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PREPRINT 2006:23

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ERIK D. SVENSSON

Department of Mathematical Sciences
Division of Mathematics
CHALMERS UNIVERSITY OF TECHNOLOGY
GÖTEBORG UNIVERSITY
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Erik D. Svensson

CHALMERS | GÖTEBORG UNIVERSITY

Department of Mathematical Sciences
Division of Mathematics
Chalmers University of Technology and Göteborg University
SE-412 96 Göteborg, Sweden
Göteborg, August 2006

Preprint 2006:23
ISSN 1652-9715

Matematiska vetenskaper
Göteborg 2006

MULTIGRID METHODS ON ADAPTIVELY REFINED TRIANGULATIONS: PRACTICAL CONSIDERATIONS

ERIK D. SVENSSON

Abstract. We outline the implementation of the finite element multigrid method on adaptively refined triangulations for Lagrange and hierarchical finite elements of degree ≤ 2 in two and three dimensions. Refining the triangulations we relax the requirement that no vertex of any n-simplex lies in the interior of an edge of another n-simplex. As a result the refinements are easy to implement and the finite element spaces can be made nested, which simplifies the multigrid implementation. The refined triangulations may however contain 'hanging' nodes which must be taken into account in order to make the finite element spaces conforming. We modify the finite elements accordingly in these situations.

1. Introduction

The finite element multigrid method is theoretically well established as outlined in for example [4, 10, 14]. In this work we consider the practical aspects implementing the method on adaptively refined triangulations for conforming linear and quadratic finite elements in two and three dimensions.

We choose to use a refinement method that produce triangulations on which we can define nested finite element spaces and thus makes the formulation of the multigrid method straight forward with well defined projection operators on the finite element spaces. This is in contrast to the situation when the finite elements spaces are non-nested [4, 13]. Moreover this choice is also motivated by the fact that the refinement algorithm becomes simple compared to the rather involved refinement algorithm proposed in [3], which also renders the finite element spaces non-nested.

Date: April 18, 2006.

2000 *Mathematics Subject Classification.* 65N55, 65N30, 65N50.

Key words and phrases. multigrid, finite elements, refinement.

The refined triangulations are irregular [8] in the sense that there will be 'hanging' nodes and the construction of conforming finite element spaces is a non-trivial task that in practice requires implementation of flexible data structures. This and the even more general aspect of *hp* refinements has already be considered in [1, 8, 12]. We partially reformulate these results using concepts from modern finite element theory.

1.1. **Preliminaries.** We assume that the underlying problem is second order linear elliptic on a polyhedral domain $\Omega \subset \mathbf{R}^n$ for $n = 2, 3$. Let $a(\cdot,\cdot) : V \times V \to \mathbf{R}$ be a continuous, symmetric and V-elliptic bilinear form, and let $f(\cdot) : V \to \mathbf{R}$ be a continuous linear form. We pose the problem as the variational formulation

$$u \in V : \quad a(u, v) = f(v) \quad \forall v \in V, \tag{1.1}$$

where we assume that $V \subset H^1(\Omega)$ is a Hilbert space such that (1.1) is well-posed.

We will use standard notation for the Lebesgue and Sobolev spaces and for any measurable set $\omega \subseteq \mathbf{R}^n$, $n = 2, 3$, we let

$$(u, v)_\omega = \int_\omega uv \, dx,$$

denote the $L^2(\omega)$ scalar product.

For vectors $\tilde{v} = (\tilde{v}_1, \ldots, \tilde{v}_N) \in \mathbf{R}^N$ we will use the Euclidean norm denoted by $\|\tilde{v}\| = (\tilde{v}_1^2 + \tilde{v}_2^2 + \ldots + \tilde{v}_N^2)^{1/2}$.

Finally, throughout this work we will use C and c_i to denote various constants, not necessarily taking the same value from time to time.

1.2. **Finite elements.** We will use the notion *finite element* to denote the triplet $(T, \mathcal{P}, \mathcal{N})$ where $T \subset \Omega$ is a non empty Lipschitz continuous set, $\mathcal{P}$ is a finite dimensional space of functions on T and $\mathcal{N} = \{N_1, N_2, \ldots, N_{m_q}\}$ is a base for $\mathcal{P}'$, the set of nodal variables [5, 6].

Remark 1.1. For a d-dimensional vector space $\mathcal{P}$ and for a subset $\{N_1, N_2, \ldots, N_d\}$ of $\mathcal{P}'$ the following two statements are equivalent [5, Lemma 3.1.4, p. 70].

(1) $\{N_1, N_2, \ldots, N_d\}$ is a basis for $\mathcal{P}'$.
(2) If $v \in \mathcal{P}$ with $N_i v = 0$ for $i = 1, \ldots, d$, then $v = 0$.

We use this to verify that a given triplet $(T, \mathcal{P}, \mathcal{N})$ is a finite element.

As for T we only consider n-simplices with vertices $a_i \in \mathbf{R}^n$ for $i = 1, \ldots, n+1$ and $n = 2, 3$ as in Figure 2.1 and 2.2a. We set $h_T = \mathrm{diam}(T)$.

Let $\mathcal{P}_q$ denote the space of polynomials of degree $\leq q$ and note that

$$\dim(\mathcal{P}_q) = \binom{n+q}{q} = \mathrm{card}(\mathcal{N}) = m_q, \tag{1.2}$$

where we use the cardinal number to count the number of elements in a set.

Let $L_q(T)$ denote the *principal lattice of order* q on T with m_q lattice points [6, Theorem 6.1, p. 70], that is,

$$L_q(T) = \Big\{ x = \sum_{i=1}^{n+1} \xi_i a_i : \sum_{i=1}^{n+1} \xi_i = 1,\ \xi_i \in \Big\{0, \frac{1}{q}, \ldots, \frac{q-1}{q}, 1\Big\}\Big\}.$$

For example, $L_1(T) = \{a_i\}_{i=1}^{n+1}$ is the vertices in the n-simplex T and $L_2(T) = \{a_i\}_{i=1}^{n+1} \cup \{a_{ij} = (a_i + a_j)/2 : 1 \leq i < j \leq n+1\}$, see Figures 2.1 and 2.2a.

We use the common practice and refer to points in $L_q(T)$ as *local nodes*.

In order to express $\mathcal{P}_q$ on T we use *barycentric coordinates*, that is, $\lambda_i \in \mathcal{P}_1$ on T such that $\lambda_i(x_j) = \delta_{ij}$ for $x_j \in L_1(T)$ and $i, j = 1, \ldots, n+1$, see for example [9].

Given a basis to $\{\varphi_1, \ldots, \varphi_{m_q}\}$ to $\mathcal{P}_q$ we choose the nodal variables such that $N_i(\varphi_j) = \delta_{ij}$ for $i, j = 1, \ldots, m_q$.

