A Connection between the Volume Fraction of the Stienen Model and the Dead Leaves Model

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Abstract

The volume fraction of the intact grains of the dead leaves model with spherical grains of equal size is $2^{-d}$ in $d$ dimensions. This is the same volume fraction as the original Stienen model has. Here we consider some variants of these models: the dead leaves model with grains of a fixed convex shape and possibly random size and random orientations and a generalisation of the Stienen model with convex grains growing at random speeds. The main result of this paper is that the volume fraction is the same for the two models also in this case, if the radius distribution in the dead leaves model equals the speed distribution in the Stienen model.

Furthermore, we show that for grains of a fixed shape and orientation, centrally symmetric sets give the highest volume fraction, while simplices give the lowest. If the grains are randomly rotated, then the volume fraction achieves its highest value only for spheres.

Keywords: Dead leaves model, Stienen model, Matérn’s hard-core process, non-intersecting grains, volume fraction, convex grains.

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1 Introduction

Two well-known random models for non-intersecting spheres are the intact grains of the dead leaves model and the Stienen model. The dead leaves
model, which was introduced by Matheron (1968), is usually described in two dimensions; one can think of leaves falling randomly to the ground from time $-\infty$ to 0. The intact grains, or leaves, are then the leaves which are intact at time zero, when you look from above. The leaves can be any compact objects, possibly of random shape, size and orientation. A proper description of this model is given in Section 4.

The Stienen model was originally proposed in a material science context by Stienen (1982). In his model, each point of a stationary Poisson process is the centre of a sphere with a diameter equal to the distance to of its closest neighbour. Another way to determine the size of a particular sphere, say A, in the Stienen model, is to let all spheres grow at unit speed until the first collision with A, without bothering about collisions between other spheres. This is the size that sphere A gets in the Stienen model. The reason for giving this description of the model is that it allows for natural generalisations of the model: one is to let the grains grow at random speeds, and another is to allow the grains to have other shapes than spherical.

The Stienen and the dead leaves models are in many respects quite different: The germs of the Stienen model constitute a Poisson process, while this is not the case with the intact grains of the dead leaves model; in the patterns of the Stienen model, typically small spheres tend to appear in touching pairs, and larger spheres tend to appear isolated, while the probability of grains touching each other is zero in the dead leaves model; if the falling leaves are of a fixed size, this is clearly also the case with the intact leaves, while the sizes of the grains in the Stienen model are determined by the underlying Poisson process and by the growth speed distribution.

In the present paper, however, we focus on the similarity rather than on the difference between the two models. This similarity concerns the volume fraction, which, for stationary germ-grain models with non-intersecting grains, can be written as

$$\rho = \lambda \overline{V},$$

where $\lambda$ is the intensity of the germs and $\overline{V}$ denotes the mean volume of a typical grain. The main result of the paper is the following: If the radii of the falling leaves in the dead leaves model have the same distribution as the growth speeds of the grains in the Stienen model, then the two models have the same volume fraction. Furthermore, we present the radii distribution of the grains in both models, and give upper and lower bounds on the volume fraction.
fraction. The grains are supposed to have a given convex shape, but can have random sizes and orientations.

## 2 Miscellaneous

The size of a sphere is often given in terms of its radius; in accordance with this we define the size of a convex grain $K \subset \mathbb{R}^d$ as half its diameter, that is

$$\sup_{x,y \in K} |x - y|/2,$$

where $| \cdot |$ denotes the Euclidean distance. Let $\mathcal{C}^d$ denote the family of compact, convex sets $K$ with interior points in $\mathbb{R}^d$, such that $o \in K$ and with the size equal to 1.

