CHALMERS | GÖTEBORG UNIVERSITY

PREPRINT 2007:26

The Formation of Black Holes in Spherically Symmetric Gravitational Collapse

HÅKAN ANDRÉASSON
MARKUS KUNZE
GERHARD REIN

Department of Mathematical Sciences
Division of Mathematics
CHALMERS UNIVERSITY OF TECHNOLOGY
GÖTEBORG UNIVERSITY
Göteborg Sweden 2007

Preprint 2007:26

The Formation of Black Holes in Spherically Symmetric Gravitational Collapse

Håkan Andréasson, Markus Kunze, Gerhard Rein

Department of Mathematical Sciences
Division of Mathematics
Chalmers University of Technology and Göteborg University
SE-412 96 Göteborg, Sweden
Göteborg, September 2007

Preprint 2007:26
ISSN 1652-9715

Matematiska vetenskaper
Göteborg 2007

The formation of black holes in spherically symmetric gravitational collapse

Håkan Andréasson
Mathematical Sciences
Chalmers University of Technology
Göteborg University
S-41296 Göteborg, Sweden
email: hand@math.chalmers.se

Markus Kunze
Fachbereich Mathematik
Universität Duisburg-Essen
D-45117 Essen, Germany
email: markus.kunze@uni-due.de

Gerhard Rein
Fakultät für Mathematik und Physik
Universität Bayreuth
D-95440 Bayreuth, Germany
email: gerhard.rein@uni-bayreuth.de

June 26, 2007

Abstract

We consider the spherically symmetric, asymptotically flat Einstein equations coupled to a suitable matter model and find explicit conditions on the initial data which guarantee the formation of a black hole in the evolution. We establish such results for general matter models characterized by general conditions on the matter quantities, and we prove that the collisionless gas as described by the Vlasov equation satisfies these conditions for a large class of initial data.

1 Introduction

One of the many striking predictions of General Relativity is the assertion that under appropriate conditions astrophysical objects like stars or galaxies undergo a gravitational collapse resulting in a spacetime singularity. This was first proven by Oppenheimer and Snyder [18] who constructed a semi-explicit example of a homogeneous spherically symmetric ball of dust, i.e., of a pressure-less fluid, which under its self-consistent, general relativistic gravitational interaction collapses. During this collapse the scalar curvature of spacetime blows up at the centre of symmetry, and the geometry of spacetime breaks down there. This is referred to as the formation of a spacetime singularity. An important feature of the Oppenheimer-Snyder solution is that during the collapse a two-dimensional spacelike sphere evolves which encloses the singularity and through which no causal curve, i.e., no light ray or particle trajectory, can pass outward. In this way the spacetime singularity is isolated from the outside part of spacetime by a so-called event horizon, and the singularity cannot be seen or in any other way be experienced by observers outside the event horizon. This configuration was later termed a black hole.

In the 1960s Penrose [19] proved that the formation of spacetime singularities from regular initial data is not restricted to spherically symmetric, especially constructed or isolated examples, but it is a genuine, stable feature of spacetimes. However, this result gives little information about the geometric structure of a spacetime with such a singularity. In particular, it is in general not known if every spacetime singularity which arises from the gravitational collapse of regular initial data is covered by an event horizon. Since the existence of so-called naked singularities (for which, by definition, the latter is not true) would violate predictability (it would not be possible to predict from the initial data what an observer would see if he could observe a singularity), the cosmic censorship conjecture was formulated which demands that any singularity which arises from the gravitational collapse of *generic* regular initial data is indeed hidden behind an event horizon. The restriction to generic data means that naked singularities are allowed to occur for a "null set" of the initial data. An important example where naked singularities do form for a null set, but for which cosmic censorship holds true, is the spherically symmetric Einstein-scalar field system, cf. [9, 10]. Actually the above is an informal statement of the so-called weak cosmic censorship conjecture [30, 12.1]; we will not be concerned with the strong version in the present paper. For a mathematical discussion and the definition of the weak cosmic censorship conjecture we refer to [11].

To deal with this conjecture in full generality is out of reach of the present level of mathematics, but under the assumption of spherical symmetry progress has been made in recent years. One important outcome of these investigations is that the answer is sensitive to which model is chosen to describe the matter. Christodoulou [6] showed that for dust, i.e., the matter model used by Oppenheimer and Snyder, cosmic censorship is violated. On the other hand, in a series of papers Christodoulou investigated a massless scalar field as matter model and showed in 1999 that weak and strong cosmic censorship hold true for this matter model; see [10] and the references therein.

