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$H^1 - L^1$-BOUNDEDNESS OF FIRST ORDER RIESZ TRANSFORMS ON A LIE GROUP OF EXPONENTIAL GROWTH

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Abstract. Let G be the Lie group $\mathbb{R}^2 \ltimes \mathbb{R}^+$ endowed with the Riemannian symmetric space structure. Let X_0, X_1, X_2 be a distinguished basis of left-invariant vector fields of the Lie algebra of G and define the Laplacian $\Delta = -(X_0^2 + X_1^2 + X_2^2)$. In this paper we consider the first order Riesz transforms $R_i = X_i \Delta^{-1/2}$ and $S_i = \Delta^{-1/2} X_i$, for $i = 0, 1, 2$. We prove that the operators R_i, but not the S_i, are bounded from the Hardy space H^1 to L^1.

1. Introduction

Let G be the Lie group $\mathbb{R}^2 \ltimes \mathbb{R}^+$ where the product rule is the following:

$$(x_1, x_2, a) \cdot (x_1', x_2', a') = (x_1 + a\,x_1', x_2 + a\,x_2', a\,a') \qquad \forall (x_1, x_2, a),\ (x_1', x_2', a') \in G\,.$$

The group G is not unimodular; the right and left Haar measures are given by

$$\mathrm{d}\rho(x_1, x_2, a) = a^{-1}\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}a \qquad \text{and} \qquad \mathrm{d}\lambda(x_1, x_2, a) = a^{-3}\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}a\,,$$

respectively. The modular function is thus $\delta(x_1, x_2, a) = a^{-2}$. Throughout this paper, unless explicitly stated, we consider the right measure ρ on G and we denote by L^p and $\|\cdot\|_p$ and $\langle \cdot, \cdot \rangle$ the L^p-space, the L^p-norm and the L^2-scalar product with respect to the measure ρ.

The group G has a Riemannian symmetric space structure, and the corresponding metric, which we denote by d, is that of the three-dimensional hyperbolic half-space. The metric d is invariant under left translation and it is given by

$$\cosh r(x_1, x_2, a) = \frac{a + a^{-1} + a^{-1}(x_1^2 + x_2^2)}{2} \qquad \forall (x_1, x_2, a) \in G\,,$$

where $r(x_1, x_2, a) = d\big((x_1, x_2, a), (0, 0, 1)\big)$ denotes the distance of the point (x_1, x_2, a) from the identity of G. The measure of a hyperbolic ball B_r, centred at the identity and of radius

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r, behaves like

$$\lambda(B_r)=\rho(B_r)\asymp\begin{cases} r^3 & \text{if } r<1\\ \mathrm{e}^{2r} & \text{if } r\geq 1\,.\end{cases}$$

Thus G is a group of exponential growth. In this context, the classical Calderón–Zygmund theory and the classical definition of the atomic Hardy space H^1 (see [CW, St]) do not apply. Recently Hebisch and Steger [HS] constructed a new Calderón–Zygmund theory which holds in some spaces of exponential growth, in particular on the space (G,d,ρ) defined above. The main idea is to replace the family of balls which is used in the classical Calderón–Zygmund theory by a family of suitable parallelepipeds which we call *Calderón–Zygmund sets* and whose definition appears in [GS] and [HS].

Definition 1.1. *A* Calderón–Zygmund set *is a parallelepiped* $R=[b_1-L/2,b_1+L/2]\times[b_2-L/2,b_2+L/2]\times[a\mathrm{e}^{-r},a\mathrm{e}^{r}]$*, where the first two intervals are intervals in* $\mathbb{R}$ *of length* L*,* $a\in\mathbb{R}^+$*,* $r>0$ *and*

$$\mathrm{e}^2 a\,r\leq L<\mathrm{e}^8 a\,r \qquad \text{if } r<1\,,$$

$$a\,\mathrm{e}^{2r}\leq L<a\,\mathrm{e}^{8r} \qquad \text{if } r\geq 1\,.$$

Given a Calderón–Zygmund set R, we define its dilated set as $R^*=\{x\in S:\ d(x,R)<r\}$. There exists a constant C_0 such that $\rho(R^*)\leq C_0\,\rho(R)$ and $R\subset B\big((b_1,b_2,a),C_0r\big)$.

Let $\mathcal{R}$ denote the family of all Calderón–Zygmund sets. In [HS] it is proved that every integrable function on G admits a Calderón–Zygmund decomposition involving the family $\mathcal{R}$, and that a new Calderón–Zygmund theory can be developed in this context. By using the Calderón–Zygmund sets, it is natural to define an atomic Hardy space H^1 on the group G, as follows (see [V] for details).

We define an *atom* as a function a in L^1 such that

(i) a is supported in a Calderón–Zygmund set R;
(ii) $\|a\|_\infty\leq\rho(R)^{-1}$;
(iii) $\int a\,\mathrm{d}\rho=0$.

The atomic Hardy space is now defined in a standard way.

Definition 1.2. *The Hardy space* H^1 *is the space of all functions* f *in* L^1 *which can be written as* $f=\sum_j\lambda_j\,a_j$*, where* a_j *are atoms and* λ_j *are complex numbers such that* $\sum_j|\lambda_j|<\infty$*. We denote by* $\|f\|_{H^1}$ *the infimum of* $\sum_j|\lambda_j|$ *over such decompositions.*

The new Calderón–Zygmund theory introduced in [HS] is used to study the boundedness of some singular integral operators related to a distinguished Laplacian on G, which is defined as follows.

Let X_0, X_1, X_2 denote the left-invariant vector fields

$$X_0 = a\,\partial_a \qquad X_1 = a\,\partial_{x_1} \qquad X_2 = a\,\partial_{x_2}\,,$$

which span the Lie algebra of G. The Laplacian $\Delta = -(X_0^2 + X_1^2 + X_2^2)$ is a left-invariant operator which is essentially selfadjoint on $L^2(\rho)$. Since Δ is positive definite and one-to-one [GQS], its powers Δ^α, $\alpha \in \mathbb{R}$, have dense domains and are self-adjoint. This makes it possible to form the the Riesz transforms of the first order associated with Δ, defined by

$$R_i = X_i\,\Delta^{-1/2} \qquad \text{and} \qquad S_i = \Delta^{-1/2}\,X_i \qquad i = 0, 1, 2\,,$$

and the Riesz transforms of the second order, defined by

$$R_{ij} = X_iX_j\,\Delta^{-1} \qquad \text{and} \qquad S_{ij} = \Delta^{-1}\,X_iX_j \qquad \text{and} \qquad T_{ij} = X_i\Delta^{-1}X_j \qquad i, j = 0, 1, 2\,. \tag{1.1}$$

The boundedness properties of the Riesz transforms associated with the distinguished Laplacian Δ defined above have been considered by many authors. Actually some results ([GQS, GS2, S]) have been proved in the context of the affine group of the real line, which is not the group G. However, even if the setting is different, the results and the arguments may be reformulated and applied also to our context, with some slight changes.