Let $l_d$ denote the $d$-dimensional Lebesgue measure and let $o$ denote the origin. Furthermore, let $B_d(o, r) = \{ x \in \mathbb{R}^d : |z - x| \leq r \}$ denote the $d$-dimensional ball centered at $z$ with radius $r$. The volume and the surface area of $B_d(o, 1)$ is denoted by $\kappa_d$ and $\omega_d$, respectively. Let $K(z, r)$ denote a set with the same shape as $K$, translated by $z$ and with size $r > 0$, that is $K(z, r) = \{ ry + z : y \in K \}$, and $l_d(K(z, r)) = r^d l_d(K)$. Note that if $K \in \mathcal{C}^d$, then $K(o, 1) = K$.

For $A, B \subset \mathbb{R}^d$, the Minkowski addition is defined by

$$A \oplus B := \{ x + y : x \in A, y \in B \}$$

which can also be written as

$$A \oplus B = \{ x : A \cap (\tilde{B} + x) \neq \emptyset \}, \quad (2.1)$$

where $\tilde{B} = -B$ denotes the reflection of $B$ at the origin. The volume of $xK \oplus yL$, where $x, y \in \mathbb{R}^+$ and $K, L \subset \mathbb{R}^d$ are non-empty convex sets, can be written as:

$$l_d(xK \oplus yL) = \sum_{i=0}^{d} \binom{d}{i} x^i y^{d-i} V_{i,d-i}(K, L), \quad (2.2)$$

where $V_{i,d-i}(K, L) := V(K, \ldots, \underline{K}, L, \ldots, L)$ are the mixed volumes (areas in $\mathbb{R}^2$) of $K$ and $L$. Note the special cases $V_{d,0}(K, L) = l_d(K)$ and $V_{0,d}(K, L) = l_d(L)$. 

3
The set
\[ K \oplus \bar{K} = \{ x \in \mathbb{R}^d : K \cap (K + \{ x \}) \neq \emptyset \} \] (2.3)
is called the *difference body* of \( K \subset \mathbb{R}^d \), and if \( K \) is convex, then
\[ 2^d l_d(K) \leq l_d(K \oplus \bar{K}) \leq \left( \frac{2^d}{d} \right) l_d(K), \] (2.4)
where the lower bound is attained if and only if \( K \) is centrally symmetric, and the upper bound is attained if and only if \( K \) is a simplex. Furthermore, if \( K \) is a convex set, then
\[ l_d(K) \leq V_{i,d-i}(K, \bar{K}) \leq d^{\min\{i,d-i\}} l_d(K), \] (2.5)
with equality on the left-hand side iff \( K \) is centrally symmetric or \( i(d-i) = 0 \). On the right-hand side, there is equality in dimension 2 and 3 if and only if \( K \) is a triangle and a tetrahedron, respectively, or \( i(d-i) = 0 \). It is furthermore conjectured by Godbersen (1938) and Makai jr. (1974) that
\[ V_{i,d-i}(K, \bar{K}) \leq \left( \frac{d}{i} \right) l_d(K), \] (2.6)
with equality if and only if \( K \) is a simplex (see Schneider (1993) p.412).

Next, we introduce the so-called *intrinsic volumes* \( V_i(K) \), \( i = 1, \ldots, d \), for compact, convex \( K \subset \mathbb{R}^d \),
\[ V_i(K) := \frac{d}{\kappa_{d-i}} V_{i,d-i}(K, B_d(o, 1)). \] (2.7)
Note that \( V_0 = 1 \), and that \( V_d \) is the volume. Furthermore \( 2V_{d-1} = \) the surface area and \( 2\kappa_{d-1}V_1/\omega_d = \) the mean width.

## 3 The Stienen model

Two generalisations of the original Stienen model, in which spherical grains grow at the same speed, were proposed in the Introduction:
- random growth speeds,
- non-spherical grains.
How the sizes of the spheres are determined in the original model can be described in two different ways: the diameter of a sphere equals the distance from the centre of the sphere to the centre of its closest neighbour; another way to put it is that a sphere gets the size it would have if it grew from a germ until it met another sphere, and all other spheres grew forever. With the latter description it is easy to generalise the model to have random growth speeds. Assume that all germs get a random speed according to distribution $F_{sp}$, independently of each other and of the Poisson process. Now the sizes of the grains are determined as in the original model, the only change being that they grow with the random speeds they are given.

