectant.** If we were able to prove that $\mathcal{T}_s(f,g) \in \mathcal{H}^1_{d,s}$, for positive s, that is generalize the analogous result for the case of Bergman spaces (see Lemma 2.7 in [Su2]), then boundedness properties and compactness properties would follow in the same way as for the case of Bergman spaces, see [Su1]. But, unfortunately, we can find $f, g \in H^2(\partial\mathbb{B})$ such that $\|\mathcal{T}_s(f,g)\|_{d,s,1} = \infty$.

Example 3.5. This example is based on the proof of Theorem II in [Ru]. First consider the case when $s = 1$ and $d = 1$. Let

$$f(z) = \sum_{k=1}^{\infty} \frac{1}{k} z^{2^k} \quad \text{and} \quad g(z) = 1 \,.$$

Then $f, g \in H^2(\partial\mathbb{D})$ and since the series $f(z)$ is lacunary then

$$\|\mathcal{T}_1(f,g)\|_{1,1,1} = \int_{\mathbb{D}} |f'(z)| \, dm(z) = \infty \,.$$

This is a consequence of a result about lacunary series by Zygmund, see [Ru]. Namely, if $n_{k+1}/n_k > \lambda$ for some $\lambda > 1$, and if $h(z) = \sum_{k=0}^{\infty} c_k z^{n_k}$ satisfies

$$\int_0^1 |h'(re^{i\theta})| \, dr < \infty$$

for some θ, then $\sum_{k=0}^{\infty} |c_k| < \infty$.

In the general case, $d \geq 1$ and $s = 1, 2, \cdots$, we just change f into

$$f(z) = \sum_{k=1}^{\infty} \frac{1}{k} z_1^{2^k} \,,$$

and still let $g(z) = 1$. Then

$$\begin{aligned}\|\mathcal{T}_s(f,g)\|_{d,s,1} &= \int_{\mathbb{B}} (1-|z_1|^2)^{s/2}(1-|z|^2)^{s/2-1}\left|\frac{\partial^s f}{\partial z_1^s}(z)\right| dm(z)\\ &\geq \int_{\mathbb{B}} (1-|z|^2)^{s-1}\left|\frac{\partial^s f}{\partial z_1^s}(z)\right| dm(z)\,.\end{aligned}$$

By Theorem 2.17 in [Zhu1] there is a constant $C > 0$ such that

$$\int_{\mathbb{B}} (1-|z|^2)^{s-1}\left|\frac{\partial^s f}{\partial z_1^s}(z)\right| dm(z) \geq C\int_{\mathbb{B}} \left|\frac{\partial f}{\partial z_1}(z)\right| dm(z)$$

and the right hand side of the inequality above is infinite, as we can see in the initial case ($s = 1$, $d = 1$).

But, what we can prove is the following Lemma.

Lemma 3.6. *Let s be a nonnegative integer and let $\varepsilon > 0$. Then there is a constant $C_\varepsilon > 0$ such that*

$$\int_{\mathbb{B}} \left\|\otimes^s B^t(z,z)^{1/2}\mathcal{T}_s(f,g)(z)\right\| (1-|z|^2)^{\varepsilon-1}\, dm(z) \leq C_\varepsilon \cdot \|f\|_{H^2}\cdot\|g\|_{H^2}\,.$$

Proof. It follows by exactly the same arguments as in the proof of Theorem 4.1 in [Su1] that

$$(10)\qquad \left(\int_{\mathbb{B}} \left\langle \otimes^k B^t(z,z)\partial^k f(z), \partial^k f(z)\right\rangle \frac{dm(z)}{(1-|z|^2)}\right)^{1/2} \leq C_{d,k}\cdot\|f\|_{H^2}\,.$$

This will be enough since $\mathcal{T}_s(f,g)$ is a linear combination of terms $\partial^k f(z)\otimes\partial^{s-k}g(z)$ and by Hölder's inequality

$$\left\|\otimes^s B^t(\cdot,\cdot)^{1/2}\partial^k f(\cdot)\otimes\partial^{s-k}g(\cdot)\right\|_{L^1((1-|z|^2)^{-1}dm)} \leq C_{d,s}\cdot\|f\|_{H^2}\cdot\|g\|_{H^2}$$

if $k \neq 0$ and for $k = 0$ it follows that

$$\begin{aligned}&\int_{\mathbb{B}} \left\|\otimes^s B^t(z,z)^{1/2} g(z)\partial^s f(z)\right\| (1-|z|^2)^{\varepsilon-1}\, dm(z)\\ \leq\ & C_{d,s}\cdot\|f\|_{H^2}\cdot\left(\int_{\mathbb{B}} |g(z)|^2(1-|z|^2)^{\varepsilon-1}\, dm(z)\right)^{1/2}\\ \leq\ & C_\varepsilon\cdot\|f\|_{H^2}\cdot\|g\|_{H^2}\,.\end{aligned}$$

□

3.3. Class $\mathcal{S}_p$ Hankel forms for $1 \le p \le 2$.

Theorem 3.7. *Let s be a positive integer. If $F \in \mathcal{H}^1_{d,s}$ then the corresponding Hankel form H^s_F is of class $\mathcal{S}_1$.*