In the present investigation the main example considered as a matter model is the so-called collisionless gas as described by the Vlasov equation. It is used extensively in astrophysics, cf. [5], to describe galaxies or globular clusters which are viewed as large ensembles of mass points which interact only through the gravitational field that the ensemble creates collectively. In a relativistic context this leads to the Einstein-Vlasov system. All results available for this system support the following

Conjecture: *Weak cosmic censorship holds for the Einstein-Vlasov system.*

We mention explicitly that, in contrast to dust, small, spherically symmetric initial data launch global solutions, i.e., the solutions are geodesically complete and hence satisfy cosmic censorship, cf. [23]. Also, the numerical simulations [4, 17, 26] which treat large initial data support the hypothesis that naked singularities do not form in the evolution. We point out a further interesting feature of Vlasov matter observed in these numerical studies: In a one-parameter family of solutions which for large parameters, i.e., large amplitudes of the initial data, collapse to a black hole the smallest black hole always has a strictly positive ADM mass, i.e., there is a mass gap. This contrasts several other matter models for which no mass gap is found, cf. [14] for a review.

The aim of the present paper is to find explicit conditions on the initial data which ensure the formation of black holes. This class of initial data has the important property that, except for "boundary cases", properly restricted small perturbations of the data lead to solutions with the same properties. In this sense the established behavior of the solutions is stable and not restricted to especially constructed solutions or initial data, respectively. It turns out that considerable parts of our argument can be formulated for a general matter model which satisfies certain specific assumptions, and in order to give a broader impact to our results we shall do so. At the same time we emphasize that the Vlasov matter model is the only one which is presently known to actually satisfy all the assumptions needed for our arguments to go through.

One aspect of our result is that there is a set of initial data which leads to gravitational collapse such that weak cosmic censorship holds. This point should be related to an earlier result by Rendall [28], where it is shown that there exists a set of initial data for the spherically symmetric Einstein-Vlasov system such that a trapped surface forms in the evolution. The occurrence of a trapped surface signals the formation of an event horizon. Indeed, Dafermos [12] has proved that if a spherically symmetric spacetime contains a trapped surface and the matter model satisfies certain hypotheses then weak cosmic censorship holds true. In [13] it was then shown that Vlasov matter does satisfy the required hypotheses. Hence, by combining these results, one obtains a set of initial data which lead to gravitational collapse and for which weak cosmic censorship holds. However, the construction in [28] rests on a continuity argument, and it is not possible to tell whether or not a given initial data set will give rise to a black hole. This is in contrast to the explicit conditions that we obtain in the present work. In this regard it is very natural to relate our results to those of Christodoulou on the spherically symmetric Einstein-scalar field system [8]. There explicit conditions on the initial data are specified which guarantee the formation of trapped surfaces. This paper played a crucial role in his proof [10] of the weak and strong cosmic censorship conjectures mentioned above. In [11] Christodoulou gave a historical review of his search for a proof of cosmic censorship, and we quote: "The work which opened up the path to the settlement of the cosmic censorship conjectures within the framework of the spherical symmetric scalar field model was [8]." We hope that the results in the present paper will lead to similar progress on the weak cosmic censorship conjecture in the case of Vlasov matter. A few more comments on the relationship between our results and the results in [8] are given before the statements of the main theorems in Section 2.

The Vlasov matter model has a further promising property as compared to other matter models. For the Vlasov-Poisson system, which arises as the Newtonian limit of the Einstein-Vlasov system in a rigorous sense [24, 27], and which is used extensively in astrophysics, there is a global existence and uniqueness result for general, smooth initial data [16, 20]. This means in particular that any breakdown of a solution of the Einstein-Vlasov system can be expected to be a genuine, general relativistic effect such as a spacetime singularity, but not a remainder of some bad behavior which the matter model exhibits already on the Newtonian level.

To be more specific, consider now a smooth spacetime manifold M

equipped with a spacetime metric $g_{\alpha\beta}$; Greek indices run from 0 to 3. Then the Einstein equations read

$$G_{\alpha\beta} = 8\pi T_{\alpha\beta}, \tag{1.1}$$

where $G_{\alpha\beta}$ is the Einstein tensor, a non-linear second order differential expression in the metric $g_{\alpha\beta}$, and $T_{\alpha\beta}$ is the energy-momentum tensor given by the matter content (or other fields) of the spacetime. To obtain a closed system, the field equations (1.1) have to be supplemented by

$$\text{evolution equation(s) for the matter} \tag{1.2}$$

and

$$\text{the definition of } T_{\alpha\beta} \text{ in terms of the matter and the metric.} \tag{1.3}$$

It is often possible to specify conditions on (1.2) and (1.3) under which one can establish geometric properties of a spacetime described by the Einstein-matter system (1.1), (1.2), (1.3). The Penrose singularity theorem mentioned above is of this nature, and part of our arguments will also be presented in this form.