1.2.1. *Lagrange finite elements.* We recall the definition of the standard Lagrange finite element which determine a finite element space of continuous piecewise polynomials of degree $q \geq 1$. In terms of the triplet $(T, \mathcal{P}, \mathcal{N})$, $\mathcal{P} = \mathcal{P}_q$ with basis functions $\varphi_i \in \mathcal{P}_q$ for $i = 1, \ldots, m_q$ such that $\varphi_i(x_j) = \delta_{ij}$ and the nodal variables are defined by $N_j(v) = v(x_j)$ for $x_j \in L_q(T)$ and $v \in C^0(T)$. For example: if $q = 1$, $\varphi_i = \lambda_i$, and if $q = 2$, $\varphi_i = \lambda_i(2\lambda_i - 1)$ for $i = 1, \ldots, n+1$ and $\varphi_{ij} = 4\lambda_i\lambda_j$ for $1 \leq i < j \leq n+1$ denoting the last $n+2, \ldots, m_2$ basis functions.

It is easily verified by Remark 1.1 that the triplet $(T, \mathcal{P}, \mathcal{N})$ is a finite element.

1.2.2. *Higher degree hierarchical finite elements.* We consider the higher degree hierarchical finite element which determine a finite element spaces of continuous piecewise polynomials of degree $q \geq 2$ as outlined in [2]. In terms of the triplet $(T, \mathcal{P}, \mathcal{N})$, $\mathcal{P} = \mathcal{P}_1 \oplus \mathcal{B}_q$, where $\mathcal{B}_q$ is the space of polynomials of degree > 1 and $\leq q$, that is, excluding the linear functions. For example, if $q = 2$, we choose the basis functions $\varphi_i = \lambda_i$ for $i = 1, \dots, n+1$ and $\varphi_{ij} = 4\lambda_i\lambda_j$ for $1 \leq i < j \leq n+1$ denoting the last $n+2, \dots, m_q$ basis functions and the nodal variables are defined by $N_i(v) = v(a_i)$ for $i = 1, \dots, n+1$ and

$$N_{ij}(v) = v(a_{ij}) - \frac{1}{2}(v(a_i) + v(a_j)) \quad \text{for } 1 \leq i < j \leq n+1.$$

In order to show that the triplet $(T, \mathcal{P}, \mathcal{N})$ is a finite element we take

$$\mathcal{P}_2 \ni v = \sum_{i=1}^{n+1} \tilde{v}_i \varphi_i + \sum_{\substack{i,j=1 \\ i<j}}^{n+1} \tilde{v}_{ij} \varphi_{ij}, \tag{1.3}$$

for constants $\tilde{v}_i, \tilde{v}_{ij} \in \mathbf{R}$. Then for $i = 1, \dots, n+1$,

$$N_i(v) = 0 \quad \Rightarrow \quad \tilde{v}_i = 0$$

and for $1 \leq i < j \leq n+1$

$$N_{ij}(v) = 0 \quad \Rightarrow \quad \frac{1}{2}\tilde{v}_i + \frac{1}{2}\tilde{v}_j + \tilde{v}_{ij} - \frac{1}{2}(\tilde{v}_i + \tilde{v}_j) = \tilde{v}_{ij} = 0.$$

Thus $v = 0$ and from Remark 1.1 we conclude that $\{N_i\}_{i=1}^{n+1} \cup \{N_{ij} : 1 \leq i < j \leq n+1\}$ is a basis for $\mathcal{P}'$ and $(T, \mathcal{P}, \mathcal{N})$ is a finite element.

1.3. **The finite element multigrid method.** We use the notation and framework outlined in [4]. Let $\mathcal{T}_1$ be a triangulation and define $\mathcal{T}_\ell$ for $\ell = 2, \dots, L$ recursively by subdividing all n-simplices in $\mathcal{T}_{\ell-1}$ as described in Section 2.1 below. We remark that all sub-tetrahedra are not congruent but on repeating the process the sub-tetrahedra will remain shape regular [3]. We note that since the n-simplices in $\mathcal{T}_\ell$ stay shape regular, the family of triangulations $\{T_\ell\}_{\ell=1}^{L}$ will be quasi-uniform.

In the usual way we define the piecewise continuous finite element spaces V_ℓ on Ω by the finite elements $(T, \mathcal{P}_T, \mathcal{N}_T)_{T \in \mathcal{T}_\ell}$ with local basis functions $\{\varphi_{1,T}, \dots, \varphi_{m_q,T}\}$ and node variables $\mathcal{N}_T = \{N_{1,T}, N_{2,T}, \dots, N_{m_q,T}\}$.

Let $\{\phi_1, \dots, \phi_{M_\ell}\}$ be a basis to V_ℓ, the *global basis*, with

$$\dim(V_\ell) := M_\ell = \operatorname{card}(\{L_q(T) : \ T \in \mathcal{T}_\ell\}), \tag{1.4}$$

and such that ϕ_i has support in S_i for $i = 1, \dots, M_\ell$ where

$$S_i := \bigcup \{T \in \mathcal{T}_\ell : x_i \in T\}, \tag{1.5}$$

for the *global nodes* $\{x_i\}_{i=1}^{M_\ell} = \{L_q(T) : T \in \mathcal{T}_\ell\}$.

For $T \in \mathcal{T}_\ell$ let I_T be an index set of the local nodes in the finite element $(T, \mathcal{P}_T, \mathcal{N}_T)$, for example, $I_T = \{1, 2, 3, 12, 13, 23\}$ for the quadratic Lagrange finite element in two dimensions. Let $i_j : I_T \to [1, M_\ell]$ be the injective map that maps the local index j to the corresponding global index i_j. We express the global basis functions in terms of the local finite element basis functions. For $i = 1, \ldots, M_\ell$ and with j so that $i_j = i$

$$\phi_i\big|_T = \begin{cases} \varphi_{j,T} & \text{if } T \in S_i, \\ 0 & \text{if } T \notin S_i. \end{cases} \tag{1.6}$$

Now $\{V_\ell\}_{\ell=1}^L$ is a nested sequence of finite element spaces, that is,

$$V_1 \subset V_2 \subset \cdots \subset V_L \subset V. \tag{1.7}$$

From equation (1.1) we obtain the finite element equation on the L:th level

$$u \in V_L : \quad a(u, v) = (f, v) \quad \forall v \in V_L, \tag{1.8}$$

where we assume that $f \in V_L$ is a finite element approximation to the linear form $f(\cdot)$ in equation (1.1).

In order to describe the multigrid method we will need the following auxiliary operators. For $\ell = 1, \ldots, L$ let $A_\ell : V_\ell \to V_\ell$ be defined by

$$(A_\ell v, \phi) = a(v, \phi) \quad \forall \phi \in V_\ell,$$

and let the projectors $P_{\ell-1} : V_\ell \to V_{\ell-1}$ and $Q_{\ell-1} : V_\ell \to V_{\ell-1}$ be defined by

$$a(P_{\ell-1} v, \phi) = a(v, \phi) \quad \forall \phi \in V_{\ell-1},$$

and

$$(Q_{\ell-1} v, \phi) = (v, \phi) \quad \forall \phi \in V_{\ell-1}.$$

We will also need a generic smoother $R_\ell : V_\ell \to V_\ell$ for $\ell = 1, \ldots, L$ and denote by R_ℓ^t the adjoint of R_ℓ with respect to $(\cdot, \cdot)$.