Also the second of the proposed generalisations is easily explained with the latter description of how the sizes are determined: Here it is non-spherical grains of possible different orientations that grow until the first collision with another grain.

If all grains have the same shape as $K \in C^d$, then the volume fraction of the Stienen model is

$$\rho = \lambda E[l_d(K(o, R))] = \lambda l_d(K)E[R^d],$$

where $R$ denotes the random size, that is half the diameter, of a typical grain and $\lambda$ is the intensity of the underlying Poisson process. Thus we need to find the distribution of $R$ in order to determine the volume fraction. This is what the rest of this section is about, and it will be split into some different cases: the first subsection handles the case of grains of fixed orientation which grow at a fixed speed, the next takes care of fixed orientation and random speeds, and in the final subsection also the orientations are random.

### 3.1 Fixed orientation, fixed growth speed

In the original $d$-dimensional Stienen model a sphere grows halfway to its nearest neighbour, and since we start with a Poisson point process it is clear that the radius of a sphere at the origin is larger than $r$ if there are no points in $B_d(o, 2r)$. Hence

$$P(R > r) = \exp\{-\lambda \kappa_d(2r)^d\},$$

which gives

$$E[R^d] = (2^d \lambda \kappa_d)^{-1}.$$
Inserting (3.2) in (3.1) gives the volume fraction \( \rho = 2^{-d} \).

This result is easily generalised to grains of the same shape and orientation as \( K \in \mathcal{C}^d \). For spherical sets, the radius of a sphere is larger than \( r \) if there are no points in the set
\[
B_d(o, 2r) = \{ x \in \mathbb{R}^d : B_d(o, r) \cap B_d(x, r) \neq \emptyset \}.
\]
Here we need to replace this set by
\[
\{ x \in \mathbb{R}^d : K(o, r) \cap K(x, r) \neq \emptyset \} = K(o, r) \oplus \tilde{K}(o, r),
\]
by (2.1). Except for this change, the same argument as above leads to the following theorem.

**Theorem 3.1** In the Stienen model with convex grains of equal shape and orientation, growing at the same speed, the distribution function of the size of a typical grain is given by
\[
F(r) = 1 - \exp\{-\lambda r^d \Lambda_{\text{fix}}(K)\},
\]
where
\[
\Lambda_{\text{fix}}(K) := l_d(K \oplus \tilde{K}) = \sum_{i=0}^d \binom{d}{i} V_{i,d-i}(K, \tilde{K}).
\]
The volume fraction is
\[
\rho = \frac{l_d(K)}{\Lambda_{\text{fix}}(K)}.
\]

### 3.2 Fixed orientation, random growth speeds

In this subsection the grains grow with random speeds, which are independent of each other, and the grains are supposed to have the shape and orientation of \( K \in \mathcal{C}^d \). Let
\[
\Lambda_{\text{fix}}(K, r, F) := \int_0^\infty l_d(K(o, r) \oplus \tilde{K}(o, s)) F(ds)
\]
\[
= \sum_{i=0}^d \binom{d}{i} r^i V_{i,d-i}(K, \tilde{K}) \int_0^\infty s^{d-i} F(ds),
\]  \( (3.3) \)

where the equality follows by (2.2), and we assume that the \( d \)th moment of \( F \) exists. First we determine the distribution function of the final size of a typical grain, and let \( F_{sp} \) denote the distribution of the growth speed.
Theorem 3.2 Assume that the grains have the same shape and orientation as $K \in \mathcal{C}^d$. Then the following results hold for the Stienen model with grains growing with random speeds.

