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Abstract

Let γ be the standard Gauss measure on $\mathbf{R}^n$, let $\Phi(t) = \int_{-\infty}^{t} \exp(-s^2/2)ds/\sqrt{2\pi}$, $-\infty \leq t \leq \infty$, and let $m \geq 2$ be an integer. Given m positive real numbers $\alpha_1, ..., \alpha_m$ this paper gives a necessary and sufficient condition such that the inequality $\Phi^{-1}(\gamma(\alpha_1 A_1 + ... + \alpha_m A_m)) \geq \alpha_1 \Phi^{-1}(\gamma(A_1)) + ... + \alpha_m \Phi^{-1}(\gamma(A_m))$ is true for all Borel sets $A_1, ..., A_m$ in $\mathbf{R}^n$ of positive γ-measure or all convex Borel sets $A_1, ..., A_m$ in $\mathbf{R}^n$ of positive γ-measure, respectively. In particular, the paper exhibits inequalities of the Brunn-Minkowski type for γ which are true for all convex sets but not for all measurable sets.

1 Introduction

The main purpose of this paper is to study inequalities of the Brunn-Minkowski type for Gaussian measures when the number of sets involved is more than two. Besides, we will show that an inequality of the Brunn-Minkowski type for Gaussian measures, valid for all convex sets, in general, will not be true for all measurable sets. First, however, we will introduce more precise definitions.

Let γ be the standard Gaussian measure on $\mathbf{R}^n$, that is

$$d\gamma(x) = e^{-|x|^2/2} \frac{dx}{\sqrt{2\pi}^n}$$

where

$$| x |= \sqrt{\sum_1^n x_k^2} \text{ if } x = (x_1, ..., x_n) \in \mathbf{R}^n$$

and let

$$\Phi(t) = \int_{-\infty}^t e^{-s^2/2} \frac{ds}{\sqrt{2\pi}}, \quad -\infty \leq t \leq \infty.$$

Moreover, if $A_1, ..., A_m$ are subsets of $\mathbf{R}^n$ and $(\alpha_1, ..., \alpha_m) \in \mathbf{R}^m$, the linear combination $\alpha_1 A_1 + ... + \alpha_m A_m$ of the sets $A_1, ..., A_m$ equals

$$\{y; \; y = \alpha_1 x_1 + ... + \alpha_m x_m \text{ where } x_i \in A_i, \; i = 1, ..., m\}.$$

Recall from the theory of analytic sets that a linear combination of Borel subsets of $\mathbf{R}^n$ is universally Borel measurable, that is μ-measurable with respect to every finite positive Borel measure on $\mathbf{R}^n$. Below $\mathcal{B}(\mathbf{R}^n)$ stands for the Borel field of $\mathbf{R}^n$ and $\mathbf{R}_+$ means the open interval $]0, \infty[$.

Now suppose $m \geq 2$ is a fixed integer and denote by $\mathcal{S}_m$ the set of all $\alpha = (\alpha_1, ..., \alpha_m) \in \mathbf{R}_+^m$ such that

$$\Phi^{-1}(\gamma(\alpha_1 A_1 + ... + \alpha_m A_m)) \geq \alpha_1 \Phi^{-1}(\gamma(A_1)) + ... + \alpha_m \Phi^{-1}(\gamma(A_m)) \quad (1.1)$$

where $A_1, ..., A_m \in \mathcal{B}(\mathbf{R}^n)$. Here, if $t_1, ..., t_m \in [-\infty, \infty]$, the sum $\Sigma_1^m t_i$ is defined to be equal to $-\infty$ if some of the $t_i = -\infty$ and in other cases has its usual meaning.

