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POINTWISE A POSTERIORI ERROR ESTIMATES FOR THE STOKES EQUATIONS IN POLYHEDRAL DOMAINS

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Abstract. We derive pointwise a posteriori residual-based error estimates for finite element solutions to the Stokes equations in polyhedral domains. The estimates relies on the regularity of the of Stokes equations and provide an upper bound for the pointwise error in the velocity field on polyhedral domains. Whereas the estimates provide upper bounds for the pointwise error in the gradient of the velocity field and the pressure only for a restricted class of polyhedral domains, convex polyhedral domains in $\mathbf{R}^2$, and polyhedral domains with angles at edges $< 3\pi/4$ in $\mathbf{R}^3$. In the cause of this study we also derive L^q a posteriori error estimates, generalizing well known L^2 estimates.

1. Introduction

Let $\Omega \subset \mathbf{R}^n$, $n = 2, 3$, be a polyhedral domain and consider the Dirichlet Stokes problem in dimensionless form

$$\begin{aligned} -\Delta u + \nabla p &= f \quad \text{in}\,\Omega, \\ \nabla \cdot u &= g \quad \text{in}\,\Omega, \\ u &= 0 \quad \text{on}\,\partial\Omega, \end{aligned} \tag{1.1}$$

where $u = (u_1, \dots, u_n)$ is the unknown velocity field, p the unknown pressure, $f = (f_1, \dots, f_n)$ is an external body force and g is a function prescribing the compressibility of the flow, for incompressible flows $g = 0$.

The purpose of this paper is to establish residual-based pointwise a posteriori error estimates for conforming finite element approximations

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(u_h, p_h) to the Stokes problem (1.1). Only requiring that the finite element mesh is regular, allowing adaptively refined meshes, we obtain a number of error estimates.

(1) For polyhedral domains we derive pointwise error estimates for the velocity field
$$\|u_h - u\|_{L^\infty(\Omega)} \le \mathcal{E}_1(u_h, p_h, f, g, \Omega, \mathcal{T}).$$
(2) For convex polyhedral domains in $\mathbf{R}^2$, and for polyhedral domains in $\mathbf{R}^3$ with angles at edges $< 3\pi/4$ we derive pointwise error estimates for the gradient of the velocity field
$$\|\nabla(u_h - u)\|_{L^\infty(\Omega)} \le \mathcal{E}_2(u_h, p_h, f, g, \Omega, \mathcal{T})$$
(3) For polyhedral domain as specified in Item 2 above we derive pointwise error estimates for the pressure
$$\|p_h - p\|_{L^\infty(\Omega)} \le \mathcal{E}_3(u_h, p_h, f, g, \Omega, \mathcal{T}).$$
(4) For polyhedral domains and for $q \in [2n/(n+1), 2n/(n-1)]$ we also derive the following L^q-estimate
$$\|\nabla(u_h - u)\|_{L^q(\Omega)} + \|p_h - p\|_{L^q(\Omega)} \le \mathcal{E}_4(u_h, p_h, f, g, \Omega, \mathcal{T}).$$

The right hand sides $\mathcal{E}_{1,2,3,4}$ in the estimates above are functions derived from the residuals, depending on the finite element solution, the data, the domain and the triangulation.

The first estimate in Item 1 relies on the fact that, for sufficiently regular data, the velocity field is Hölder continuous in polyhedral domains. Similarly, the pointwise estimates for the gradient of the velocity field, Item 2, and the pressure, Item 3, require continuity. This is generally not obtained in polyhedral domains without imposing extra constraints, convexity for polyhedral domains in $\mathbf{R}^2$ and a minimum inner angle condition, $< 3\pi/4$ at edges, for polyhedral domains in $\mathbf{R}^3$ [13]. We note that estimating the gradient of the velocity field is somewhat more involved since ∇u_h is discontinuous at the $(n-1)$-faces of the triangulation.

The fourth estimate in Item 4 relies on L^q-regularity estimates stated in [3] for Lipschitz domains and also in [13] for polyhedral domains. It is a straightforward generalization of the L^2-based estimates in [19].

The techniques used to prove the pointwise error estimate is inspired by [14], where an a posteriori residual-based pointwise error estimate was derived for Poisson's equation in two dimensions, later this analysis was also done in three dimensions [4]. We remark that the gradient of the solution was not considered in these studies. The pointwise a priori error analysis for the Stokes problem was worked out in two dimensions for convex domains and quasiuniform triangulations [5], and in three dimensions for polyhedral domains with the similar type of constraints as mentioned above and for quasiuniform triangulations [10].

1.1. **Assumptions and notation.** We only consider functions defined on bounded domains $\omega \subseteq \Omega \subset \mathbf{R}^n$, $n = 2, 3$, with measure denoted by $|\omega|$, and where Ω is associated with the Stokes problem (1.1) and the dual problem (1.4).

Let $\{e_i\}_{i=1}^n$ denote the canonical unit vectors, $e_1 = (1, 0)$ and $e_2 = (0, 1)$ for $n = 2$ and $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$ and $e_3 = (0, 0, 1)$ for $n = 3$.

We denote the i:th partial derivative by

$$D_i := \frac{\partial}{\partial x_i}, \quad i = 1, \dots, n,$$

and the gradient by

$$\nabla := (D_1, \dots, D_n),$$

and the matrix of second order derivatives

$$\nabla^2 := (D_i D_j)_{i,j=1}^n.$$

We use standard notation for spaces of smooth functions, for example, $C^m(\omega)$, $C_0^\infty(\omega)$ and $C^{m,\gamma}(\overline{\omega})$, and for Lebesgue and Sobolev spaces, $L^q(\omega) = W^{0,q}(\omega)$, $W^{k,q}(\omega)$ and $W_0^{k,q}(\omega)$, see for example [1]. For $u \in L^q(\omega)$ or $u \in W^{k,q}(\omega)$ we use the following notation for the norm

$$\|u\|_{L^q(\omega)} = \|u\|_{q,\omega} \qquad \text{and} \qquad \|u\|_{W^{q,k}(\omega)} = \|u\|_{q,k,\omega},$$

and likewise for the corresponding seminorms $|u|_{q,k,\omega}$.

When $q = 2$ $L^q(\omega) = L^2(\omega)$ becomes a Hilbert space and we denote the scalar product by

$$(u, v)_\omega := \int_\omega uv \, dx.$$

For $u \in W_0^{1,q}(\omega)$ or for $u \in W^{1,q}(\omega)$ with $\int_{\omega_0} u \, dx = 0$ for some non empty $\omega_0 \subset \omega$, the norm is equivalent to the seminorm, $\|u\|_{1,q,\omega} \approx |u|_{1,q,\omega}$, see for example [18, Lemma 1.1.1–2, pp. 43–44]. We will use this equivalence without further notice throughout this work.

We denote the dual exponent to q by $q' = q/(q-1)$ and the dual space to $W_0^{k,q}(\omega)$ by $W^{-k,q'}(\omega)$ with the dual norm

$$\|u\|_{-k,q',\omega} := \sup_{\varphi \in C_0^\infty(\omega)} \frac{|\langle u, \varphi\rangle|}{\|\varphi\|_{k,q,\omega}}, \tag{1.2}$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing.

Generally, for a vector space V we denote its dual space by V' with dual norm

$$\|u\|_{V'} := \sup_{\varphi \in V} \frac{|\langle u, \varphi\rangle|}{\|\varphi\|_V},$$

for example, $W_0^{k,q}(\omega)' := W^{-k,q'}(\omega)$.

When $\omega = \Omega$ we sometimes write L^q instead of $L^q(\Omega)$ and $\|\cdot\|_q$ instead of $\|\cdot\|_{q,\Omega}$ and likewise for Sobolev spaces and their norms and the L^2 scalar product.

We use the quotient space $W^{k,q}/\mathbf{R}$ with the norm

$$\|v\|_{W^{k,q}/\mathbf{R}} := \inf_{c \in \mathbf{R}} \|v + c\|_{k,q}.$$

For vector fields

$$\Omega \ni x \mapsto u(x) = (u_1(x), \ldots, u_n(x)) \in \mathbf{R}^n$$

we set

$$\nabla u := (D_i u_j)_{i,j=1}^n,$$
$$\nabla^2 u := (D_i D_j u_k)_{i,j,k=1}^n,$$

and for $u = (u_1, \ldots, u_n) \in W^{k,q}(\Omega)^n$ we use the Sobolev (Lebesgue) norm

$$\|u\|_{k,q} := \left(\sum_{i=1}^n \|u_i\|_{k,q}^q \right)^{1/q},$$

and the corresponding seminorms, the maximum norms

$$\|u\|_\infty := \max_i \|u_i\|_\infty,$$
$$\|\nabla u\|_\infty := \max_{i,j} \|D_i u_j\|_\infty,$$

and the scalar product

$$(u, v) = \sum_{i=1}^n (u_i, v_i).$$

We also use the product spaces $\mathcal{W}^{1,q} := W_0^{1,q}(\Omega)^n \times L^q(\Omega)/\mathbf{R}$ with the norm

$$\|(u,p)\|_{\mathcal{W}^{1,q}} := \|u\|_{1,q} + \|p\|_{L^q/\mathbf{R}},$$

and $\mathcal{W}^{2,q} := (W^{2,q}(\Omega)^n \times W^{1,q}(\Omega)) \cap \mathcal{W}^{1,s}$ where $s = nq/(n-q)$, see Theorem 1.3, with the norm

$$\|(u,p)\|_{\mathcal{W}^{2,q}} := \|u\|_{2,q} + \|p\|_{W^{1,q}/\mathbf{R}}.$$

Finally, throughout this work we use C or C_i, $i = 1, 2, \ldots$, to denote various constants, not necessarily with the same value from time to time.

1.2. **Weak formulation.** We follow the standard notation, *cf.* [11, 19], and define the bilinear form

$$\mathcal{L}((u,p),(\phi,\lambda)) := a(u,\phi) + b(\phi,p) - b(u,\lambda),$$

for test functions (ϕ,λ) and where

$$a(u,\phi) := \int_\Omega \sum_{i,j=1}^n \frac{\partial u_i}{\partial x_j}\frac{\partial \phi_i}{\partial x_j}\,dx \qquad \text{and} \qquad b(\phi,p) := -\int_\Omega (\nabla\cdot\phi)p\,dx.$$

For data $f \in W^{-1,q}$ and $g \in L^q$ such that $\int_\Omega g\,dx = 0$ and for $2n/(n+1) < q < 2n/(n-1)$ there is a unique weak solution to (1.1), see Theorem 1.1 for a more precise statement. The weak formulation of (1.1) now reads. Find $(u,p) \in \mathcal{W}^{1,q}(\Omega)$ such that

$$\mathcal{L}((u,p),(\phi,\lambda)) = \langle f,\phi\rangle + (g,\lambda) \quad \forall(\phi,\lambda) \in \mathcal{W}^{1,q'}(\Omega), \tag{1.3}$$

where $\langle\cdot,\cdot\rangle$ denotes the appropriate duality pairing.