(i) The distribution function of the final size of a typical grain is

$$F(r) = 1 - \int_0^\infty \exp\{-\lambda(r/s)^d \Lambda_{fix}(K, s, F_{sp})\} F_{sp}(ds).$$

(ii) Then the volume fraction is

$$\rho = l_d(K) \int_0^\infty s^d \Lambda_{fix}(K, s, F_{sp})^{-1} F_{sp}(ds).$$

Proof. (i) Assume that there is a grain at the origin which grows at speed $s$. The size of this grain is smaller than $r$ if it stops growing before the time $r/s$. This means that there must be at least one point of the original Poisson process with some speed $h$ and some position $z$ which collides with the grain at the origin before time $r/s$, that is

$$K(o, r) \cap K(z, hr/s) \neq \emptyset.$$  

Since the speeds are given independently of each other, the positions of the grains which can prevent the size of the grain at the origin which grows at speed $s$ to be larger than $r$ can be seen as an independent thinning of the original Poisson process, which results in an inhomogeneous Poisson process on $\mathbb{R}^d \times \mathbb{R}^+$ with the intensity

$$\lambda I\{x \in \{z : K(o, r) \cap K(z, hr/s) \neq \emptyset\}\} dx F_{sp}(dh) =$$

$$\lambda I\{x \in K(o, r) \oplus \tilde{K}(o, hr/s)\} dx F_{sp}(dh),$$

where the equality follows by (2.3). The expected total number of points of this process is

$$\lambda \int_0^\infty \int_{K(o, r) \oplus K(o, hr/s)} dx F_{sp}(dh) = \lambda \sum_{i=0}^d \binom{d}{i} r^i V_{i,d-i}(K, \tilde{K}) \int_0^\infty (hr/s)^{d-i} F_{sp}(dh)$$

$$= \lambda (r/s)^d \sum_{i=0}^d \binom{d}{i} s^i V_{i,d-i}(K, \tilde{K}) \int_0^\infty h^{d-i} F_{sp}(dh).$$


using (2.2) in the first equality. The radius of a grain at the origin growing at speed $s$ is larger than $r$ if there are no points at all in this Poisson process. The proof is concluded by integrating over all possible growth speeds.

(ii) By (i) it follows that

$$E[R^d] = \frac{1}{\lambda} \int_0^\infty s^d \Lambda_{fix}(K, s, F_{sp})^{-1} F_{sp}(ds),$$

and we get the volume fraction by inserting this expectation in (3.1).

Note that if all grains grow at the same speed, say $s_0$, then

$$\Lambda_{fix}(K, s_0, F_{sp}) = s_0^d l_d(K \oplus \hat{K}),$$

and the size distribution and volume fraction is the same as in the previous subsection.

### 3.3 Random orientations

In addition to letting the grains grow at different speeds, they can also have different orientations. Let $SO(d)$ denote the group of rotations about the origin. By a uniformly distributed rotation we mean an element from $SO(d)$, chosen according to the Haar measure $\nu$, with $\nu(SO(d)) = 1$ (see e.g. Schneider and Wieacker (1993) for details). Here we let the grains be rotated according to this uniform distribution, and we let the rotations be independent of each other and of the positions. Let the sizes of the grains be determined as previously.

Now we get another inhomogeneous Poisson process consisting of those germs which can prevent the size of the grain at the origin to be larger than $r$. This process is defined on $\mathbb{R}^d \times \mathbb{R}^+ \times SO(d)$ and has the intensity measure

$$\lambda \{ x \in K(o, r) \oplus \vartheta K(o, hr/s) \} \, dx \, F_{sp}(dh) \, \nu(d\vartheta).$$