However, in order to verify such general conditions, in particular with respect to the existence of local or global solutions to the corresponding initial value problem, a specific matter model must be chosen, and in the present paper this is a collisionless gas. All the particles in the gas are assumed to have the same rest mass, normalized to unity, and to move forward in time. Hence, their number density f is a non-negative function supported on the mass shell

$$PM := \left\{ g_{\alpha\beta} p^\alpha p^\beta = -1,\ p^0 > 0 \right\},$$

a submanifold of the tangent bundle TM of the spacetime manifold M; p^α are the canonical momenta corresponding to general coordinates $x^\alpha = (t, x^a)$ on M. We use coordinates (t, x^a) with zero shift, and Latin indices run from 1 to 3. On the mass shell PM the variable p^0 becomes a function of the remaining variables (t, x^a, p^b) as

$$p^0 = \sqrt{-g^{00}} \sqrt{1 + g_{ab} p^a p^b}.$$

The number density $f = f(t, x^a, p^b)$ satisfies a continuity equation, the so-called Vlasov equation, which says that f is constant along the geodesics of the spacetime metric,

$$\partial_t f + \frac{p^a}{p^0} \partial_{x^a} f - \frac{1}{p^0} \Gamma^a_{\beta\gamma} p^\beta p^\gamma \partial_{p^a} f = 0, \tag{1.4}$$

where $\Gamma^\alpha_{\beta\gamma}$ are the Christoffel symbols induced by the metric $g_{\alpha\beta}$. The energy-momentum tensor is given by

$$T_{\alpha\beta} = \int p_\alpha p_\beta f \, |g|^{1/2} \, \frac{dp^1 dp^2 dp^3}{-p_0}, \tag{1.5}$$

where $|g|$ denotes the determinant of the metric. The system (1.1), (1.4), (1.5) is the Einstein-Vlasov system in general coordinates. For an introduction to relativistic kinetic theory and the Einstein-Vlasov system we refer to [1] and [29].

If, for comparison, the matter is to be described as a perfect fluid with density $\mathcal{R}$, four-velocity field U^α, and pressure P, then the matter evolution equations are the Euler equations

$$U^\alpha \nabla_\alpha \mathcal{R} + (\mathcal{R} + P) \nabla^\alpha U_\alpha = 0,$$

$$(\mathcal{R} + P) U^\alpha \nabla_\alpha U_\beta + (g_{\alpha\beta} + U_\alpha U_\beta) \nabla^\alpha P = 0,$$

where ∇_α is the covariant derivative corresponding to the metric $g_{\alpha\beta}$. The energy-momentum tensor in this case is

$$T_{\alpha\beta} = \mathcal{R} U_\alpha U_\beta + P(g_{\alpha\beta} + U_\alpha U_\beta).$$

To close the Einstein-Euler system it has to be supplemented by an equation of state $P = P(\mathcal{R})$. The choice $P = 0$ yields the dust matter model referred to above.

Due to the complexity of the field equations (1.1) very little can be said about the questions at hand for these equations in their general form. Since on the other hand these questions are of considerable interest also in spacetimes satisfying simplifying symmetry assumptions, we from now on focus on asymptotically flat, spherically symmetric spacetimes and write down the metric

$$ds^2 = -e^{2\mu(t,r)} dt^2 + e^{2\lambda(t,r)} dr^2 + r^2 (d\theta^2 + \sin^2\theta \, d\varphi^2)$$

in Schwarzschild coordinates. Here $t \in \mathbb{R}$ is the time coordinate, $r \in [0, \infty[$ is the area radius, i.e., $4\pi r^2$ is the area of the orbit of the symmetry group SO(3) labelled by r, and the angles $\theta \in [0, \pi]$ and $\varphi \in [0, 2\pi]$ parameterize these orbits. Asymptotic flatness means that the metric quantities λ and μ have to satisfy the boundary conditions