The main concern in this paper is to find an explicit form of the set $\mathcal{S}_m$. In the special case $m = 2$ the above problem was recently solved by the author who proved that

$$\mathcal{S}_2 = \left\{\alpha \in \mathbf{R}_+^2; \; \alpha_1 + \alpha_2 \geq 1 \text{ and } | \alpha_1 - \alpha_2 | \leq 1\right\} \quad (1.2)$$

(see [3]). By applying this result we will show that

$$\mathcal{S}_m = \left\{\alpha \in \mathbf{R}_+^m; \; \sum_1^m \alpha_i \geq \max(1, -1 + 2 \max_{1 \leq i \leq m} \alpha_i)\right\}. \quad (1.3)$$

Given $(\alpha_1, ..., \alpha_m) \in \mathbf{R}_+^m$, clearly equality occurs in (1.1) if $A_1, ..., A_m$ are parallel affine half-spaces and if, in addition, $\alpha_1 + ... + \alpha_m = 1$ equality occurs in (1.1) if $A_1, ..., A_m$ are equal and convex.

We will also consider the inequality (1.1) restricted to convex Borel sets. To be more precise let m be as above and denote by $\mathcal{C}_m$ the set of all $\alpha = (\alpha_1, ..., \alpha_m) \in \mathbf{R}_+^m$ such that the inequality (1.1) holds for arbitrary convex sets $A_1, ..., A_m \in \mathcal{B}(\mathbf{R}^n)$. Clearly, $\mathcal{S}_m \subseteq \mathcal{C}_m$. Moreover, it will be proved that

$$\mathcal{C}_m = \left\{ \alpha \in \mathbf{R}_+^m;\ \sum_1^m \alpha_i \geq 1 \right\}. \tag{1.4}$$

In particular, from (1.3) and (1.4), $\mathcal{S}_m \neq \mathcal{C}_m$ and consequently $\mathcal{S}_m$ is a proper subset of $\mathcal{C}_m$. From this we conclude that an inequality of the Brunn-Minkowski type for γ valid for all convex sets, in general, will not extend to all measurable sets. As far as we know a similar phenomenon has never been reported on before.

Note that the relations (1.3) and (1.4) show that the sets $\mathcal{S}_m$ and $\mathcal{C}_m$ are independent of n. In Section 4 we will point out that all the results in Sections 2 and 3 extend to centred Gaussian measures on real, separable Fréchet spaces. Thus, in particular, the results apply to Wiener measure, the probability law of Brownian motion.

Finally, in this section let us repeat more on the history of the problems we face here. Ehrhard [4] proved in 1983 that $\{\alpha \in \mathbf{R}_+^2;\ \alpha_1 + \alpha_2 = 1\} \subseteq \mathcal{C}_2$ and in 1996 the Ehrhard result was generalized by Latała [5] who established (1.1) when $m = 2$ and one of the Borel sets A_1 and A_2 is convex. In particular, this result has the isoperimetric inequality in Gauss space, independently due to Sudakov and Tsirelson [7] and the author [1] as an immediate corollary. Moreover, in 2003 the author [2] proved that $\{\alpha \in \mathbf{R}_+^2;\ \alpha_1 + \alpha_2 = 1\} \subseteq \mathcal{S}_2$ and, as already mentioned above, the description of the class $\mathcal{S}_2$ given by (1.2) goes back to my forth-coming paper [3] .

2 Characterization of $\mathcal{S}_m$ and $\mathcal{C}_m$

Theorem 2.1

$$\mathcal{C}_m = \left\{ \alpha \in \mathbf{R}_+^2;\ \sum_1^m \alpha_i \geq 1 \right\}.$$

The proof of Theorem 2.1 is based on the following

Lemma 2.1
$$\left\{\alpha \in \mathbf{R}_{+}^{m};\ \sum_{1}^{m} \alpha_i = 1\right\} \subseteq \mathcal{S}_m.$$

PROOF It follows from (1.2) that Lemma 2.1 holds for $m = 2$. If $m \geq 3$ and $\alpha \in \mathbf{R}_{+}^{m}$ satisfies $\alpha_1 + ... + \alpha_m = 1$, then

$$\Phi^{-1}(\gamma(\alpha_1 A_1 + ... + \alpha_m A_m))$$
$$= \Phi^{-1}(\gamma(\sigma(\frac{\alpha_1}{\sigma}A_1 + ... + \frac{\alpha_{m-1}}{\sigma}A_{m-1}) + \alpha_m A_m))$$

where $\sigma = \alpha_1 + ... + \alpha_{m-1}$. Now $(\sigma, \alpha_m) \in \mathcal{S}_2$ and we get

$$\Phi^{-1}(\gamma(\alpha_1 A_1 + ... + \alpha_m A_m))$$
$$\geq \sigma\Phi^{-1}(\gamma(\frac{\alpha_1}{\sigma}A_1 + ... + \frac{\alpha_{m-1}}{\sigma}A_{m-1})) + \alpha_m\Phi^{-1}(\gamma(A_m))$$

and the proof of Lemma 2.1 can be completed by induction on m.