If we let

$$\Lambda_{rod}(K, r, F) := \int_0^\infty \int_{SO(d)} l_d(K(o, r) \oplus \vartheta K(o, y)) \nu(d\vartheta) F(dy),$$

we have

$$\Lambda_{rod}(K, r, F) = \frac{1}{\kappa_d} \sum_{k=0}^d \frac{\kappa_k \kappa_{d-k}}{\binom{d}{k}} r^k V_k(K) V_{d-k}(K) \int_0^\infty y^{d-k} F(dy), \quad (3.4)$$
where we used the generalised Steiner formula (see eg. Weil and Wieacker (1993), p. 1407) in the second equality, the expected total number of points of this Poisson process is

$$\lambda \int_0^{\infty} \int_{SO(d)} \int_{K(a,r) \ominus \partial K(a,hr/s)} dx \nu(d\partial) F_{sp}(dh) = \lambda (r/s)^d \Lambda_{rot}(K, s, F_{sp}).$$

Since the number of points that beat the typical one is Poisson distributed also when the orientation is random, we get the following theorem.

**Theorem 3.3** Theorem 3.2 hold true in the case of random orientations if $\Lambda_{fix}$ is replaced by $\Lambda_{rot}$, defined in (3.4).

**Corollary 3.4** If all grains grow at the same speed, then

$$\rho = \frac{l_d(K)}{\Lambda_{rot}(K)},$$

where

$$\Lambda_{rot}(K) := \int_{SO(d)} l_d(K \ominus \partial K) \nu(d\partial) = \frac{1}{\kappa_d} \sum_{k=0}^{d} \frac{\kappa_d \kappa_{d-k}}{\binom{d}{k}} V_k(K)V_{d-k}(K).$$

### 4 The dead leaves model

The original dead leaves model was introduced by Matheron (1968) and is a random tessellation of the space, as well as a model for non-intersecting sets. It can be defined as follows. Consider a stationary Poisson process $\{(x_i, t_i)\}$ with unit intensity in $\mathbb{R}^d \times (-\infty, 0]$. Interpret $t_i$ as the arrival time of the point $x_i \in \mathbb{R}^d$. Let $d$-dimensional, possibly random, compact grains be implanted at the points $x_i$ sequentially in time, so that a new grain deletes portions of the “older” ones. At time $t = 0$ the space $\mathbb{R}^d$ is completely occupied, and the grains which are not completely deleted constitute a tessellation of $\mathbb{R}^d$ which is called the dead leaves model.

The grains which are intact, that is not intersected by any later grains, constitute a model of non-intersecting grains. Note that the union of the intact grains constitute a random closed set in $\mathbb{R}^d$.

The intact grains of the dead leaves model can also be considered as the limit of a generalisation of one of Matérn’s hard-core models introduced in Månsson and Rudemo (2002). This generalisation is constructed as follows:
1. First we generate a Poisson process with constant intensity in $\mathbb{R}^d$. At the points of this process independent grains of a given shape, but possibly of random size and orientation, are implanted. Furthermore, to each grain a weight is given, according to a continuous distribution. The weights are independent of each other and of everything else.

2. Then we thin the process by letting all pairs of grains which intersect ‘compete’. A grain is kept if it has the higher weight in all pairwise comparisons.

The original Poisson process together with the grains can be seen as a *proposal* model. In accordance with this, we let $\lambda_{pr}$ and $F_{pr}$ denote the *proposal* intensity and *proposal* distribution function of the sizes of the grains, respectively, before thinning. The intensity measure and distribution function of the sizes in the thinned process, will be denoted by $\lambda$ and $F$, respectively.

Now let $\Psi_T$ denote this model with $\lambda_{pr} = T$, and with weights uniformly distributed on $[-T, 0]$. Then the points of the original Poisson process together with the weights can be regarded as a Poisson process in $\mathbb{R}^d \times [-T, 0]$ with unit intensity. If we let $\Psi_i, i = 1, 2, \ldots$, be based on the same Poisson process in $\mathbb{R}^d \times (-\infty, 0]$, and think of the weights as arrival times, it follows that $\Psi_i \subseteq \Psi_{i+1}$, and the limiting random set $\bigcup_{i=1}^{\infty} \Psi_i$ equals the intact grains of the dead leaves model. Results for $\Psi_T$ can now be used to achieve results for the intact grains of the dead leaves model.

