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COMPUTATIONAL CHARACTERIZATION OF FLOWS WITH SOME HYPERBOLICITY

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Abstract. Studying flows in general we do not know if the flow is hyperbolic in a strict sense. Instead we vaguely assume that the flow is dominated by contractions and expansions and say that flow have some hyperbolicity. We compare a posteriori and shadowing error estimates for computed orbits in flows with some hyperbolicity. Principal to the estimates are the stability factors which we estimate in two examples for orbits generated by velocity fields modelled by the Stokes equations and computed by a finite element method.

1. Introduction

We consider domains $\Omega \subseteq \mathbf{R}^3$ and Lipschitz continuous vector fields $\Omega \ni x \mapsto f(x) \in \mathbf{R}^3$ so that the dynamical system

$$\partial_t u(t,x) = f(u(t,x)), \quad t > 0; \quad u(0,x) = x, \tag{1.1}$$

defines a flow $(t,x) \mapsto u(t,x) \in \Omega$ describing the motion of a fluid particle starting at x and moving in the velocity field f.

Generally we can not find a closed expression for the flow and in order to study the properties of the flow we may instead analyse a limited number of numerically computed of orbits $u_k(t,x_i)$ for $i = 1, 2, \ldots, I$, where k refers to the time discretization. For a reliable analysis we will have to control the error

$$e(t,x) := u_k(t,x_i) - u(t,x), \tag{1.2}$$

and make is small. From now on we consider a fixed x and set $e(t) = e(t,x)$. We are lead to the following classic question. Given a dynamical system (1.1) and a number Tol > 0, is there a threshold time T so that $\|e(t)\| \leq$ Tol for all $t \in [0,T]$, i.e., so the error is uniformly bounded on $[0,T]$?

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For dynamical systems that are dynamically unstable, that is, sensitive to perturbations, we anticipate that the error will grow, possibly at an exponential rate, and we will only expect to be able to compute $u_k(t, x_i)$ with a small error for small T. However, if Ω is uniformly hyperbolic for (1.1) and $u_k(t, x_i)$ is computed with sufficient accuracy there is a shadow orbit $u(t, y)$ such that $\|u_k(t, x_i) - u(t, y)\| < \text{Tol}$ for arbitrary t [16].

However, in practice we probably do not know if Ω is uniformly hyperbolic for (1.1) and also this requirement seems to be too strong and mainly of theoretical interest. If we instead alleviate on the uniform hyperbolicity and require Ω to have some hyperbolicity meaning that the flow is dominated by contractions and expansions in a less strict sense we may still obtain shadowing results similar to the aforementioned. In this case we will expect the shadowing to hold for finite but large t, see for example [5, 11, 12, 19] and the book [16].

As a concrete example we consider the Lorenz system

$$\partial_t u = (\sigma(u_2 - u_1), \rho u_1 - u_2 - u_1 u_3, u_1 u_2 - \beta u_3), \quad t > 0;$$
$$u(0) = (1, 0, 0); \quad \text{for} \quad (\sigma, \rho, \beta) = (10, 28, 8/3).$$

In [14] this problem was solved accurately, in the sense that $\|e(T)\|$ is small, up to $T = 50$ which is predicted to be the threshold beyond which $\|e(T)\|$ becomes too large to be represented with double precision arithmetics (from the same work $T = 100$ for quadruple precision is predicted).

This result should be compared to [5] where the same problem is solved accurately up to $T = 9 \times 10^6$ in the sense that $\|u_k(t, u(0)) - u(t, y)\|$ is small for $t \in [0, T]$, that is, very close to the computed orbit $u_k(t, u(0))$ there is an exact orbit $u(t, y)$.

This example obviously suggest that long time error control for problems that are dynamically unstable will fail with the first method but could possibly be archived with the last method, provided the structure of the problem is sufficiently hyperbolic-like.

1.1. **About this work.** In this work we consider the case when the vector field f is not given in closed form but rather defined by a model, e.g., a partial differential equation, and approximated by computed numerical data f_h, where h refers to the space discretization. We solve (1.1) numerically with f_h as right hand side and estimate the error (1.2), where we now also have to take the error in the velocity field

$$e_f := f_h - f \tag{1.3}$$

into account.

We assume that u_k and f_h are finite element approximations obtained by solving appropriate finite element problems, which depend on the choice of finite element method and the type of model defining f.

Provided e_f is small enough and that we solve u_k accurately enough we have the following a posterior error estimate, see for example [6],

$$\sup_{t\in[0,T]} \|e(t)\| \leq S(T)\mathcal{E}(f_h, f, x), \tag{1.4}$$

where $S(T)$ is a stability factor and $\mathcal{E}(f_h, f, x)$ is a function depending on the data and made small as e_f is made small and is u_k solved more accurately. The dependence on initial data for a particular problem will be reflected in the stability factor and for dynamically unstable problems this factor may grow exponentially in T, rendering the estimate useless after some rather small time.

If we in addition to the requirements on e_f and u_k made for the estimate above also require that the flow (1.1) is sufficient hyperbolic then we have the following shadowing error estimate, see for example [5],

$$\sup_{t\in[0,T]} \|u_k(t, x) - u(t, y)\| \leq \widetilde{S}(T)\mathcal{E}_1(f_h, f, x), \tag{1.5}$$

where $u(t, y)$ is an exact solution to (1.1) with different initial data, $\widetilde{S}(T)$ a stability factor and $\mathcal{E}(f_h, f, x)$ is the same function as in the a posteriori estimate above. Depending on the contractive and expansive directions in the flow the stability factor may be subject to a mild growth over time and the estimate will be valid for a rather large time.

In the present work we derive the finite time shadowing error estimate (1.5). The overall idea is from [5] but now expressed using a finite element framework. This work also differs in the way we estimate $\widetilde{S}(T)$ and that we use numeric data f_h in the right hand side to (1.1). We also remark that the overall framework in this paper has been inspired by [13] where shadowing was considered in a more abstract setting, for parabolic partial differential equations.

Finally, we describe a numerical experiment where we obtain f_h as the solution to a Stokes flow and with this data we compute and compare $S(T)$ and $\widetilde{S}(T)$ in (1.4) and (1.5). The experiment is inspired by the experimental work [18] on a micro fluid mixing devise.

2. Notation and preliminaries

For real valued functions $u, v \in \mathbf{R}^3$ we denote their scalar product by $u \cdot v = u_1 v_1 + u_2 v_2 + u_3 v_3$.

For matrixes A and linear operators L we denote their transpose and adjoint by A^* and L^*. We let I denote the identity matrix or identity operator.

We will use $\|\cdot\|$ to denote the appropriate matrix and vector norms.

We only consider bounded domains $\omega \subseteq \Omega \subset \mathbf{R}^3$ with measure denoted by $|\omega|$, and where Ω is associated with the flow (1.1).

We will denote piecewise smooth functions by C^m and use standard notation for Sobolev spaces $W^{k,q}(\omega)$ and $W_0^{k,q}(\omega)$.

For vector fields

$$\Omega \ni x \mapsto f(x) = (f_1(x), \dots, f_n(x)) \in \mathbf{R}^n$$

we set

$$\nabla u := (D_i u_j)_{i,j=1}^n,$$

where

$$D_i := \frac{\partial}{\partial x_i} \quad i = 1, \dots, n,$$

denote the i:th partial derivative.

Finally, throughout this work we will use C or C_i, $i = 1, 2, \dots$, to denote various constants, not necessarily taking the same value from time to time.

2.1. Hyperbolic sets.

A compact set $\omega \subset \Omega$ is said, see for example [17, p. 8], to be *uniformly hyperbolic* for the flow $u(t, x)$ if there is a continuous decomposition

$$\mathbf{R}^3 = E^0(x) \oplus E^s(x) \oplus E^u(x) \quad \forall x \in \omega, \tag{2.1}$$

and constants $c > 0$ and $0 < \lambda < 1 < \mu$ such that for each $x \in \omega$,

(1) $E^0(x)$ is the one-dimensional subspace generated by $f(x)$;
(2) $\nabla u(t, x) E^s(x) = E^s(u(t, x))$ and $\nabla u(t, x) E^u(x) = E^u(u(t, x))$;
(3) $\|\nabla u(t, x)\xi\| \leq c\lambda^t \|\xi\|$ for all $\xi \in E^s(x)$ and $t \geq 0$;
(4) $\|(\nabla u(t, x))^{-1}\xi\| \leq c\mu^{-t}\|\xi\|$ for all $\xi \in E^u(x)$ and $t \geq 0$.