The dual problem to (1.1) is

$$\begin{aligned} -\Delta\tilde{u} - \nabla\tilde{p} &= \tilde{f} \quad \text{in } \Omega, \\ -\nabla\cdot\tilde{u} &= \tilde{g} \quad \text{in } \Omega, \\ \tilde{u} &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1.4}$$

where $\tilde{f} \in W^{-1,q'}$ and $\tilde{g} \in L^{q'}$ such that $\int_\Omega \tilde{g}\,dx = 0$ and for $2n/(n+1) < q' < 2n/(n-1)$. The corresponding weak formulation is. Find $(\tilde{u},\tilde{p}) \in \mathcal{W}^{1,q'}(\Omega)$ such that

$$\mathcal{L}((\phi,\lambda),(\tilde{u},\tilde{p})) = \langle\phi,\tilde{f}\rangle + (\lambda,\tilde{g}) \quad \forall(\phi,\lambda) \in \mathcal{W}^{1,q}(\Omega). \tag{1.5}$$

1.3. **Existence and regularity in non-smooth domains.** For any domain $\Omega \subset \mathbf{R}^n$, $n = 2, 3$, and data $f \in W^{-1,2}(\Omega)^n$ and $g \in L^2(\Omega)$ such that $\int_\Omega g \, dx = 0$, it is well known that there exists a unique weak solution $(u, p) \in W_0^{1,2}(\Omega)^n \times L^2(\Omega)/\mathbf{R}$ to (1.1), see for example [18, Chaper 3] and references therein. For sufficiently regular domains and data there are several extensions such that $(u, p) \in W_0^{1,q}(\Omega)^n \times L^q(\Omega)/\mathbf{R}$, see Remark 1.1 below. In Theorem 1.1 we quote one example of such an extension where the Stokes problem is formulated on Lipschitz domains. This is a slight modification of [3, Theorem 2.9] where it was provided with $g = 0$. However the case $g \neq 0$ is readily included.

Theorem 1.1. *Let $\Omega \subset \mathbf{R}^n$, $n = 2, 3$, be a bounded Lipschitz domain. There exist $\varepsilon > 0$ such that if $(3+\varepsilon)/(2+\varepsilon) < q < 3+\varepsilon$ and $f \in W^{-1,q}(\Omega)^n$ and $g \in L^q(\Omega)$ with $\int_\Omega g \, dx = 0$, then there exist a unique weak solution $(u, p) \in W_0^{1,q}(\Omega)^n \times L^q(\Omega)/\mathbf{R}$ to (1.1). Moreover, the solution satisfies the inequality*

$$\|u\|_{1,q} + \|p\|_{L^q/\mathbf{R}} \leq C\big(\|f\|_{-1,q} + \|g\|_q\big), \tag{1.6}$$

for some $C = C(n, q, \Omega)$.

Proof. For $g = 0$ this is [3, Theorem 2.9]. For $g \neq 0$ we use the method of *subtracting the divergence*, see for example [18, Theorem 1.4.1, p. 114], to handle the non-homogenous compressibility constraint.

For Ω and g as stated there exists $v \in W_0^{1,q}(\Omega)^n$ such that

$$\nabla \cdot v = g \quad \text{and} \quad \|v\|_{1,q} \leq C\|g\|_q, \tag{1.7}$$

see, for example, [18, Lemma 2.1.1, p. 68]. Taking $w = u - v$ we see that (1.1) is equivalent to

$$-\Delta w + \nabla p = f + \Delta v, \qquad \nabla \cdot w = 0, \qquad \text{in } \Omega,$$

and $w|_{\partial\Omega} = 0$. Now [3, Theorem 2.9] implies that there exist a unique pair $(w, p) \in W_0^{1,q}(\Omega)^n \times L^q(\Omega)/\mathbf{R}$ satisfying the above equations and the inequality

$$\|w\|_{1,q} + \|p\|_{L^q/\mathbf{R}} \leq C\|f + \Delta v\|_{-1,q},$$

for some $C = C(n, q, \Omega)$.

Thus, $(u, p) \in W_0^{1,q}(\Omega)^n \times L^q(\Omega)/\mathbf{R}$ is a unique solution to (1.1) and the estimate above implies that

$$\|u\|_{1,q} + \|p\|_{L^q/\mathbf{R}} \leq C\big(\|f\|_{-1,q} + \|v\|_{1,q} + \|\Delta v\|_{-1,q}\big).$$

The inequality (1.6) now follows from the estimate in (1.7) and the fact that $\|\Delta v\|_{-1,q} \leq \|v\|_{1,q}$. □

Remark 1.1. (1) For $n = 2$ the results of the theorem actually holds with $(4+\varepsilon)/(3+\varepsilon) < q < 4+\varepsilon$. This is provided in the same way as for $n = 3$ [17]. (2) For polyhedral domains a similar theorem was established in [13], in particular, for convex polyhedral domains the result holds with $1 < q < \infty$. (3) For C^1-domains there is a similar theorem again with $1 < q < \infty$, see for example [8].

As a consequence of Theorem 1.1 and Remark 1.1 we obtain the following inf-sup like estimate.

Corollary 1.2. *For q and Ω as in Theorem 1.1 we have*

$$\|(u,p)\|_{\mathcal{W}^{1,q}} \leq C \sup_{(\phi,\lambda)\in\mathcal{W}^{1,q'}} \frac{|\mathcal{L}((u,p),(\phi,\lambda))|}{\|(\phi,\lambda)\|_{\mathcal{W}^{1,q'}}} \qquad \forall (u,p) \in \mathcal{W}^{1,q}(\Omega), \tag{1.8}$$

where $C = C(n, q', \Omega)$.

Proof. Let (ϕ_i, λ_i) be the solutions to the following problems

$$\begin{aligned} -\Delta\phi_1 - \nabla\lambda_1 &= \tilde{f}, \quad \nabla\cdot\phi_1 = 0, \quad \text{in } \Omega; \qquad \phi_1|_{\partial\Omega} = 0, \\ -\Delta\phi_2 - \nabla\lambda_2 &= 0, \quad \nabla\cdot\phi_2 = \tilde{g} - \tilde{g}_0, \quad \text{in } \Omega; \qquad \phi_2|_{\partial\Omega} = 0, \end{aligned}$$

where $\tilde{f} \in W^{-1,q'}(\Omega)^n$ and $\tilde{g} \in L^{q'}(\Omega)$ with the mean $\tilde{g}_0 = |\Omega|^{-1}\int_\Omega \tilde{g}\,dx$.

With Theorem 1.1 applied to the above problems and with (1.5) we get

$$\begin{aligned} &\sup_{(\phi,\lambda)\in\mathcal{W}^{1,q'}} \frac{|\mathcal{L}((u,p),(\phi,\lambda))|}{\|(\phi,\lambda)\|_{\mathcal{W}^{1,q'}}} \\ &\geq \frac{1}{2}\left(\frac{|\mathcal{L}((u,p),(\phi_1,\lambda_1))|}{\|(\phi_1,\lambda_1)\|_{\mathcal{W}^{1,q'}}} + \frac{|\mathcal{L}((u,p),(\phi_2,\lambda_2))|}{\|(\phi_2,\lambda_2)\|_{\mathcal{W}^{1,q'}}} \right) \\ &\geq C\left(\frac{|\langle u,\tilde{f}\rangle|}{\|\tilde{f}\|_{-1,q'}} + \frac{|(p,\tilde{g}-\tilde{g}_0)|}{\|\tilde{g}-\tilde{g}_0\|_{q'}} \right) \end{aligned} \tag{1.9}$$

Since $W^{1,q}$ and L^q are reflexive for $1 < q < \infty$ we get

$$\sup_{\tilde{f}\in W^{-1,q'}(\Omega)^n} \frac{|\langle u,\tilde{f}\rangle|}{\|\tilde{f}\|_{-1,q'}} = \|u\|_{1,q},$$

and since $(p, \tilde{g} - \tilde{g}_0) = (p - p_0, \tilde{g})$, where $p_0 = |\Omega|^{-1} \int_\Omega p \, dx$, we have

$$\sup_{\tilde{g} \in L^{q'}(\Omega)} \frac{|(p, \tilde{g} - \tilde{g}_0)|}{\|\tilde{g} - \tilde{g}_0\|_{q'}} \geq \frac{1}{2} \inf_{c \in \mathbf{R}} \sup_{\tilde{g} \in L^{q'}(\Omega)} \frac{|(p + c, \tilde{g})|}{\|\tilde{g}\|_{q'}} = \frac{1}{2} \|p\|_{L^q/\mathbf{R}},$$

where we also used the estimate $\|\tilde{g} - \tilde{g}_0\|_{q'} \leq 2\|\tilde{g}\|_{q'}$.

Now since (1.9) is valid for any $\tilde{f} \in W^{-1,q'}(\Omega)^n$ and for any $\tilde{g} \in L^{q'}(\Omega)$ we may take the supremum with respect to $\tilde{f}$ and $\tilde{g}$, which together with the last two estimates above completes the proof. □

The next theorem concerns the $W^{2,q}(\Omega)^n \times W^{1,q}(\Omega)$-regularity of the solution to (1.1) in polyhedral domains. The theorem is due to [13], for a review see [12], although it is formulated somewhat differently here.

Theorem 1.3. *Let $\Omega \subset \mathbf{R}^n$, $n = 2, 3$, be a polyhedral domain and let $1 < q \leq 4/3$. Suppose $f \in L^q(\Omega)^n$ and $g \in W^{1,q}(\Omega)$ such that $\int_\Omega g \, dx = 0$. Then there exist a unique weak solution $(u, p) \in W_0^{1,s}(\Omega)^n \times L^s(\Omega)/\mathbf{R}$ to (1.1) for $s = nq/(n-q)$ such that $(u, p) \in W^{2,q}(\Omega)^n \times W^{1,q}(\Omega)$. Moreover, the solution satisfies the inequality*

$$\|u\|_{2,q} + \|p\|_{W^{1,q}/\mathbf{R}} \leq C\big(\|f\|_q + |g|_{1,q}\big), \tag{1.10}$$

for some $C = C(n, q, \Omega)$.

Proof. By virtue of Theorem 1.1 and Remark 1.1 we obtain the existence, since by Sobolev's imbedding theorem we have $L^q \subset W^{-1,s}$ and $W^{1,q} \subset L^s$ for $s = nq/(n-q)$, $1 < q \leq 4/3$ and we readily check that $2 \leq s \leq 4$ for $n = 2$ and $(3 + \varepsilon)/(2 + \varepsilon) < s < 3 + \varepsilon$ for $n = 3$ and any $\varepsilon > 0$.

The regularity $(u, p) \in W^{2,q}(\Omega)^n \times W^{1,q}(\Omega)$ follows from [13, Theorem 5.3] which is also true provided $(u, p) \in W_0^{1,s}(\Omega)^n \times L^s(\Omega)/\mathbf{R}$ [15]. The estimate (1.10) is then as consequence of the open mapping theorem, see for example [6, Corollary 5.11, p. 162]. □

Remark 1.2. (1) For $n = 2$ and if the maximum inner angle in the polyhedral domain is less than $\pi - \delta$ for some $\delta > 0$, then the result can be extended to hold with $1 < q \leq 2 + \varepsilon$ for some $\varepsilon > 0$ [15] and *cf.* [13, §5.5]. (2) For $n = 3$ and if the maximum inner angle at the edges in the polyhedral domain is less than $3\pi/4 - \delta$ for some $\delta > 0$, then the result can be extended to hold with $1 < q \leq 3 + \varepsilon$ for some $\varepsilon > 0$ [15] and *cf.* [13, §5.5]. (3) For C^1-domains there is a similar theorem with $1 < q < \infty$, see for example [8]. In cases (1) and (2) the existence is also true since for convex domains Theorem 1.1 is modified as in Remark 1.1.

We now state a corollary where we assume that we have the higher regularity in Remark 1.2.