For $i = 0, 1, 2$, the operators R_i are of weak type 1 and bounded on L^p when $1 < p \leq 2$. This result was obtained in [S] for the operator $X\Delta^{-1/2}$, where Δ is a distinguished Laplacian and X is a distinguished vector field, in the context of the affine group of the real line. Subsquently the result was proved in [HS, Theorem 6.4] in a more general setting including the group G, as an application of the Calderón–Zygmund theory.

The operators S_i are bounded on L^2, for $i = 0, 1, 2$. Moreover if $i = 1, 2$, then S_i is of weak type 1 and bounded on L^p when $1 < p \leq 2$. This result was proved in [GS2] in the context of the affine group of the real line and may be generalized to the group G. The operator S_0 is not of weak type 1 (W. Hebisch, private communication).

Since R_i and S_i are bounded on L^p, for $p < 2$, by duality it follows that R_i and S_i are also bounded on L^p when $2 < p < \infty$.

The Riesz transforms of the second order defined by (1.1) have been studied first in [GQS] in the context of the affine group of the real line, then in [GS1] in the general setting of NA groups of rank 1, which includes the group G. The operators T_{ij} are of weak type 1 and bounded on L^p, when $1 < p < \infty$. The operators R_{ij} and S_{ij} are not of weak type p, for any $1 \leq p < \infty$.

In this paper we study the $H^1 - L^1$ boundedness of the Riesz transforms on the group G. Our main results are the following:

(1) the operators R_i, $i = 0, 1, 2$, are bounded from H^1 to L^1 (Section 3);
(2) the operators S_i, $i = 0, 1, 2$, are not bounded from H^1 to L^1 (Sections 4, 5).

In a forthcoming paper, the authors will give analogous boundedness properties of the second-order operators defined in (1.1). It turns out that the operators T_{ij} are bounded from H^1 to L^1, but that the R_{ij} and S_{ij} are not. The proofs rely on a partition of the operators into local and global parts.

The problem of the boundedness of the Riesz transforms on the Hardy space H^1 have been studied on various Lie groups and Riemannian manifolds. Many results in the literature concern "doubling spaces", i.e., measured metric spaces where the volume of balls satisfies the doubling condition. In this context, the Hardy space H^1 is defined as in [CW].

In the classical setting of $\mathbb{R}^n$, the Riesz transforms are bounded from H^1 to H^1 [St, III.3].

For a nilpotent Lie group, N. Lohoué and N. Varopoulos [LV] proved that given left-invariant vector fields X_i, $i = 1, \ldots, k$, which generate the Lie algebra of the group and the sublaplacian $\Delta = -\sum_{i=1}^{k} X_i^2$, the Riesz transforms of the first order $R_i = X_i \Delta^{-1/2}$ are bounded from H^1 to H^1. Subsequently L. Saloff-Coste [SC] generalized this result to all connected Lie groups of polynomial growth.

On Riemannian manifolds with nonnegative Ricci curvature the Riesz transforms of the first order $\nabla \Delta^{-1/2}$, where Δ is the Laplace-Beltrami operator, are bounded from H^1 to L^1 [B, CL]. Subsequently E. Russ generalized the same results to all Riemannian manifolds satisfying the doubling condition and the Poincaré inequality [R].

The previous results do not apply to the space (G, d, ρ) since it is a space of exponential growth.

Our paper is organized as follows: in Section 2 we find explicit formulae for the kernels of the Riesz transforms of the first order. In Section 3 we prove the $H^1 - L^1$-boundedness of the operators R_i as a consequence of a more general boundedness theorem for integral operators . In Section 4 we prove that the operators S_1 and S_2 are not bounded from H^1 to L^1. In Section 5 we show the unboundeness from H^1 to L^1 of the operator S_0.

In the following, C denotes a positive, finite constant which may vary from line to line and may depend on parameters according to the context.

2. The convolution kernels of the Riesz transforms

In this section, we compute the convolution kernels of the Riesz transforms of the first order. First recall that the definition of the convolution of two functions f, g on G is

$$f * g(x) = \int_G f(xy^{-1})\, g(y)\, \mathrm{d}\rho(y) \qquad \forall x \in G\,.$$

Let V denote the space $\{\Delta u : \ u \in C_c^\infty(G)\}$. In [GS1] it is verified that V is a dense subspace of L^2 and that $V \subset D(\Delta^{-1}) \subset D(\Delta^{-1/2})$. We denote by U_α the convolution kernel of $\Delta^{-\alpha/2}$, in the sense that $\Delta^{-\alpha/2} f = f * U_\alpha$, for all $f \in V$. Since

$$\Delta^{-\alpha/2} = \frac{1}{\Gamma(\alpha/2)} \int_0^\infty t^{\alpha/2-1} \mathrm{e}^{-t\Delta}\, \mathrm{d}t\,,$$

we have that

$$U_\alpha = \frac{1}{\Gamma(\alpha/2)} \int_0^\infty t^{\alpha/2-1} p_t\, \mathrm{d}t\,,$$

where p_t denotes the heat kernel of Δ. It is well known [CGGM, Theorem 5.3, Proposition 5.4], [ADY, Formula (5.7)] that

$$p_t(x) = \frac{1}{8\pi^{3/2}}\, \delta^{1/2}(x)\, \frac{r(x)}{\sinh r(x)}\, t^{-3/2}\, \mathrm{e}^{-\frac{r^2(x)}{4t}} \qquad \forall x \in G\,,$$