We consider the V-cycle multigrid algorithm. Given initial data $u^0 \in V_L$ the algorithm generates a sequence approximating u, the solution to equation (1.8), by

$$u^{m+1} = \mathrm{VMG}_L(u^m, f) \quad m = 0, 1, \ldots, \tag{1.9}$$

where $\mathrm{VMG}_L(\cdot, \cdot) : V_L \times V_L \to V_L$ is defined by the following Algorithm 1 [4].

Algorithm 1: $\mathrm{VMG}_\ell(v, f)$

Input: multigrid level ℓ, initial value $v = u^0$ as in (1.9) and right hand side f.
Output: u^1 in (1.9).

if $\ell = 0$ **then**
  return $A_0^{-1} f$ `/* exact solution */`
else
  $v' = v + R_\ell^t (f - A_\ell v)$ `/* presmoothing */`
  $v'' = v' + \mathrm{VMG}_{\ell-1}(0, Q_{\ell-1}(f - A_\ell v'))$ `/* error correction */`
  return $v'' + R_\ell (f - A_\ell v'')$ `/* postsmoothing */`

2. Irregular notions

We now relax one of the requirements in the definition of the triangulation, namely, the property that *no vertex of any n-simplex lies in the interior of an edge or face of another n-simplex* [5]. As for conforming finite element spaces we then will have to modify the finite elements accordingly.

2.1. **Irregular triangulations.** Inspired by [8] we say that a *1-irregular triangulation* is a partition of Ω into n-simplices such that at the most one vertex lies in the interior of any edge of all the n-simplices in the partition. Moreover, we say that a vertex is an *irregular vertex* if it lies in the interior of another edge and that an n-simplex is an *irregular n-simplex* if it contains irregular vertices.

We note that the definition is readily generalized to *m-irregular triangulation*, for $m \geq 1$, but we will not consider this type of triangulation in this work. When we in the sequel sometimes write irregular we thus means 1-irregular.

For an irregular n-simplex there may be one or several local nodes in $L_q(T)$ that do not represent true degrees of freedom, since the nodal variables in these points will be evaluated in other, possibly global, nodes in such way that the finite element space is made conforming. In the literature this kind of nodes are called hanging, constrained or slaved, in this work we call them *hanging nodes*, see Figures 3.1 and 3.2.

Now let $\mathcal{T}_\ell$ be an 1-irregular triangulation and suppose S is a subset of n-simplices $T \in \mathcal{T}_\ell$ that we want to refine, for example, S could be the set of n-simplices where an error estimator is larger than a certain threshold. In order to refine $\mathcal{T}_\ell$ we need to check the consistency of S, that is, the refined triangulation $\mathcal{T}_{\ell+1}$ must also be a 1-irregular triangulation and for this reason we cannot refine irregular n-simplices. We must check and modify S by adding n-simplices intersecting irregular vertices. Since some of the added n-simplices may also be irregular we must repeat the checking on the added n-simplices recursively. We describe this procedure in Algorithm 2.

Algorithm 2: CheckConsistency($\mathcal{T}$,S)

Input: a 1-irregular triangulation $\mathcal{T}$ and a set S of n-simplices $T \in \mathcal{T}$.
Output: S, possibly modified.

$S_{\text{new}} = \emptyset$
forall *irregular* $T \in S$ **do**
  forall *irregular vertices* $a_i \in T$ **do**
    forall $T' \in \mathcal{T}$ *such that* $a_i \in T'$ **do**
      $S_{\text{new}} = S_{\text{new}} \cup \{T'\}$
if $S_{new} \neq \emptyset$ **then**
  CheckConsistency($\mathcal{T}$,S_{new})
$S = S \cup S_{\text{new}}$

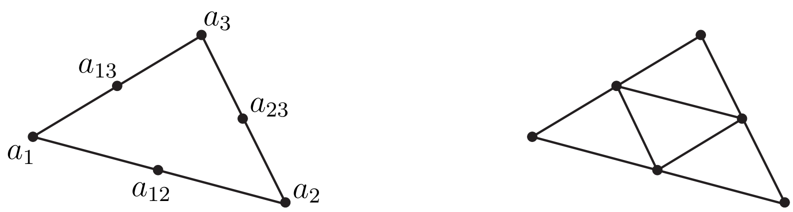
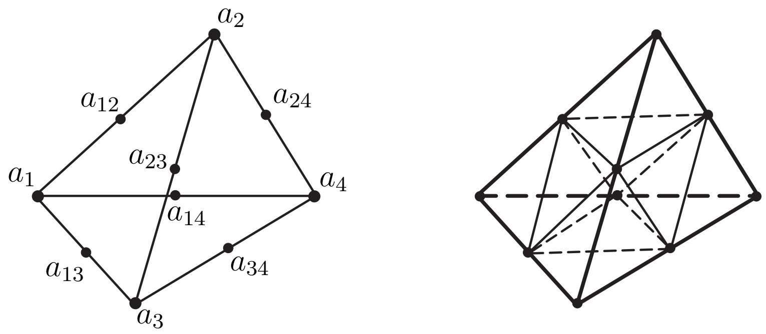


Figure 2.1: Regular triangle refinement. Original and refined triangles.

When we have checked the consistency of S we proceed with the refinement of all $T \in S$ and hence create the refined triangulation $\mathcal{T}_{\ell+1}$. We use the regular refinement algorithm from [3], where the two-dimensional case is trivial but included here for completeness. For any n-simplex let $a_{ij} = (a_i + a_j)/2$ for $1 \leq i < j \leq n+1$ denote the midpoint of the

(a)



(b)

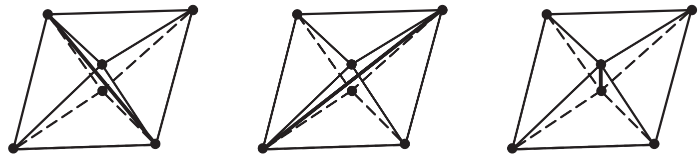


Figure 2.2: Regular tetrahedron refinement due to [3]. **(a)** Original and refined tetrahedron. **(b)** The interior octahedron is divided in one out of three ways as specified in [3].

edge connecting the vertices a_i and a_j. Now triangles are subdivided into four congruent subtriangles connecting the edge midpoints as in Figure 2.1 and as described in Algorithm 3. Tetrahedra are subdivided into eight subtetrahedra as depicted in Figure 2.2 and as described in Algorithm 4. We remark that all subtetrahedra are not congruent but on repeating the procedure the subtetrahedra will stay shape-regular [3].