Corollary 1.4. *Suppose that the solution $(\tilde{u}, \tilde{p})$ to (1.4) with data as in Theorem 1.3 belongs to $W^{2,q'}(\Omega)^n \times W^{1,q'}(\Omega)$ for some $q' > n$. Then the solution (u, p) to (1.1) satisfies*

$$\|u\|_q + \|p\|_{W^{1,q'}(\Omega)'/\mathbf{R}} \leq C\big(\|f\|_{-2,q} + \|g\|_{W^{1,q'}(\Omega)'}\big), \tag{1.11}$$

for some $C = C(n, q', \Omega)$ and where $1/q + 1/q' = 1$ and $W^{1,q'}(\Omega)'/\mathbf{R}$ is the dual space to $W^{1,q'}(\Omega)/\mathbf{R}$.

Proof. We use the same technique as in the proof of Corollary 1.2. With (1.3) we estimate

$$\begin{aligned}\|f\|_{-2,q} + \|g\|_{W^{1,q'}(\Omega)'} &= \sup_{\phi \in C_0^\infty(\Omega)^n} \frac{|\langle f, \phi\rangle|}{\|\phi\|_{2,q'}} + \sup_{\lambda \in W^{1,q'}/\mathbf{R}} \frac{|\langle g, \lambda\rangle|}{\|\lambda\|_{W^{1,q'}/\mathbf{R}}} \\ &\geq \sup_{(\phi,\lambda)\in \mathcal{W}^{2,q'}} \frac{|\mathcal{L}((u,p),(\phi,\lambda))|}{\|(\phi,\lambda)\|_{\mathcal{W}^{2,q'}}}.\end{aligned}$$

Let (ϕ_i, λ_i) be the solutions to the following problems

$$\begin{aligned}-\Delta\phi_1 - \nabla\lambda_1 = \tilde{f}, \quad \nabla\cdot\phi_1 = 0, \quad \text{in } \Omega; \qquad \phi_1|_{\partial\Omega} = 0,\\ -\Delta\phi_2 - \nabla\lambda_2 = 0, \quad \nabla\cdot\phi_2 = \tilde{g} - \tilde{g}_0, \quad \text{in } \Omega; \qquad \phi_2|_{\partial\Omega} = 0,\end{aligned}$$

where $\tilde{f} \in L^{q'}(\Omega)^n$ and $\tilde{g} \in W^{1,q'}(\Omega)$ with the mean $\tilde{g}_0 = |\Omega|^{-1}\int_\Omega \tilde{g}\, dx$. We assumed that $(\phi_i, \lambda_i) \in W^{2,q'}(\Omega)^n \times W^{1,q'}(\Omega)$ and thus we estimate

$$\begin{aligned}&\sup_{(\phi,\lambda)\in \mathcal{W}^{2,q'}} \frac{|\mathcal{L}((u,p),(\phi,\lambda))|}{\|(\phi,\lambda)\|_{\mathcal{W}^{2,q'}}} \\ &\qquad \geq \frac{1}{2}\left(\frac{|\mathcal{L}((u,p),(\phi_1,\lambda_1))|}{\|(\phi_1,\lambda_1)\|_{\mathcal{W}^{2,q'}}} + \frac{|\mathcal{L}((u,p),(\phi_2,\lambda_2))|}{\|(\phi_2,\lambda_2)\|_{\mathcal{W}^{2,q'}}}\right) \\ &\qquad \geq C\left(\frac{|(u,\tilde{f})|}{\|\tilde{f}\|_{q'}} + \frac{|\langle p, \tilde{g} - \tilde{g}_0\rangle|}{|\tilde{g}|_{1,q'}}\right),\end{aligned} \tag{1.12}$$

for some $C = C(n, q', \Omega)$.

Since L^q and $W^{1,q'}(\Omega)'$ are reflexive for $1 < q < \infty$ we get

$$\sup_{\tilde{f}\in L^{q'}(\Omega)^n} \frac{|\langle u, \tilde{f}\rangle|}{\|\tilde{f}\|_{q'}} = \|u\|_q,$$

and since $(p, \tilde{g} - \tilde{g}_0) = (p - p_0, \tilde{g})$, where $p_0 = |\Omega|^{-1} \int_\Omega p \, dx$, we have

$$\sup_{\tilde{g} \in W^{1,q'}(\Omega)} \frac{|\langle p, \tilde{g} - \tilde{g}_0 \rangle|}{|\tilde{g}|_{1,q'}} \geq \inf_{c \in \mathbf{R}} \sup_{\tilde{g} \in W^{1,q'}(\Omega)} \frac{|\langle p + c, \tilde{g} \rangle|}{\|\tilde{g}\|_{1,q'}} = \|p\|_{W^{1,q'}(\Omega)'/\mathbf{R}}.$$

Now since (1.12) is valid for any $\tilde{f} \in L^{q'}(\Omega)^n$ and any $\tilde{g} \in W^{1,q'}(\Omega)$ we may take the supremum with respect to $\tilde{f}$ and $\tilde{g}$, which together with the last two estimates above completes the proof. □

1.4. **Finite element formulation.** Let $\{\mathcal{T}\}_{h>0}$ denote a family of regular triangulations of Ω and let h_T denote the diameter of an n-simplex $T \in \mathcal{T}$ and set $h_{\min} = \min_{T \in \mathcal{T}_h} h_T$.

We only consider conforming finite element spaces, $X_h \subset W_0^{1,q}(\Omega)^n$ for the velocity and, $M_h/\mathbf{R} \subset L^q(\Omega)/\mathbf{R}$ for the pressure and define the product space $\mathcal{W}_h = X_h \times M_h/\mathbf{R}$. From (1.3) we obtain the finite element formulation. Find $(u_h, p_h) \in \mathcal{W}_h$ such that

$$\mathcal{L}((u_h, p_h), (\phi_h, \lambda_h)) = \langle f, \phi_h \rangle + (g, \lambda_h) \quad \forall (\phi_h, \lambda_h) \in \mathcal{W}_h. \tag{1.13}$$

As usual we also require that $\mathcal{W}_h$ satisfies the inf-sup condition [11], that is,

$$\|(u_h, p_h)\|_{\mathcal{W}^{1,2}} \leq C \sup_{(\phi_h, \lambda_h) \in \mathcal{W}_h} \frac{|\mathcal{L}((u_h, p_h), (\phi_h, \lambda_h))|}{\|(\phi_h, \lambda_h)\|_{\mathcal{W}^{1,2}}}, \tag{1.14}$$

for all $(u_h, p_h) \in \mathcal{W}_h$, which implies that (1.13) is well posed.

We particularly have in mind the family of Taylor-Hood finite elements, see fore example [11], which satisfy the above requirement.

We recall a few standard results from interpolation theory, see for example [16]. Let S_T denote the union of all simplices adjacent to T and let $\mathcal{I}_{X_h}$ and $\mathcal{I}_{M_h}$ denote interpolation operators $\mathcal{I}_{X_h} : W_0^{m,q}(\Omega)^n \to X_h$ and $\mathcal{I}_{M_h} : W^{m-1,q}(\Omega)/\mathbf{R} \to M_h/\mathbf{R}$. For integers $\ell = 0, 1$, $m = 1, \ldots$, and $(\phi, \lambda) \in W^{m,q}(S_T)^n \times W^{m-1,q}(S_T)/\mathbf{R}$, we have

$$\|\nabla^\ell (\phi - \mathcal{I}_{X_h} \phi)\|_{q,T} \leq C h_T^{m-\ell} |\phi|_{m,q,S_T}, \tag{1.15}$$

and

$$\|\lambda - \mathcal{I}_{M_h} \lambda\|_{L^q(T)/\mathbf{R}} \leq C h_T^{m-1} |\lambda|_{W^{m-1,q}(S_T)/\mathbf{R}}. \tag{1.16}$$

On the boundary, ∂T, we use the trace inequality [8, Theorem 3.3, p. 43] and scale it appropriately, *i.e.*, for $w \in W^{1,q}(T)$ we obtain the estimate

$$\|w\|_{q,\partial T} \leq C\big(h_T^{-1/q} \|w\|_{q,T} + h_T^{1-1/q} |w|_{1,q,T}\big),$$

and hence

$$\|\phi - \mathcal{I}_{X_h}\phi\|_{q,\partial T} \le C h_T^{m-1/q} |\phi|_{m,q,S_T}. \tag{1.17}$$

We also use inverse estimates, see for example [2, Theorem 4.5.3, p. 111]. For any $T \in \mathcal{T}$, let V be a finite dimensional subspace of $W^{k,q}(T) \cap W^{m,s}(T)$, where $1 \le q \le \infty$ and $1 \le s \le \infty$ and $0 \le m \le k$. Then there exist a constant C such that for all $v \in V$

$$\|v\|_{k,q,T} \le C h_T^{m-k+n/q-n/s} \|v\|_{m,s,T}. \tag{1.18}$$

2. Error analysis

We consider the error in the finite element solution to (1.13),

$$e_u := u_h - u \quad \text{and} \quad e_p := p_h - p,$$

and note that $(e_u, e_p) \in \mathcal{W}^{1,q}$, since the finite elements are conforming.

Define the residual in the momentum equation (me) by

$$R_{\text{me}} := f + \Delta u_h - \nabla p_h \in W^{-1,q}(\Omega)^n, \tag{2.1}$$

and the residual in the compressibility constraint (cc) by

$$R_{\text{cc}} := g - \nabla \cdot u_h \in L^q(\Omega), \tag{2.2}$$

where we note that $\int_\Omega R_{\text{cc}}\, dx = 0$.

In weak form the residual becomes

$$\mathcal{R}((u_h, p_h), (\phi, \lambda)) := \langle f, \phi \rangle + (g, \lambda) - \mathcal{L}((u_h, p_h), (\phi, \lambda)), \tag{2.3}$$

for all $(\phi, \lambda) \in \mathcal{W}^{1,q'}$.

From (1.3) we obtain the identity

$$\mathcal{L}((e_u, e_p), (\phi, \lambda)) = \mathcal{R}((u_h, p_h), (\phi, \lambda)) \quad \forall (\phi, \lambda) \in \mathcal{W}^{1,q'} \tag{2.4}$$

and from (1.13) and it follows

$$\mathcal{R}((u_h, p_h), (\phi_h, \lambda_h)) = 0 \quad \forall (\phi_h, \lambda_h) \in \mathcal{W}_h, \tag{2.5}$$

which is the classical Galerkin orthogonality.

Inspired by [7, Lemma 3.1] we now provide the following lemma.