Proof. Let $F \in \mathcal{H}^1_{d,s}$. Then, by Theorem 3.4 we can write

$$F(w) = \sum_{j=1}^{\infty} F_j(w)$$

where $F_j(w) = (1-|z_j|^2)^d K_{d,s}(w, z_j) \otimes^s B^t(z_j, z_j)^{1/2} a_j$. As a consequence of Lemma 7.1 in [Su1],

$$\|F_j\|' = \sup_{w\in\mathbb{B}} \| \otimes^s B^t(w,w)^{1/2} F_j(w)\| \le 2^d (1-|z_j|^2)^{-d} \|a_j\| \,. \tag{11}$$

By Lemma 3.6 and (11) it then follows that

$$\begin{aligned} &\int_{\mathbb{B}} \left|\left\langle \otimes^s B^t(w,w) \mathcal{T}_s(f,g)(w), F_j(w)\right\rangle\right| (1-|w|^2)^{d-1}\, dm(w) \\ \le\ & \|F_j(w)\|' \cdot \int_{\mathbb{B}} \left\|\otimes^s B^t(w,w)^{1/2} \mathcal{T}_s(f,g)(w)\right\| (1-|w|^2)^{d-1}\, dm(w) \\ \le\ & C_d \cdot (1-|z_j|^2)^{-d} \cdot \|a_j\| \cdot \|f\|_{H^2} \cdot \|g\|_{H^2} < \infty \,. \end{aligned}$$

Hence, by the reproducing property,

$$H_{F_j}(f,g) = c\left\langle \mathcal{T}_s(f,g)(z_j), (1-|z_j|^2)^d \otimes^s B^t(z,z)^{1/2} a_j \right\rangle \,.$$

The bilinear form $(f,g) \to \mathcal{T}_s(f,g)(z_j)$ is a sum of finitely many rank one forms where the number of summands M_s only depends on s. We see this by writing $f(z_j) = c\langle f, K_{z_j}\rangle_{H^2}$, where $K_{z_j}(w) = (1-\langle w, z_j\rangle)^{-d}$, so that

$$\partial^{s-k} f(z_j) \otimes \partial^k g(z_j) = c^2 \langle f, \overline{\partial^{s-k}} K_{z_j}\rangle_{H^2} \otimes \langle g, \overline{\partial^k} K_{z_j}\rangle_{H^2} \,.$$

Hence

$$\|H^s_{F_j}\|_{\mathcal{S}_1} \le \sqrt{M_s} \cdot \|H_{F_j}\|_{\mathcal{S}_2}$$

for all $j = 1, 2, \cdots$ so by Theorem 3.2 it follows that

$$\|H^s_F\|_{\mathcal{S}_1} \le \sum_{j=1}^{\infty} \|H^s_{F_j}\|_{\mathcal{S}_1} \le \sqrt{M_s} \cdot \sum_{j=1}^{\infty} \|H^s_{F_j}\|_{\mathcal{S}_2} = C \cdot \sum_{j=1}^{\infty} \|F_j\|_{\mathcal{H}^2_{d,s}}$$

and

$$\|F_j\|^2_{\mathcal{H}^2_{d,s}} = c' \cdot \|a_j\|^2$$

by the reproducing property. Thus

$$\|H_F\|_{\mathcal{S}_1} \leq C' \cdot \sum_{j=1}^{\infty} \|a_j\| < \infty .$$

This completes the proof. □

Corollary 3.8. *The map* $\Gamma : \mathcal{H}^1_{d,s} \to \mathcal{S}_2$, $\Gamma(F) = H^s_F$, *is bounded.*

Proof. This follows immediately from the last inequality in the proof of the theorem above and from the fact that $\|F\|_{d,s,1}$ is equivalent to

$$\|F\|_{\text{inf}} = \inf \left\{ \sum_{j=1}^{\infty} \|a_j\| : \{a_j\}_{j=1}^{\infty} \text{ defines } F \text{ by } (9) \right\}. \tag{12}$$

We need to prove that $\|F\|_{d,s,1}$ is equivalent to $\|F\|_{\text{inf}}$. If we let $\mathcal{B}$ be the Banach space of holomorphic $F : \mathbb{B} \to \odot^s V'$ such that $\|F\|_{\text{inf}} < \infty$, then the bijection $I : \mathcal{H}^1_{d,s} \to \mathcal{B}$, $F \mapsto F$, is bounded. Hence, by the Open Mapping Theorem $I : \mathcal{B} \to \mathcal{H}^1_{d,s}$ is also bounded, and thus we get equivalent norms. □

Theorem 3.9. *Let s be a positive integer and let $1 \leq p \leq 2$. Then* $\Gamma : \mathcal{H}^p_{d,s} \to \mathcal{S}_p$, $\Gamma(F) = H^s_F$, *is bounded.*

Proof. By Corollary 3.8 and by Theorem 3.2 it follows that $\Gamma : \mathcal{H}^i_{d,s} \to \mathcal{S}_i$ is bounded for $i = 1, 2$ respectively. Then the theorem follows by interpolation and Theorem 3.3. □

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