We remark that $u(t, x)$ is called an *Anosov flow* if Ω is uniformly hyperbolic for $u(t, x)$.

The requirements in this definition are rather strong and there are many examples of dynamical systems with non-trivial and interesting properties that do not meet these requirements [20]. We therefore relax this requirement and instead vaguely think of the flow as being dominated by contractive and expansive direction in a less strict sense. An example of one such relaxation is the notion of *nonuniform hyperbolicity*, where loosely speaking, "for every" in the definition is replace by "for almost all", see for example [1, 20]. We will not discuss and specify this in more detail. Instead we consider an example that to some extent motivates the reasoning.

Example 2.1. Suppose f is incompressible, that is,

$$\nabla \cdot f(x) = 0 \quad \forall x \in \Omega.$$

Then for every open set $A \subset \Omega$ the volume of A is preserved in the flow (1.1), that is,

$$|A| = |u(t, A)| \quad \text{for } t > 0$$

see for example [4, p. 10].

Now for a small ball $B(x, \varepsilon)$ at $x \in \Omega$ and with radios ε and for small t we consider the deformation of the ball in the flow, $B(x, \varepsilon) \to u(t, B(x, \varepsilon))$. Since the volume of the ball is preserved and since the flow leaves the ball unchanged in the direction of the flow we are left with two possibilities (1) the ball is contracted and expanded in some directions such that the volume is unchanged and (2) the ball is unchanged.

Consequently, it seems reasonable to assume that in large parts of Ω there is a splitting (2.1). If we assume that the case where the ball is not deformed only happen in isolated points then the splitting (2.1) will exist for almost all $x \in \omega$ but now E^s and E^u must not be continuous.

2.2. **Finite element approximation.** The finite element formulation of (1.1) is derived from the following variational formulation of (1.1). Find $u \in C^1([0, T])^3$ with $u(0, x) = x$ such that

$$\int_0^T \big(\partial_t u - f(u)\big) \cdot v \, dt = 0 \quad \forall v \in C^1([0, T])^3. \tag{2.2}$$

As the functions u and v are replaced by piecewise polynomials we obtain the Galerkin finite element approximation.

For simplicity we only consider continuous finite elements although this work is readily generalized to discontinuous finite elements. Partition $[0, T]$ into intervals $I_i = [t_{i-1}, t_i]$ for $i = 1, 2, \ldots, N$ such that $0 = t_0 < t_1 < \ldots < t_N = T$ and set $k_i = t_{i-1} - t_i$. Let $P_q(I_i)$ denote the polynomials of degree less or equal to q on I_i and set

$$V_q([0,T]) := \{v \in C^0([0,T]) : v|_{I_i} \in P_q(I_i), \text{ for } i = 1 \dots N\}$$
$$W_q([0,T]) := \{v \in C^0(\bigcup_{i=1}^{N}(t_{i-1}, t_i)) : v|_{I_i} \in P_q(I_i), \text{ for } i = 1 \dots N\}$$

which is the finite element spaces of continuous and discontinuous piecewise polynomials of degree q.

For $q \geq 1$ and from (2.2) we obtain the finite element formulation. Find $u_k \in V_q([0,T])^d$ with $u_k(0, x_i) = x_i$ such that

$$\int_0^T (\partial_t u_k - f(u_k)) \cdot v \, dt = 0 \quad \forall v \in W_{q-1}([0,T])^3, \tag{2.3}$$

where we note that now $v \in W_{q-1}([0,T])^3$. This is the continuous Galerkin method of order q, referred to as the cG(q) method in [7, p. 210].

There are $q+1$ points in the interval I_i, where the piecewise polynomials are evaluated, referred to as local nodes. In the same way there are $N(q+1)-1$ points in the interval $[0,T]$ referred to as global nodes.

We recall the following interpolation estimate, see for example [7, Theorem 5.1, p. 79]. For a smooth function v on I_i let $\mathcal{I}_{i,q}v \in P_q(I_i)$ interpolate v at the local nodes. Then $\mathcal{I}_{i,q}v$ satisfies

$$\|\mathcal{I}_{i,q}v - v\|_{L^\infty(I_i)} \leq C\|k_i^{q+1} D^{q+1} v\|_{L^\infty(I_i)}.$$

In the same way, for a smooth function v on $[0,T]$, let $\mathcal{I}_q v \in W_q([0,T])$ interpolate v at the global nodes. For a global estimate we let $k = k(t)$ denote the piecewise constant function so that $k|_{I_i} = k_i$. Then

$$\|\mathcal{I}_q v - v\|_{L^\infty([0,T])} \leq C\|k^{q+1} D^{q+1} v\|_{L^\infty([0,T])}. \tag{2.4}$$

Finally we recall the following inverse estimate. Let $\mathcal{T}$ be a finite element triangulation of Ω and set $h_T = \mathrm{diam}\,(T)$ for all $T \in \mathcal{T}$. For any $T \in \mathcal{T}$, let V be a finite-dimensional subspace of $W^{k,q}(T) \cap W^{m,s}(T)$, where $1 \leq q \leq \infty$, $1 \leq s \leq \infty$ and $0 \leq m \leq k$. Then there exists a constant C such that for all $v \in V$

$$\|v\|_{W^{k,q}(T)} \leq C h_T^{m-k+n/q-n/s} \|v\|_{W^{m,s}(T)}, \tag{2.5}$$

see for example [2, Theorem 4.5.3, p. 111].

2.3. **Linearization.** Let $\bar{u} \in C^0([0,T])^3$ and rewrite (1.1) by linearization around $\bar{u}$

$$\partial_t u(t,x) + A(t)u = F(t,u), \tag{2.6}$$

where we define the linear part of f,

$$A(t) := -\nabla f(\bar{u}(t)) \tag{2.7}$$

and the nonlinear part,

$$F(t,u) := f(u) + A(t)u. \tag{2.8}$$

Let $L(t,s)$ for $0 \le s \le t \le T$ be the solution operator to the linearized homogeneous problem

$$\partial_t u + A(t)u = 0, \quad t > s; \quad u(s,x) = x. \tag{2.9}$$

Thus, $u(t,x) = L(t,s)x$ is the solution of (2.9). We note that $L(t,s)$ satisfies the following properties: $L(s,s) = I$ and $L(t,r)L(r,s) = L(t,s)$ for $0 \le s \le r \le t \le T$. Consequently we may regard $L(t,s)$ as the inverse to $L(s,t)$.

For $t \in [0,T]$ we consider the following weak formulation of (2.9). Find $u \in C^1([s,t])^3$ with $u(s,x) = x$ such that

$$\int_s^t (\partial_\tau u + A(\tau)u) \cdot v \, d\tau = 0 \quad \forall v \in C^1([s,t])^3. \tag{2.10}$$

We also introduce the dual problem to (2.10). Find $\varphi \in C^1([s,t])^3$ with $\varphi(t,\psi) = \psi$ such that

$$\int_s^t \phi \cdot (-\partial_\tau \varphi + A^*(\tau)\varphi) \, d\tau = 0 \quad \forall \phi \in C^1([s,t])^3, \tag{2.11}$$

which is the weak formulation of the following problem,

$$-\partial_s \varphi + A^*(s)\varphi = 0, \quad s < t; \quad \varphi(t,\psi) = \psi. \tag{2.12}$$

Let $K(s,t)$ denote the solution operator to (2.12), that is, $\varphi(s) = K(s,t)\psi$. Note that $K(s,t) = L^*(t,s)$ since, by integration by parts in (2.11),

$$\int_s^t (\partial_\tau \phi + A(\tau)\phi) \cdot \varphi \, d\tau = \phi(s^+) \cdot \varphi(s^+) - \phi(t^-) \cdot \varphi(t^-),$$

and thus, with $\phi = u$ in the above identity and $v = \varphi$ in (2.10) we get

$$0 = \int_s^t (\partial_\tau u + A(\tau)u) \cdot \varphi \, d\tau = x \cdot K(s,t)\psi - L(t,s)x \cdot \psi.$$

Finally, we consider (1.1) with $f(x)$ replaced by $f_h(x)$ and linearize around $\bar{u} \in C^0([0,T])^3$:

$$\partial_t u(t,x) + A_h(t)u = F_h(t,u), \tag{2.13}$$

where we define the linear part of f_h,

$$A_h(t) := -\nabla f_h(\bar{u}) \tag{2.14}$$

and the nonlinear part,

$$F_h(t,u) := f_h(u) + A_h(t)u. \tag{2.15}$$

Let $L_h(t,s)$ for $0 \leq s \leq t \leq T$ be the solution operator the the linearized homogeneous problem

$$\partial_t u + A_h(t)u = 0, \quad t > s; \quad u(s,x) = x. \tag{2.16}$$

Thus, $u(t,x) = L_h(t,s)x$ is the solution to (2.16). In analogy to (2.9) there is a weak form and a dual problem to (2.16) with solution operator $L_h^*(t,s)$.