$$\lim_{r\to\infty} \lambda(t, r) = \lim_{r\to\infty} \mu(t, r) = 0. \tag{1.6}$$

For a metric of this form the 00, 11, and 01 components of the Einstein equations are found to be

$$e^{-2\lambda}(2r\lambda_r - 1) + 1 = 8\pi r^2 e^{-2\mu} T_{00}, \tag{1.7}$$

$$e^{-2\lambda}(2r\mu_r + 1) - 1 = 8\pi r^2 e^{-2\lambda} T_{11}, \tag{1.8}$$

$$\lambda_t = 4\pi r T_{01}, \tag{1.9}$$

where subscripts indicate partial derivatives. The 22 and 33 components are also nontrivial, but they are not needed for our analysis, and the remaining components vanish identically due to the symmetry assumption.

Our aim is to find explicit conditions on the initial data such that the corresponding solutions of the spherically symmetric, asymptotically flat version of the system (1.1), (1.2), (1.3) have the following property: There is an outgoing radial null geodesic γ^+ originating from $r = r_0 > 0$, i.e., γ^+ is the solution of

$$\frac{d\gamma^+(s)}{ds} = e^{(\mu-\lambda)(s,\gamma^+(s))},\ \gamma^+(0) = r_0, \tag{1.10}$$

such that the solution exists on the outer region

$$D := \{(t,r) \in [0,\infty[^2 \mid r \geq \gamma^+(t)\}, \tag{1.11}$$

and has the properties that

$$\lim_{s\to\infty} \gamma^+(s) < \infty. \tag{1.12}$$

Furthermore, as $t \to \infty$ there remains matter in the outer region D. Thus the matter distribution undergoes a gravitational collapse and a black hole forms.

In the next section we state our main results for the Einstein-Vlasov system, where we specify classes of spherically symmetric initial data which lead to solutions showing the above behavior. The Vlasov equation and the corresponding energy-momentum tensor components in the case of spherical symmetry are stated there. In Section 3 we give a general formulation of one of our results where no particular matter model is considered. The reason for this is that most steps in the proof of Theorem 2.2 below are of a general character and—besides the fact that for the Einstein-Vlasov system there is an existence theory for the initial value problem which guarantees the existence of solutions on D—the specific properties of Vlasov matter are used only in one key lemma. Hence it is natural to precisely single out the required conditions on the level of the macroscopic matter quantities. This clarifies the main mechanism in our method, and it may lead to applications of our method to other matter models. Using an additional feature of Vlasov matter we construct an alternative, and in some respects larger, class of initial data which ensure the formation of black holes, cf. Theorem 2.1.

The proofs of our results then proceed as follows. After stating some general auxiliary results in Section 4 we prove Theorem 3.1, which is the general-matter version of Theorem 2.2, in Section 5. The latter result is then established in Section 6 by showing that Vlasov matter satisfies the required general conditions on the matter for a suitable class of initial data. Theorem 2.1 is established in Section 7. For all these results it is essential to make sure that in the outer region D all the matter moves inward. In the case of general matter this is a condition which we have to impose on the solution, whereas in the case of Vlasov matter we can specify conditions on the initial data such that this is true. In the results discussed so far the solutions have the property that all the matter which is initially in the outer region D remains there for all future coordinate time t, i.e., no matter is swallowed by the $r = \gamma^+(t)$ surface. In a final section we show that in the case of Vlasov matter initial data do exist where a small piece of the matter originally outside $r = \gamma^+(t)$ is indeed swallowed.

2 Main results for Vlasov matter

In this section Eqns. (1.6)–(1.9) will be supplemented by the spherically symmetric version of the Vlasov equation together with expressions for the relevant components of the energy-momentum tensor so that a closed system is obtained, known as the spherically symmetric, asymptotically flat Einstein-Vlasov system.

In order to exploit the symmetry it is useful to introduce non-canonical variables on momentum space and write $f = f(t, r, w, L)$. For a detailed derivation of the corresponding equations we refer to [22]; here we just state the result. The Vlasov equation is

$$\partial_t f + e^{\mu-\lambda}\frac{w}{E}\partial_r f - \left(\lambda_t w + e^{\mu-\lambda}\mu_r E - e^{\mu-\lambda}\frac{L}{r^3 E}\right)\partial_w f = 0, \tag{2.1}$$

where

$$E = E(r, w, L) := \sqrt{1 + w^2 + L/r^2} = e^{\mu} p^0.$$

The variables $w \in]-\infty, \infty[$ and $L \in [0, \infty[$ can be thought of as the radial component of the momentum and the square of the angular momentum respectively. Notice that the latter is conserved along characteristics of the Vlasov equation. The matter quantities are given by