PROOF OF THEOREM 2.1 We first prove that

$$\left\{\alpha \in \mathbf{R}_{+}^{2};\ \sum_{1}^{m} \alpha_i \geq 1\right\} \subseteq \mathcal{C}_m.$$

To this end let $\alpha \in \mathbf{R}_{+}^{m}$ be such that $\sigma =_{def} \alpha_1 + ... + \alpha_m \geq 1$. Futhermore, suppose $A_1, ..., A_m \in \mathcal{B}(\mathbf{R}^n)$ are convex and define

$$C = \frac{\alpha_1}{\sigma}A_1 + ... + \frac{\alpha_m}{\sigma}A_m$$

so that

$$\Phi^{-1}(\gamma(\alpha_1 A_1 + ... + \alpha_m A_m)) = \Phi^{-1}(\gamma(\sigma C)).$$

Now since C is convex an inequality by Sudakov and Tsirelson [7] states that

$$\Phi^{-1}(\gamma(\sigma C)) \geq \sigma\Phi^{-1}(\gamma(C))$$

(note that this inequality follows from the relations $(\sigma/2)C + (\sigma/2)C = \sigma C$ and $(\sigma/2, \sigma/2) \in \mathcal{S}_2$, see [3]). Hence

$$\Phi^{-1}(\gamma(\alpha_1 A_1 + ... + \alpha_m A_m)) \geq \sigma\Phi^{-1}(\gamma(\frac{\alpha_1}{\sigma}A_1 + ... + \frac{\alpha_m}{\sigma}A_m))$$

and by Lemma 2.1 the member in the right-hand side does not fall below

$$\alpha_1\Phi^{-1}(\gamma(A_1)) + ... + \alpha_m\Phi^{-1}(\gamma(A_m)).$$

Accordingly from this $\alpha \in \mathcal{C}_m$.

Next we claim that

$$\mathcal{C}_m \subseteq \left\{ \alpha \in \mathbf{R}^m_+;\ \sum_1^m \alpha_i \geq 1 \right\}.$$

To see this, let $C \in \mathcal{B}(\mathbf{R}^n)$ be a convex symmetric set such that $0 < \gamma(C) < \frac{1}{2}$. Then, if $\alpha \in \mathcal{C}_m$,

$$\alpha_1 C + ... + \alpha_m C = (\alpha_1 + ... + \alpha_m)C$$

and we get

$$\Phi^{-1}(\gamma((\alpha_1 + ... + \alpha_m)C)) \geq \alpha_1\Phi^{-1}(\gamma(C)) + ... + \alpha_m\Phi^{-1}(\gamma(C)).$$

Here, if $\alpha_1 + ... + \alpha_m < 1$ it follows that

$$\Phi^{-1}(\gamma(C)) \geq (\alpha_1 + ... + \alpha_m)\Phi^{-1}(\gamma(C))$$

or

$$0 \geq (\alpha_1 + ... + \alpha_m - 1)\Phi^{-1}(\gamma(C))$$

which is a contradiction. This proves Theorem 2.1.

Theorem 2.2

$$\mathcal{S}_m = \left\{ \alpha \in \mathbf{R}^m_+;\ \sum_{i=1}^m \alpha_i \geq \max(1, -1 + 2\max_{1\leq i\leq m} \alpha_i) \right\}.$$

The proof of Theorem 2.2 is based on two lemmas.

Lemma 2.2 *For any* $I \subseteq \{1, ..., m\}$ *the set*

$$\mathcal{P}_I = \left\{ \alpha \in \mathbf{R}^m_+;\ \sum_1^m \alpha_i \geq 1 \text{ and } \mid \sum_I \alpha_i - \sum_{I^c} \alpha_i \mid \leq 1 \right\}$$

is contained in $\mathcal{S}_m$.