2.4. **Exponential dichotomies.** If Ω is uniformly hyperbolic for $u(t,x)$ then the following definition is meaningful, cf. [13].

Definition 2.1. The solution operator $L(t,s)$ is said to have an exponential dichotomy in the interval $[0,T]$ if there are projections $P(t)$, $t \in [0,T]$ and constants $M \geq 1$, $\beta > 0$ such that, for $0 \leq s \leq t \leq T$,

(1) $L(t,s)P(s) = P(t)L(t,s)$;
(2) $\|L(t,s)P(s)\| \leq Me^{-\beta(t-s)}$;
(3) $\|L(s,t)(I-P(t))\| \leq Me^{-\beta(t-s)}$.

The range $\mathcal{R}(P(t))$ is called the stable subspace and the complementary space $\mathcal{R}(I-P(t)) = \mathcal{N}(P(t))$ (the null space of $P(t)$) is called the unstable subspace.

If $L(t,s)$ has an exponential dichotomy on the interval $[0,T]$ then for sufficiently smooth f the following boundary value problem is well posed,

$$\begin{gathered} \partial_t \varphi + A(t)\varphi = f(t), \quad t \in (0,T), \\ P(0)\varphi(0) = \varphi_0, \quad (I-P(T))\varphi(T) = \varphi_T, \end{gathered} \tag{2.17}$$

where $\varphi_0 \in \mathcal{R}(P(0))$ and $\varphi_T \in \mathcal{R}(I-P(T))$.

The solution is given by

$$\varphi(t) = G(t,0)\varphi_0 - G(t,T)\varphi_T + \int_0^T G(t,s)f(s)\,ds, \tag{2.18}$$

where $G(t,s)$ is the operator

$$G(t,s) = \begin{cases} L(t,s)P(s), & 0 \le s \le t, \\ -L(t,s)(I - P(s)), & t < s \le T. \end{cases} \tag{2.19}$$

This is readily verified by the following calculations. By Duhamel's principle on the interval $(0,t)$

$$\varphi(t) = L(t,0)\varphi_0 + \int_0^t L(t,s)f(s)\,ds,$$

and by Property 1 in Definition 2.1,

$$P(t)\varphi(t) = L(t,0)P(0)\varphi_0 + \int_0^t L(t,s)P(s)f(s)\,ds.$$

In the same way on the interval (t,T)

$$(I - P(T))\varphi(T) = L(T,t)(I - P(t))\varphi(t) + \int_t^T L(T,s)(I - P(s))f(s)\,ds.$$

By applying the operator $L(t,T)$ and rearranging the terms,

$$(I - P(t))\varphi(t) = L(t,T)(I - P(T))\varphi(T) - \int_t^T L(t,s)(I - P(s))f(s)\,ds,$$

since $L(t,T)L(T,s) = L(t,s)$ for $s \le T \le t$. The above result now follows by considering

$$\varphi(t) = P(t)\varphi(t) + (I - P(t))\varphi(t).$$

We also see that the solution satisfies the estimate

$$\sup_{t\in[0,T]} \|\varphi(t)\| \le M\big(\|\varphi_0\| + \|\varphi_T\| + 2\beta^{-1} \sup_{t\in[0,T]} \|f(t)\|\big), \tag{2.20}$$

which follows from Property 2 and 3 in Definition 2.1 and the estimates

$$\|\varphi(t)\| \le \|G(t,0)\|\,\|\varphi_0\| + \|G(t,T)\|\,\|\varphi_T\| + \sup_{t\in[0,T]} |f(t)| \int_0^T |G(t,s)|\,ds,$$

and

$$\int_0^T |G(t,s)|\,ds \le \int_0^T e^{-\beta|t-s|}\,ds \le \frac{2M}{\beta}.$$

Note that with $f(t) = -\psi\delta(t-\tau)$ for some $\psi \in \mathbf{R}^n$ and $\tau \in [0,T]$, where δ is the Dirac distribution, we obtain the estimate

$$\sup_{t\in[0,T]} \|\varphi(t)\| \le M \max\{\|\varphi_0\|, \|\varphi_T\|, \|\psi\|\}. \tag{2.21}$$

3. Error analysis

Subtracting (2.3) from (2.2) we obtain the weak representation of the error $e := u_k(t, x_i) - u(t, x)$. Find $e \in C^1([0, T])^3$ with $e(0, x) = x_i - x$ such that

$$\int_0^T \partial_t e \cdot v \, dt = \int_0^T (f(u) - \partial_t u_h) \cdot v \, dt \quad \forall v \in C^1([0, T])^3. \tag{3.1}$$

With $A_h(t)$ as in (2.14) we linearize around u_k and let

$$f(u) - \partial_t u_k = e_f(u) + A_h(t)e + \eta(u_k, u) + R(u_k),$$

where we define the error in the computed velocity field,

$$e_f(u) := f(u) - f_h(u), \tag{3.2}$$

the non-linear part,

$$\eta(u_k, u) := f_h(u) - f_h(u_k) + A_h(t)e \tag{3.3}$$

and the residual to (2.3),

$$R(u_k) := f_h(u_k) - \partial_t u_k. \tag{3.4}$$

We note that the residual is orthogonal to functions in the finite element space $W_{q-1}([0, T])^3$ in the following sense,

$$\int_0^T R(u_k) \cdot v \, dt = 0 \quad \forall v \in W_{q-1}([0, T])^3. \tag{3.5}$$

We rewrite (3.1) according to the linearization above. Find $e \in C^1([0, T])^3$ with $e(0, x) = x_i - x$ such that

$$\int_0^T (\partial_t e + A_h(t)e) \cdot v \, dt = \int_0^T (e_f(u) + \eta(u_k, u) + R(u_k)) \cdot v \, dt, \tag{3.6}$$

for all $v \in C^1([0, T])^3$.

The following lemma will be useful characterizing the function $\eta(\cdot, \cdot)$. Note that ∇f_h is discontinuous across $\partial T \setminus \partial\Omega$ for $T \in \mathcal{T}$.

Lemma 3.1. *Let $u, v, w \in \Omega$ and suppose the convex hull K of $\{u, v, w\}$ is contained in Ω. Then a finite element function $f_h : \Omega \ni x \mapsto f_h(x) \in \mathbf{R}^3$*

satisfies

$$\begin{aligned}&\|f_h(u) - f_h(v) + \nabla f_h(w)(u-v)\| \\ &\quad \le C\|u-v\|\Big(h_{\min}^{-1-n/p}\big(\|u-w\| + \|v-w\|\big)\|\nabla f_h\|_{L^p(\Omega)} \\ &\qquad + \max_{T\in\mathcal{T}}\|[\nabla f_h]\|_{L^\infty(\partial T\setminus\partial\Omega)}\Big),\end{aligned}$$

for some $1 \le p \le \infty$ *and where the constant* C *depends on card*$(K \cap \mathcal{T})^2$ *and the constant in* (2.5), *and* $[\cdot]$ *denotes the jump across* ∂T.

We remark that the exponent p in practice is determined by available error estimates.