Algorithm 3: RegularRefinement2D(T)

Input: a triangle T.
Output: 4 subtriangles $T_i \subset T$ for $i = 1, \ldots, 4$ such that $\bigcup_i T_i = T$.

divide $T = \{a_1, a_2, a_3\}$ into 4 subtriangles

$$T_1 = \{a_1, a_{12}, a_{13}\}, \quad T_2 = \{a_2, a_{23}, a_{12}\},$$
$$T_3 = \{a_3, a_{13}, a_{23}\}, \quad T_4 = \{a_{12}, a_{23}, a_{13}\}.$$

Algorithm 4: RegularRefinement3D(T)

Input: a tetrahedron T.
Output: 8 tetrahedra $T_i \subset T$ for $i = 1, \ldots, 8$ such that $\bigcup_i T_i = T$.

divide $T = \{a_1, a_2, a_3, a_4\}$ into 8 subtetrahedra

$$
\begin{aligned}
T_1 &= \{a_1, a_{12}, a_{13}, a_{14}\}, \quad & T_2 &= \{a_{12}, a_2, a_{23}, a_{24}\},\\
T_3 &= \{a_{13}, a_{23}, a_3, a_{34}\}, \quad & T_4 &= \{a_{14}, a_{24}, a_{34}, a_4\},\\
T_5 &= \{a_{12}, a_{13}, a_{14}, a_{24}\}, \quad & T_6 &= \{a_{12}, a_{23}, a_{23}, a_{24}\},\\
T_7 &= \{a_{13}, a_{14}, a_{24}, a_{34}\}, \quad & T_2 &= \{a_{13}, a_{23}, a_{34}, a_{34}\}.
\end{aligned}
$$

2.2. **Irregular finite elements.** In order to construct a conforming finite element space from the finite elements $(T, \mathcal{P}_T, \mathcal{N}_T)_{T\in\mathcal{T}_\ell}$ where $\mathcal{T}_\ell$ is a 1-irregular triangulation we need to define a new type of finite elements on irregular n-simplices.

We say that a finite element is a *q-irregular finite element* if we evaluate one or more of the nodal variables N_i at points $x_j \in L_q(T) \pm p$ where $p = a_i - a_j$ such that the line between a_i and a_j is an edge in T. For irregular n-simplices we define q-irregular finite elements so that the generated finite element space becomes conforming. We describe this in a few examples below.

2.2.1. *1-irregular Lagrange finite elements.* In $\mathbf{R}^2$ there are finite elements with 1–3 hanging nodes as in Figure 2.3. The basis functions are as defined in Sections 1.2.1 but the nodal variables are slightly different. We consider the finite element in the case of one hanging node, the other cases are defined in the same way. The nodal variables are defined by

$$N_i(v) = v(a_i), \quad i = 1, 2, \tag{2.1}$$

and

$$N_3(v) = \frac{1}{2}(v(a_2) + v(a_4)), \tag{2.2}$$

where $a_4 = 2a_3 - a_2$ and N_3 is eliminated as a global degree of freedom.

In order to show that the triplet $(T, \mathcal{P}_1, \mathcal{N})$ is a finite element we take $\mathcal{P}_1 \ni v = \sum_{i=1}^{3} \tilde{v}_i \varphi_i$ for constants $\tilde{v}_i \in \mathbf{R}$. Then for $i = 1, 2$,

$$N_i(v) = 0 \quad \Rightarrow \quad \tilde{v}_i = 0$$

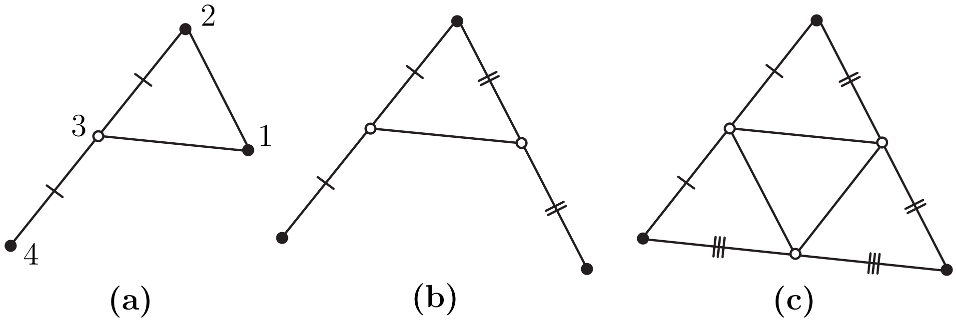


Figure 2.3: Three types of finite elements for the 1-irregular Lagrange finite elements in $\mathbf{R}^2$. $\bullet$ denotes a regular node and $\circ$ denotes a hanging node. **(a)** One hanging node. **(b)** Two hanging nodes. **(c)** Three hanging nodes.

and

$$N_3(v) = 0 \quad \Rightarrow \quad 1/2(\tilde{v}_2 - \tilde{v}_2 + 2\tilde{v}_3) = \tilde{v}_3 = 0.$$

Thus $v = 0$ and from Remark 1.1 we conclude that $\{N_i\}_{i=1}^3$ is a basis for $\mathcal{P}_1'$ and $(T, \mathcal{P}_1, \mathcal{N})$ is a finite element. Note that this construction of $\mathcal{N}$ guarantees that the global finite element functions are conforming.

In $\mathbf{R}^3$ there are finite elements with 1–4 hanging nodes and the treatment is analogous to the $\mathbf{R}^2$ case. In the case of one hanging node as in Figure 2.4 the nodal variables are defined by

$$N_i(v) = v(a_i), \quad i = 1, 2, 3, \tag{2.3}$$

and

$$N_4(v) = \frac{1}{2}(v(a_2) + v(a_5)), \tag{2.4}$$

where $a_5 = 2a_4 - a_2$, and it follows that $(T, \mathcal{P}_1, \mathcal{N})$ is a finite element.

2.2.2. *2-irregular hierarchical finite elements.* In $\mathbf{R}^2$ there are finite elements with 1–2 hanging nodes as in Figure 2.5, note that the hanging nodes now are on the the edges instead of in the vertices as for the 1-irregular Lagrange finite elements. The basis functions are as defined in Subsection 1.2.2 but the nodal variables are slightly different. We consider the finite element in the case of one hanging node, the other case is defined in the same way. The nodal variables are defined by

$$N_i(v) = v(a_i), \quad i = 1, 2, 3, \tag{2.5}$$

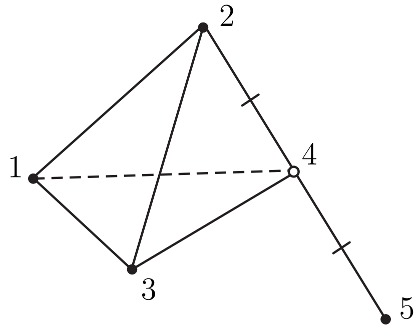


Figure 2.4: One hanging node for the 1-irregular Lagrange finite elements in $\mathbf{R}^3$. $\bullet$ denotes a regular node and $\circ$ denotes a hanging node.

and

$$\begin{aligned} N_{12}(v) &= v(a_{12}) - \frac{1}{2}(v(a_1) + v(a_2)), \\ N_{13}(v) &= v(a_{13}) - \frac{1}{2}(v(a_1) + v(a_3)), \\ N_{23}(v) &= \frac{1}{4}v(a_3) - \frac{1}{8}(v(a_2) + v(a_4)), \end{aligned} \tag{2.6}$$

where $a_4 = 2a_3 - a_2$.