Lemma 2.1. *For $q \in [1, \infty]$, and $m = 1, 2$, there is a constant C such that*

$$|\mathcal{R}((u_h, p_h), (\phi, \lambda))| \le C \eta_{m,q} \big(|\phi|_{m,q'} + |\lambda|_{W^{m-1,q'}/\mathbf{R}} \big),$$

for all $(\phi,\lambda)\in\mathcal{W}^{m,q'}$ *where*

$$\eta_{m,q}=\begin{cases}\Big(\sum_{T\in\mathcal{T}}\eta_{m,q,T}^{q}\Big)^{1/q} & \textit{for } q\in[1,\infty),\\ \max_{T\in\mathcal{T}}\eta_{m,\infty,T} & \textit{for } q=\infty,\end{cases}$$

with

$$\eta_{m,q,T}=h_T^m\|R_{\mathrm{me}}\|_{q,T}+\frac{1}{2}h_T^{m-1/q'}\|[\partial_\nu u_h]\|_{q,\partial T\setminus\partial\Omega}+h_T^{m-1}\|R_{\mathrm{cc}}\|_{q,T}.$$

Here $[\partial_\nu u_h]$ *denotes the jump across* ∂T *in the normal derivative,* $\partial_\nu u_h=\nu\cdot\nabla u_h$*, where* ν *denotes the outward normal to* ∂T*.*

Proof. By (2.5) and by integration by parts

$$\begin{aligned}\mathcal{R}((u_h,p_h),(\phi,\lambda))&=\mathcal{R}((u_h,p_h),(\phi-\mathcal{I}_{X_h}\phi,\lambda-\mathcal{I}_{M_h}\lambda))\\&=\sum_{T\in\mathcal{T}}\big((f+\Delta u_h-\nabla p_h,\phi-\mathcal{I}_{X_h}\phi)_T\\&\qquad+\frac{1}{2}([\partial_\nu u_h],\phi-\mathcal{I}_{X_h}\phi)_{\partial T\setminus\partial\Omega}\\&\qquad+(g-\nabla\cdot u_h,\lambda-\mathcal{I}_{M_h}\lambda)_T\big).\end{aligned}$$

Since $\int_\Omega(g-\nabla\cdot u_h)\,dx=0$, we have

$$(g-\nabla\cdot u_h,\lambda-\mathcal{I}_{M_h}\lambda)_T=\inf_{c\in\mathbf{R}}(g-\nabla\cdot u_h,\lambda-\mathcal{I}_{M_h}\lambda+c)_T$$

and hence by Hölder's inequality,

$$\begin{aligned}&|\mathcal{R}((u_h,p_h),(\phi,\lambda))|\\&\quad\le\sum_{T\in\mathcal{T}}\big(\|f+\Delta u_h-\nabla p_h\|_{q,T}\|\phi-\mathcal{I}_{X_h}\phi\|_{q',T}\\&\qquad+\frac{1}{2}\|[\partial_\nu u_h]\|_{q,\partial T\setminus\partial\Omega}\|\phi-\mathcal{I}_{X_h}\phi\|_{q',\partial T\setminus\partial\Omega}\\&\qquad+\|g-\nabla\cdot u_h\|_{L^q(T)}\|\lambda-\mathcal{I}_{M_h}\lambda\|_{L^{q'}(T)/\mathbf{R}}\big).\end{aligned}\tag{2.6}$$

Thus, with the interpolation estimates (1.15)–(1.17) in (2.6) we get

$$
\begin{aligned}
&|\mathcal{R}((u_h,p_h),(\phi,\lambda))| \\
&\quad\le C\sum_{T\in\mathcal{T}}\Big(h_T^m\big(\|f+\Delta u_h-\nabla p_h\|_{q,S_T} \\
&\qquad+\frac{1}{2}h_T^{1/q'}\|[\partial_n u_h]\|_{q,\partial T\setminus\partial\Omega}|\phi|_{m,q',S_T} \\
&\qquad+h_T^{m-1}\|g-\nabla\cdot u_h\|_{L^q(T)}|\lambda|_{W^{m-1,q'}(S_T)/\mathbf{R}}\Big).
\end{aligned}
\tag{2.7}
$$

Finally, we conclude the proof by using Hölder's inequality for sums and the notation in (2.1) and (2.2). □

Let $(\tilde u,\tilde p)$ be the solution to the dual problem (1.5). By choosing $(\phi,\lambda)=(\tilde u,\tilde p)$ in (2.4) we get

$$\mathcal{L}((e_u,e_p),(\tilde u,\tilde p))=\mathcal{R}((u_h,p_h),(\tilde u,\tilde p)),$$

and by choosing $(\phi,\lambda)=(e_u,e_p)$ in (1.5) we obtain

$$\mathcal{L}((e_u,e_p),(\tilde u,\tilde p))=\langle e_u,\tilde f\rangle+(e_p,\tilde g).$$

Thus

$$\langle e_u,\tilde f\rangle+(e_p,\tilde g)=\mathcal{R}((u_h,p_h),(\tilde u,\tilde p)). \tag{2.8}$$

In order to proceed in the error analysis we need to choose the data in the dual problem in a certain way. Let $\delta=\delta_{x_0,\rho/2}\in C_0^\infty(\Omega)$ be a regularization of the Dirac distribution at $x_0\in\Omega$, that is, let

$$\operatorname{supp}(\delta)\subset\mathcal{B}(x_0;\rho/2),\qquad \int_{\mathbf{R}^n}\delta\,dx=1,\qquad 0\le\delta\le C\rho^{-n}, \tag{2.9}$$

where $\mathcal{B}(x_0;\rho/2)$ denotes the ball with center in x_0 and radius $\rho/2$ chosen such that

$$\rho\le h_{\min}^\sigma, \tag{2.10}$$

where $\sigma>0$ will be specified in the proofs of Lemmas 2.2–2.4 below. For $q\in[1,\infty]$ it follows that

$$|\delta|_{k,q}\le C\rho^{-n(1-1/q)-k}. \tag{2.11}$$

In the remainder of this section we state and prove three lemmas providing estimates of the following kind

$$\|e_u\|_\infty \lesssim |(e_{u_i}, \delta_{x_0,\rho/2})|,$$
$$\|\nabla e_u\|_\infty \lesssim |(e_u, D_i\delta_{x_0,\rho/2}e_j)|,$$
$$\|e_p\|_\infty \lesssim |(e_p, \delta_{x_0,\rho/2})|,$$

where e_{u_i} denotes the i:th component of e_u and where e_j is the j:th unit vector. We stress that x_0 may be different in the there estimates. With these estimates we will be able to make a connection to the estimate in Lemma 2.1, which in turn is crucial for the final pointwise error analysis.

In order to obtain these estimates we will have to assume that e_u and e_p are continuous. This will be the case for e_u provided the data is sufficient regular due to Theorem 1.1, whereas for e_p we also have to impose further constraints on the domain Ω, see Remark 1.2. We note that ∇e_u is not continuous since ∇u_h is discontinuous. However, with the same assumptions as for e_p we derive an estimate that includes jump terms of the same type as in the right hand side of the estimate in Lemma 2.1.

Lemma 2.2. *Let $\Omega \subset \mathbf{R}^n$, $n = 2, 3$, be a polyhedral domain and let $x_0 \in \Omega$ and i be such that $\|e_u\|_\infty = |e_{u_i}(x_0)|$. Then for data to* (1.1) *as in Theorem 1.1 and for some $q > n$ there is a constant C such that*

$$\|e_u\|_\infty \le |(e_{u_i}, \delta)| + Ch_{\min}^{\beta}\big(\|f\|_{-1,q} + \|g\|_q\big),$$

where $\delta = \delta_{x_0,\rho/2}$ is the regularized Dirac distribution (2.9) and β may be chosen arbitrarily large.

We note that the lemma is meaningful since due to Theorem 1.1 and Remark 1.1 there is $q > n$ such that $e_u \in W_0^{1,q}(\Omega)^n$.

Proof. By Sobolev's imbedding theorem, see [1, p. 98], $W_0^{1,q}(\Omega)^n \subset C^{0,\gamma}(\overline{\Omega})^n$ for some γ such that $0 < \gamma \le 1 - n/q$. Consequently, by the mean value theorem there is $x_1 \in \mathcal{B}(x_0, \rho/2) \cap \overline{\Omega}$ such that $(e_{u_i}, \delta) = e_{u_i}(x_1)$ and thus

$$\|e_u\|_\infty \le |(e_{u_i}, \delta)| + |e_{u_i}(x_0) - e_{u_i}(x_1)|.$$

We estimate the last term in the right hand side above. By Sobolev's inequality

$$|e_{u_i}(x_0) - e_{u_i}(x_1)| \le C\rho^\gamma \|e_{u_i}\|_{C^{0,\gamma}(\mathcal{B}(x_0,\rho/2)\cap\overline{\Omega})} \le C\rho^\gamma\|e_u\|_{1,q}.$$

By the triangle inequality,

$$\|e_u\|_{1,q} \le \|u\|_{1,q} + \|u_h\|_{1,q},$$

and by Theorem 1.1,

$$\|u\|_{1,q} \le C\big(\|f\|_{-1,q} + \|g\|_q\big),$$

and by the inverse estimate (1.18) and the inf-sup condition (1.14),

$$\|u_h\|_{1,q} \le C h_{\min}^{n(1/q-1/2)} \|u_h\|_{1,2} \le C h_{\min}^{n(1/q-1/2)} \big(\|f\|_{-1,q} + \|g\|_q\big).$$

Thus, with (2.10) we obtain

$$|e_{u_i}(x_0) - e_{u_i}(x_1)| \le C h_{\min}^{\beta} \big(\|f\|_{-1,q} + \|g\|_q\big),$$

where $\beta = \gamma\sigma + n(1/q - 1/2)$ may be chosen arbitrarily large by taking σ large. □

Lemma 2.3. *Let $\Omega \subset \mathbf{R}^n$, $n = 2, 3$, be a polyhedral domain such that the solution to (1.1) with data as in Theorem 1.3 is continuous in the sense that $(u, p) \in \mathcal{W}^{2,q}$, for $q > n$. Let $x_0 \in \Omega$, i and j be such that $\|\nabla e_u\|_\infty = |D_i e_{u_j}(x_0)|$. Then there are constants $C_{1,2}$ such that*

$$\|\nabla e_u\|_\infty \le |(e_u, D_i \delta e_j)| + C_1 h_{\min}^{\beta} \big(\|f\|_q + |g|_{1,q}\big) + C_2 \max_{T\in\mathcal{T}} \|[\partial_\nu u_h]\|_{\infty,\partial T\setminus\partial\Omega},$$

where $\delta = \delta_{x_0,\rho/2}$ is the regularized Dirac distribution (2.9)*, β may be chosen arbitrarily large, and $[\partial_\nu u_h]$ is the jump as described in Lemma 2.1.*

We note that the lemma is meaningful since with additional constraints on the domain Ω as in Remark 1.2 there is $q > n$ such that $u \in W^{2,q}(\Omega)^n$ so that $u \in W^{1,\infty}(\Omega)^n$. Note also that ∇u_h is discontinuous across ∂T for $T \in \mathcal{T}$ which need to be taken into account proving Lemma 2.3. However, ∇u_h is continuous in the interior of each $T \in \mathcal{T}$.

Proof. The idea of the proof is the same as for Lemma 2.2. Let

$$\mathcal{B}_{\mathcal{T}} = \bigcup \{T \in \mathcal{T} : T \cap \mathcal{B}(x_0, \rho/2) \ne \emptyset\},$$

where we for simplicity assume that $\mathcal{B}_{\mathcal{T}}$ is convex and note that $\operatorname{card}(\mathcal{B}_{\mathcal{T}}) \le C$ due to the regularity in the triangulation.

By the mean value theorem there are $x_T \in \mathcal{B}(x_0, \rho/2) \cap T$ for $T \in \mathcal{B}_{\mathcal{T}}$ such that

$$(D_i e_{u_j}, \delta) = \sum_{T\in\mathcal{B}_{\mathcal{T}}} (D_i e_{u_j}, \delta)_{\mathcal{B}(x_0,\rho/2)\cap T} = \sum_{T\in\mathcal{B}_{\mathcal{T}}} D_i e_{u_j}(x_T) \int_{\mathcal{B}(x_0,\rho/2)\cap T} \delta \, dx,$$

where $\int_{\mathcal{B}(x_0,\rho/2)\cap T} \delta\, dx < 1$ and thus

$$\|\nabla e_u\|_\infty \le |(e_u, D_i\delta e_j)| + \sum_{T\in\mathcal{B}_\mathcal{T}} |D_i e_{u_j}(x_0) - D_i e_{u_j}(x_T)|, \tag{2.12}$$

since by integration by parts $(D_i e_{u_j}, \delta) = -(e_u, D_i\delta e_j)$.