Proof. Consider the line $l : [0,1] \ni s \mapsto su + (1-s)v \in \mathbf{R}^n$ and let

$$l_{\mathcal{T}} = \bigcup_{T\in\mathcal{T}} T \cap l \neq \emptyset.$$

From the identity

$$\begin{aligned}&f_h(u) - f_h(v) - \nabla f_h(w)(u-v) \\ &\qquad = \int_0^1 \big(\nabla f_h(su + (1-s)v) - \nabla f_h(w)\big)(u-v)\, ds,\end{aligned}$$

and by the mean value theorem there are points $\xi_T \in T$ for $T \in l_{\mathcal{T}}$ such that

$$\begin{aligned}\int_0^1 \nabla f_h(su+(1-s)v)\, ds &= \sum_{T\in l_{\mathcal{T}}} \int_{l\cap T} \nabla f_h(su+(1-s)v)\, ds \\ &= \sum_{T\in l_{\mathcal{T}}} \nabla f_h(\xi_T) \int_{l\cap T} ds.\end{aligned}$$

Hence, since $\int_{l\cap T} ds < 1$

$$\|f_h(u) - f_h(v) - \nabla f_h(w)(u-v)\| \le \|u-v\| \sum_{T\in l_{\mathcal{T}}} \|\nabla(f_h(\xi_T) - f_h(w))\|$$

For each point ξ_T consider the line between ξ_T and w. Suppose this line crosses m_T boundaries ∂T for $T \in \mathcal{T}$ at points $\xi_{T,i}$ for $i = 1, \dots, m_T$. Let $\xi_{T,i}^-$ and $\xi_{T,i}^+$ be the limits at $\xi_{T,i}$ going from ξ_T and w respectively, and set

$\xi_{T,0}^{+} = \xi_T$ and $\xi_{T,m_T+1}^{-} = w$. Estimate the terms in the sum above

$$\|\nabla(f_h(\xi_T) - f_h(w))\| \le \sum_{i=0}^{m_T} \|\nabla(f_h(\xi_{T,i}^{+}) - f_h(\xi_{T,i+1}^{-}))\| + \sum_{i=1}^{m_T} \|[\nabla f_h(\xi_{T,i})]\|,$$

where $[\nabla f_h(\xi_{T,i})] = \nabla(f_h(\xi_{T,i}^{-}) - f_h(\xi_{T,i}^{+}))$ denotes the jump at $\xi_{T,i}$.

By the mean value theorem and an inverse estimate

$$\begin{aligned}\|\nabla(f_h(\xi_{T,i}^{+}) - f_h(\xi_{T,i+1}^{-}))\| &\le \big(\|u - v\| + \|v - w\|\big)\|\nabla^2 f_h\|_{L^\infty(T)} \\ &\le C h_{\min}^{-1-n/p}\big(\|u - w\| + \|v - w\|\big)\|\nabla f_h\|_{L^p(\Omega)},\end{aligned}$$

since $\|\xi_{T,i}^{+} - \xi_{T,i+1}^{-}\| \le \|\xi_T - w\| \le \|u - v\| + \|v - w\|$.

For the jump terms we estimate

$$\|[\nabla f_h(\xi_{T,i})]\| \le \max_{T\in\mathcal{T}} \|[\nabla f_h]\|_{L^\infty(\partial T\setminus\partial\Omega)}.$$

Collecting the estimates above concludes the proof. ☐

For fixed u_k we consider $\eta = \eta(u_k, u_k - e)$ and $e_f = e_f(u_k - e)$ as a functions of e. Set

$$\begin{aligned} N_{0,T}(e, v) &:= \int_0^T \eta(u_k, u_k - e)\cdot v\,dt, \\ E_{0,T}(e, v) &:= \int_0^T e_f(u_k - e)\cdot v\,dt, \qquad (3.7) \\ R_{0,T}(u_k, v) &:= \int_0^T R(u_k)\cdot v\,dt, \end{aligned}$$

and estimate $N_{0,T}$, $E_{0,T}$ and $R_{0,T}$. Let

$$\mathcal{B}_\rho := \{e \in C^1([0,T]) : \|e\|_{L^\infty([0,T])} \le \rho\}. \tag{3.8}$$

With $u = u_k - e$ and $v = w = u_k$ in Lemma 3.1 we get

$$\|N_{0,T}(e, v)\| \le C\|v\|_{L^1([0,T])} r_N(f_h, \rho)\rho \quad \text{for } e \in \mathcal{B}_\rho \tag{3.9}$$

where we defined

$$r_N(f_h, \rho) := \rho h_{\min}^{-1-n/p}\|\nabla f_h\|_{L^p(\Omega)} + \max_{T\in\mathcal{T}}\|[\nabla f_h]\|_{L^\infty(\partial T\setminus\partial\Omega)}. \tag{3.10}$$

Now $N_{0,T}$ is Lipschitz continuous, that is,

$$\|N_{0,T}(e_1, v) - N_{0,T}(e_2, v)\| \le C\|v\|_{L^1([0,T])} r_N(f_h, \rho)\|e_1 - e_2\|_{L^\infty([0,T])}, \tag{3.11}$$

for $e_1, e_2 \in \mathcal{B}_\rho$ and where $r_N(f_h, \rho)$ is as in (3.10).

To see this, suppose $e_1, e_2 \in \mathcal{B}_\rho$. By Hölder's inequality,

$$|N_{0,T}(e_1, v) - N_{0,T}(e_2, v)| \\ \leq \|\eta(u_k, u_k - e_1) - \eta(u_k, u_k - e_2)\|_{L^\infty([0,T])}\|v\|_{L^1([0,T])},$$

where

$$\eta(u_k, u_k - e_1) - \eta(u_k, u_k - e_2) \\ = f_h(u_k - e_1) - f_h(u_k - e_2) - \nabla f_h(u_k)(e_1 - e_2).$$

With $u = u_k - e_1$, $v = u_k - e_2$ and $w = u_k$ in Lemma 3.1, (3.11) follows.

As for $E_{0,T}$ we will use the uniform estimate

$$E_{0,T}(e, v) \leq C\|e_f\|_{L^\infty(\Omega)}\|v\|_{L^1([0,T])}. \tag{3.12}$$

We also note by taking $u = u_k - e_1$, $v = u_k - e_2$ and $w = 0$ in Lemma 3.1 that $E_{0,T}$ is Lipschitz continuous, that is,

$$\|E_{0,T}(e_1, v) - E_{0,T}(e_2, v)\| \leq C\|v\|_{L^1([0,T])}r_E(e_f)\|e_1 - e_2\|_{L^\infty([0,T])}, \tag{3.13}$$

for $e_1, e_2 \in \mathcal{B}_\rho$ and where $r_E(e_f)$ is defined by

$$r_E(e_f) := h_{\min}^{-n/p}\|\nabla e_f\|_{L^p(\Omega)}. \tag{3.14}$$

Finally, due to the Galerkin orthogonality (3.5) we may add $\mathcal{I}_{q-1}v$

$$\int_0^T R(u_k) \cdot v \, dt = \int_0^T R(u_k) \cdot (v - \mathcal{I}_{q-1}v) \, dt,$$

and hence by (2.4)

$$R_{0,T}(u_k, v) \leq C\|k^q R(u_k)\|_{L^\infty([0,T])}\|D^q v\|_{L^1([0,T])}. \tag{3.15}$$

3.1. **A posteriori error analysis.** Consider the dual problem to (3.6). Find $\varphi \in C^1([0,T])^3$ with $\varphi(T, x) = \varphi_T$ such that

$$\int_0^T \phi \cdot (-\partial_t \varphi + A_h^*(t)\varphi) \, dt = 0 \quad \forall \phi \in C^1([0,T])^3. \tag{3.16}$$

With $v = \varphi$ in (3.6) and $\phi = e$ in (3.16) subtracting the equations we get

$$\int_0^T \partial_t(e \cdot \varphi) \, dt = \int_0^T (e_f(u) + \eta(u_k, u) + R(u_k)) \cdot \varphi \, dt,$$

or with the notation in (3.7) we get

$$e(T) \cdot \varphi_T = e(0) \cdot \varphi(0) + R_{0,T}(u_k, \varphi) + E_{0,T}(e, \varphi) + N_{0,T}(e, \varphi), \tag{3.17}$$

which is a fixed point problem in e that admits a unique solution provided $N_{0,T}$ and $E_{0,T}$ has sufficiently small Lipschitz constants.