where $r(x)$ denotes as before the distance of x from the identity. Hence,

$$\begin{aligned} U_\alpha(x) &= \frac{1}{\Gamma(\alpha/2)}\, \frac{1}{8\pi^{3/2}}\, \delta^{1/2}(x)\, \frac{r(x)}{\sinh r(x)} \int_0^\infty t^{\alpha/2-1} t^{-3/2}\, \mathrm{e}^{-\frac{r^2(x)}{4t}}\, \mathrm{d}t \\ &= \frac{1}{\Gamma(\alpha/2)}\, \frac{2^{1-\alpha}}{\pi^{3/2}}\, \delta^{1/2}(x)\, \frac{r(x)}{\sinh r(x)} \int_0^\infty r(x)^{\alpha-3}(x) v^{2-\alpha} \mathrm{e}^{-v^2}\, \mathrm{d}v \\ &= C_\alpha\, \delta^{1/2}(x)\, \frac{r^{\alpha-2}(x)}{\sinh r(x)} \qquad \forall x \in G\,, \end{aligned}$$

if $\alpha < 3$. When $\alpha = 1$ we get that $C_1 = \frac{1}{2\pi^2}$. We denote by $U = U_1$ the convolution kernel of $\Delta^{-1/2}$ given by

$$U(x) = \frac{1}{2\pi^2}\, \delta^{1/2}(x)\, \frac{1}{r(x)\, \sinh r(x)} \qquad \forall x \in G. \tag{2.1}$$

Since $R_i = X_i\, \Delta^{-1/2}$, we get for all $f \in V$ and $x \in G$

$$\begin{aligned} R_i f(x) = X_i(f * U)(x) &= \int X_{i,x} f(xy^{-1})\, U(y)\, \mathrm{d}\rho(y) \\ &= \lim_{\varepsilon \to 0} \int_{r(y) > \varepsilon} X_{i,x} f(xy^{-1})\, U(y)\, \mathrm{d}\rho(y) \\ &= -\lim_{\varepsilon \to 0} \int_{r(y) > \varepsilon} X_{i,y} f(xy^{-1})\, U(y)\, \mathrm{d}\rho(y) \end{aligned}$$

$$= \lim_{\varepsilon\to 0} \int_{r(y)>\varepsilon} f(xy^{-1})\, X_{i,y}U(y)\, \mathrm{d}\rho(y)\,, \tag{2.2}$$

where the last step follows by integration by parts, as in [S, Section 3]. Thus the convolution kernel of R_i is the distribution pv k_i, where $k_i = X_i U$. Moreover for $f \in V$ and $x \notin \mathrm{supp} f$

$$\begin{aligned} R_i f(x) &= \int_G f(xy)\, k_i(y^{-1})\, \mathrm{d}\lambda(y) \\ &= \int_G f(y)\, k_i(y^{-1}x)\, \mathrm{d}\lambda(y) \\ &= \int_G f(y)\, k_i(y^{-1}x)\, \delta(y)\, \mathrm{d}\rho(y) \\ &= \int_G f(y)\, R_i(x,y)\, \mathrm{d}\rho(y)\,, \end{aligned} \tag{2.3}$$

where $R_i(\cdot,\cdot)$ denotes the integral kernel of R_i, related to k_i by

$$R_i(x,y) = \delta(y)\, k_i(y^{-1}x) \qquad \forall x,\, y \in G, \qquad x \neq y\,. \tag{2.4}$$

We now consider the operators S_i. By arguing as in [GS2, page 246-247], it is easy to see that if $f \in C_c^\infty(G)$, then $X_i f \in D(\Delta^{-1/2})$, so that S_i is well defined on $C_c^\infty(G)$. Moreover for all $f \in C_c^\infty(G)$ and $g \in V$

$$\langle S_i f, g\rangle = \langle \Delta^{-1/2} X_i f, g\rangle = \langle X_i f, \Delta^{-1/2} g\rangle = -\langle f, X_i\, \Delta^{-1/2} g\rangle = -\langle f, R_i g\rangle\,.$$

Thus by (2.4) we deduce that the integral kernel of S_i is given by

$$S_i(x,y) = -\overline{R_i(y,x)} = -\delta(x)\, k_i(x^{-1}y) \qquad \forall x,\, y \in G, \qquad x \neq y\,. \tag{2.5}$$

We now compute k_i explicitly. To do so we shall need the following simple lemma.

Lemma 2.1. *At any point $(x_1, x_2, a) \neq (0,0,1)$ in G, the derivatives of r along the vector fields X_i are given by*

$$X_i r(x_1,x_2,a) = \begin{cases} \frac{a-a^{-1}-a^{-1}(x_1^2+x_2^2)}{2\,\sinh r(x_1,x_2,a)} = \frac{a-\cosh r}{\sinh r} & \text{if } i=0 \\ \frac{x_1}{\sinh r(x_1,x_2,a)} & \text{if } i=1,2\,. \end{cases}$$

Proof. It suffices to differentiate the expression

$$\cosh r(x_1,x_2,a) = \frac{a + a^{-1} + a^{-1}(x_1^2+x_2^2)}{2}\,, \tag{2.6}$$

with respect to X_i. For $X_0 = a\,\partial_a$ we obtain

$$\sinh r(x_1,x_2,a)\, X_0 r(x_1,x_2,a) = a\, \frac{1 - a^{-2} - a^{-2}(x_1^2+x_2^2)}{2}\,,$$

which gives the result for $i=0$. The cases of $X_i = a\,\partial_{x_i}$, $i=1,2$, are similar. $\square$

By (2.1) and Lemma 2.1 for $i = 1, 2$ and $(x_1, x_2, a) \neq (0, 0, 1)$, we get

$$\begin{aligned} k_i(x_1, x_2, a) &= X_i U(x_1, x_2, a) \\ &= -\frac{1}{2\,\pi^2}\, a^{-1}\, \frac{\sinh r + r\,\cosh r}{r^2\,\sinh^2 r}\, X_i r(x_1, x_2, a) \\ &= -\frac{1}{2\,\pi^2}\, a^{-1}\, x_i\, \frac{\sinh r + r\,\cosh r}{r^2\,\sinh^3 r}\,. \end{aligned} \tag{2.7}$$

For $i = 0$ and $(x_1, x_2, a) \neq (0, 0, 1)$ we get

$$\begin{aligned} k_0(x_1, x_2, a) &= X_0 U(x_1, x_2, a) \\ &= \frac{1}{2\,\pi^2}\Big[-\, a\, a^{-2}\, \frac{1}{r\,\sinh r} - \, a^{-1}\, \frac{\sinh r + r\,\cosh r}{r^2\,\sinh^2 r}\, X_0 r(x_1, x_2, a)\Big] \\ &= \frac{1}{2\,\pi^2}\Big[-\, a^{-1}\, \frac{1}{r\,\sinh r} - \, a^{-1}\, \frac{a - a^{-1} - a^{-1}(x_1^2 + x_2^2)}{2}\, \frac{\sinh r + r\,\cosh r}{r^2\,\sinh^3 r}\Big] \\ &= -U\big((x_1, x_2, a)\big) + \frac{1}{2\,\pi^2}\, \frac{-1 + a^{-2} + a^{-2}(x_1^2 + x_2^2)}{2}\, \frac{\sinh r + r\,\cosh r}{r^2\,\sinh^3 r}\,. \end{aligned} \tag{2.8}$$

3. $H^1 - L^1$-BOUNDEDNESS OF R_i

In this section we prove that the Riesz transforms R_i are bounded from H^1 to L^1, for $i = 0, 1, 2$.