$$\rho(t,r) = e^{-2\mu} T_{00}(t,r) = \frac{\pi}{r^2} \int_{-\infty}^{\infty} \int_0^{\infty} E f(t,r,w,L)\, dL\, dw, \tag{2.2}$$

$$p(t,r) = e^{-2\lambda} T_{11}(t,r) = \frac{\pi}{r^2} \int_{-\infty}^{\infty} \int_0^{\infty} \frac{w^2}{E} f(t,r,w,L)\, dL\, dw, \tag{2.3}$$

$$j(t,r) = -e^{-(\lambda+\mu)} T_{01}(t,r) = \frac{\pi}{r^2} \int_{-\infty}^{\infty} \int_0^{\infty} w\, f(t,r,w,L)\, dL\, dw. \tag{2.4}$$

Notice that the quantities ρ, p, j appear on the right hand sides of the field equations (1.7)–(1.9), and they are given in terms of f alone, which is the main reason for using the non-canonical variables w and L. The system (1.6)–(1.9), (2.1)–(2.4) is the spherically symmetric Einstein-Vlasov system in Schwarzschild coordinates. As initial data we need to prescribe an initial distribution function $\mathring{f} = \mathring{f}(r,w,L) \geq 0$, which should be compactly supported in $]0,\infty[\times]-\infty,\infty[\times]0,\infty[$, and such that

$$\int_0^r 4\pi\eta^2 \mathring{\rho}(\eta)\, d\eta = 4\pi^2 \int_0^r \int_{-\infty}^{\infty} \int_0^{\infty} E \mathring{f}(\eta,w,L)\, dL\, dw\, d\eta < \frac{r}{2}. \tag{2.5}$$

The origin $r=0$ is excluded from the support for technical reasons, but this could be avoided by using Cartesian coordinates. The condition (2.5) implies that the equations (1.7) and (1.8) have solutions λ and μ, cf. Section 4, and since $\mathring{f}$ has compact support, a property which is inherited by $f(t)$, the matter terms are well defined. If in addition $\mathring{f}$ is C^1 we say that the initial data is *regular*. As is shown in [23] or [22], regular initial data launch a unique local solution for which all the derivatives which appear in the system exist classically. In Section 6 we discuss in more detail that this local solution extends to the whole outer region D defined in (1.11).

To state our main results let $0 < r_0 < r_1$ be given, put $M = r_1/2$ (this is going to be the ADM mass of the solution), and fix $0 < M_{\rm out} < M$ such that

$$\frac{2(M - M_{\rm out})}{r_0} < \frac{8}{9}. \tag{2.6}$$

Remark. The value $8/9$ is chosen for definiteness, and any number less than one would do, effecting the values of some of the constants below.

Two different theorems will be stated below, corresponding to the following two situations.

(i) Let $R_1 > r_1$ be such that

$$R_1 - r_1 < \frac{r_1 - r_0}{6}, \tag{2.7}$$

or

(ii) let $R_1 > r_1$ be such that

$$\sqrt{\frac{R_1 - r_1}{R_1}} < \min\left\{\frac{1}{6}, \frac{r_0^2}{12\kappa R_1 M}, \frac{r_1 - r_0}{8\kappa R_1}\right\}, \tag{2.8}$$

where the (explicit) constant $\kappa > 0$ will be specified in Theorems 2.2 and 3.1 below.

Finally, we define

$$R_0 := \frac{1}{2}(r_1 + R_1).$$

Denote by $\mathring{\rho}$ the energy density induced by the initial distribution function $\mathring{f}$. We require that all the matter in the outer region $[r_0, \infty[$ is initially located in the strip $[R_0, R_1]$, with M_{out} being the corresponding fraction of the ADM mass M, i.e.,

$$\int_{r_0}^{\infty} 4\pi r^2 \mathring{\rho}(r) dr = \int_{R_0}^{R_1} 4\pi r^2 \mathring{\rho}(r) dr = M_{\text{out}}. \tag{2.9}$$

Furthermore, the remaining fraction $M - M_{\text{out}}$ should be initially located within the ball of area radius r_0, i.e.,

$$\int_0^{r_0} 4\pi r^2 \mathring{\rho}(r) dr = M - M_{\text{out}}. \tag{2.10}$$

Remark. The set up described above is quite similar to the set up in [8] for a scalar field. In [8] it is not required to have matter in an "inner" strip $[0, r_0]$, as is the case here in view of (2.10) and the condition $M_{\text{out}} < M$. The reason why we need some matter in the region $r \leq r_0$ is to ensure that initially ingoing matter continues to be ingoing for all times, cf. Lemma 6.1 below. If one only considers purely radially ingoing particles, i.e., with no angular momentum (which results in a non-smooth distribution function f), then we could allow for $M_{\text{out}} = M$. It is interesting to note that $p = \rho$ holds for Vlasov matter, if the particles have no angular momentum and their rest mass is zero, which is the case for the scalar field considered in [8].