Estimating the right hand side in (3.17) we use Cauchy's inequality for the fist term and for the remaining terms we use the estimates (3.15), (3.12) and (3.9). As is usual we define the stability factors

$$\begin{aligned} S_0(T) &:= \|\varphi(0)\|, \\ S_1(T) &:= \|D^q\varphi\|_{L^1([0,T])}, \\ S_2(T) &:= \|\varphi\|_{L^1([0,T])}. \end{aligned} \tag{3.18}$$

We remark that the stability factor mentioned in (1.4) now is $S(T) = \max\{S_0, S_1, S_2\}$.

Theorem 3.2 (A priori error estimate)**.** *Let ρ, f_h and u_k be such that*

$$\begin{aligned} &S_0(T) \leq 1/6, \\ &CS_2(T)r_E(e_f) \leq 1/6, \\ &CS_2(T)r_N(f_h,\rho) \leq 1/6, \end{aligned} \tag{3.19}$$

where C is as in Lemma 3.1, $r_N(f_h,\rho)$ and $r_E(e_f)$ as in (3.10) *and* (3.14)*, and suppose*

$$\begin{aligned} &e(0)\cdot\varphi(0) \leq S_0(T)\|e(0)\| \leq \frac{1}{6}\rho, \\ &R_{0,T}(u_k,\varphi) \leq CS_1(T)\|k^qR(u_k)\|_{L^\infty([0,T])} \leq \frac{1}{6}\rho, \\ &E_{0,T}(u,\varphi) \leq S_2(T)\|e_f\|_{L^\infty(\Omega)} \leq \frac{1}{6}\rho. \end{aligned} \tag{3.20}$$

Then the error $e(T) = u_k(T) - u(T)$ is bounded from above by

$$\begin{aligned} e(T)\cdot\varphi_T \leq &S_0(T)\|e(0)\| + S_1(T)\|k^qR(u_k)\|_{L^\infty([0,T])} \\ &+ S_2(T)\|e_f\|_{L^\infty(\Omega)} \leq \rho. \end{aligned} \tag{3.21}$$

Proof. From (3.11), (3.13) and (3.19) it follows that (3.17) is a contraction mapping on $\mathcal{B}_\rho$. From (3.19) and (3.20) we also see that the mapping is into $\mathcal{B}_\rho$. Therefore there is a unique solution $e \in \mathcal{B}_\rho$ to (3.17) that satisfies (3.21). $\square$

We note that

$$\|\varphi(T)\| \leq \|\varphi(0)\| + \|A_h^*\|_{L^\infty([0,T])}\int_0^T \|\varphi(s)\|\, ds,$$

and by Gronwall's lemma, see for example [8, p. 625] we estimate

$$\|\varphi(T)\| \le \|\varphi(0)\|\big(1 + T\|A_h^*\|_{L^\infty([0,T])}e^{T\|A_h^*\|_{L^\infty([0,T])}}\big).$$

For flows that are dynamically unstable we do not expect any better estimates than this. Thus (3.19) and (3.20) will be very difficult or impossible to achieve in these situations.

3.2. **Shadowing.** In this section we assume that $L(t,s)$ has an exponential dichotomy on the interval $[0,T]$. We note the connection between $L(t,s)$ and $L_h(t,s)$ provided in the following roughness result. From [16, Lemma 7.4, p.133] we know that if $L(t,s)$ has an exponential dichotomy on $[0,T]$ and if

$$\|A_h(t) - A(t)\| \le \delta \le \delta_0(M,\beta).$$

Then $L_h(t,s)$ also has an exponential dichotomy on $[0,T]$ with constants M_h, β_h and projection $P_h(t)$ satisfying

$$0 < \beta_h < \beta \qquad \text{and} \qquad \|P_h(t) - P(t)\| \le C\delta,$$

where M_h, β_h and C are constants only depending on M and β.

We now assume that $L_h(t,s)$ has an exponential dichotomy on the interval $[0,T]$ in the sense given in the paragraph above. It then follows that $L_h^*(s,t)$ also has an exponential dichotomy on $[0,T]$ with projection $I - P_h^*(t)$ and constants M_h and β_h. By taking the adjoint in Property 1 of Definition 2.1 and subtracting the identity we get

$$(I - P_h^*(s))L_h^*(t,s) = L^*(t,s)(I - P_h^*(t)),$$

and multiplying from left and right with $L_h^*(s,t)$ and $L_h^*(s,t)$ we obtain Property 1 for $L_h^*(s,t)$

$$L_h^*(s,t)(I - P_h^*(s)) = (I - P_h^*(t))L_h^*(s,t).$$

The other properties now follow using the identity above.

Consider the following boundary value problem related to (2.17)

$$\begin{aligned} &-\partial_s\varphi + A_h^*(s)\varphi = -\psi\delta(s-t), \quad s \in ([0,T]);\\ &(I - P_h^*(0))\varphi(0) = 0, \quad P_h^*(T)\varphi(T) = 0, \end{aligned} \tag{3.22}$$

where $\psi \in \mathbf{R}^3$ and δ is the Dirac delta distribution and thus the solution $\varphi(s)$ will have a jump $-\psi = \varphi(t)^+ - \varphi(t)^-$ at time $s = t$.

This problem is also well posed by the same arguments as for (2.17) and the solution is

$$\varphi(s,t) = -G_h^*(s,t)\psi,$$

where we explicitly added t as an argument in the solution and where $G_h^*(s,t)$ now is the Green operator

$$G_h^*(s,t) = \begin{cases} (I - P_h^*(t))L_h^*(s,t), & 0 \le t \le s, \\ -P_h^*(t)L_h^*(s,t), & s < t \le T. \end{cases} \tag{3.23}$$

In weak form (3.22) reads. Find $\varphi \in C^1([0,t))^3 \cup C^1((t,T])^3$:

$$\int_0^T \phi \cdot (-\partial_s \varphi + A_h^*(s)\varphi)\, ds = \phi(t) \cdot \psi \quad \forall \phi \in C^1([0,T])^3, \tag{3.24}$$

and by integration by parts

$$\phi(t) \cdot \psi = \int_0^T (\partial_s \phi + A_h(s)\phi) \cdot \varphi\, ds + \phi(T) \cdot \varphi(T) - \phi(0) \cdot \varphi(0), \tag{3.25}$$

where we stress that $\varphi(0)$ and $\varphi(T)$ are not equal to zero, in fact only $(I - P_h^*(0))\varphi(0) = 0$ and $P_h^*(T)\varphi(T) = 0$ ($P_h^*(0)\varphi(0)$ and $(I - P_h^*(T))\varphi(T)$ are determined by the differential equation).

Suppose $e(t) = u_k(t, x_i) - u(t, y) \in \mathcal{B}_\rho$, where $\mathcal{B}_\rho$ is the ball (3.8), and such that $P_h(0)e(0) = 0$ and $(I - P_h(T)e(T) = 0$ which implies that

$$e(T) \cdot \varphi(T) = e(T) \cdot (I - P_h^*(T))\varphi(T) = (I - P_h(T))e(T) \cdot \varphi(T) = 0,$$

and likewise $e(0) \cdot \varphi(0) = 0$.

Taking $\phi = e$ in (3.25) and with (3.6) and (3.7) we get

$$e(t) \cdot \psi = R_{0,T}(u_k, \varphi) + E_{0,T}(u, \varphi) + N_{0,T}(u, \varphi), \tag{3.26}$$

which is a fixed point problem with a similar right hand side as in (3.17) although the problem defining φ is not the same in this case. Note that the right hand side does not have any derivative in φ and hence is well defined even when φ is discontinuous as in the present case.

Estimating the right hand side in (3.26) we use Cauchy's inequality for the fist two terms and for the remaining terms we use the estimates (3.15) (with care), (3.12) and (3.9), now tanking into account that φ is discontinuous at $s = t$. As is usual we define the stability factors

$$\begin{aligned} \widetilde{S}_1(T) &:= \sup_{t \in [0,T]} \max\big\{ \|D^q \varphi(\cdot, t)\|_{L^1([0,t))}, \|D^q \varphi(\cdot, t)\|_{L^1((t,T])} \big\} \\ \widetilde{S}_2(T) &:= \sup_{t \in [0,T]} \|\varphi(\cdot, t)\|_{L^1([0,T])}, \end{aligned} \tag{3.27}$$

where now φ is the solution to the boundary value problem (3.22). We remark that the stability factor in (1.5) now is $\widetilde{S}(T) = \max\{\widetilde{S}_1, \widetilde{S}_2\}$.