The result is a consequence of the following boundedness theorem for integral operators. Note that the hypotheses of the following proposition are the same as those of [HS, Theorem 2.1].

Proposition 3.1. *Let T be a linear operator bounded on L^2 such that $T = \sum_{j\in\mathbb{Z}} T_j$, where*

(i) *the series converges in the strong operator topology of L^2;*
(ii) *every T_j is an integral operator with integral kernel T_j;*
(iii) *there exist positive constants a, A, ε and $c > 1$ such that*

$$\int_G |T_j(x, y)|\,\big(1 + c^j d(x, y)\big)^{\varepsilon}\, \mathrm{d}\rho(x) \leq A \qquad \forall y \in G; \tag{3.1}$$

$$\int_G |T_j(x, y) - T_j(x, z)|\, \mathrm{d}\rho(x) \leq A\, \big(c^j d(y, z)\big)^{a} \qquad \forall y, z \in G\,. \tag{3.2}$$

Then T is bounded from H^1 to L^1.

Proof. It is enough to show that there exists a constant C such that $\|Ta\|_1 \leq C$ for any atom a.

Let R be the support of the atom a centred at the point c_R. We estimate the integral of Ta on R^* by the Cauchy–Schwarz inequality:

$$\begin{aligned}\int_{R^*} |Ta| \, \mathrm{d}\rho &\leq \|Ta\|_2 \, \rho(R^*)^{1/2} \\ &\leq C \, \|T\|_{2,2} \, \|a\|_2 \, \rho(R)^{1/2} \\ &\leq C \, \|T\|_{2,2} \, . \end{aligned} \tag{3.3}$$

It is easy to show that from the estimates (3.1) and (3.2) it follows that

$$\sup_{R \in \mathcal{R}} \sup_{y, z \in R} \int_{(R^*)^c} |T(x,y) - T(x,z)| \, \mathrm{d}\rho(x) < \infty \, , \tag{3.4}$$

where T is the integral kernel of T. Thus the integral of Ta on the complementary set of R^* is estimated as follows:

$$\begin{aligned}\int_{R^{*c}} |Ta| \, \mathrm{d}\rho &\leq \int_{(R^*)^c} \Big| \int_R T(x,y) \, a(y) \, \mathrm{d}\rho(y) \Big| \, \mathrm{d}\rho(x) \\ &= \int_{(R^*)^c} \Big| \int_R [T(x,y) - T(x,c_R)] \, a(y) \, \mathrm{d}\rho(y) \Big| \, \mathrm{d}\rho(x) \\ &\leq \int_{(R^*)^c} \int_R |T(x,y) - T(x,c_R)| \, |a(y)| \, \mathrm{d}\rho(y) \, \mathrm{d}\rho(x) \\ &= \int_R |a(y)| \Big(\int_{(R^*)^c} |T(x,y) - T(x,c_R)| \, \mathrm{d}\rho(x) \Big) \, \mathrm{d}\rho(y) \\ &\leq \|a\|_1 \sup_{y \in R} \int_{(R^*)^c} |T(x,y) - T(x,c_R)| \, \mathrm{d}\rho(x) \\ &\leq C \, . \end{aligned}$$

This concludes the proof of the proposition. □

We now easily obtain the following theorem.

Theorem 3.2. *The Riesz transforms R_i, for $i = 0, 1, 2$, are bounded from H^1 to L^1.*

Proof. In the proof of [HS, Theorem 6.4], it is shown that the kernel of the operator R_i satisfies the estimates (3.1) and (3.2). Thus by Proposition 3.1, the operator R_i is bounded from H^1 to L^1. □

4. Unboundedness of S_1 and S_2

In this section we prove that the operators S_1 and S_2 are not bounded from H^1 to L^1. To do so, we shall define an atom a on G such that the images of a under the Riesz transforms S_i, $i = 1, 2$, are not integrable in a region far from the support of the atom (see Theorem 4.2). By symmetry it suffices to consider the case $i = 1$.

By differentiating the expression (2.7) of k_1 along the vector field X_2 and applying Lemma 2.1, we obtain that

$$\begin{aligned} X_2k_1(x_1,x_2,a) &= -\frac{1}{2\,\pi^2}\,a^{-1}\,x_1\,X_2r(x_1,x_2,a)\Big[\frac{r^2\,\sinh^3 r(2\cosh r + r\sinh r)}{r^4\,\sinh^6 r} - \\ &\quad - \frac{(\sinh r + r\cosh r)(2r\sinh^3 r + 3r^2\,\sinh^2 r\cosh r)}{r^4\,\sinh^6 r}\Big] \\ &= \frac{1}{2\,\pi^2}\,a^{-1}\,\frac{x_1\,x_2}{\sinh r}\,\frac{2\,r^2\,\cosh^2 r + r^2 + 2\sinh^2 r + 3r\,\sinh r\cosh r}{r^3\,\sinh^4 r}\,. \end{aligned} \tag{4.1}$$

We now define three regions Γ, Γ' and Γ'' of G where we shall estimate and integrate the derivative X_2k_1, for reasons which will become clear later on. Set

$$\begin{aligned} \Gamma'' &= \{(x_1,x_2,a)\in G:\ x_1/4 < x_2 < (1+\mathrm{e}^2)\,x_1,\ x_1 > a > 2\,\cosh 1\}\,, \\ \Gamma' &= \{(x_1,x_2,a)\in \Gamma'':\ x_2 < x_1\}\,, \\ \Gamma &= \{(x_1,x_2,a)\in \Gamma'':\ x_1/4 + a/2 < x_2 < x_1 - 5a/4,\ x_1 > 9a/2,\ a > 4\,\cosh 1\}\,. \end{aligned} \tag{4.2}$$

Obviously $\Gamma\subset\Gamma'\subset\Gamma''$.