Now we are in the position to formulate our main results for Vlasov matter. Corresponding to Case (i) above, we prove

Theorem 2.1 *Let r_0, r_1, M, and M_{out} be given as above, and let R_1 satisfy (2.7). Then there exists a set $\mathcal{I}_1$ of regular initial data for the spherically symmetric Einstein-Vlasov system such that if $\mathring{f} \in \mathcal{I}_1$, then (2.9) and (2.10) hold, the corresponding solution exists on D, and*

$$\lim_{s\to\infty} \gamma^+(s) < \infty, \quad \lim_{s\to\infty} \int_{\gamma^+(s)}^{\infty} 4\pi r^2 \rho(s,r)\, dr > 0,$$

where γ^+ satisfies (1.10).

By abuse of notation we denote by D both the outer region in spacetime defined by (1.11) and the part of the mass shell with $(t, r) \in D$.

The next theorem addresses Case (ii) above, assuming the stronger condition (2.8). This allows for a more straightforward proof, and the constraints on the momentum variables of the initial distribution function $\mathring{f}$ which are used to specify the set $\mathcal{I}_1$ will be slightly relaxed. Hence, the initial data set $\mathcal{I}_1$ does not contain $\mathcal{I}_2$ in Theorem 2.2 below, but it is larger in the sense that data in $\mathcal{I}_2$ are quite close to containing a trapped surface, which is not necessarily the case for data in $\mathcal{I}_1$. The precise form of $\mathcal{I}_1$ and $\mathcal{I}_2$ is specified in the proofs.

Theorem 2.2 *Let r_0, r_1, M, and M_{out} be given as above and let R_1 satisfy (2.8) with $\kappa = 6$. Then there exists a set $\mathcal{I}_2$ of regular initial data for the spherically symmetric Einstein-Vlasov system such that if $\mathring{f} \in \mathcal{I}_2$, then (2.9) and (2.10) hold, the corresponding solution exists on D, and*

$$\lim_{s\to\infty} \gamma^+(s) < \infty, \quad \lim_{s\to\infty} \int_{\gamma^+(s)}^{\infty} 4\pi r^2 \rho(s,r)\, dr > 0,$$

where γ^+ satisfies (1.10).

In the next section we formulate a version of Theorem 2.2 for quite general matter models. One reason for this is that the main mechanism behind our method becomes very transparent by posing sufficient conditions on the macroscopic matter terms rather than conditions on the initial distribution function $\mathring{f}$ as we did in the theorems above. Theorem 2.2 will then be a consequence of this generalization, cf. Section 6, whereas Theorem 2.1 is established in Section 7.

In these proofs it turns out that for the classes of initial data that we have specified we can obtain somewhat sharper asymptotic information on γ^+ and the mass in the outer region; see (5.7) below.

3 The result for general matter models

In this section we specify the general assumptions on a matter model sufficient for our method to be applied. In order to keep the discussion consistent with the Vlasov part of our arguments, and in view of the right hand sides of the field equations (1.7), (1.8), (1.9), it is convenient to use the notation

$$\rho := e^{-2\mu} T_{00},\ p := e^{-2\lambda} T_{11},\ j := -e^{-\mu-\lambda} T_{01}. \tag{3.1}$$

Firstly, we assume that the following two conditions are satisfied.

- The dominant energy condition holds. (DEC)
- The radial pressure p is non-negative. (NNP)

The dominant energy condition (DEC) plays a central role in general relativity and is the main criterion that a matter model should satisfy to be considered realistic. We refer to [15] for its definition. The non-negative pressure condition (NNP) is restrictive in the sense that it rules out, for example, a Maxwell field as matter model. However, for most astrophysical models it is a standard assumption, with e.g. fluid models satisfying this condition. For the purpose of this paper we only need to focus on two consequences of these two criteria, cf. [15] and [21]. The (DEC) condition implies, together with the (NNP) condition, that

$$0 \leq p \leq \rho \text{ and } |j| \leq \rho. \tag{3.2}$$

Furthermore, by (DEC) any geodesic $(s, R(s))$ of a material particle or a light ray satisfies

$$\left| \frac{dR(s)}{ds} \right| \leq e^{(\mu-\lambda)(s,R(s))}. \tag{3.3}$$

The meaning of the latter condition is that locally the speed of energy flow is less than or equal to the speed of light.