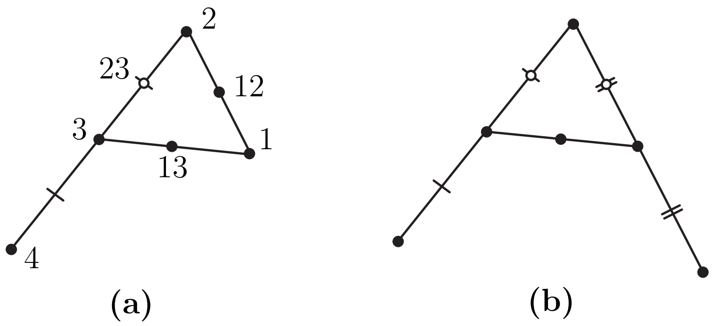


Figure 2.5: Two types of finite elements for the 2-irregular hierarchical finite element in $\mathbf{R}^2$. $\bullet$ denotes a regular node and $\circ$ denotes a hanging node. **(a)** One hanging node. **(b)** Two hanging nodes.

In order to show that the triplet $(T, \mathcal{P}_2, \mathcal{N})$ is a finite element we take v as in (1.3). Then for $i = 1, 2, 3$

$$N_i(v) = 0 \quad \Rightarrow \quad \tilde{v}_i = 0$$

and

$$\begin{aligned}
N_{12}(v) = 0 \quad &\Rightarrow \quad \frac{1}{2}\tilde{v}_1 + \frac{1}{2}\tilde{v}_2 + \tilde{v}_{12} - \frac{1}{2}(\tilde{v}_1 + \tilde{v}_2) = \tilde{v}_{12} = 0,\\
N_{13}(v) = 0 \quad &\Rightarrow \quad \frac{1}{2}\tilde{v}_1 + \frac{1}{2}\tilde{v}_3 + \tilde{v}_{13} - \frac{1}{2}(\tilde{v}_1 + \tilde{v}_3) = \tilde{v}_{13} = 0,\\
N_{23}(v) = 0 \quad &\Rightarrow \quad \frac{1}{4}\tilde{v}_3 - \frac{1}{8}(\tilde{v}_2 - \tilde{v}_2 + 2\tilde{v}_3 - 8\tilde{v}_{23}) = \tilde{v}_{23} = 0.
\end{aligned}$$

Thus $v = 0$ and from Remark 1.1 we conclude that $\{N_i\}_{i=1}^3 \cup \{N_{ij} : 1 \leq i < j \leq 3\}$ is a basis for $\mathcal{P}_2'$ and $(T, \mathcal{P}_2, \mathcal{N})$ is a finite element.

In $\mathbf{R}^3$ it is a bit more involved to maintain the continuity. There are tetrahedra with 1–5 hanging nodes. The basis functions are as defined in Sections 1.2.2 but the nodal variables are slightly different. We consider the finite element in the case of three hanging node as in Figure 2.6. The first nodal variables are defined by

$$N_i(v) = v(a_i), \quad i = 1, 2, 3, 4,$$

and

$$\begin{aligned}
N_{12}(v) &= v(a_{12}) - \frac{1}{2}(v(a_1) + v(a_2)),\\
N_{13}(v) &= v(a_{13}) - \frac{1}{2}(v(a_1) + v(a_3)),\\
N_{14}(v) &= v(a_{14}) - \frac{1}{2}(v(a_1) + v(a_4)),\\
N_{23}(v) &= \frac{1}{4}v(a_3) - \frac{1}{8}(v(a_2) + v(a_5)),\\
N_{24}(v) &= \frac{1}{4}v(a_4) - \frac{1}{8}(v(a_2) + v(a_6)),\\
N_{34}(v) &= \frac{1}{4}v(a_7) - \frac{1}{8}(v(a_5) + v(a_6)),
\end{aligned}$$

where

$$a_5 = 2a_3 - a_2, \qquad a_6 = 2a_4 - a_2, \qquad a_7 = a_3 + a_4 - a_2.$$

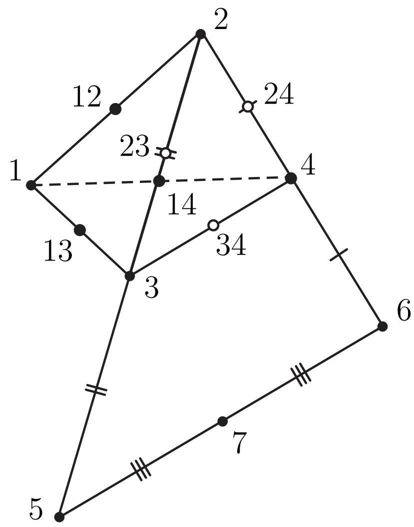


Figure 2.6: Three hanging nodes for the 2-irregular hierarchical finite element in $\mathbf{R}^3$. $\bullet$ denotes a regular node and $\circ$ denotes a hanging node.

In order to show that the triplet $(T, \mathcal{P}_2, \mathcal{N})$ is a finite element we take v as in (1.3). Then for $i = 1, 2, 3, 4$

$$N_i(v) = 0 \quad \Rightarrow \quad \tilde{v}_i = 0$$

and

$$\begin{aligned}
N_{12}(v) = 0 \quad &\Rightarrow \quad \frac{1}{2}\tilde{v}_1 + \frac{1}{2}\tilde{v}_2 + \tilde{v}_{12} - \frac{1}{2}(\tilde{v}_1 + \tilde{v}_2) = \tilde{v}_{12} = 0,\\
N_{13}(v) = 0 \quad &\Rightarrow \quad \frac{1}{2}\tilde{v}_1 + \frac{1}{2}\tilde{v}_3 + \tilde{v}_{13} - \frac{1}{2}(\tilde{v}_1 + \tilde{v}_3) = \tilde{v}_{13} = 0,\\
N_{14}(v) = 0 \quad &\Rightarrow \quad \frac{1}{2}\tilde{v}_1 + \frac{1}{2}\tilde{v}_4 + \tilde{v}_{14} - \frac{1}{2}(\tilde{v}_1 + \tilde{v}_4) = \tilde{v}_{14} = 0,\\
N_{23}(v) = 0 \quad &\Rightarrow \quad \frac{1}{4}\tilde{v}_3 - \frac{1}{8}(\tilde{v}_2 - \tilde{v}_2 - 2\tilde{v}_3 - 8\tilde{v}_{23}) = \tilde{v}_{23} = 0,\\
N_{24}(v) = 0 \quad &\Rightarrow \quad \frac{1}{4}\tilde{v}_4 - \frac{1}{8}(\tilde{v}_2 - \tilde{v}_2 - 2\tilde{v}_4 - 8\tilde{v}_{24}) = \tilde{v}_{24} = 0,\\
N_{34}(v) = 0 \quad &\Rightarrow \quad \frac{1}{4}(-\tilde{v}_2 + \tilde{v}_3 + \tilde{v}_4 - 4\tilde{v}_{23} - 4\tilde{v}_{24} + 4\tilde{v}_{34}),\\
&\qquad - \frac{1}{8}(-2\tilde{v}_2 + 2\tilde{v}_3 + 2\tilde{v}_4 - 8\tilde{v}_{23} - 8\tilde{v}_{24}) = \tilde{v}_{34} = 0.
\end{aligned}$$

Thus $v = 0$ and from Remark 1.1 we conclude that $\{N_i\}_{i=1}^4 \cup \{N_{ij} : 1 \leq i < j \leq 4\}$ is a basis for $\mathcal{P}_2'$ and $(T, \mathcal{P}_2, \mathcal{N})$ is a finite element.