Lemma 4.1. *There exist a positive continuous function Φ on Γ'' and a positive constant C such that*

(i) $X_2k_1 \geq C\,\Phi$ *in* Γ'';

(ii) *for any* (x_1,x_2,a) *in* Γ' *and* τ *in* $[0,\mathrm{e}^2]$, *the point* $(x_1,x_2,a)\cdot(0,\tau,1)$ *is in* Γ'' *and*

$$\Phi\big((x_1,x_2,a)\cdot(0,\tau,1)\big) \geq \Phi(x_1,x_2,a)\,;$$

(iii) $\int_\Gamma \Phi\,\mathrm{d}\rho = \infty$.

Let E be the parallelepiped $(-1/2,1/2)\times(-1/4,0)\times(1,2)$. Then

$$\Gamma\cdot E^{-1}\cdot E \subseteq \Gamma'\,. \tag{4.3}$$

Proof. For any (x_1,x_2,a) in Γ''

$$\cosh r(x_1,x_2,a) = \frac{a + a^{-1} + a^{-1}(x_1^2+x_2^2)}{2} > \frac{a^{-1}x_1^2}{2} > \frac{a^{-1}a^2}{2} > \cosh 1\,,$$

and

$$\cosh r(x_1,x_2,a) = \frac{a+a^{-1}+a^{-1}(x_1^2+x_2^2)}{2} < C\,a^{-1}\,x_1^2\,.$$

Thus for any (x_1,x_2,a) in Γ'' we have that $r(x_1,x_2,a)>1$ and, since $\mathrm{e}^r < 2\,\cosh r < C\,a^{-1}\,x_1^2$,

$$r(x_1,x_2,a) \leq C\,\log(a^{-1}\,x_1^2)\,.$$

By the formula (4.1) it is clear that X_2k_1 is positive on Γ'' and that for all (x_1,x_2,a) in Γ''

$$\begin{aligned} X_2k_1(x_1,x_2,a) &\geq C\,a^{-1}\,x_1\,x_2\,\frac{1}{\cosh r}\Big(\frac{1}{r\,\sinh^2 r}+\frac{1}{r\,\sinh^4 r}+\frac{1}{r^3\,\sinh^2 r}+\frac{1}{r^2\,\sinh^2 r}\Big)\\ &\geq C\,a^{-1}\,x_1\,x_2\,\frac{1}{r\,\cosh^3 r}\\ &\geq C\,\frac{a^{-1}\,x_1\,x_2}{\log(a^{-1}\,x_1^2)\,(a^{-1}\,x_1^2)^3}\,. \end{aligned}$$

We define $\Phi(x_1,x_2,a) = \frac{a^{-1}\,x_1\,x_2}{\log(a^{-1}\,x_1^2)\,(a^{-1}\,x_1^2)^3}$. The condition (i) is verified. Let (x_1,x_2,a) be a point in Γ' and τ in $[0,\mathrm{e}^2]$. Then $(x_1,x_2,a)\cdot(0,\tau,1)=(x_1,x_2+a\,\tau,a)$. Since (x_1,x_2,a) is in Γ', we have $x_1>a>2\cosh 1$ and $x_1/4<x_2<x_2+a\,\tau<x_1+a\mathrm{e}^2<(1+\mathrm{e}^2)\,x_1$, so that $(x_1,x_2,a)\cdot(0,\tau,1)$ is in Γ''. Moreover,

$$\Phi\big((x_1,x_2,a)\cdot(0,\tau,1)\big) = a^{-1}\,x_1\,(x_2+a\,\tau)\,\frac{1}{\log(a^{-1}\,x_1^2)\,(a^{-1}\,x_1^2)^3} \geq \Phi(x_1,x_2,a)\,,$$

as required in (ii). To prove (iii), we integrate Φ over Γ and obtain

$$\begin{aligned} \int_\Gamma \Phi\,\mathrm{d}\rho &= \int_{4\,\cosh 1}^\infty \int_{9a/2}^\infty \int_{x_1/4+a/2}^{x_1-5a/4} \frac{a^{-1}\,x_1\,x_2}{\log(a^{-1}\,x_1^2)\,(a^{-1}\,x_1^2)^3}\,\mathrm{d}x_2\,\mathrm{d}x_1\frac{\mathrm{d}a}{a}\\ &= C\int_{4\,\cosh 1}^\infty \int_{9a/2}^\infty \frac{x_1}{(a^{-1}\,x_1^2)^2\,\log(a^{-1}\,x_1^2)}\,\mathrm{d}x_1\frac{\mathrm{d}a}{a}\\ &= C\int_{4\,\cosh 1}^\infty \int_{81a/4}^\infty \frac{\mathrm{d}u}{u^2\,\log u}\,\mathrm{d}a\\ &\geq C\int_{4\,\cosh 1}^\infty \frac{1}{a\,\log a}\,\mathrm{d}a\\ &= \infty\,. \end{aligned}$$

Given $(x_1,x_2,a)\in\Gamma$ and (y_1,y_2,b), $(z_1,z_2,c)\in E$ we have that

$$(x_1,x_2,a)\cdot(y_1,y_2,b)^{-1}\cdot(z_1,z_2,c) = \big(x_1+ab^{-1}(z_1-y_1),x_2+ab^{-1}(z_2-y_2),ab^{-1}c\big)\,,$$

where $ab^{-1}c > 4\,\cosh 1/2 = 2\cosh 1$ and

$$x_1+ab^{-1}(z_1-y_1) > 9a/2 - ab^{-1}\cdot 2\cdot 1/2 > 7ab^{-1}/2 > 2ab^{-1} > ab^{-1}c\,.$$

Moreover

$$\begin{aligned}
x_2 + ab^{-1}(z_2 - y_2) &> x_1/4 + a/2 - ab^{-1}/4 \\
&= \frac{x_1 + ab^{-1}}{4} - ab^{-1}/4 + a/2 - ab^{-1}/4 \\
&> \frac{x_1 + ab^{-1}(z_1 - y_1)}{4} \,,
\end{aligned}$$

and

$$\begin{aligned}
x_2 + ab^{-1}(z_2 - y_2) &< x_1 - 5a/4 + ab^{-1}/4 \\
&= x_1 - 2ab^{-1}/2 + 2ab^{-1}/2 - 5a/4 + ab^{-1}/4 \\
&< x_1 + ab^{-1}(z_1 - y_1) \,.
\end{aligned}$$