Theorem 3.3 (Shadowing)**.** *Let ρ, f_h and u_k be such that*

$$C\widetilde{S}_2(T)r_E(e_f) \leq 1/4, \qquad (3.28)$$
$$C\widetilde{S}_2(T)r_N(f_h,\rho) \leq 1/4,$$

where C is as in Lemma 3.1, $r_N(f_h,\rho)$ and $r_E(e_f)$ as in (3.10) *and* (3.14)*, and suppose*

$$R_{0,T}(u_k,\varphi) \leq C\widetilde{S}_1(T)\|k^q R(u_k)\|_{L^\infty([0,T])} \leq \frac{1}{4}\rho, \qquad (3.29)$$
$$E_{0,T}(u,\varphi) \leq \widetilde{S}_2(T)\|e_f\|_{L^\infty(\Omega)} \leq \frac{1}{4}\rho.$$

Then the numerical solution $u_k(t,x_i)$ is shadowed by an exact solution $u(t,y_i)$ and the error $e(t) = u_k(t,x_i) - u(t,y_i)$ is bounded from above for all $t \in [0,T]$

$$|e(t)| \leq \widetilde{S}_1(T)\|k^q R(u_k)\|_{L^\infty([0,T])} + \widetilde{S}_2(T)\|e_f\|_{L^\infty(\Omega)} \leq \rho. \qquad (3.30)$$

Proof. Set $\psi = 1$. From (3.11), (3.13) and (3.28) it follows that (3.26) is a contraction mapping on $\mathcal{B}_\rho$. From (3.28) and (3.29) we also see that the mapping is into $\mathcal{B}_\rho$. Therefore there is a unique solution $e \in \mathcal{B}_\rho$ to (3.26) that satisfies (3.30) and we get $u(t,y_i) = u_k(t,x_i) - e(t)$. □

We note that provided $L_h^*(s,t)$ has an exponential dichotomy φ will stay bounded by (2.21) and in contrast to the error estimate (3.21) the estimate in this case (3.30) will remain valid for large T. However we must show that $L_h^*(s,t)$ has an exponential dichotomy or by some means estimate $\varphi(t,\cdot)$. We discuss this matter in the next section.

3.3. **Finite time shadowing.** In this section we discuss the finite time shadowing results from [5]. We first assume that $L(t,s)$ has an exponential dichotomy as described in Sections 2.4 and 3.2.

We consider the boundary value problem (3.22) and the solution operator (3.23). From now on set $\psi = 1$.

Partition $[0,T]$ into M sub intervals $[T_m, T_{m+1}]$ for $m = 0,1,\ldots,M-1$ and where $T_0 = 0$ and $T_M = T$. Let $L_m = L(T_{m+1},T_m)$ be a sequence of operators and set

$$L_{mn} = L_{m-1}\cdots L_n, \quad m > n, \qquad \text{and} \qquad L_{mm} = I.$$

If we choose $s = T_m$ and $t = T_n$ in (3.23) we get

$$\varphi(T_m,T_n) = -\mathcal{G}^*_{mn}, \qquad (3.31)$$

where

$$\mathcal{G}^*_{mn} = \begin{cases} (I - P^*(T_n))L^*_{mn}, & 0 \le n \le m, \\ -P^*(T_n)L^*_{mn}, & m < n \le M. \end{cases}$$

This is the solution to the recurrence problem cf. [13, Section 3.2]

$$\begin{aligned} -\delta_{m+1,n} &= \varphi_{m+1} - L^*_m\varphi_m, \quad m = 0, \ldots, M-1; \\ (I - P^*(0))\varphi_0 &= 0, \quad P^*(T_M)\varphi_M = 0, \end{aligned} \tag{3.32}$$

for $n \in [0, M-1]$ and where $\delta_{m,n} = 1$ if $m = n$ and $\delta_{m,n} = 0$ if $m \neq n$.

Let $\widehat{f} = f/\|f\|$ denote the normalization of f. Choose one (3×2) matrix Z_0 such that the (3×3) matrix

$$\begin{pmatrix} \widehat{f}_h(u(0,x)) & Z_0 \end{pmatrix}$$

is orthonormal and by QR-factorization define recursively for $m = 0, 1, \ldots, M-1$

$$\begin{pmatrix} \widehat{f}_h(u(T_{m+1},x)) & L^*_m Z_m \end{pmatrix} = \begin{pmatrix} \widehat{f}_h(u(T_{m+1},x)) & Z_{m+1} \end{pmatrix} \begin{pmatrix} \cdots & \cdots \\ 0 & A_m \end{pmatrix},$$

where

$$A_m := \begin{pmatrix} a_m & b_m \\ 0 & c_m \end{pmatrix} = Z^*_{m+1} L^*_m Z_m \tag{3.33}$$

is upper triangular and with positive diagonal entries as long as matrix on the left hand side has full rank [10, Theorem 5.2.2, p. 217]. Note that $Z^*_m Z_m = I$.

Set $\varphi_m = Z_m \phi_m$ and transform (3.32)

$$\begin{aligned} -\delta_{m+1,n} Z^*_{m+1} &= \phi_{m+1} - A_m\phi_m, \quad m = 0, \ldots, M-1; \\ Z^*_0(I - P^*(0))Z_0\phi_0 &= 0, \quad Z^*_M P^*(T) Z_M \phi_M = 0. \end{aligned} \tag{3.34}$$

In most situations we do not know the projections $P(0)$ and $P(T)$. Nevertheless we may solve (3.34) by taking a good guess. With $\phi_m = (\phi_{m,1}, \phi_{m,2})$ and (3.33) we rewrite (3.34)

$$\begin{aligned} \phi_{m+1,1} &= a_m\phi_{m,1} + b_m\phi_{m,2} + \delta_{m+1,n} z_{m+1,1}, \\ \phi_{m+1,2} &= c_m\phi_{m,2} + \delta_{m+1,n} z_{m+1,2}, \end{aligned} \tag{3.35}$$

where $z_{m+1,i}$ is the sum of the i:th row in Z^*_{m+1}.

Considering the sequences $\{a_m\}_{m=0}^M$ and $\{c_m\}_{m=0}^M$ we distinguish six different cases and solve (3.35) accordingly. Set $a = \prod_{m=0}^M a_m$ and $c = \prod_{m=0}^M c_m$.

(1) If $a > 1.0$ and $c < 1.0$. Set $\phi_{0,2} = 0$ and solve the second equation forwards obtaining $\phi_{m,2}$, and set $\phi_{m,1} = 0$, substitute $\phi_{m,2}$ into the first equation and solve backwards obtaining $\phi_{m,1}$.
(2) If $a < 1.0$ and $c > 1.0$. Set $\phi_{0,2} = 0$ and solve the second equation backwards obtaining $\phi_{m,2}$, and set $\phi_{m,1} = 0$, substitute $\phi_{m,2}$ into the first equation and solve forwards obtaining $\phi_{m,1}$.
(3) If $a < 1.0$ and $c < 1.0$ and $a > c$. Do as in the first case.
(4) If $a < 1.0$ and $c < 1.0$ and $a < c$. Do as in the second case.
(5) If $a > 1.0$ and $c > 1.0$ and $a > c$. Do as in the first case.
(6) If $a > 1.0$ and $c > 1.0$ and $a < c$. Do as in the second case.

Cases (1) and (2) are considered as ideal and imply that $\|\phi\|$ is small. The remanding cases are not ideal and the solution may blow up and $\|\phi\|$ may be large.

Since we only guess the projections we may expect to mix the stable and unstable subspaces when solving according to the steps above. The computed solution will serve as an estimate for the true solution and hopefully this solution will be small or have a mild growth over time.

3.3.1. *Computing $\widetilde{S}_i(T)$, $i = 2, 3$, in practice.* We now substitute $L^*(s,t)$ by $L_h^*(s,t)$ in the analysis above and compute the norm to $\{\phi_m\}_{m=0}^M$ in (3.35) in two different ways.

Case I. In the first case we solve $M-2$ problems (3.35) for $n = 1, 2, \ldots, M-2$ and compute the norms from this set of solutions. The amount of work for this procedure will scale like $O(M^2)$.