2.3. **Finite element approximations on 1-irregular triangulations.** Let $\mathcal{T}_\ell$ be a 1-irregular triangulation. We define continuous finite element spaces V_ℓ on Ω by the finite elements $(T, \mathcal{P}_T, \mathcal{N}_T)_{T\in\mathcal{T}_\ell}$ as in Section 1.3, although we now use the 1 or 2-irregular finite elements defined in Section 2.2. Note that the index set I_T also changes, for example, $I_T = \{1, 2, 3, 12, 13, 4\}$ for the second order hierarchical finite element in two dimensions, Figure 2.5a.

With

$$V_\ell \ni u = \sum_{i=1}^{M_\ell} \tilde{u}_i \phi_i,$$

where $(\tilde{u}_1, \dots, \tilde{u}_{M_\ell}) \in \mathbf{R}^{M_\ell}$ is the coordinate vector with respect to the basis $\{\phi_1, \dots, \phi_{M_\ell}\}$ and taking $v = \phi_j$ we express (1.8), now with $u, v \in V_\ell$, as

$$\sum_{i=1}^{M_\ell} \tilde{u}_i a(\phi_i, \phi_j) = f(\phi_j) \quad \text{for } j = 1, \dots, M_\ell. \tag{2.7}$$

Locally on each $T \in \mathcal{T}_\ell$ we have

$$\phi_i|_T = \sum_{k\in I_T} N_{k,T}(\phi_i)\varphi_{k,T},$$

and hence (2.7) is equivalent to

$$\begin{aligned} &\sum_{i=1}^{M_\ell} \sum_{T\in S_i} \sum_{k,l\in I_T} \tilde{u}_i N_{k,T}(\phi_i) a(\varphi_{k,T}, \varphi_{l,T}) N_{l,T}(\phi_j) \\ &\quad = \sum_{T\in S_i} \sum_{l\in I_T} N_{l,T}(\phi_j) f(\varphi_{l,T}) \quad \text{for } j = 1, \dots, M_\ell, \end{aligned} \tag{2.8}$$

where we identify $a(\varphi_{k,T}, \varphi_{l,T})$ as a local stiffness matrix and $f(\varphi_{l,T})$ as the local load vector.

The rather involved formula (2.8) in fact expresses the distribution mapping defined in [1, 12], which is useful in practice implementing finite element problems. We note that assembling the stiffness matrix and load vector we only need to know a few things for each $T \in \mathcal{T}_\ell$: (1) where to put the elements from the local stiffness matrix and load vector into the global stiffness matrix and load vector, and (2) the weights $N_{k,T}(\phi_i)$ on the local elements. We represent this information in a set of arrays holding three numbers, $(i, k, N_{k,T}(\phi_i))_{T\in\mathcal{T}_\ell}$, where i is a global index, k is a local index on T and $N_{k,T}(\phi_i)$ is a weight, and likewise for $(j, l, N_{l,T}(\phi_j))_{T\in\mathcal{T}_\ell}$. More precisely we define the representation

$$\mathrm{Rep}(T) = \{(i, k, N_{k,T}(\phi_i)) : i \in i_{k'},\ k, k' \in I_T,\ N_{k,T}(\phi_i) \neq 0\}. \tag{2.9}$$

Thus, provided the finite elements $(T, \mathcal{P}_T, \mathcal{M}_{\ell T})_{T\in\mathcal{T}_\ell}$ are well defined we express the finite element problem as in (2.8) and use the representation (2.9) for assembling the problem in practice.

We remark that when $V_\ell \ni f = \sum_{i=1}^{M_\ell} \tilde{f}_i\phi_i$ and (2.8) becomes

$$\begin{aligned}
&\sum_{i=1}^{M_\ell}\sum_{T\in S_i}\sum_{k,l\in I_T} \tilde{u}_i N_{k,T}(\phi_i) a(\varphi_{k,T}, \varphi_{l,T}) N_{l,T}(\phi_j)\\
&\quad = \sum_{T\in S_i}\sum_{k,l\in I_T} \tilde{f}_i N_{k,T}(\phi_i)(\varphi_{k,T}, \varphi_{l,T}) N_{l,T}(\phi_j) \quad \text{for } j = 1, \dots, M_\ell.
\end{aligned} \tag{2.10}$$

where we identify $(\varphi_{k,T}, \varphi_{l,T})$ as the local mass matrix.

In the next four sections we explicitly compute $\mathrm{Rep}(T)$ for the finite elements in Sections 1.2 and 2.2.

2.3.1. *Lagrange finite elements.* In this case since $N_k(v) = v(x_k)$ for $x_k \in L_q(T)$ and $k \in I_T$ as defined in Section 1.2.1, the representation (2.9) is particularly simple:

$$\mathrm{Rep}(T) = \begin{pmatrix} i_k \\ k \\ 1 \end{pmatrix} \quad \forall\, k \in I_T,$$

which probably anyone that have implemented the Lagrange finite elements recognizes.

2.3.2. *Higher degree hierarchical finite elements.* We evaluate (2.9) for the quadratic hierarchical finite element in two dimensions. With $N_k(\cdot)$ and I_T as defined in Section 1.2.2 we get

$$\mathrm{Rep}(T) =$$
$$\begin{pmatrix} i_1 & i_2 & i_3 & i_{12} & i_{13} & i_{23} & i_1 & i_1 & i_2 & i_2 & i_3 & i_3 \\ 1 & 2 & 3 & 12 & 13 & 23 & 12 & 13 & 12 & 23 & 13 & 23 \\ 1 & 1 & 1 & 1 & 1 & 1 & -1/2 & -1/2 & -1/2 & -1/2 & -1/2 & -1/2 \end{pmatrix}.$$

The three-dimensional case is analogous.

2.3.3. *1-irregular Lagrange finite elements.* We evaluate (2.9) for the 1-irregular finite element in two dimensions. With $N_k(\cdot)$ for $x_k \in L_1(T)$ and I_T as defined in Section 2.2.1 we get

$$\mathrm{Rep}(T) = \begin{pmatrix} i_1 & i_2 & i_2 & i_4 \\ 1 & 2 & 3 & 3 \\ 1 & 1 & 1/2 & 1/2 \end{pmatrix},$$

The three-dimensional case is analogous.

2.3.4. *2-irregular hierarchical finite elements.* We evaluate (2.9) for the 2-irregular finite element in two dimensions. With $N_k(\cdot)$ and I_T as defined in Section 2.2.2 we get

$$\mathrm{Rep}(T) =$$

$$\begin{pmatrix} i_1 & i_2 & i_3 & i_{12} & i_{13} & i_1 & i_1 & i_2 & i_3 & i_3 & i_2 & i_4 \\ 1 & 2 & 3 & 12 & 13 & 12 & 13 & 12 & 13 & 23 & 23 & 23 \\ 1 & 1 & 1 & 1 & 1 & -1/2 & -1/2 & -1/2 & -1/2 & 1/4 & -1/8 & -1/8 \end{pmatrix}.$$

The three-dimensional case is analogous.