The dead leaves model and generalisations of it, for instance the colour dead leaves, are studied in a number of papers by Jeulin, see e.g. Jeulin (1997). Results on the intensity and size distribution of the intact grains can be found in Jeulin (1989). The connection between Matérn’s second hard-core model and the dead leaves model in the case of fixed-sized spheres was noted by Stoyan and Schlather (2000). In Andersson, Häggström and Månsson (2005) some aspects of the volume fraction of the dead leaves model are considered.

The following results for the generalisation of Matérn’s model can be found in, or follow directly from, Månsson and Rudemo (2002):

**Lemma 4.1** Assume that the grains have the same shape as $K \in \mathcal{C}^d$, and let $\Lambda(K, r) = \Lambda_{fix}(K, r)$ if the orientation is fixed, and $\Lambda(K, r, F) = \Lambda_{rot}(K, r, F)$ otherwise, where $\Lambda_{fix}(K, r, F)$ and $\Lambda_{rot}(K, r, F)$ are given by (3.3) and (3.4), respectively.
(i) The intensity is
\[
\lambda = \int_0^\infty 1 - \exp\{-\lambda_{pr} \Lambda(K, r, F_{pr})\} \frac{\Lambda(K, r, F_{pr})}{F_{pr}(dr)}. 
\]

(ii) The size distribution of the grains is
\[
F(r) = 1 - \lambda^{-1} \int_r^\infty 1 - \exp\{-\lambda_{pr} \Lambda(K, s, F_{pr})\} \frac{\Lambda(K, s, F_{pr})}{F_{pr}(ds)}. 
\]

(iii) The volume fraction is
\[
\rho = \int_0^\infty r^d 1 - \exp\{-\lambda_{pr} \Lambda(K, r, F_{pr})\} \frac{\Lambda(K, r, F_{pr})}{F_{pr}(dr)}. 
\]

Letting \(\lambda_{pr}\) tend to infinity in the above lemma immediately gives the following theorem.

**Theorem 4.2** Assume that the grains have the same shape as \(K \in C^d\), and let \(\Lambda(K, r, F) = \Lambda_{fix}(K, r, F)\) if the orientation is fixed, and \(\Lambda(K, r, F) = \Lambda_{rot}(K, r, F)\) otherwise, where \(\Lambda_{fix}(K, r, F)\) and \(\Lambda_{rot}(K, r, F)\) are given by (3.3) and (3.4), respectively. The following results hold for the intact grains of the dead leaves model.

(i) The intensity is
\[
\lambda = \int_0^\infty (\Lambda(K, r, F_{pr}))^{-1} F_{pr}(dr). 
\]

(ii) The size distribution of the grains is
\[
F(r) = 1 - \lambda^{-1} \int_r^\infty (\Lambda(K, s, F_{pr}))^{-1} F_{pr}(ds). 
\]

If the original size distribution is continuous with density \(f_{pr}\), then the distribution of the sizes is also continuous with density
\[
f(r) = \frac{f_{pr}(r)}{\Lambda(K, r, F_{pr}) \lambda}. 
\]

(iii) The volume fraction is
\[
\rho = l_d(K) \int_0^\infty r^d (\Lambda(K, r, F_{pr}))^{-1} F_{pr}(dr). 
\]
In particular, if the grain size is fixed, then
\[ \rho = \frac{l_d(K)}{\Lambda(K)}, \]
where \( \Lambda(K) \) equals \( \Lambda_{\text{fix}}(K) \) if the orientation is fixed, and \( \Lambda_{\text{rot}}(K) \) if the orientations are random. \( \Lambda_{\text{fix}}(K) \) and \( \Lambda_{\text{rot}}(K) \) are defined in (3.1) and (3.4), respectively.

Comparing Theorems 3.2 and 4.2, we have the following result.

**Theorem 4.3** If the growth speed distribution in the Stienen model, \( F_{sp} \), and the grain size distribution before thinning in the dead leaves model, \( F_{pr} \), are equal, then the volume fractions are equal in the two models.