Thus the point $(x_1, x_2, a) \cdot (y_1, y_2, b)^{-1} \cdot (z_1, z_2, c)$ is in Γ', proving (4.3). □

Theorem 4.2. *The operators S_1 and S_2 are not bounded from H^1 to L^1.*

Proof. By symmetry, it is enough to treat the case of S_1. We shall construct an atom a such that $S_1 a$ does not belong to L^1. Let R be the parallelepiped $\big(-\mathrm{e}^2 \log 2/2, \mathrm{e}^2 \log 2/2\big) \times \big(-\mathrm{e}^2 \log 2/2, \mathrm{e}^2 \log 2/2\big) \times (1/2, 2)$; it is easy to check that R is a Calderón–Zygmund set centred at the identity. Now let E be the parallelepiped defined in Lemma 4.1 and consider the right translate E^σ of E by the point $\exp(\sigma X_2) = (0, \sigma, 1)$ for some $\sigma > 0$, i.e.,

$$\begin{aligned}
E^\sigma = E \cdot (0, \sigma, 1) &= \{(y_1, y_2 + b\,\sigma, b) : \ (y_1, y_2, b) \in E\} \\
&\subset (-1/2, 1/2) \times \big(-1/4 + \sigma, 2\,\sigma\big) \times (1, 2) \,.
\end{aligned}$$

With $\sigma = 1/4$, E and E^σ are disjoint and contained in R.

Let us consider the function $a := \rho(R)^{-1} \big(\mathbf{1}_E - \mathbf{1}_{E^\sigma}\big)$. It is obvious that a is supported in the Calderón–Zygmund set R and $\|a\|_\infty \le \rho(R)^{-1}$. Moreover $\int a \, \mathrm{d}\rho = \rho(R)^{-1} \big(\rho(E) - \rho(E^\sigma)\big) = 0$. Thus a is an atom. We now compute $S_1 a$ outside the support of a. For all $x \notin \overline{E \cup E^\sigma}$

$$\begin{aligned}
S_1 a(x) &= \int S_1(x, y)\, a(y) \, \mathrm{d}\rho(y) \\
&= \rho(R)^{-1} \int_E S_1(x, y) \, \mathrm{d}\rho(y) - \rho(R)^{-1} \int_{E^\sigma} S_1(x, y) \, \mathrm{d}\rho(y) \,.
\end{aligned}$$

Changing variable $y = v \cdot (0, \sigma, 1)$ in the last integral, this transforms into

$$\rho(R)^{-1} \int_E S_1(x, y) \, \mathrm{d}\rho(y) - \rho(R)^{-1} \int_E S_1(x, v \cdot (0, \sigma, 1)) \, \mathrm{d}\rho(v)$$

$$= \rho(R)^{-1} \int_E \big[S_1(x,y) - S_1\big(x, y\cdot(0,\sigma,1)\big)\big]\,\mathrm{d}\rho(y)\,.$$

By (2.5) we know that

$$\begin{aligned} S_1(x,y) - S_1\big(x, y\cdot(0,\sigma,1)\big) &= \delta(x)\,\big(-k_1(x^{-1}y) + k_1(x^{-1}y\exp(\sigma\,X_2))\big) \\ &= \delta(x)\,\sigma\,\frac{d}{dt}\Big|_{t=\tau(x,y)} k_1\big(x^{-1}y\exp(t\,X_2)\big) \\ &= \delta(x)\,\sigma\,X_2k_1\big(x^{-1}y\exp(\tau(x,y)\,X_2)\big)\,, \end{aligned}$$

for some $\tau(x,y)$ in $(0,\sigma)$. It follows that for all $x\notin \overline{E\cup E^\sigma}$

$$S_1a(x) = \rho(R)^{-1}\,\sigma\,\delta(x)\int_E X_2k_1\big(x^{-1}y\exp(\tau(x,y)\,X_2)\big)\,\mathrm{d}\rho(y)\,. \tag{4.4}$$

To prove that S_1a is not in L^1, we integrate $|S_1a|$ in the region $E\,\Gamma^{-1}$, where Γ is defined by (4.2). It is easy to check that if $x\in E\,\Gamma^{-1}$, then $x\notin\overline{E\cup E^\sigma}$, so that we can apply (4.4) in the region $E\,\Gamma^{-1}$ and obtain

$$\begin{aligned} \int_{E\,\Gamma^{-1}} \big|S_1a(x)\big|\,\mathrm{d}\rho(x) &= \rho(R)^{-1}\,\sigma\int_{E\,\Gamma^{-1}}\delta(x)\,\Big|\int_E X_2k_1\big(x^{-1}y\exp(\tau(x,y)\,X_2)\big)\,\mathrm{d}\rho(y)\Big|\,\mathrm{d}\rho(x) \\ &= \rho(R)^{-1}\,\sigma\int_{\Gamma\,E^{-1}}\Big|\int_E X_2k_1\big(xy\exp(\tau(x,y)\,X_2)\big)\,\mathrm{d}\rho(y)\Big|\,\mathrm{d}\rho(x)\,. \end{aligned}$$

If $x\in\Gamma\,E^{-1}$ and $y\in E$, then $xy\in\Gamma'$, in view of (4.3). Since $0<\tau(x,y)<\sigma<\mathrm{e}^2$, by Lemma 4.1 the point $xy\exp(\tau(x,y)\,X_2)$ is in Γ'' and

$$X_2k_1\big(xy\exp(\tau(x,y)\,X_2)\big)\ge C\,\Phi(xy\exp(\tau(x,y)\,X_2))\ge C\,\Phi(xy)\,.$$

Hence, applying Fubini's theorem and using $w=xy$ instead of x, we get

$$\begin{aligned} \int_{E\,\Gamma^{-1}}\big|S_1a(x)\big|\,\mathrm{d}\rho(x) &\ge C\,\rho(R)^{-1}\,\sigma\int_{\Gamma\,E^{-1}}\int_E\Phi(xy)\,\mathrm{d}\rho(y)\,\mathrm{d}\rho(x) \\ &= C\,\rho(R)^{-1}\,\sigma\int_E\mathrm{d}\rho(y)\int_{\Gamma\,E^{-1}y}\Phi(w)\,\mathrm{d}\rho(w) \\ &\ge C\,\rho(R)^{-1}\,\sigma\int_E\mathrm{d}\rho(y)\int_\Gamma\Phi(w)\,\mathrm{d}\rho(w)\,. \end{aligned}$$

Lemma 4.1 (iii) implies that this integral diverges. □

5. Unboundedness of S_0

To prove that the operator S_0 is not bounded from H^1 to L^1, we follow the same idea as in the previous section. The only difference is that we consider the derivative of the kernel k_0 along the vector field X_0 in a different region of G.