Case II. In the second case we proceed as proposed in [5]. Instead of (3.35) we consider

$$\begin{aligned} \eta_{m+1,1} &= a_m \eta_{m,1} \mp |b_m| \eta_{m,2} \mp |z_{n,1}|, \\ \eta_{m+1,2} &= c_m \eta_{m,2} \pm |z_{m,2}|, \end{aligned} \tag{3.36}$$

where the $\mp$ and $\pm$ depend on whether we solve according to case (1) or (2) as described above. This procedure will imply that $|\phi_{m,1}| \leq \eta_{m,1}$ and $|\phi_{m,2}| \leq \eta_{m,2}$. The amount of work for this procedure will scale like $O(M)$.

4. Finite time shadowing in Stokes flow

Inspired by [18] where laminar fluid mixing was experimentally studied in small channels we set up the following model. Let $\Omega \subset \mathbf{R}^3$, be a polyhedral domain with periodic boundaries Γ_A and Γ_B, see Figures 4.1

and 4.2, and consider the Dirichlet Stokes problem with periodic boundary conditions in dimensionless form

$$\begin{aligned} -\Delta U + \nabla P &= 0 \quad \text{in}\, \Omega, \\ \nabla \cdot U &= 0 \quad \text{in}\, \Omega, \\ U &= 0 \quad \text{on}\, \partial\Omega \setminus (\Gamma_A \cup \Gamma_B), \\ U|_{\Gamma_A} &= U|_{\Gamma_B}, \\ P|_{\Gamma_A} &= P|_{\Gamma_B} + R, \end{aligned} \tag{4.1}$$

where $U = (U_1, U_2, U_3)$ is the unknown velocity field, P the unknown pressure and R is a constant modelling the pressure drop.

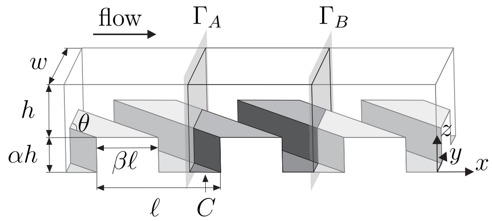

Figure 4.1: Three juxtaposed Ridge Domains. The shaded planes Γ_A and Γ_B are periodic boundaries. We choose the following values for the parameters: $\ell = w = 1$, $h = 0.3$, $\theta = 45°$, $\alpha = 2/3$, $\beta = 0.5$, and the length of the unit cell is $= 1$.

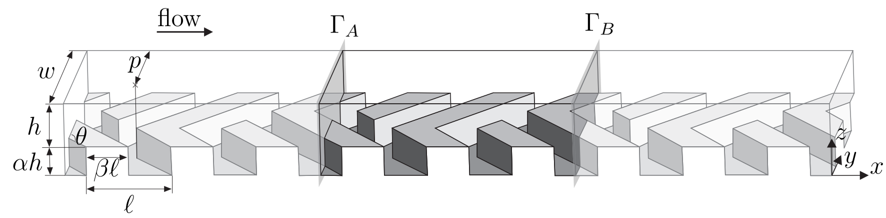

Figure 4.2: Three juxtaposed Herringbone Domains. The shaded planes Γ_A and Γ_B are periodic boundaries. We choose the following values for the parameters: $\ell = 2/3$, $w = 1$, $h = 1/5$, $\theta = 45°$, $\alpha = 2/3$, $\beta = 9/16$, $p = 2/3$, and the length of the unit cell is $= 14/9$.

From [3] and [15] we know that $U \in W^{2,4/3}(\Omega)^3 \cap W_0^{1,3}$ and thus U is continuous although not Lipschitz continuous. There will be singularities in ∇U and P along the edges and vertices of Ω. However, if we let $\Omega' \subset \Omega$ such that $\mathrm{dist}(\Omega', \partial\Omega)$ is not too small, then we may argue that U is Lipschitz continuous in Ω' by an interior estimate as in for example [9, Theorem 4.2, p. 209]. Thus when we compute orbits using $f = U$ (or in practice $f = U_h$) in (1.1) we only consider orbits that are not too close to $\partial\Omega$.

We refer to the domains in Figures 4.1 and 4.2 as Ridge and Herringbone respectively, the names are from [18]. Accurate solutions to (4.1) in the two domains are computed by a finite element method, Hood-Taylor P_2P_1 on fine triangulations. We illustrate the solutions in Figures 4.3 and 4.4.

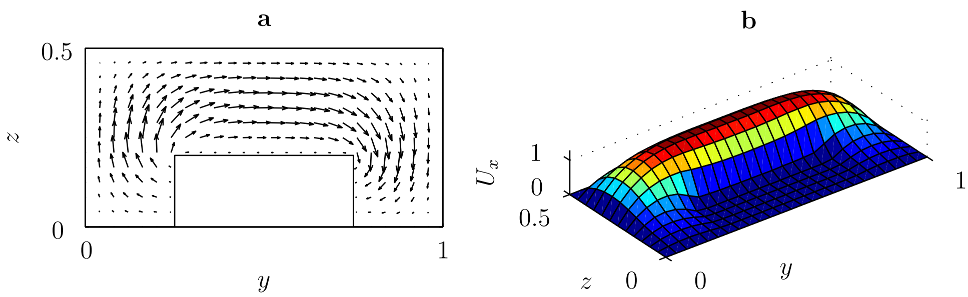

Figure 4.3: Velocity field for (4.1) solved in the Ridge Domain, Figure 4.1, at $x = 0.0$. **(a)** The y and z components of the velocity field. **(b)** The x component of the velocity field.

We compute orbits to (1.1) using the simple cG(1) method described in Section 2.2, with $f = U_h$ where U_h now is the computed solution to (4.1) . The time steps k_i for $i = 1, 2, \ldots, N$ is chosen adaptively so that the local residual is less than a small tolerance, for more details see [7]. We plot two typical orbits in Figure 4.5 for the Ridge Domain and in Figure 4.6 for the Herringbone Domain.

The dual problem (3.16) is solved by the same means but with time steps k_i for $i = 1, 2, \ldots, 2N - 1$ obtained by refining the partition of $[0, T]$ used for computing the orbits to (1.1). As φ_T we choose either of the canonical unit vectors, e.g., $(1, 0, 0)$. The stability factors $S_i(T)$ for $i = 1, 2, 3$ are then readily computed, see Figure 4.7.

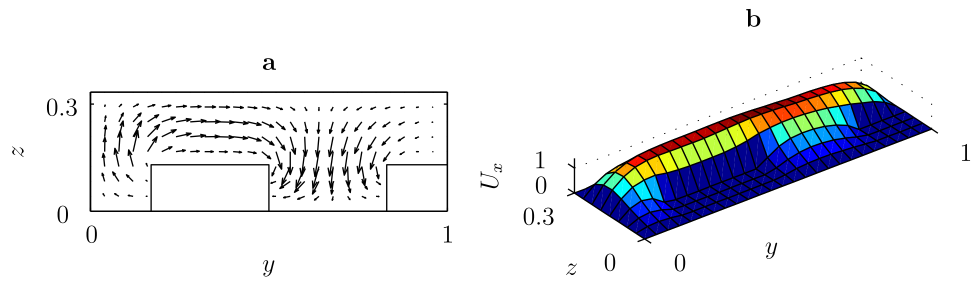

Figure 4.4: Velocity field for (4.1) solved in the Herringbone domain, Figure 4.2, at $x = 0.0$. **(a)** The y and z components of the velocity field. **(b)** The x component of the velocity field.