PROOF If $I = \phi$ or $\{1, ..., m\}$, Lemma 2.1 shows that $\mathcal{P}_I \subseteq \mathcal{S}_m$. Therefore assume I is a non-empty proper subset of $\{1, ..., m\}$ and set $\sigma_0 = \Sigma_I \alpha_i$ and $\sigma_1 = \Sigma_{I^c} \alpha_i$. Now since $\sigma_0, \sigma_1 > 0$,

$$\Phi^{-1}(\gamma(\alpha_1 A_1 + ... + \alpha_m A_m))$$
$$= \Phi^{-1}(\gamma(\sigma_0 \sum_I \frac{\alpha_i}{\sigma_0} A_i + \sigma_1 \sum_{I^c} \frac{\alpha_i}{\sigma_1} A_i))$$

and the description of $\mathcal{S}_2$ given in (1.2) implies that

$$\Phi^{-1}(\gamma(\alpha_1 A_1 + ... + \alpha_m A_m))$$
$$\geq \sigma_0 \Phi^{-1}(\gamma(\sum_I \frac{\alpha_i}{\sigma_0} A_i)) + \sigma_1 \Phi^{-1}(\gamma(\sum_{I_c} \frac{\alpha_i}{\sigma_1} A_i)).$$

Here by Lemma 2.1 the last expression does not fall below

$$\sigma_0 \sum_I \frac{\alpha_i}{\sigma_0} \Phi^{-1}(\gamma(A_i)) + \sigma_1 \sum_{I^c} \frac{\alpha_i}{\sigma_1} \Phi^{-1}(\gamma(A_i))$$

$$= \alpha_1 \Phi^{-1}(\gamma(A_1)) + ... + \alpha_m \Phi^{-1}(\gamma(A_m))$$

which proves Lemma 2.2.

Lemma 2.3

$$\left\{ \alpha \in \mathbf{R}^m_+;\ \alpha = (t, t, ..., t) \text{ and } t \geq \frac{1}{m} \right\} \subseteq \mathcal{S}_m.$$

PROOF Let d denote the integer part of $m/2$. If $1/m \leq t \leq 1$ the m-tuple $(t, ..., t) \in \mathcal{P}_{\{1,...,d\}}$ and $(t, ..., t) \subseteq \mathcal{S}_m$ by Lemma 2.2. In addition, if $t \geq 1$, then

$$\Phi^{-1}(\gamma(tA_1 + ... + tA_m)) = \Phi^{-1}(\gamma(t(A_1 + ... + A_{m-1}) + tA_m))$$
$$\geq t\Phi^{-1}(\gamma(A_1 + ... + A_{m-1})) + t\Phi^{-1}(\gamma(A_m))$$

due to (1.2) and by induction on m it follows that $(t, ..., t) \in \mathcal{S}_m$. This proves Lemma 2.3.

PROOF OF THEOREM 2.2 The relation (1.2) shows that Theorem 2.2 is true if $m = 2$ and there is no loss of generality in assuming that $m \geq 3$.

We first prove that

$$\mathcal{S}_m \subseteq \left\{ \alpha \in \mathbf{R}_+^m;\ \sum_{i=1}^{m} \alpha_i \geq \max(1, -1 + 2 \max_{1\leq i\leq m} \alpha_i) \right\}.$$

To this end let $\alpha \in \mathcal{S}_m$ and first assume

$$\alpha_m - \alpha_1 - ... - \alpha_{m-1} > 1.$$

Then, if C is as in the proof of Theorem 2.1,

$$\mathbf{R}^n \setminus C \supseteq \alpha_1 C + ... + \alpha_{m-1} C + \alpha_m (\mathbf{R}^n \setminus C)$$

and we get

$$\Phi^{-1}(\gamma(\mathbf{R}^n \setminus C)) \geq (\alpha_1 + ... + \alpha_{m-1})\Phi^{-1}(\gamma(C)) + \alpha_m \Phi^{-1}(\gamma(\mathbf{R}^n \setminus C))$$