We first compute the derivative of the expression (2.8) for k_0 along the vector field X_0:

$$
\begin{aligned}
X_0 k_0(x_1,x_2,a) = {} & \frac{1}{2\,\pi^2}\,\frac{a^{-1}}{r\,\sinh r} + \frac{1}{2\,\pi^2}\,\frac{1-a^{-2}-a^{-2}(x_1^2+x_2^2)}{2}\,\frac{\sinh r + r\,\cosh r}{r^2\,\sinh^3 r} + \\
& - \frac{1}{2\,\pi^2}\,\big[a^{-2}+a^{-2}(x_1^2+x_2^2)\big]\,\frac{\sinh r + r\,\cosh r}{r^2\,\sinh^3 r} + \\
& + \frac{1}{2\,\pi^2}\,\frac{-1+a^{-2}+a^{-2}(x_1^2+x_2^2)}{2}\,\frac{a-a^{-1}-a^{-1}(x_1^2+x_2^2)}{2\,\sinh r} \times \\
& \times \Big[\frac{(2\cosh r + r\,\sinh r)r^2\,\sinh^3 r}{r^4\sinh^6 r} - \\
& - \frac{(\sinh r + r\,\cosh r)(2r\,\sinh^3 r + 3\,r^2\,\sinh^2 r\,\cosh r)}{r^4\sinh^6 r}\Big] \\
= {} & \frac{1}{2\,\pi^2}\,\frac{a^{-1}}{r\,\sinh r} + \frac{1}{2\,\pi^2}\,\frac{1-3\,a^{-2}-3\,a^{-2}(x_1^2+x_2^2)}{2}\,\frac{\sinh r + r\,\cosh r}{r^2\,\sinh^3 r} + \\
& + \frac{1}{2\,\pi^2}\,a^{-1}\,\frac{\big[a-a^{-1}-a^{-1}(x_1^2+x_2^2)\big]^2}{4} \times \\
& \times \frac{2\,r^2\cosh^2 + r^2 + 2\,\sinh^2 r + 3\,r\,\sinh r\,\cosh r}{r^3\sinh^5 r}\,. \qquad (5.1)
\end{aligned}
$$

We shall estimate and integrate the function X_0k_0 in the regions

$$
\begin{aligned}
\Omega' &= \{(x_1,x_2,a)\in G:\ x_1^2+x_2^2 < a^2/4,\ a > 4\,\cosh 1\}\,, \\
\Omega &= \{(x_1,x_2,a)\in G:\ x_1^2+x_2^2 < a^2/64,\ a > 4\,\sqrt{2}\,\cosh 1\}\,. \qquad (5.2)
\end{aligned}
$$

Lemma 5.1. *There exist a positive continuous function Ψ on Ω' and a positive constant C such that*

(i) $X_0k_0 \geq C\,\Psi$ *in Ω';*

(ii) *for any (x_1,x_2,a) in Ω' and τ in $[0,1]$, the point $(x_1,x_2,a)\cdot(0,0,\mathrm{e}^\tau)$ is in Ω' and*

$$\Psi\big((x_1,x_2,a)\cdot(0,0,\mathrm{e}^\tau)\big) \geq C\,\Psi(x_1,x_2,a)\,;$$

(iii) $\int_\Omega \Psi\,\mathrm{d}\rho = \infty\,.$

Let F be the parallelepiped $(-1/16,1/16)\times(-1/16,1/16)\times(1,\sqrt{2})$. Then

$$\Omega\cdot F^{-1}\cdot F \subseteq \Omega'\,. \qquad (5.3)$$

Proof. Note that for all (x_1, x_2, a) in Ω'

$$\cosh 1 < \frac{a}{2} < \cosh r(x_1, x_2, a) < C\, a\,,$$

so that $r(x_1, x_2, a) > 1$ and, since $\mathrm{e}^r \le 2\cosh r \le C\, a$, we have $r \le C\, \log a$.

It is easy to show that in the region Ω' all the summands which appear in the last expression in (5.1) are positive and that for all (x_1, x_2, a) in Ω'

$$X_0 k_0(x_1, x_2, a) \ge C\, \frac{a^{-1}}{r\, \sinh r} \ge \frac{C}{a^2\, \log a}\,.$$

We define $\Psi(x_1, x_2, a) = \frac{1}{a^2\, \log a}$. The condition (i) is satisfied.

Let $(x_1, x_2, a) \in \Omega'$ and $\tau \in [0, 1]$. It is easy to check that the point $(x_1, x_2, a) \cdot (0, 0, \mathrm{e}^\tau) = (x_1, x_2, a\, \mathrm{e}^\tau)$ is in Ω'. Moreover,

$$\Psi\big((x_1, x_2, a) \cdot (0, 0, \mathrm{e}^\tau)\big) = \frac{1}{a^2\, \mathrm{e}^{2\tau}\, \log(a\, \mathrm{e}^\tau)} \ge C\, \frac{1}{a^2\, \log a} = C\, \Psi(x_1, x_2, a)\,,$$

as claimed in (ii). To prove (iii), we integrate Ψ over Ω and obtain

$$\begin{aligned}
\int_\Omega \Psi\, \mathrm{d}\rho &= \int_{4\sqrt{2}\, \cosh 1}^\infty \frac{1}{a^2 \log a} \int\!\!\int_{x_1^2 + x_2^2 \le a^2/64} \mathrm{d}x_1\, \mathrm{d}x_2 \frac{\mathrm{d}a}{a} \\
&= C \int_{4\sqrt{2}\, \cosh 1}^\infty \frac{1}{a^2 \log a}\, a^2 \frac{\mathrm{d}a}{a} \\
&= \infty\,.
\end{aligned}$$