We estimate the terms in sum above. For $T \in \mathcal{B}_\mathcal{T}$ consider the line from x_0 to x_T and for $T_\ell \in \mathcal{B}_\mathcal{T}$ suppose this line intersect $m+1$ n-simplices T_ℓ and m boundaries ∂T_ℓ at points x_ℓ for $\ell = 1,\dots,m$. Note that m is bounded from above since $\operatorname{card}(\mathcal{B}_\mathcal{T}) \le C$. Let x_ℓ^- and x_ℓ^+ be the limits at x_ℓ going from x_0 and x_T respectively. Set $x_0^+ = x_0$ and $x_{m+1}^- = x_T$. We estimate

$$\begin{aligned} |D_i e_{u_j}(x_0) - D_i e_{u_j}(x_T)| \le &\sum_{\ell=0}^{m} |D_i e_{u_j}(x_\ell^+) - D_i e_{u_j}(x_{\ell+1}^-)| \\ &+ \sum_{\ell=1}^{m} |D_i e_{u_j}(x_\ell^-) - D_i e_{u_j}(x_\ell^+)|. \end{aligned} \tag{2.13}$$

For each term in the first sum above we may now proceed as in the proof of Lemma 2.2. By Sobolev's and the triangle inequality we get

$$\begin{aligned} |D_i e_{u_j}(x_\ell^+) - D_i e_{u_j}(x_{\ell+1}^-)| &\le C\rho^\gamma \|D_i e_{u_j}\|_{C^{0,\gamma}(\mathcal{B}(x_0,\rho/2)\cap T_\ell)} \\ &\le C\rho^\gamma \|e_u\|_{2,q,T_\ell} \\ &\le C\rho^\gamma \big(\|u\|_{2,q} + \|u_h\|_{2,q,T_\ell}\big). \end{aligned}$$

By Theorem 1.3 we have

$$\|u\|_{2,q} \le C\big(\|f\|_q + |g|_{1,q}\big),$$

and by the inverse estimate (1.18) and the inf-sup condition (1.14)

$$\|u_h\|_{2,q,T_\ell} \le C h_{T_\ell}^{-1+n(1/q-1/2)} \|u_h\|_{1,2,T_\ell} \le C h_{\min}^{-1+n(1/q-1/2)} \big(\|f\|_{-1,q} + \|g\|_q\big),$$

since $q > n$.

Thus, with (2.10) and for $T_\ell \in \mathcal{B}_\mathcal{T}$ we obtain the uniform estimate

$$|D_j e_{u_i}(x_\ell^+) - D_j e_{u_i}(x_\ell^-)| \le C h_{\min}^\beta \big(\|f\|_q + |g|_{1,q}\big), \tag{2.14}$$

where $\beta = \gamma\sigma - 1 + n(1/q - 1/2)$ may be chosen arbitrarily large by taking σ large.

As for the terms in the second sum in (2.13) and for $T_\ell \in \mathcal{B}_\mathcal{T}$ we use the following uniform estimate

$$|D_j e_{u_i}(x_\ell^-) - D_j e_{u_i}(x_\ell^+)| \leq \max_{T\in\mathcal{T}} \|[\partial_\nu u_h]\|_{\infty,\partial T\setminus\partial\Omega}. \tag{2.15}$$

Finally, (2.13) – (2.15) in (2.12) concludes the proof. □

Lemma 2.4. *Let $\Omega \subset \mathbf{R}^n$, $n = 2, 3$, be a polyhedral domain such that the solution to (1.1) with data as in Theorem 1.3 is continuous in the sense that $(u, p) \in \mathcal{W}^{2,q}$, for some $q > n$. Let e_p be such that $\int_\Omega e_p\, dx = 0$ and let $x_0 \in \Omega$ be such that $\|e_p\|_\infty = |e_p(x_0)|$. Then there is a constant C such that*

$$\|e_p\|_\infty \leq |(e_p, \delta)| + Ch_{\min}^\beta\big(\|f\|_q + |g|_{1,q}\big),$$

where $\delta = \delta_{x_0,\rho/2}$ is the regularized Dirac distribution (2.9) and β may be chosen arbitrarily large.

We note that the lemma is meaningful since with additional constraints on the domain Ω as in Remark 1.2 there is $q > n$ such that $e_p \in W^{1,q}(\Omega)$ so that $e_p \in L^\infty(\Omega)$.

Proof. The idea of the proof is the same as for Lemma 2.2. By assumption $p \in W^{1,q}(\Omega)$ for $q > n$ and hence it follows by Sobolev's imbedding theorem that e_p is continuous. Consequently, by the mean value theorem there is $x_1 \in \mathcal{B}(x_0, \rho/2) \cap \overline{\Omega}$ such that $(e_p, \delta) = e_p(x_1)$ and thus

$$\|e_p\|_\infty \leq |(e_p, \delta)| + |e_p(x_0) - e_p(x_1)|.$$

We estimate the last term above. By Sobolev's inequality

$$|e_p(x_0) - e_p(x_1)| \leq C\rho^\gamma \|e_p\|_{C^{0,\gamma}(\mathcal{B}(x_0,\rho/2)\cap\overline{\Omega})} \leq C\rho^\gamma \|e_p\|_{1,q}.$$

By the triangle inequality

$$\|e_p\|_{1,q} \leq \|p\|_{1,q} + \|p_h\|_{1,q},$$

and Theorem 1.3

$$\|p\|_{1,q} \leq C\big(\|f\|_q + |g|_{1,q}\big),$$

and by the inverse estimate and the inf-sup condition (1.14)

$$\|p_h\|_{1,q} \leq Ch_{\min}^{-1+n(1/q-1/2)}\|p_h\|_2 \leq Ch_{\min}^{-1+n(1/q-1/2)}\big(\|f\|_{-1,q} + \|g\|_q\big).$$

Thus with (2.10) we obtain

$$|e_p(x_0) - e_p(x_1)| \leq Ch_{\min}^\beta\big(\|f\|_q + |g|_{1,q}\big),$$

where $\beta = \gamma\sigma - 1 + n(1/q - 1/2)$ may be chosen arbitrarily large by taking σ large. □

3. A priori estimates of the dual solution

We consider the dual problem (1.4) for specific choices of data so that we may estimate the scaling of the constants in (1.6) and (1.10) as $q \downarrow 1$. For (1.6) we will consider $(\tilde{f}, \tilde{g}) = (D_i \delta e_j, 0)$ or $(\tilde{f}, \tilde{g}) = (0, \delta - |\Omega|^{-1})$ and for (1.10) we will consider $(\tilde{f}, \tilde{g}) = (\delta e_i, 0)$, where δ is the regularized Dirac distribution (2.11). We proceed as in [14, Theorem 3.1] and [4, Lemma 2.2]. The analysis relies on the explicit knowledge of how the constant in Sobolev's inequality scales as $q \downarrow 1$, which can be estimated by using the the best constant in the Sobolev inequality, where the dependence on the dimension n and the exponent q appear explicitly. We quote Sobolev's inequality from [9, Theorem 7.10, p. 155]. Let ω be a bounded domain in $\mathbf{R}^n$, $n = 2, 3$. Then there is a constant C such that for any $v \in W_0^{1,s}(\omega)^d$, $d = 1, \ldots, n$, and for $1 \le s < n$

$$\|v\|_{ns/(n-s),\omega} \le C|v|_{1,s,\omega}, \tag{3.1}$$

where $C = C(n, s)$ scales like

$$C \le \gamma \left(n \frac{s-1}{n-s} \right)^{1-1/s}, \tag{3.2}$$

and where $\gamma = \gamma(n, s) < \infty$ as $s \uparrow n$.

In the analysis below we will find it useful to have (3.1) and (3.2) formulated somewhat differently. By rearranging the exponents in (3.1) and estimating the constant (3.2) accordingly we conclude that, for any $v \in W_0^{1,nr/(n+r)}(\omega)^d$ and for $n/(n-1) \le r < \infty$,

$$\|v\|_{r,\omega} \le C r^{1-1/n} |v|_{1,nr/(n+r),\omega}. \tag{3.3}$$

The following lemma is a consequence of (3.3).

Lemma 3.1. *Let $\omega \subset \mathbf{R}^n$, $n = 2, 3$, be a bounded domain. Then there is a constant C such that, if $v \in L^q(\omega)^d$, $d = 1, \ldots, n$,*

$$\|\nabla^{k-1} v\|_{-k,\tilde{q},\omega} \le C(q-1)^{-1+1/n} \|v\|_{q,\omega}, \tag{3.4}$$

for $\tilde{q} = nq/(n-q)$ and $1 < q \le n$.

Proof. By integration by parts and with Hölder's inequality in the definition of the dual norm (1.2) we estimate

$$\begin{aligned} \|\nabla^{k-1} v\|_{-k,\tilde{q},\omega} &= \sup_{\varphi \in C_0^\infty(\omega)^d} \frac{|\langle v, \nabla^{k-1}\varphi\rangle|}{\|\varphi\|_{k,\tilde{q}',\omega}} \\ &\le \|v\|_{q,\omega} \sup_{\varphi \in C_0^\infty(\omega)^n} \frac{|\varphi|_{k-1,q',\omega}}{\|\varphi\|_{k,\tilde{q}',\omega}}. \end{aligned} \tag{3.5}$$

Since $1 < q \le n$ implies $n/(n-1) \le q' < \infty$, we may use Sobolev's inequality (3.3) to estimate,

$$|\varphi|_{k-1,q',\omega} \le C q'^{1-1/n} |\varphi|_{k,\tilde{q}',\omega}, \tag{3.6}$$

because $nq'/(n+q') = \tilde{q}'$. Thus, inserting (3.6) in (3.5) concludes the proof. □

As in [14, 4] we introduce a dyadic partition of Ω. Let $d_j = 2^j \rho$ for $j \in \mathbf{N}$ and $d_{-1} = 0$. Define the partition of Ω,

$$A_j = \{x \in \Omega : d_{j-1} \le |x - x_0| \le d_j\}, \tag{3.7}$$

and the supersets to A_j,

$$B_j = \{x \in \Omega : 2^{-1} d_{j-1} \le |x - x_0| \le 2 d_j\}. \tag{3.8}$$

From this definition we get the simple estimate

$$|B_j| \le C d_j^n = C 2^{jn} \rho^n. \tag{3.9}$$

Moreover, let $\eta_j \in C_0^\infty(B_j)$ be a mollifier such that, $\eta_j = 1$ in a neighborhood of A_j and such that for $s \in [1, \infty]$,

$$|\eta_j|_{k,s,B_j} \le C d_j^{n/s-k}. \tag{3.10}$$

Generalizing the last estimate in [14, Proof of Theorem 3.1] we get. For $a > 1$ and as $q \downarrow 1$ we have,

$$\sum_{j=0}^{\infty} 2^{-ja(1-1/q)} = \frac{1}{1 - 2^{-a(1-1/q)}} \le \frac{C}{q-1}. \tag{3.11}$$

Finally, we recall the following two generalizations of Hölder's inequality. Let $1 \le q \le \infty$, $q \le r \le \infty$ and $q \le s \le \infty$ such that

$$\frac{1}{q} = \frac{1}{r} + \frac{1}{s}$$

and let $u \in L^r(\omega)$ and $v \in L^s(\omega)$. Then $uv \in L^q(\omega)$ and

$$\|uv\|_{q,\omega} \le \|u\|_{r,\omega}\|v\|_{s,\omega}. \tag{3.12}$$

In the second generalization we estimate the duality pairing. For a vector space V let $u \in V'$ and $v \in V$. Then

$$|\langle u, v\rangle| \le \|u\|_{V'}\|v\|_{V}. \tag{3.13}$$

In particular, when $u \in W^{-k,q}(\omega)$ and $v \in W_0^{k,q'}(\omega)$ we get

$$|\langle u, v\rangle| \le \|u\|_{-k,q,\omega}\|v\|_{k,q',\omega}. \tag{3.14}$$

3.1. $\mathcal{W}^{1,q}$**-estimates as** $q \downarrow 1$**.** In the following theorem we assume that we have the higher regularity in Remark 1.2.