## 5 Bounds on the volume fraction

The volume fraction depends on the shape of the grains. For instance in two dimensions the area fraction is 1/4 for discs and 1/6 for triangles of a fixed size and orientation. In the formulas in Theorems 3.2 and 4.2 it can be seen that the volume fraction depends on the shape through the mixed volumes if the orientation is fixed. If the orientations are random, it is the intrinsic volumes rather than the mixed volumes which involve the shape.

### 5.1 Fixed orientation

We will now use the inequalities concerning mixed volumes in Section 2 to give upper and lower bounds on the volume fraction.

**Theorem 5.1** (i) Let the grains have the same shape and orientation as \( K \in \mathcal{C}^d \). If \( F_{pr} \) denotes the proposal distribution of the grain sizes in the dead leaves model and the growth speed distribution in the Stienen model, then the volume fraction in these models has the following bounds:

\[
\rho \geq \int_0^{\infty} \left( \sum_{i=0}^{d} \binom{d}{i} r^{d-i} d^{\min\{i,d-i\}} \int_0^{\infty} s^{d-i} F_{pr}(ds) \right)^{-1} F_{pr}(dr),
\]
\[
\rho \leq \int_0^{\infty} \left( \sum_{i=0}^{d} \binom{d}{i} r^{d-i} \int_0^{\infty} s^{d-i} F_{pr}(ds) \right)^{-1} F_{pr}(dr),
\]
where the upper bound is attained if and only if $K$ is centrally symmetric. In $d = 2$ and $d = 3$ the lower bound is attained if and only if $K$ is a triangle and a tetrahedron, respectively. Furthermore, if the conjectured inequality (2.6) is true, then

$$\rho \geq \int_0^\infty \left( \sum_{i=0}^{d} \binom{d}{i} \frac{2}{r^{i-d}} \int_0^\infty s^{d-i} F_{pr}(ds) \right)^{-1} F_{pr}(dr),$$

with equality if and only if $K$ is centrally symmetric.

(ii) If the grain sizes in the dead leaves model and the growth speed in the Stienen model, respectively, are fixed, then the volume fraction has the following bounds:

$$\frac{1}{\left( \frac{2d}{d} \right)} \leq \rho \leq \frac{1}{2^d},$$

(5.1)

where the upper bound is attained if and only if $K$ is centrally symmetric, and the lower bound is attained if and only if $K$ is a simplex.

**Proof.** (i) Follows from Theorem 4.2, (2.5) and (2.6).
(ii) Follows from Theorem 4.2 and (2.4). □

5.2 Random orientations

Now we present upper bounds on the volume fraction when the orientations of the grains are random. The lower bound is zero, which in two dimensions can be seen by noting that for a fixed area there are sets with an arbitrary large perimeter.

**Theorem 5.2** (i) Assume that the grains are independently and uniformly rotated and have the same shape as $K \in C^d$. If $F_{pr}$ denotes the proposal distribution for the grain sizes in the dead leaves model and the growth speed distribution in the Stienen model, then the volume fraction in these models have the following bound

$$\rho \leq \int_0^\infty \left( \sum_{i=0}^{d} \binom{d}{i} \frac{1}{r^{i-d}} \int_0^\infty s^{d-i} F_{pr}(ds) \right)^{-1} F_{pr}(dr),$$

with equality if and only if $K$ is a sphere.

13
(ii) If the grain sizes in the dead leaves model and the growth speed in the Stienen model, respectively, are fixed, then
\[ \rho \leq \frac{1}{2^d}, \]
with equality if and only if \( K \) is a sphere.