or

$$-\Phi^{-1}(\gamma(C)) \geq (\alpha_1 + ... + \alpha_{m-1})\Phi^{-1}(\gamma(C)) - \alpha_m \Phi^{-1}(\gamma(C))$$

since $\Phi^{-1}(1-y) = -\Phi^{-1}(y)$ for all $0 < y < 1$. Thus

$$0 > (\alpha_1 + ... + \alpha_{m-1} - \alpha_m + 1)\Phi^{-1}(\gamma(C))$$

which is a contradiction and we conclude that $\alpha_m - \alpha_1 - ... - \alpha_{m-1} \leq 1$. In a similar way it follows that

$$\alpha_k - \sum_{i\in\{1,...,k-1,k+1,...,m\}} \alpha_i \leq 1 \text{ if } k = 1, ..., m-1$$

and we get

$$\sum_{i=1}^{m} \alpha_i \geq -1 + 2 \max_{1 \leq i \leq m} \alpha_i .$$

Finally since $\mathcal{S}_m \subseteq \mathcal{C}_m$ Theorem 2.1 implies that

$$\sum_{i=1}^{m} \alpha_i \geq \max(1, -1 + 2 \max_{1 \leq i \leq m} \alpha_i) \ . \tag{2.1}$$

We next show that

$$\left\{ \alpha \in \mathbf{R}_+^m ; \ \sum_{i=1}^{m} \alpha_i \geq \max(1, -1 + 2 \max_{1 \leq i \leq m} \alpha_i) \right\} \subseteq \mathcal{S}_m .$$

Therefore suppose $\alpha \in \mathbf{R}_+^m$ satisfies (2.1) so that, in particular,

$$\alpha_k - \sum_{i \in \{1,...,k-1,k+1,...,m\}} \alpha_i \leq 1 \text{ if } k = 1, ..., m.$$

Now, if

$$\sum_{i \in \{1,...,k-1,k+1,...,m\}} \alpha_i - \alpha_k \leq 1$$

for some $k \in \{1, ..., m\}$ Lemma 2.2 proves that $\alpha \in \mathcal{S}_m$. On the other hand if

$$\sum_{i \in \{1,...,k-1,k+1,...,m\}} \alpha_i - \alpha_k > 1$$

for every $k \in \{1, ..., m\}$ we proceed as follows. Since $(\alpha_i)_{i=1}^m \in \mathcal{S}_m$ if and only if $(\alpha_{\sigma(i)})_{i=1}^m \in \mathcal{S}_m$ for a suitable permutation $\sigma : \{1, ..., m\} \to \{1, ..., m\}$ it can be assumed that

$$\alpha_1 \leq \alpha_2 \leq ... \leq \alpha_m$$

and, hence,

$$\sum_{1}^{m-1} \alpha_i > 1 + \alpha_m$$

and

$$\sum_{1}^{m-1} \frac{\alpha_i}{\alpha_m} > 1. \tag{2.2}$$

Next we claim there exists a subset I of $\{1, ..., m-1\}$ such that

$$| \sum_I \alpha_i - \sum_{\{1,...,m-1\}\setminus I} \alpha_i |\leq \alpha_m.$$

In fact, if m is odd

$$0 \leq (\alpha_2 - \alpha_1) + (\alpha_4 - \alpha_3) + ... + (\alpha_{m-1} - \alpha_{m-2)} \leq \alpha_m$$

and we get

$$| \alpha_2 + \alpha_4 + ... + \alpha_{m-1} - \alpha_1 - \alpha_3 ... - \alpha_{m-2} |\leq \alpha_m.$$

On the other hand if m is even

$$0 \leq \alpha_{m-1} - (\alpha_2 - \alpha_1) - (\alpha_4 - \alpha_3) - ...(\alpha_{m-2} - \alpha_{m-3}) \leq \alpha_m$$

and we get

$$| \alpha_1 + \alpha_3 + ... + \alpha_{m-1} - \alpha_2 - \alpha_4 - ... - \alpha_{m-2} |\leq \alpha_m.$$

Consequently, for each positive integer $m \geq 3$ there exists a subset I of $\{1, ..., m-1\}$ such that

$$| \sum_I \frac{\alpha_i}{\alpha_m} - \sum_{\{1,...,m-1\}\setminus I} \frac{\alpha_i}{\alpha_m} |\leq 1. \tag{2.3}$$