Theorem 3.2. *Let $\Omega \subset \mathbf{R}^n$, $n = 2, 3$, be a polyhedral domain such that the solution to* (1.4) *with data as in Theorem 1.3 is continuous in the sense that $(\tilde{u}, \tilde{p}) \in \mathcal{W}^{2,q}$ for some $q > n$. Then for $1 < q < 2$ there is a constant C such that the solution $(\tilde{u}, \tilde{p})$ to (1.4) with $(\tilde{f}, \tilde{g}) = (D_i\delta e_j, 0)$ or $(\tilde{f}, \tilde{g}) = (0, \delta - |\Omega|^{-1})$ satisfies the inequality*

$$\|\tilde{u}\|_{1,q} + \|\tilde{p}\|_{L^q/\mathbf{R}} \le C(q-1)^{-2+1/n}\rho^{-n(1-1/q)}.$$

Proof. Let A_j, B_j and η_j be as in (3.7)–(3.10). Choose a fixed value $\tilde{q} = n/(n-1)$. Let $\bar{p} = \tilde{p} + c$ for a fixed $c \in \mathbf{R}$. By Hölder's inequality

$$\begin{aligned}
\|\tilde{u}\|_{1,q} + \|\tilde{p}\|_{L^q/\mathbf{R}} &\le \sum_{j=0}^{\infty}\big(\|\tilde{u}\|_{1,q,A_j} + \|\bar{p}\|_{q,A_j}\big) \\
&\le \sum_{j=0}^{\infty}\big(\|\eta_j\tilde{u}\|_{1,q,B_j} + \|\eta_j\bar{p}\|_{q,B_j}\big) \\
&\le \sum_{j=0}^{\infty}|B_j|^{1/q-1/\tilde{q}}\big(\|\eta_j\tilde{u}\|_{1,\tilde{q},B_j} + \|\eta_j\bar{p}\|_{\tilde{q},B_j}\big).
\end{aligned} \tag{3.15}$$

Notice that $\eta_j\tilde{u}$ and $\eta_j\bar{p}$ satisfy (1.4) in Ω with right hand side $\tilde{f} = \tilde{f}_j = \Delta(\eta_j\tilde{u}) + \nabla(\eta_j\bar{p})$ and $\tilde{g} = \tilde{g}_j = \nabla \cdot (\eta_j\tilde{u})$, where $\tilde{f}_j$ and $\tilde{g}_j$ vanish outside

B_j. Hence, for each term in (3.15) we can apply Theorem 1.1,

$$
\begin{aligned}
\|\eta_j\tilde{u}\|_{1,\tilde{q},B_j} + \|\eta_j\bar{p}\|_{\tilde{q},B_j} &= \|\eta_j\tilde{u}\|_{1,\tilde{q},\Omega} + \|\eta_j\bar{p}\|_{\tilde{q},\Omega} \\
&\le C\big(\|\Delta(\eta_j\tilde{u}) + \nabla(\eta_j\bar{p})\|_{-1,\tilde{q},B_j} + \|\nabla\cdot(\eta_j\tilde{u})\|_{\tilde{q},B_j}\big) \\
&\le C\big(\|\eta_j(\Delta\tilde{u} + \nabla\bar{p}) + 2\nabla\eta_j\cdot\nabla\tilde{u} + \Delta\eta_j\tilde{u} + \nabla\eta_j\bar{p}\|_{-1,\tilde{q},B_j} \\
&\quad + \|\nabla\eta_j\cdot\tilde{u} + \eta_j\nabla\cdot\tilde{u}\|_{\tilde{q},B_j}\big) \\
&\le C\big(\|\eta_j\tilde{f}\|_{-1,\tilde{q},B_j} + \|\eta_j\tilde{g}\|_{\tilde{q},B_j} + \|\nabla\eta_j\bar{p}\|_{-1,\tilde{q},B_j} \\
&\quad + \|\nabla\eta_j\cdot\tilde{u}\|_{\tilde{q},B_j} + \|2\nabla\eta_j\cdot\nabla\tilde{u} + \Delta\eta_j\tilde{u}\|_{-1,\tilde{q},B_j}\big),
\end{aligned}
\tag{3.16}
$$

where $C = C(n,\tilde{q},\Omega)$.

We estimate the right hand side of (3.16) in a few steps. By integration by parts

$$
\|2\nabla\eta_j\cdot\nabla\tilde{u} + \Delta\eta_j\tilde{u}\|_{-1,\tilde{q},B_j} \le \|\nabla\eta_j\cdot\nabla\tilde{u}\|_{-1,\tilde{q},B_j} + \sup_{\varphi\in C_0^\infty(B_j)^n} \frac{(\nabla\eta_j,\tilde{u}\cdot\nabla\varphi)_{B_j}}{\|\varphi\|_{1,\tilde{q}',B_j}}.
$$

Since $(\nabla\eta_j\bar{p},\varphi) \le \|\bar{p}\|_{W^{1,n}(B_j)'}|\nabla\eta_j\cdot\varphi|_{1,n,B_j}$, notice that the dual exponent to $\tilde{q}$ is $\tilde{q}' = n$,

$$
\|\nabla\eta_j\tilde{p}\|_{-1,\tilde{q},B_j} \le \|\bar{p}\|_{W^{1,n}(B_j)'} \sup_{\varphi\in C_0^\infty(B_j)^n} \frac{|\nabla\eta_j\cdot\varphi|_{1,n,B_j}}{\|\varphi\|_{1,n,B_j}},
$$

and since $(\nabla\eta_j\cdot\nabla\tilde{u},\varphi) = -(\tilde{u},\nabla(\nabla\eta_j\cdot\varphi))$,

$$
\|\nabla\eta_j\cdot\nabla\tilde{u}\|_{-1,\tilde{q},B_j} \le \|\tilde{u}\|_{\tilde{q},B_j} \sup_{\varphi\in C_0^\infty(B_j)^n} \frac{|\nabla\eta_j\cdot\varphi|_{1,n,B_j}}{\|\varphi\|_{1,n,B_j}}.
$$

Now by Hölder's inequality

$$
|\nabla\eta_j\cdot\varphi|_{1,n,B_j} \le |\eta_j|_{1,\infty,B_j}|\varphi|_{1,n,B_j} + \|\nabla^2\eta_j\varphi\|_{n,B_j},
$$

and moreover by (3.12) with s such that $1/n = 1/s + 1/q'$, (3.3), and Hölder's inequality

$$
\begin{aligned}
\|\nabla^2\eta_j\varphi\|_{n,B_j} &\le |\eta_j|_{2,s,B_j}\|\varphi\|_{q',B_j} \\
&\le C(q')^{-1+1/n}|\eta_j|_{2,s,B_j}|\varphi|_{1,nq'/(n+q'),B_j} \\
&\le C|B_j|^{1-1/q}(q-1)^{-1+1/n}|\eta_j|_{2,s,B_j}|\varphi|_{1,n,B_j}.
\end{aligned}
\tag{3.17}
$$

Finally, by Hölder's inequality

$$
\|\nabla\eta_j\cdot\tilde{u}\|_{\tilde{q},B_j} + \sup_{\varphi\in C_0^\infty(B_j)^n} \frac{(\nabla\eta_j,\tilde{u}\cdot\nabla\varphi)}{\|\varphi\|_{1,\tilde{q}',B_j}} \le 2|\eta_j|_{1,\infty,B_j}\|\tilde{u}\|_{\tilde{q},B_j}.
$$

Thus, with the above estimates in (3.16) we obtain

$$\begin{aligned}\|\eta_j\tilde u\|_{1,\tilde q,B_j} + \|\eta_j\bar p\|_{L^{\tilde q}(B_j)/\mathbf{R}} &\le C_I\|\eta_j\tilde f\|_{-1,\tilde q,B_j} + C_{II}\|\eta_j\tilde g\|_{\tilde q,B_j}\\ &\quad + C_{III}\big(|\eta_j|_{1,\infty,B_j} + |B_j|^{1-1/q}(q-1)^{-1+1/n}|\eta_j|_{2,s,B_j}\big)\\ &\quad\times\big(\|\tilde u\|_{\tilde q,B_j} + \|\bar p\|_{W^{1,n}(B_j)'}\big)\\ &= I_j + II_j + III_j.\end{aligned} \tag{3.18}$$

With (3.18) we now estimate (3.15) in three steps. Recall (3.9) that will repeatedly be used in the estimates below.

I. For data $\tilde f = D_i\delta e_\ell$ and by integration by parts we obtain by the same argument as in (3.17) and with the same exponents

$$\begin{aligned}\|\eta_j D_i\delta e_\ell\|_{-1,\tilde q,B_j} &\le C\|\delta\|_{\tilde q,B_j}\sup_{\varphi\in C_0^\infty(B_j)^n}\frac{|\eta_j\varphi|_{1,n,B_j}}{\|\varphi\|_{1,n,B_j}}\\ &\le C\|\delta\|_{\tilde q,B_j}\big(\|\eta_j\|_{\infty,B_j} + |B_j|^{1-1/q}(q-1)^{-1+1/n}|\eta_j|_{1,s,B_j}\big).\end{aligned}$$

Since $\mathrm{supp}(\delta)\cap B_j = \emptyset$ for $j\ge 1$ and with (2.11) and for ρ sufficiently small

$$\begin{aligned}\sum_{j=0}^\infty |B_j|^{1/q-1/\tilde q}I_j &\le C\rho^{n(1/q-1/\tilde q)}(q-1)^{-1+1/n}\|\delta\|_{\tilde q}\\ &\le C\rho^{-n(1-1/q)}(q-1)^{-1+1/n},\end{aligned} \tag{3.19}$$

where we used $n/q - n/\tilde q - n(1-1/\tilde q) + n(1-1/q) + n/s - 1 = -n(1-1/q)$.

II. For data $\tilde g = \delta - |\Omega|^{-1}$ and since $\mathrm{supp}(\delta)\cap B_j = \emptyset$ for $j\ge 1$ and with (3.1) and (2.11)

$$\sum_{j=0}^\infty |B_j|^{1/q-1/\tilde q}II_j \le C\rho^{n/q-n/\tilde q}\|\nabla\delta\|_1 \le C\rho^{-n(1-1/q)}, \tag{3.20}$$

where we used $n/q - n/\tilde q - 1 = -n(1-1/q)$.

III. By Hölder's inequality and since $q<2$

$$\begin{aligned}|B_j|^{1/q-1/\tilde q}III_j &\le Cd_j^{n/q-n/\tilde q}\big(|\eta_j|_{1,\infty,B_j} + d_j^{n(1-1/q)}(q-1)^{-1+1/n}|\eta_j|_{2,s,B_j}\big)\\ &\quad\times\big(\|\tilde u\|_{\tilde q,B_j} + \|\bar p\|_{W^{1,n}(B_j)'}\big)\\ &\le Cd_j^{-n(1-1/q)}\big(1 + (q-1)^{-1+1/n}\big)\big(\|\tilde u\|_{\tilde q,B_j} + \|\bar p\|_{W^{1,n}(B_j)'}\big)\\ &\le Cd_j^{-n(1-1/q)}(q-1)^{-1+1/n}\big(\|\tilde u\|_{\tilde q,B_j} + \|\bar p\|_{W^{1,n}(B_j)'}\big),\end{aligned}$$

where we used $n/q - n/\tilde{q} - 1 = -n(1-1/q)$ and $n/q - n/\tilde{q} + n - n/q + n/s - 2 = -n(1-1/q)$.