Let λ, μ, ρ, p, j correspond to a solution of the spherically symmetric Einstein-matter equations (1.6)–(1.9), (1.2), (1.3) in Schwarzschild coordinates, launched by initial data from a class $\mathcal{I}$. In order to investigate the global structure of the solutions it is necessary that they exist globally in an appropriate sense. In the situation at hand they need to exist on the outer region D defined in (1.11). In the spherical symmetric case the main obstruction for obtaining global solutions arises from the difficulties related to the centre of symmetry $r = 0$. For example, for a massless scalar field or a collisionless gas as matter model it has been shown that solutions remain regular away from $r = 0$ for general initial data, cf. [11, 2, 25]. On the other hand, for dust a singularity of shell crossing type can also occur at some $r > 0$. Although in that case there are no true geometric spacetime singularities, such behavior has to be ruled out in order not to interfere with the analysis of the solution on D. This can be achieved by proper assumptions on the initial data, cf. [6]. In view of (3.3) a possible break down of solutions at $r = 0$ will have no influence on the outer domain D. Hence we formulate a third condition, concerning global existence of solutions in the outer domain, as follows.

- For solutions launched by data from the set $\mathcal{I}$, γ^+ defined by (1.10) exists on $[0, \infty[$, and λ, μ, ρ, p, $j \in C^1(D)$. (GLO)

The three conditions above are of a quite general nature. The fourth and final condition however, is tightly connected to our method of proof.

- There exists a constant $c_1 > 0$ such that $\rho \leq -c_1 j$ in D. (EHC)

The acronym (EHC) stands for "event horizon condition", and this condition plays a crucial role for our method of proof. We emphasize that our main results show that for Vlasov matter there are initial data sets such that (EHC) holds. As a first consequence of (EHC) and (3.2), note that $j \leq 0$ in D, i.e., the matter is ingoing for all times. In this respect our present results complement [3], where purely outgoing matter is considered.

Let us now assume that our matter model satisfies (DEC) and (NNP), and that there exists an initial data set $\mathcal{I}$ such that (GLO) and (EHC) hold as well. Then we have the following result, which should be viewed as a version of Theorem 2.2 for general matter.

Theorem 3.1 *Let r_0, r_1, M, and M_{out} be given as above and let R_1 satisfy (2.8) with $\kappa = 2c_1$. Assume that there exists an initial data set $\mathcal{I}_3 \subset \mathcal{I}$ such that (2.9) and (2.10) hold for all initial data in $\mathcal{I}_3$. Then for any solution launched by initial data in $\mathcal{I}_3$,*

$$\lim_{s \to \infty} \gamma^+(s) < \infty, \quad \lim_{s \to \infty} \int_{\gamma^+(s)}^{\infty} 4\pi r^2 \rho(s, r)\, dr > 0,$$

where γ^+ satisfies (1.10).

Remark. For a spherically symmetric perfect fluid with density $\mathcal{R}$, pressure $P = P(\mathcal{R})$, and radial velocity field u, the (DEC) and (NNP) conditions and Eqn. (3.2) respectively are satisfied provided that $0 \leq P(\mathcal{R}) \leq \mathcal{R}$, which restricts the equation of state. The (EHC) condition holds for example with $c_1 = \sqrt{2}$ if $-e^{\lambda}u \geq 1$ on D. In the kinetic context of the Vlasov model we derive analogous estimates on the particle level from conditions on the initial data.

4 Preliminaries

In this section we collect some general facts concerning the spherically symmetric Einstein-matter equations under the assumptions (DEC) and (NNP) that have been specified in the previous section.