Given $(x_1, x_2, a) \in \Omega$ and (y_1, y_2, b), $(z_1, z_2, c) \in F$ we have that $(x_1, x_2, a) \cdot (y_1, y_2, b)^{-1} \cdot (z_1, z_2, c) = \big(x_1 + ab^{-1}(z_1 - y_1), x_2 + ab^{-1}(z_2 - y_2), ab^{-1}c\big)$, where $ab^{-1}c > 4\cosh 1$ and

$$\begin{aligned}
&\big[x_1 + ab^{-1}(z_1 - y_1)\big]^2 + \big[x_2 + ab^{-1}(z_2 - y_2)\big]^2 \\
&< x_1^2 + x_2^2 + a^2 b^{-2}(z_1 - y_1)^2 + a^2 b^{-2}(z_2 - y_2)^2 + 2ab^{-1}x_1(z_1 - y_1) + 2ab^{-1}x_2(z_2 - y_2) \\
&< a^2/64 + a^2(1/8)^2 + a^2(1/8)^2 + 2a|x_1|/8 + 2a|x_2|/8 \\
&< a^2(1/64 + 1/32 + 1/16) \\
&< a^2/8 \\
&< (ab^{-1}c)^2/4\,.
\end{aligned}$$

Thus $(x_1, x_2, a) \cdot (y_1, y_2, b)^{-1} \cdot (z_1, z_2, c) \in \Omega'$ and (5.3) is proved. □

Theorem 5.2. *The operator S_0 is not bounded from H^1 to L^1.*

Proof. Following closely the proof of Theorem 4.2, we shall construct an atom a such that S_0a does not belong to L^1. With R as in the proof of Theorem 4.2, we let F be the parallelepiped defined in Lemma 5.1 and consider the right translate F^σ of F by the point $\exp(\sigma X_0) = (0,0,\mathrm{e}^\sigma)$, i.e.,

$$\begin{aligned} F^\sigma &= F \cdot (0,0,\mathrm{e}^\sigma) = \{(y_1,y_2,a\mathrm{e}^\sigma):\ (y_1,y_2,b) \in F\} \\ &= \big(-1/16, 1/16\big) \times \big(-1/16,1/16\big) \times (\mathrm{e}^\sigma, \mathrm{e}^\sigma\sqrt{2}). \end{aligned}$$

With $\sigma = (\log 2)/2$, F and F^σ are disjoint and contained in R.

Let us consider the atom $a := \rho(R)^{-1}\big(\mathbf{1}_F - \mathbf{1}_{F^\sigma}\big)$. We compute S_0a outside the support of a. For all $x \notin \overline{F \cup F^\sigma}$

$$\begin{aligned} S_0a(x) &= \int S_0(x,y)\, a(y)\, \mathrm{d}\rho(y) \\ &= \rho(R)^{-1} \int_F S_0(x,y)\, \mathrm{d}\rho(y) - \rho(R)^{-1} \int_{F^\sigma} S_0(x,y)\, \mathrm{d}\rho(y), \end{aligned}$$

which, by the change of variable $y = v \cdot (0,0,\mathrm{e}^\sigma)$ in the last integral, transforms into

$$= \rho(R)^{-1} \int_F \big[S_0(x,y) - S_0\big(x, y\cdot(0,0,\mathrm{e}^\sigma)\big)\big]\, \mathrm{d}\rho(y).$$

By (2.5) we know that

$$\begin{aligned} S_0(x,y) - S_0\big(x, y \cdot (0,0,\mathrm{e}^\sigma)\big) &= \delta(x)\, \big(-k_0(x^{-1}y) + k_0(x^{-1}y\exp(\sigma X_0))\big) \\ &= \delta(x)\, \sigma \frac{d}{dt}\Big|_{t=\tau(x,y)} k_0\big(x^{-1}y \exp(t X_0)\big) \\ &= \delta(x)\, \sigma\, X_0 k_0\big(x^{-1}y \exp(\tau(x,y)\, X_0)\big), \end{aligned}$$

for some $\tau(x,y)$ in $(0,\sigma)$. It follows that for all $x \notin \overline{F \cup F^\sigma}$

$$S_0a(x) = \rho(R)^{-1}\, \sigma\, \delta(x) \int_F X_0 k_0\big(x^{-1}y\exp(\tau(x,y)\, X_0)\big)\, \mathrm{d}\rho(y). \tag{5.4}$$

To prove that S_0a is not in L^1, we integrate S_0a in the region $F\,\Omega^{-1}$. It is easy to verify that if $x \in F\,\Omega^{-1}$, then $x \notin \overline{F \cup F^\sigma}$, so that we can apply (5.4) and obtain

$$\begin{aligned} \int_{F\,\Omega^{-1}} \big|S_0a(x)\big|\, \mathrm{d}\rho(x) &= \rho(R)^{-1}\, \sigma \int_{F\,\Omega^{-1}} \delta(x) \Big| \int_F X_0 k_0\big(x^{-1}y \exp(\tau(x,y)\, X_0)\big)\, \mathrm{d}\rho(y)\Big|\, \mathrm{d}\rho(x) \\ &= \rho(R)^{-1}\, \sigma \int_{\Omega\, F^{-1}} \Big| \int_F X_0 k_0\big(xy \exp(\tau(x,y)\, X_0)\big)\, \mathrm{d}\rho(y)\Big|\, \mathrm{d}\rho(x). \end{aligned}$$

If $x \in \Omega\, F^{-1}$ and $y \in F$, then $xy \in \Omega'$, in view of (5.3). Since $0 < \tau(x,y) < \sigma < 1$, by Lemma 5.1(ii) the point $xy \exp(\tau(x,y)\, X_0)$ is in Ω' and

$$X_0 k_0\big(xy \exp(\tau(x,y)\, X_0)\big) \geq C\, \Psi(xy \exp(\tau(x,y)\, X_0)) \geq C\, \Psi(xy)\,.$$

As in the proof of Theorem 4.2, we get

$$\begin{aligned}\int_{F\,\Omega^{-1}} \big|S_0 a(x)\big|\, \mathrm{d}\rho(x) &\geq C\, \rho(R)^{-1}\, \sigma \int_{\Omega\, F^{-1}} \int_F \Psi(xy)\, \mathrm{d}\rho(y)\, \mathrm{d}\rho(x) \\ &= C\, \rho(R)^{-1}\, \sigma \int_F \mathrm{d}\rho(y) \int_{\Omega\, F^{-1} y} \Psi(w)\, \mathrm{d}\rho(w) \\ &\geq C\, \rho(R)^{-1}\, \sigma \int_F \mathrm{d}\rho(y) \int_{\Omega} \Psi(w)\, \mathrm{d}\rho(w)\,.\end{aligned}$$

Lemma 5.1 (iii) implies that the last integral diverges. □

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