Adding all the terms and by Hölder's inequality in the sum with exponent $\tilde{q}$, with conjugate exponent $\tilde{q}' = n$, estimating the geometric sum as in (3.11) and by Corollary 1.4

$$\begin{aligned}\sum_{j=0}^{\infty} |B_j|^{1/q-1/\tilde{q}} III_j &\le C(q-1)^{-1+1/n}\left(\sum_{j=0}^{\infty} d_j^{-n^2(1-1/q)}\right)^{1/n} \\ &\quad\times \left(\sum_{j=0}^{\infty}\big(\|\tilde{u}\|_{\tilde{q},B_j} + \|\bar{p}\|_{W^{1,n}(B_j)'}\big)^{\tilde{q}}\right)^{1/\tilde{q}} \\ &\le C\rho^{-n(1-1/q)}(q-1)^{-1}\big(\|\tilde{u}\|_{\tilde{q}} + \|\tilde{p}\|_{W^{1,n}(\Omega)'/\mathbf{R}}\big) \\ &\le C\rho^{-n(1-1/q)}(q-1)^{-1}\big(\|\tilde{f}\|_{-2,\tilde{q}} + \|\tilde{g}\|_{W^{1,n}(\Omega)'}\big),\end{aligned} \tag{3.21}$$

since $\bar{p} = p + c$ for arbitrary $c \in \mathbf{R}$ we may take the infimum over all c.

For $\tilde{f} = D_i\delta e_j$ and since $\|D_i\delta e_j\|_{-2,\tilde{q}} \le C\|D_i\delta e_j\|_{-2,nq/(n-q)}$ we obtain by Lemma 3.1,

$$\|D_i\delta e_j\|_{-2,\tilde{q}} \le C(q-1)^{-1+1/n}\|\delta\|_q \le C\rho^{-n(1-1/n)}(q-1)^{-1+1/q}, \tag{3.22}$$

For $\tilde{g} = \delta - |\Omega|^{-1}$ we note that $(\delta - |\Omega|^{-1}, \varphi) = (\delta, \varphi - \varphi_0)$ where $\varphi_0 = |\Omega|^{-1}\int_\Omega \varphi\, dx$. Using Sobolev's inequality as in the proof of Lemma 3.1

$$\|\delta - |\Omega|^{-1}\|_{W^{1,n}(\Omega)'} \le C\rho^{-n(1-1/n)}(q-1)^{-1+1/n}. \tag{3.23}$$

Collecting the results in (3.19)–(3.23) concludes the proof. □

Corollary 3.3. *Let $\Omega \subset \mathbf{R}^n$, $n = 2, 3$, be a polyhedral domain such that the solution to* (1.4) *with data as in Theorem 1.3 is continuous in the sense that $(\tilde{u}, \tilde{p}) \in \mathcal{W}^{2,q}$ for some $q > n$. Then there is a constant C such that the solution, $(\tilde{u}, \tilde{p})$ to* (1.4) *with $(\tilde{f}, \tilde{g}) = (D_i\delta e_j, 0)$ or $(\tilde{f}, \tilde{g}) = (0, \delta - |\Omega|^{-1})$ satisfies the inequality,*

$$\|\tilde{u}\|_{1,1} + \|\tilde{p}\|_{L^1/\mathbf{R}} \le C|\log\rho|^{2-1/n}.$$

Proof. By Hölder's inequality,

$$\|\tilde{u}\|_{1,1} + \|\tilde{p}\|_{L^1/\mathbf{R}} \le |\Omega|^{1/q'}\big(\|\tilde{u}\|_{1,q} + \|\tilde{p}\|_{L^q/\mathbf{R}}\big).$$

Thus, with Theorem 3.2, taking $q - 1 = 1/|\log\rho|$, we finish the proof. □

3.2. $\mathcal{W}^{2,q}$**-estimates as** $q \downarrow 1$**.**

Theorem 3.4. *Let* $\Omega \subset \mathbf{R}^n$*,* $n = 2, 3$*, be a polyhedral domain. Then for* $q \in (1, 4/3]$ *there is a constant* C *such that the solution* $(\tilde{u}, \tilde{p})$ *to (1.4) with* $(\tilde{f}, \tilde{g}) = (\delta e_i, 0)$ *satisfies the inequality*

$$\|\tilde{u}\|_{2,q} + \|\tilde{p}\|_{W^{1,q}/\mathbf{R}} \leq C(q-1)^{-\alpha_n} \rho^{-2(n+1)(1-1/q)}$$

where $\alpha_2 = 2$*,* $\alpha_3 = 4/3$*.*

Proof. We proceed as in the proof of Theorem 1.1. Let A_j, B_j and η_j be as in (3.7)–(3.10). Let $\bar{p} = \tilde{p} + c$ for a fixed $c \in \mathbf{R}$. Choose a fixed value $q_0 \in (1, 4/3]$. Then for $1 < q < q_0$ by Hölder's inequality

$$\begin{aligned} \|\tilde{u}\|_{2,q} + \|\tilde{p}\|_{W^{1,q}/\mathbf{R}} &\leq \sum_{j=0}^{\infty} \big(\|\tilde{u}\|_{2,q,A_j} + \|\bar{p}\|_{1,q,A_j}\big) \\ &\leq \sum_{j=0}^{\infty} \big(\|\eta_j \tilde{u}\|_{2,q,B_j} + \|\eta_j \bar{p}\|_{1,q,B_j}\big) \\ &\leq \sum_{j=0}^{\infty} |B_j|^{1/q - 1/q_0} \big(\|\eta_j \tilde{u}\|_{2,q_0,B_j} + \|\eta_j \bar{p}\|_{1,q_0,B_j}\big). \end{aligned} \tag{3.24}$$

We note that $\eta_j \tilde{u}$ and $\eta_j \bar{p}$ satisfy (1.4) in Ω with $\tilde{f} = \tilde{f}_j = \Delta(\eta_j \tilde{u}) + \nabla(\eta_j \bar{p})$ and $\tilde{g} = \tilde{g}_j = \nabla \cdot (\eta_j \tilde{u})$, where $\tilde{f}_j$ and $\tilde{g}_j$ vanish outside B_j for each j. Hence, for each term in (3.24) we can apply Theorem 1.3,

$$\begin{aligned} \|\eta_j \tilde{u}\|_{2,q_0,B_j} + \|\eta_j \bar{p}\|_{1,q_0,B_j} &= \|\eta_j \tilde{u}\|_{2,q_0,\Omega} + \|\eta_j \bar{p}\|_{1,q_0,\Omega} \\ &\leq C\big(\|\Delta(\eta_j \tilde{u}) + \nabla(\eta_j \bar{p})\|_{q_0,B_j} + |\nabla \cdot (\eta_j \tilde{u})|_{1,q_0,B_j}\big) \\ &\leq C\big(\|\eta_j(\Delta \tilde{u} + \nabla \bar{p}) + 2\nabla \eta_j \cdot \nabla \tilde{u} + \Delta \eta_j \tilde{u} + \nabla \eta_j \bar{p}\|_{q_0,B_j} \\ &\quad + |\nabla \eta_j \cdot \tilde{u} + \eta_j \nabla \cdot \tilde{u}|_{1,q_0,B_j}\big) \\ &\leq C_I \|\eta_j \delta e_i\|_{q_0,B_j} + C_{II} \|\nabla^2 \eta_j \tilde{u}\|_{q_0,B_j} \\ &\quad + C_{III}\big(\|\nabla \eta_j \cdot \nabla \tilde{u}\|_{q_0,B_j} + \|\nabla \eta_j \bar{p}\|_{q_0,B_j}\big) \\ &= I_j + II_j + III_j, \end{aligned} \tag{3.25}$$

where $C = C(n, q_0, \Omega)$ and with $-\Delta \tilde{u} - \nabla \bar{p} = \delta e_i$ and $\nabla \cdot \tilde{u} = 0$, and where we also used $|\nabla \eta_j \cdot \tilde{u}|_{1,q_0,B_j} \leq \|\nabla^2 \eta_j \tilde{u}\|_{q_0,B_j} + \|\nabla \eta_j \cdot \nabla \tilde{u}\|_{q_0,B_j}$.

With (3.25) we now estimate (3.24) in three steps. Recall (3.9) that will repeatedly be used in the estimates below.

I. Since $\operatorname{supp}(\delta) \cap B_j = \emptyset$ for $j \geq 1$ and with (2.11)

$$\sum_{j=0}^{\infty} |B_j|^{1/q-1/q_0} I_j \leq C\rho^{n/q-n/q_0} \|\delta\|_{q_0} \leq C\rho^{-n(1-1/q)}. \tag{3.26}$$

II. By Hölder's inequality with exponent $\tilde{q} = q/(q-2/n)$ and s such that $1/q_0 = 1/s + 1/\tilde{q}$ and with (3.10)

$$|B_j|^{1/q-1/q_0} II_j \leq C d_j^{n/q-n/q_0} |\eta_j|_{2,s,B_j} \|\tilde{u}\|_{\tilde{q},B_j} \leq C d_j^{-(n+2)(1-1/q)} \|\tilde{u}\|_{\tilde{q},B_j},$$

where we used $n/q - n/q_0 + n/s - 2 = -(n+2)(1-1/q)$.

Adding all the terms and by Hölder's inequality in the sum with exponent $\tilde{q}$, with conjugate exponent $\tilde{q}' = nq/2$, and estimating the geometric sum as in (3.11)

$$\begin{aligned} &\sum_{j=0}^{\infty} |B_j|^{1/q-1/q_0} II_j \\ &\leq C \left(\sum_{j=0}^{\infty} d_j^{-(n+2)(1-1/q)nq/2} \right)^{2/nq} \left(\sum_{j=0}^{\infty} \|\tilde{u}\|_{\tilde{q},B_j}^{\tilde{q}} \right)^{1/\tilde{q}} \\ &\leq C\rho^{-(n+2)(1-1/q)} (q-1)^{-2/nq} \|\tilde{u}\|_{\tilde{q}}. \end{aligned} \tag{3.27}$$

With (3.3), Hölder's inequality ($n\tilde{q}/(n+\tilde{q}) \leq nq/(n-q)$), Theorem 1.1, Lemma 3.1 and (2.11)

$$\begin{aligned} \|\tilde{u}\|_{\tilde{q}} &\leq C\tilde{q}^{1-1/n} \|\tilde{u}\|_{1,n\tilde{q}/(n+\tilde{q})} \\ &\leq C\tilde{q}^{1-1/n} \|\delta\|_{-1,nq/(n-q)} \\ &\leq C\tilde{q}^{1-1/n} (1-q)^{-1+1/n} \|\delta\|_q \\ &\leq C\rho^{-n(1-1/q)} (q-2/n)^{-1+1/n} (1-q)^{-1+1/n}, \end{aligned} \tag{3.28}$$

where we remark that

$$2n/(n+1) \leq nq/(n-q) \leq 2n/(n-1),$$

for $n = 2, 3$ and $1 < q < 4/3$ and thus we may use Theorem 1.1.