**Proof.** (i) From the Brunn-Minkowski theorem it follows that
\[ l_d(\{ x : K(o, r) \cap \vartheta K(x, y) \neq \emptyset \}) \geq (r + y)^d l_d(K), \]
with equality if and only if \( K \) and \( \vartheta(K) \) are translates. Hence
\[
\int_0^\infty \int_{SO(d)} l_d(\{ x : K(o, r) \cap \vartheta(K)(x, y) \neq \emptyset \}) \nu(d\vartheta) F_{pr}(dy) \\
\geq \int_0^\infty (r + y)^d l_d(K) F_{pr}(dy) \\
= l_d(K) \sum_{i=0}^d \binom{d}{i} r^i \int_0^\infty y^{d-i} F_{pr}(dy),
\]
with equality if and only if \( K \) is a sphere, since \( K \) and \( \vartheta(K) \) are translates for all \( \vartheta \in SO(d) \) if and only if \( K \) is a sphere. From Theorem 4.2 and (3.4) follows the result in (i).

(ii) Follows immediately from (i).

Contrary to the case of fixed orientations, the centrally symmetric sets do not all behave in the same way now – here spheres are the only sets for which the upper bound of the volume fraction is attained. Furthermore, all triangles and tetrahedra do not give the same volume fraction when the orientations are random.

### 5.3 Discs, triangles and other extremal sets

Finally, we calculate the volume fraction for the Stienen model with grains growing at equal speed and for the dead leaves model with fixed-sized grains of the following shapes: discs, squares and equilateral triangles in two dimensions, and spheres, cubes and regular tetrahedra in three dimensions.
For fixed orientations we have, according to (5.1),
\[
\frac{1}{6} \leq \rho \leq \frac{1}{4}, \quad \text{if } d = 2,
\frac{1}{20} \leq \rho \leq \frac{1}{8}, \quad \text{if } d = 3,
\]
with equalities on the right-hand sides if and only if \( K \) is centrally symmetric (for instance for discs, spheres, squares and cubes), and equalities on the left-hand sides if and only if \( K \) is a triangle and a tetrahedron, respectively.

In the case of random orientations it follows from Theorem 4.2 that
\[
\rho = \begin{cases} \\
\frac{l_2(K)}{l_2(K)^2 + S_1(K)^2/(2\pi)}, & \text{if } d = 2, \\
\frac{l_3(K)}{l_3(K)^2 + S_2(K)b(K)}, & \text{if } d = 3,
\end{cases}
\]
where \( S_1 \) is the perimeter, \( S_2 \) is the surface area and \( b \) is the mean width. All quantities are straightforward to calculate, except maybe for those of the tetrahedron, which are given in Månsson and Rudemo (2002): \( l_3(K) = 2\sqrt{2}/3, \quad S_2(K) = 4\sqrt{3} \) and \( b(K) = \frac{3}{\pi}(\pi - \arccos 3^{-1}) \). The volume fractions are summarized in Table 1.

Note that for squares and cubes the volume fraction is lower if the grains are randomly rotated than if they have a fixed orientation, while for triangles and tetrahedra it is the other way around.

Table 1: The volume fraction in some special cases.

<table>
<thead>
<tr>
<th>Dimension 2</th>
<th>Fixed orientation</th>
<th>Random orientation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Disc</td>
<td>1/4</td>
<td>1/4</td>
</tr>
<tr>
<td>Square</td>
<td>1/4</td>
<td>((2 + 8/\pi)^{-1} \approx 0.22)</td>
</tr>
<tr>
<td>Equilateral triangle</td>
<td>1/6</td>
<td>(\sqrt{3}(2\sqrt{3} + 18/\pi)^{-1} \approx 0.19)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Dimension 3</th>
<th>Fixed orientation</th>
<th>Random orientation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sphere</td>
<td>1/8</td>
<td>1/8</td>
</tr>
<tr>
<td>Cube</td>
<td>1/8</td>
<td>1/11</td>
</tr>
<tr>
<td>Regular tetrahedron</td>
<td>1/20</td>
<td>((2 + 18\sqrt{1.5}(1 - \arccos 3^{-1}/\pi))^{-1} \approx 0.068)</td>
</tr>
</tbody>
</table>
References