2.4. **Multigrid on 1-irregular triangulations.** We need to find the projection $Q_{\ell-1} : V_\ell \to V_{\ell-1}$ as defined in Section 1.3. Since $V_{\ell-1} \subset V_\ell$ we can express the basis functions in $V_{\ell-1}$, $\{\phi_i^{\ell-1}\}_{i=1}^{M_{\ell-1}}$, in terms of the base functions in V_ℓ, $\{\phi_i^\ell\}_{i=1}^{M_\ell}$. Hence, with the definition of $Q_{\ell-1}$,

$$(Q_{\ell-1} v_\ell, \phi_i^{\ell-1}) = (v_\ell, \phi_i^{\ell-1}) = \sum_{j=J_i^\ell} \alpha_{ij}^\ell (v_\ell, \phi_j^\ell),$$

for $v_\ell \in V_\ell$ and where $J_i^\ell := \{j : \mathrm{supp}\,(\phi_j^\ell) \cap S_i^{\ell-1} \neq \emptyset\}$.

We use the nodal variables to express α_{ij}^ℓ

$$\alpha_{ij}^\ell = N_{k,T}^\ell(\phi_i^{\ell-1}),$$

for all $T \in \mathcal{T}_\ell$ such that $T \cap S_i^{\ell-1} \neq \emptyset$ and for all $k \in I_T$ where $j_k = j$ is the local to global mapping defined in Section 1.3.

3. Numerical experiments

In matrix form (1.8) becomes

$$\mathcal{A}\tilde{u} = \mathcal{F},$$

where $\mathcal{A}$ denotes the matrix $[\mathcal{A}]_{ij} = (A_L \phi_i, \phi_j)$ for $i = 1, \ldots, M_\ell$ and $\tilde{u} \in \mathbf{R}^{M_\ell}$ denotes the coordinate vector with respect to the finite element basis and $\mathcal{F} = (\mathcal{F}_1, \ldots, \mathcal{F}_{M_\ell})$ where $\mathcal{F}_i = (f, \phi_i)$. We solve this linear system using the V-cycle Algorithm 1 with five iterations of a point Gauss-Seidel smoother. With $\tilde{u}^0 = 0$ we iterate $m = 1, 2, \ldots$ until the relative residual

$$\text{Res} := \frac{\|\mathcal{F} - \mathcal{A}\tilde{u}^m\|}{\|\mathcal{F}\|}$$

is less than a specified tolerance 'Tol' set to 10^{-6} in this work. Note that the relative tolerance times a constant is always greater than $\|u - u^m\|_{1,\Omega}$ where $u \in V_L$ is the finite element solution we are approximating, cf. [9, Proposition 9.19, p. 393].

3.1. **The Poisson equation.** We consider the following Poisson equation with mixed Dirichlet-Neumann boundary conditions on bounded polyhedral domains $\Omega \subset \mathbf{R}^n$ for $n = 2, 3$,

$$-\Delta u = f \quad \text{in } \Omega, \qquad u = g \quad \text{on} \quad \partial\Omega_D, \quad \text{and} \quad \nu \cdot \nabla u = 0 \quad \text{on} \quad \partial\Omega_N,$$

where the boundary is partitioned such that $\partial\Omega_D \cup \partial\Omega_N = \partial\Omega$, g is a constant, ν is the outward normal to the boundary and we assume $f \in H^{-1}(\Omega)$ and thus the problem is a well posed. Let

$$V = \{u \in H^1(\Omega) : u = 0 \text{ on } \partial\Omega_D\}.$$

Now the bilinear and linear forms in Section 1.1 are

$$a(u, v) = \int_\Omega \nabla u \cdot \nabla v \, dx,$$

and

$$f(v) = \int_\Omega f v \, dx.$$

With u_g denoting the extension of g to $H^1(\Omega)$, the weak formulation to the above Poisson problem follows as usual and reads: find $u \in H^1(\Omega)$ such that

$$\begin{aligned} u &= u_g + \phi, \qquad \phi \in V, \\ a(\phi, v) &= f(v) - a(u_g, v) \qquad \forall v \in V. \end{aligned} \tag{3.1}$$

We use the maximum norm error estimator derived in [7, 11] to adaptively refine the triangulations. For the solution u to (1.8) and $T \in \mathcal{T}_\ell$ we compute

$$\eta_T = h_T \|f + \Delta u\|_{L^\infty(T)} + \frac{1}{2} \|[\partial_{\nu_T} u]\|_{L^\infty(\partial T \setminus \partial\Omega)},$$

where $[\partial_{\nu_T} u]$ denotes the jump across ∂T in the normal derivative, $\partial_{\nu_T} u = \nu_T \cdot \nabla u$ where ν_T denotes the outward normal to ∂T.

We define $\mathcal{T}_{\ell+1}$ by refining those $T \in \mathcal{T}_\ell$ where

$$\eta_T > \overline{\eta}_T + s,$$

where $\overline{\eta}_T$ is the mean and s is the standard deviation of η_T.

3.1.1. *Model problem.* In this case we let Ω be the L-shaped domain with one re-entrant edge, $\Omega = \{(x, y) \in [0, 2]^2 \setminus [1, 2] \times [0, 1]\}$ for $n = 2$ and $\Omega = \{(x, y, z) \in [0, 2]^2 \times [0, 0.5] \setminus [1, 2] \times [0, 1] \times [0, 0.5]\}$ for $n = 3$, see Figure 3.1 and 3.2.

Let $\partial\Omega_D = \partial\Omega_{D_0} \cup \partial\Omega_{D_1}$ where $\partial\Omega_{D_0} = \{(x, y) : x = 1, y \in [1, 2]\}$ and $\partial\Omega_{D_1} = \{(x, y) : x \in [0, 1], y = 0\}$ for $n = 2$ and $\partial\Omega_{D_0} = \{(x, y, z) : x = 2, (y, z) \in [1, 2] \times [0, 0.5]\}$ and $\partial\Omega_{D_1} = \{(x, y, z) : (x, z) \in [0, 1] \times [0, 0.5], y = 0\}$ for $n = 3$. Set $f = 0$, $g = 0$ on $\partial\Omega_{D_0}$ and $g = 1$ on $\partial\Omega_{D_1}$.

We solve the problem for different finite element approximations and refine the triangulations eight times. The results from these experiments are summarized in and Tables 3.1 and 3.2.

Table 3.1: Convergence data for the V-cycle multigrid Algorithm 1 applied to the Model Problem for $n = 2$ and the finite elements in Section 2.2.