Collecting the estimates in (3.27) and (3.28) we obtain

$$\sum_{j=0}^{\infty} |B_j|^{1/q-1/q_0} II_j \leq C\rho^{-2(n+1)(1-1/q)} (q-2/n)^{-1+1/n} (q-1)^{-1-1/n}. \tag{3.29}$$

III. By Hölder's inequality with exponent $\tilde q = n/(n-1)$ and s such that $1/q_0 = 1/s + 1/\tilde q$

$$\begin{aligned}|B_j|^{1/q-1/q_0} III_j &\le C d_j^{n/q-n/q_0} |\eta_j|_{1,s,B_j} \big(\|\tilde u\|_{1,\tilde q,B_j} + \|\bar p\|_{\tilde q,B_j}\big)\\ &\le C d_j^{-n(1-1/q)} \big(\|\tilde u\|_{1,\tilde q,B_j} + \|\bar p\|_{\tilde q,B_j}\big),\end{aligned}$$

where we used $n/q - n/q_0 + n/s - 1 = -n(1-1/q)$.

Adding all the terms and by Hölder's inequality in the sum with exponent $\tilde q$, with conjugate exponent $\tilde q' = n$, and estimating the geometric sum as in (3.11)

$$\begin{aligned}&\sum_{j=0}^{\infty} |B_j|^{1/q-1/q_0} III_j\\ &\le C\left(\sum_{j=0}^{\infty} d_j^{-n^2(1-1/q)}\right)^{1/n} \left(\sum_{j=0}^{\infty} \big(\|\tilde u\|_{1,\tilde q,B_j} + \|\bar p\|_{L^{\tilde q}(B_j)}\big)^{\tilde q}\right)^{1/\tilde q}\\ &\le C\rho^{-n(1-1/q)}(q-1)^{-1/n}\big(\|\tilde u\|_{1,\tilde q} + \|\tilde p\|_{L^{\tilde q}/\mathbf{R}}\big),\end{aligned} \tag{3.30}$$

since $\bar p = p + c$ for arbitrary $c \in \mathbf{R}$ we may take the infimum of all c.

With Theorem 1.1, Hölder's inequality ($\tilde q \le nq/(n-q)$), Lemma 3.1 and (2.11)

$$\begin{aligned}\|\tilde u\|_{1,\tilde q} + \|\tilde p\|_{L^{\tilde q}/\mathbf{R}} &\le C\|\delta\|_{-1,\tilde q}\\ &\le C(1-q)^{-1+1/n}\|\delta\|_q\\ &\le C\rho^{-n(1-1/q)}(1-q)^{-1+1/n},\end{aligned} \tag{3.31}$$

where Theorem 1.1 is applicable in analogy to the remark at (3.28).

Collecting the estimates in (3.30) and (3.31) we obtain

$$\sum_{j=0}^{\infty} |B_j|^{1/q-1/q_0} III_j \le C\rho^{-2n(1-1/q)}(q-1)^{-1}. \tag{3.32}$$

Finally adding (3.26), (3.29) and (3.32) concludes the proof. □

Corollary 3.5. *Let $\Omega \subset \mathbf{R}^n$, $n = 2, 3$, be a polyhedral domain. Then there is a constant C such that the solution, $(\tilde u, \tilde p)$ to (1.4) with $\tilde f = \delta e_i$ and $\tilde g = 0$ satisfies the inequality,*

$$\|\tilde u\|_{2,1} + \|\tilde p\|_{W^{1,1}/\mathbf{R}} \le C|\log\rho|^{\alpha_n},$$

with α_n as in Theorem 3.4.

Proof. See the proof of Corollary 3.3. □

4. Main results

We now make a precise statement of the main results and begin with the pointwise error estimate of the velocity field.

Theorem 4.1. *Let $\Omega \subset \mathbf{R}^n$, $n = 2, 3$, be a polyhedral domain. Suppose the data to (1.1) is as in Theorem 1.1 for some $q > n$. Then the error e_u in the finite element solution to (1.13) satisfies*

$$\|e_u\|_\infty \leq C|\log h_{\min}|^{\alpha_n}\eta_{2,\infty} + C_1 h_{\min}^\beta,$$

where $\alpha_2 = 2$, $\alpha_3 = 4/3$ and with $\eta_{2,\infty}$ as in Lemma 2.1 and where β can be chosen arbitrarily large.

Proof. Let $x_0 \in \Omega$ and i be such that $\|e_u\|_{L^\infty} = |e_{u_i}(x_0)|$ and let $(\tilde{u}, \tilde{p})$ be the solution to (1.4) with data $\tilde{f} = \delta e_i$ and $\tilde{g} = 0$. With Lemma 2.2, the identity (2.8), Lemma 2.1 with $q = \infty$, and Corollary 3.5, we obtain

$$\begin{aligned}
\|e_u\|_\infty &\leq (e_u, \delta e_i) + C_1 h_{\min}^\beta\big(\|f\|_{-1,q} + \|g\|_q\big) \\
&\leq |\mathcal{R}((u_h, p_h), (\tilde{u}, \tilde{p}))| + C_1 h_{\min}^\beta \\
&\leq C\eta_{2,\infty}\big(\|\tilde{u}\|_{2,1} + \|\tilde{p}\|_{W^{1,1}/\mathbf{R}}\big) + C_1 h_{\min}^\beta \\
&\leq C|\log\rho|^{\alpha_n}\eta_{2,\infty} + C_1 h_{\min}^\beta.
\end{aligned}$$

Choosing $\rho = h_{\min}^\sigma$ for σ sufficiently large such that β becomes large as in Lemma 2.2 concludes the proof. □

For the gradient of the velocity field and the pressure we only obtain pointwise error estimates on a restricted class of polyhedral domains, namely convex domains when $n = 2$ and under an inner angle condition when $n = 3$, see Remark 1.2.

Theorem 4.2. *Let $\Omega \subset \mathbf{R}^n$, $n = 2, 3$, be a polyhedral domain such that the solution to (1.1) with data as in Theorem 1.3 is continuous in the sense that $(u, p) \in \mathcal{W}^{2,q}$ for some $q > n$. Then the error ∇e_u in the finite element solution to (1.13) satisfies*

$$\|\nabla e_u\|_\infty \leq C|\log h_{\min}|^{2-1/n}\eta_{1,\infty} + C_1 h_{\min}^\beta,$$

with $\eta_{1,\infty}$ as in Lemma 2.1 and where β can be chosen arbitrarily large.

Proof. Let $x_0 \in \Omega$, i and j be such that $\|\nabla e_u\|_\infty = |D_i e_{u_j}(x_0)|$ and let $(\tilde{u}, \tilde{p})$ be the solution to (1.4) with data $\tilde{f} = D_i \delta e_j$ and $\tilde{g} = 0$. With Lemma 2.3, the identity (2.8), Lemma 2.1 with $q = \infty$, and Corollary 3.3, we obtain

$$\begin{aligned}
\|\nabla e_u\|_\infty &\le (e_u, D_i \delta e_j) + C_1 h_{\min}^{\beta}\big(\|f\|_q + |g|_{1,q}\big) + C_2 \max_{T\in\mathcal{T}} \|[\partial_\nu u_h]\|_{\infty,\partial T\setminus\partial\Omega} \\
&\le |\mathcal{R}((u_h,p_h),(\tilde{u},\tilde{p}))| + C_1 h_{\min}^{\beta} + C_2 \max_{T\in\mathcal{T}} \|[\partial_\nu u_h]\|_{\infty,\partial T\setminus\partial\Omega} \\
&\le C\eta_{1,\infty}\big(\|\tilde{u}\|_{1,1} + \|\tilde{p}\|_{L^1/\mathbf{R}}\big) + C_1 h_{\min}^{\beta} + C_2 \max_{T\in\mathcal{T}} \|[\partial_\nu u_h]\|_{\infty,\partial T\setminus\partial\Omega} \\
&\le C|\log\rho|^{2-1/n}\eta_{1,\infty} + C_1 h_{\min}^{\beta}.
\end{aligned}$$

Note that the jump term $[\partial_\nu u_h]$ from Lemma 2.3 is incorporated into the error estimator $\eta_{1,\infty}$ in Lemma 2.1.

Choosing $\rho = h_{\min}^{\sigma}$ for σ sufficiently large such that β becomes large as in Lemma 2.3 concludes the proof. □

Theorem 4.3. *Let $\Omega \subset \mathbf{R}^n$, $n = 2,3$, be a polyhedral domain such that the solution to (1.1) with data as in Theorem 1.3 is continuous in the sense that $(u,p) \in \mathcal{W}^{2,q}$ for some $q > n$. Then the error e_p in the finite element solution to (1.13) satisfies*

$$\|e_p\|_\infty \le C|\log h_{\min}|^{2-1/n}\eta_{1,\infty} + C_1 h_{\min}^{\beta},$$

with $\eta_{1,\infty}$ as in Lemma 2.1 and where β can be chosen arbitrarily large.

Proof. Let $x_0 \in \Omega$ be such that $|e_p(x_0)| = \|e_p\|_{L^\infty}$ and let $(\tilde{u}, \tilde{p})$ be the solution to (1.4) with data $\tilde{f} = 0$ and $\tilde{g} = \delta - |\Omega|^{-1}$. With Lemma 2.4, the identity (2.8) and choosing e_p such that $\int_\Omega e_p \, dx = 0$, Lemma 2.1 with $q = \infty$, and Corollary 3.3, we obtain

$$\begin{aligned}
\|e_p\|_\infty &\le (e_p, \delta) + C_1 h_{\min}^{\beta}\big(\|f\|_q + |g|_{1,q}\big) \\
&\le |\mathcal{R}((u_h,p_h),(\tilde{u},\tilde{p}))| + C_1 h_{\min}^{\beta} \\
&\le C\eta_{1,\infty}\big(\|\tilde{u}\|_{1,1} + \|\tilde{p}\|_{L^1/\mathbf{R}}\big) + C_1 h_{\min}^{\beta} \\
&\le C|\log\rho|^{2-1/n}\eta_{1,\infty} + C_1 h_{\min}^{\beta}.
\end{aligned}$$

Choosing $\rho = h_{\min}^{\sigma}$ for σ sufficiently large, such that β becomes large as in Lemma 2.4 concludes the proof. □

Finally we obtain L^q-estimates of the velocity gradient and the pressure.

Theorem 4.4. *Let $\Omega \subset \mathbf{R}^n$, $n = 2, 3$, be a polyhedral domain. Suppose the data to (1.1) is as in Theorem 1.1 for some $2n/(n+1) \le q \le 2n/(n-1)$. Then the error (e_u, e_p) in the finite element solution to (1.13) satisfies*

$$\|e_u\|_{1,q} + \|e_p\|_{L^q/\mathbf{R}} \le C\eta_{1,q},$$

where $\eta_{1,q}$ is as in lemma 2.1.

Proof. With Corollary 1.2, the identity (2.4), and Lemma 2.1 we get

$$\begin{aligned}\|(e_u, e_p)\|_{\mathcal{W}^q} &\le C \sup_{(\phi,\lambda)\in\mathcal{W}^{q'}} \frac{|\mathcal{L}((e_u, e_p), (\phi, \lambda))|}{\|(\phi, \lambda)\|_{\mathcal{W}^{q'}}} \\ &= C \sup_{(\phi,\lambda)\in\mathcal{W}^{q'}} \frac{|\mathcal{R}((u_h, u_h), (\phi, \lambda))|}{\|(\phi, \lambda)\|_{\mathcal{W}^{q'}}} \\ &\le C\eta_{1,q} \sup_{(\phi,\lambda)\in\mathcal{W}^{q'}} \frac{\|\phi\|_{1,q'} + \|\lambda\|_{L^{q'}/\mathbf{R}}}{\|(\phi, \lambda)\|_{\mathcal{W}^{q'}}} \\ &\le C\eta_{1,q}.\end{aligned}$$

□

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