Now let $A_1, ..., A_m \in \mathcal{B}(\mathbf{R}^n)$. Using the inequality (2.1) we get $\alpha_m \geq 1/m$ and Lemma 2.3 yields

$$\Phi^{-1}(\gamma(\alpha_1 A_1 + ...\alpha_m A_m)$$

$$= \Phi^{-1}(\gamma(\alpha_m(\frac{\alpha_1}{\alpha_m} A_1 + ... + \frac{\alpha_{m-1}}{\alpha_m} A_{m-1}) + \alpha_m A_m))$$

$$\geq \alpha_m \Phi^{-1}(\gamma(\frac{\alpha_1}{\alpha_m} A_1 + ... + \frac{\alpha_{m-1}}{\alpha_m} A_{m-1})) + \alpha_m \Phi^{-1}(\gamma(A_m)).$$

We now use (2.2), (2.3), and Lemma 2.2 to obtain

$$\Phi^{-1}(\gamma(\frac{\alpha_1}{\alpha_m} A_1 + ... + \frac{\alpha_{m-1}}{\alpha_m} A_{m-1}))$$

$$\geq \frac{\alpha_1}{\alpha_m} \Phi^{-1}(\gamma(A_1)) + ... + \frac{\alpha_{m-1}}{\alpha_m} \Phi^{-1}(\gamma(A_{m-1}))$$

and the inequality (1.1) follows at once. This completes the proof of Theorem 2.2.

As a consequence of Lemma 2.2 let us point out the following

Corollary 1.1

$$\left\{\alpha \in \mathbf{R}_+^m;\ \sum_1^m \alpha_i \geq 1 \text{ and } \sum_1^m \alpha_i^2 \leq 1\right\} \subseteq \mathcal{S}_m.$$

PROOF Suppose $\alpha \in \mathbf{R}_+^m$, $\alpha_1 + ... + \alpha_m \geq 1$, and $\alpha_1^2 + ... + \alpha_m^2 \leq 1$. If

$$\alpha \notin \bigcup_{I \subseteq \{1,...,m\}} \mathcal{P}_I$$

that is

$$\mid \sum_1^m \varepsilon_i \alpha_i \mid > 1 \text{ a.s.}$$

where $(\varepsilon_i)_{i=1}^m$ denotes a sequence of independent random variables such that

$$P\left[\varepsilon_i = -1\right] = P\left[\varepsilon_i = 1\right],\ i = 1, ..., m$$

the Jensen inequality forces

$$\sqrt{\sum_1^m \alpha_i^2} \geq E\left[\mid \sum_1^m \varepsilon_i \alpha_i \mid\right] > 1.$$

From this contradiction we conclude that $\alpha \in \bigcup_{I \subseteq \{1,...,m\}} \mathcal{P}_I$ and Lemma 2.2 implies that $\alpha \in \mathcal{S}_m$, which proves Corollary 1.1.

3 Other descriptions of the classes $\mathcal{S}_m$ and $\mathcal{C}_m$

In this section $\mathcal{FS}_m$ denotes the set of all $\alpha = (\alpha_1, ..., \alpha_m) \in \mathbf{R}_+^m$ such that the inequality

$$\Phi^{-1}(\int_{\mathbf{R}^n} f_0 d\gamma) \geq \alpha_1 \Phi^{-1}(\int_{\mathbf{R}^n} f_1 d\gamma) + ... + \alpha_m \Phi^{-1}(\int_{\mathbf{R}^n} f_m d\gamma) \qquad (3.1)$$

is valid for all Borel functions $f_k : \mathbf{R}^n \to [0,1]$, $k = 0, 1, ..., m$, satisfying the inequality

$$\Phi^{-1}(f_0(\alpha_1 x_1 + ... + \alpha_m x_m)) \geq \alpha_1 \Phi^{-1}(f_1(x_1)) + ... + \alpha_m \Phi^{-1}(f_m(x_m)) \quad (3.2)$$

for all $x_1, ..., x_m \in \mathbf{R}^n$.