Lagrange $(n, q) = (2, 1)$

ℓ	M_1	m	Res
1	272	2	$1.3 \cdot 10^{-7}$
2	368	2	$8.3 \cdot 10^{-8}$
3	524	2	$7.1 \cdot 10^{-8}$
4	767	3	$2.2 \cdot 10^{-7}$
5	1072	2	$3.9 \cdot 10^{-8}$
6	1642	2	$2.5 \cdot 10^{-8}$
7	2415	3	$3.0 \cdot 10^{-8}$
8	3577	3	$5.7 \cdot 10^{-7}$

Hierarchical $(n, q) = (2, 2)$

ℓ	M_2	m	Res
1	299	2	$1.2 \cdot 10^{-7}$
2	459	2	$2.7 \cdot 10^{-8}$
3	644	2	$2.0 \cdot 10^{-8}$
4	890	2	$4.3 \cdot 10^{-8}$
5	1447	3	$3.1 \cdot 10^{-8}$
6	2066	3	$2.7 \cdot 10^{-7}$
7	3158	3	$2.4 \cdot 10^{-7}$
8	4546	4	$3.8 \cdot 10^{-8}$

4. Conclusions

We outlined a methodology for implementing the finite element multigrid method on adaptively refined triangulations for various finite elements,

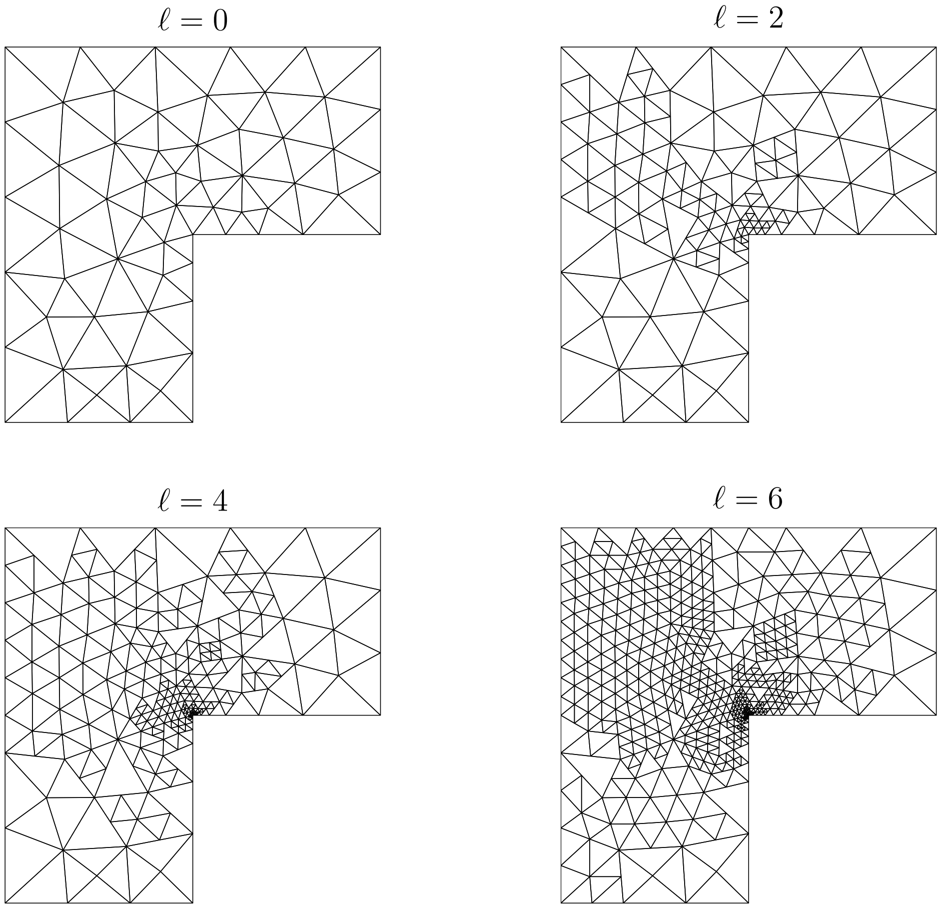


Figure 3.1: Adaptively refined triangulations $\mathcal{T}_\ell$ of the L-shaped domain in two dimensions.

Lagrange $q = 1, 2$ and hierarchical $q = 2$ for $n = 2, 3$. In a few numerical experiments we demonstrated that the methodology works in practice by solving a number of problems in two and three dimensions.

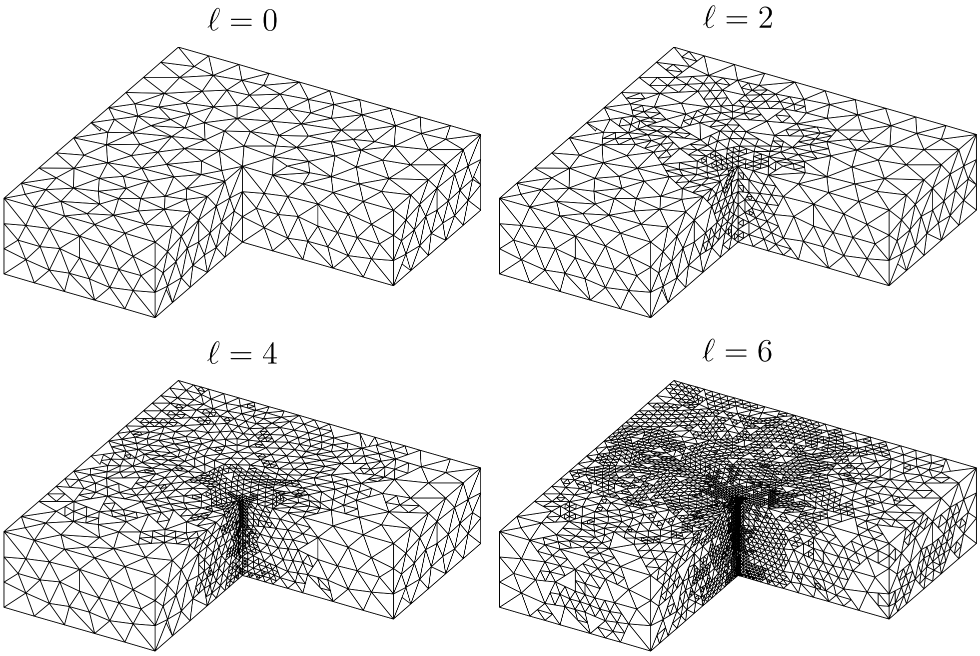


Figure 3.2: Adaptively refined triangulations $\mathcal{T}_\ell$ of the L-shaped domain in three dimensions.

Table 3.2: Convergence data for the V-cycle multigrid Algorithm 1 applied to Model Problem for $n = 3$ and the Lagrange finite element in Section 2.2.1.

Lagrange $(n, q) = (3, 1)$			
ℓ	M_1	m	Res
1	1382	2	$9.8 \cdot 10^{-9}$
2	3207	2	$1.2 \cdot 10^{-7}$
3	5378	2	$1.8 \cdot 10^{-7}$
4	11542	2	$5.2 \cdot 10^{-7}$
5	24185	2	$4.4 \cdot 10^{-7}$
6	46834	3	$2.7 \cdot 10^{-7}$
7	106711	3	$2.9 \cdot 10^{-7}$
8	225353	3	$2.3 \cdot 10^{-7}$

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Department of Mathematical Sciences, Chalmers University of Technology, SE-412 96 Göteborg, Sweden

E-mail address: erik.svensson@math.chalmers.se