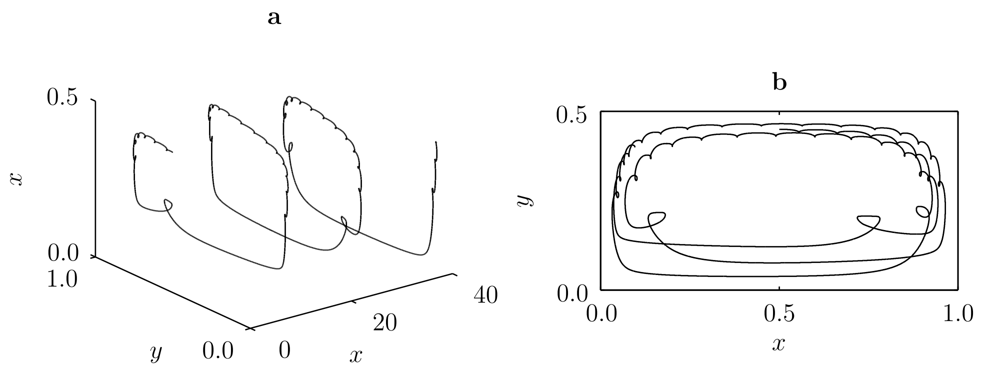

Figure 4.5: Computed orbit for $x = (0, 1/2, 9/20)$ in the velocity field U_h computed on the Ridge Domain. **(a)** Three dimensional plot. **(b)** Projection on the xy-plane.

We compute the projection matrices S_m as explained in Section 3.3 by approximating the action of L_m^* using the same method and the same time steps as for the dual problem (3.16). The recurrence problem is solved in the two different ways as described in Section 3.3, and depicted in Figures 4.8 and 4.9.

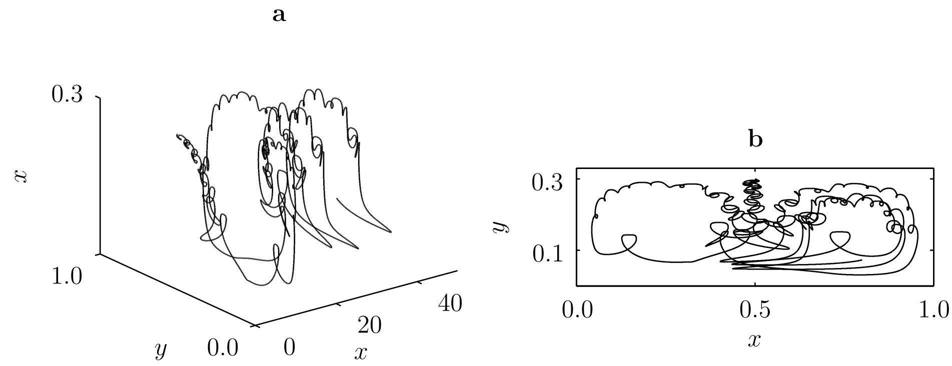

Figure 4.6: Computed orbit for $x = (0,\ 1/2,\ 1/3)$ in the velocity field U_h computed on the Herringbone Domain. **(a)** Three dimensional plot. **(b)** Projection on the xy-plane.

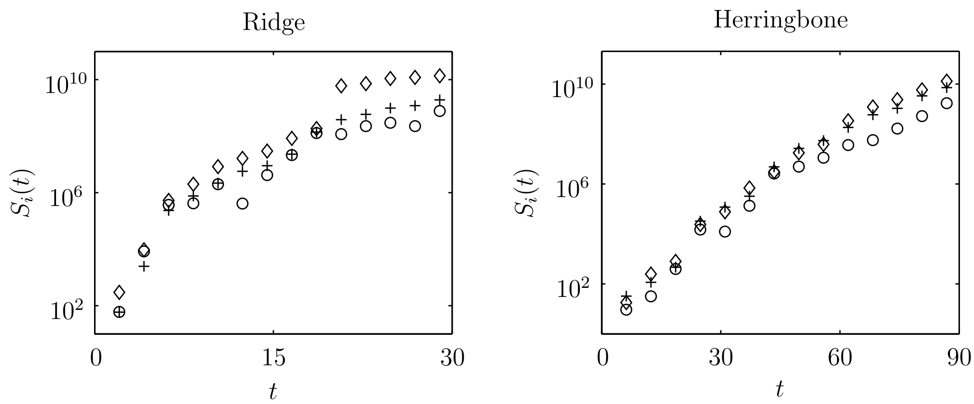

Figure 4.7: $(\circ, +, \diamond) = (S_0, S_1, S_2)$ Stability factors (3.18) for orbits in Figures 4.5 and 4.6.

5. Discussion

We have derived a shadowing error estimate (1.5) for computed orbits $u_k(t, x_i)$ to (1.1) with f replaced by a finite elements approximation f_h. Principal to the error estimate is the stability factors $\widetilde{S}_1(t)$ and $\widetilde{S}_2(t)$ which for sufficiently hyperbolic problems do not grow at any considerable rate as a function of the time t, in contrast to the stability factors S_i for the a posteriori error estimate where the stability factor grow at an exponential

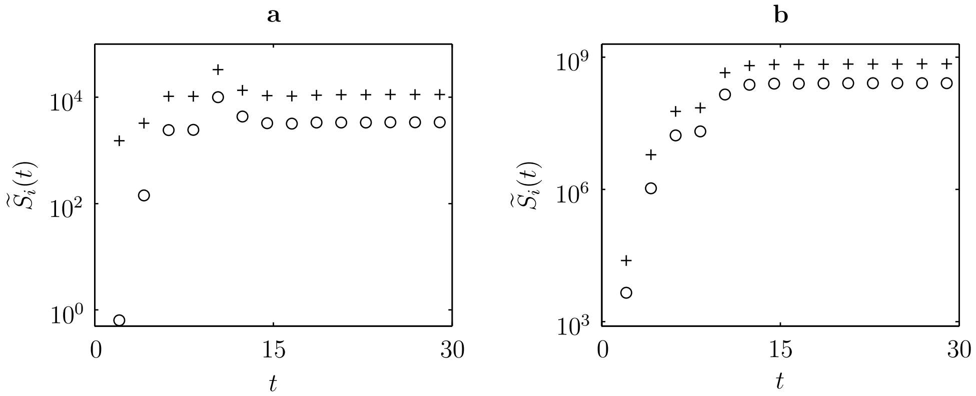

Figure 4.8: $(\circ, +) = (\widetilde{S}_1, \widetilde{S}_2)$ Stability factors (3.27) for orbit in Figure 4.5 computed as suggested in Section 3.3.1 **(a)** Case I **(b)** Case II.

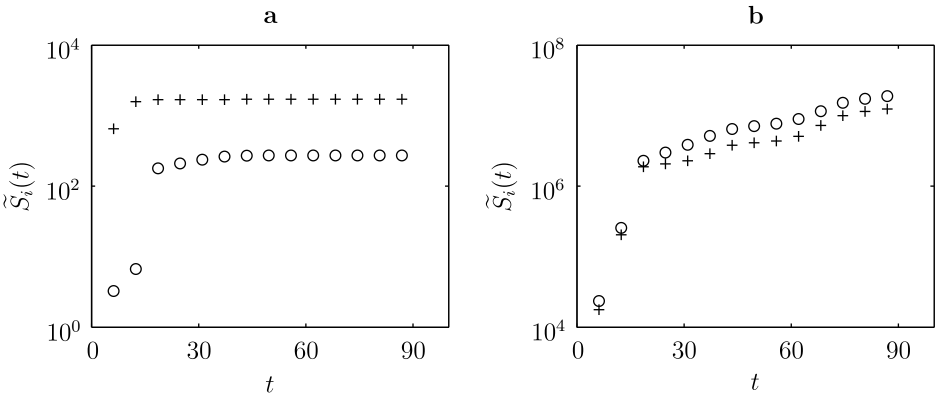

Figure 4.9: $(\circ, +) = (\widetilde{S}_1, \widetilde{S}_2)$ Stability factors (3.27) for the orbit in Figure 4.6 computed as suggested in Section 3.3.1 **(a)** Case I **(b)** Case II.

rate. We demonstrate this for orbits generated from the finite element velocity field modelled by the Stokes equations on two different domains, the Ridge Domain and the Herringbone Domain.

We note that there is an quite large difference in the way we choose to estimate the stability factors $\widetilde{S}_1(t)$ and $\widetilde{S}_2(t)$, either as in Case I or as in Case II as explained in Section 3.3.1, see Figures 4.8 and 4.9.

It is fare to say that the shadowing error estimate (1.5) is not rigorous as long as we do not control all constants in the estimate. At this stage we are not able to completely control the error in the finite element approximation

f_h. We only can provide asymptotic error estimates of e_f, that is, there is an unknown but bounded constant in the right hand side of the estimate and we can only deduce that the error goes to zero as $h \to 0$.

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