Theorem 3.1

$$\mathcal{S}_m = \mathcal{FS}_m.$$

PROOF We first prove that

$$\mathcal{S}_m \subseteq \mathcal{FS}_m$$

Therefore let $\alpha \in \mathcal{S}_m$, let $f_k : \mathbf{R}^n \to [0,1]$, $k = 0, 1, ...m$, be Borel functions such that (3.2) holds for all $x_1, ..., x_m \in \mathbf{R}^n$, and as in the Latała paper [6] define

$$B_k = \left\{(x,t) \in \mathbf{R}^n \times \mathbf{R};\ t \leq \Phi^{-1}(f_k(x)\right\},\ k = 0, 1, ..., m.$$

Then

$$B_0 \supseteq \alpha_1 B_1 + ... + \alpha_m B_m$$

and as

$$(\gamma \times \gamma_1)(B_k) = \int_{\mathbf{R}^n} f_k d\gamma,\ k = 0, 1, ..., m$$

it follows that $\alpha \in \mathcal{FS}_m$. Here note that the class $\mathcal{S}_m$ is independent of n.

We next claim that

$$\mathcal{FS}_m \subseteq \mathcal{S}_m.$$

To see this let $\alpha \in \mathcal{FS}_m$, let $A_1, ..., A_m \subseteq E$ be compact and define $A_0 = \alpha_1 A_1 + ... + \alpha_m A_m$ and $f_k = 1_{A_k}$, $k = 0, 1, ..., m$. Then (3.2) is true and as $\alpha \in \mathcal{FS}_m$, (3.1) is true which implies (1.1). Since any finite positive measure on $\mathbf{R}^n$ is inner regular with respect to compact sets it is obvious that $\alpha \in \mathcal{S}_m$. Summing up we have proved Theorem 3.1.

Next let $\mathcal{FC}_m$ denote the set of all $\alpha = (\alpha_1, ..., \alpha_m) \in \mathbf{R}_+^m$ such that the inequality (3.1) is valid for all Borel functions $f_k : \mathbf{R}^n \to [0,1]$, $k = 0, 1, ..., m$, satisfying (3.2) and such that the functions $\Phi^{-1}(f_k)$, $k = 0, 1, ..., m$, are concave. Here we say that a function $g : \mathbf{R}^n \to [-\infty, \infty]$ is concave if the set $\{(x,t) \in \mathbf{R}^n \times \mathbf{R};\ t \leq g(x)\}$ is convex.

The proof of the following theorem is very similar to the proof of Theorem 3.1 and will not be included here.

Theorem 3.1

$$\mathcal{C}_m = \mathcal{FC}_m.$$

4 Extension to infinite dimension

A Borel probability measure μ on a real, locally convex Hausdorff vector space E is said to be a centred Gaussian measure if the image measure $\mu\xi^{-1}$ is a centred Gaussian measure on the real line for each bounded linear functional ξ in E. For simplicity, we here restrict ourselves to a centred, non-degenerate Gaussian measure γ on a real, separable Fréchet space F. To say that γ is non-degenerate means that γ is not the Dirac measure at the origin. Recall that a finite positive Borel measure μ on F is inner regular with respect to compact sets and moreover, if A is a convex Borel subset of F there are compact sets K_n, $n \in \mathbf{N}$, such that $K_1 \subseteq K_2 \subseteq ...$, $\cup_0^\infty K_n$ is convex, and $A \setminus \cup_0^\infty K_n$ a μ-null set. Note also that the theory of analytic sets implies that a linear combination of Borel subsets of F is universally Borel measurable.

We define the classes $\mathcal{S}_m$, $\mathcal{C}_m$, $\mathcal{FS}_m$, and $\mathcal{FC}_m$ as above with $\mathbf{R}^n$ replaced by F and where γ now stands for a centred, non-degenerate Gaussian measure on F. The paper [3] shows that (1.2) still holds and it is obvious that the results in Sections 2 and 3 extend to this more general situation. Note here that the Sudakov and Tsirelson inequality, which is important in the proof of Theorem 2.1, extends to convex (universally) Borel measurable sets on F (see [3]). The details are omitted here.

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