A quantity which plays an important role is the quasi-local mass $m(t,r)$. Typically, the spherically symmetric Einstein-matter system is supplemented by the requirement of a regular centre, i.e., $\lambda(t,0) = 0$. Using this boundary condition the field equation (1.7) implies that

$$e^{-2\lambda} = 1 - \frac{2m}{r}, \tag{4.1}$$

where the quasi-local mass would be given by $m(t,r) := \int_0^r 4\pi\eta^2\rho(t,\eta)\,d\eta$. Then $m(t,\infty)$ is a conserved quantity, the ADM mass. However, in the present context we want to investigate the system on the outer domain D, regardless of whether or not the solution remains regular in the region $r < \gamma^+(t)$. Hence we do not use the usual boundary condition at $r = 0$. Instead, we assume that the ADM mass $M > 0$ is given and redefine the quasi-local mass by

$$m(t,r) = M - \int_r^\infty 4\pi\eta^2\rho(t,\eta)\,d\eta. \tag{4.2}$$

Then $\lim_{r\to\infty} m(t,r) = M$, $0 \leq m \leq M$, and $m_r = 4\pi r^2\rho$ holds. Defining λ by (4.1), (3.1) shows that (1.7) and the boundary condition in (1.6) are satisfied. In addition, we need to modify (2.5) to

$$\mathring{m}(r) < \frac{r}{2}, \quad r \in]0,\infty[, \tag{4.3}$$

a condition that once again will be included in the notion of regular initial data.

By (1.7) and (1.8),

$$\lambda_r = \left(4\pi r\rho - \frac{m}{r^2}\right)e^{2\lambda}, \quad \mu_r = \left(\frac{m}{r^2} + 4\pi r p\right)e^{2\lambda}.$$

In view of (1.6), $\mu = \hat{\mu} + \check{\mu}$, where we define

$$\hat{\mu}(t,r) := -\int_r^\infty \frac{m(t,\eta)}{\eta^2}\, e^{2\lambda(t,\eta)}\, d\eta, \tag{4.4}$$

$$\check{\mu}(t,r) := -\int_r^\infty 4\pi\eta\, p(t,\eta)\, e^{2\lambda(t,\eta)}\, d\eta. \tag{4.5}$$

Lemma 4.1 *The following assertions hold.*

(a) $2\hat{\mu} \le \mu - \lambda \le \hat{\mu} \le \hat{\mu} + \lambda$.

(b) $\mu + \lambda \le \hat{\mu} + \lambda$.

(c) $(\mu - \lambda)(t,r) = 2\hat{\mu}(t,r) + \int_r^\infty 4\pi\eta\,(\rho - p)(t,\eta)\, e^{2\lambda(t,\eta)}\, d\eta$.

(d) $\hat{\mu}_t(t,r) = \int_r^\infty 4\pi j(t,\eta)\, e^{(\mu+\lambda)(t,\eta)} e^{2\lambda(t,\eta)}\, d\eta$. *In particular, if* $j \le 0$, *then also* $\hat{\mu}_t \le 0$.

Proof: In view of (1.6),

$$\lambda(t,r) = -\int_r^\infty \Big(4\pi\eta\,\rho(t,\eta) - \frac{m(t,\eta)}{\eta^2}\Big)\, e^{2\lambda}\, d\eta = -\int_r^\infty 4\pi\eta\,\rho(t,\eta)\, e^{2\lambda}\, d\eta - \hat{\mu},$$

and by (3.2) the relation $\mu - \lambda \ge 2\hat{\mu}$ follows. On the other hand, by (4.1), $\lambda \ge 0$. Thus $\check{\mu} \le 0$ leads to $\mu - \lambda \le \mu \le \hat{\mu} \le \hat{\mu} + \lambda$, and part (a) is established. Part (b) follows from $\check{\mu} \le 0$. As to (c), we observe that

$$\hat{\mu} + \lambda + \int_r^\infty 4\pi\eta\,(\rho - p)\, e^{2\lambda}\, d\eta = \check{\mu},$$

which gives the claim. By (4.1) and (1.9), $(e^{2\lambda}\frac{m}{r^2})_t = \frac{1}{2r}(e^{2\lambda} - 1)_t = -4\pi\, e^{\mu+\lambda} e^{2\lambda} j$. Hence (d) follows from (4.4). □

Lemma 4.2 *For* $r \in [0,\infty[$ *the following holds:*

$$\int_r^\infty 4\pi\eta\,(\rho + p)(t,\eta)\, e^{(\mu+\lambda)(t,\eta)} e^{2\lambda(t,\eta)}\, d\eta = 1 - e^{(\mu+\lambda)(t,r)} \le 1,$$

$$\int_r^\infty 4\pi\eta\,\rho(t,\eta)\, e^{(\hat{\mu}+\lambda)(t,\eta)} e^{2\lambda(t,\eta)}\, d\eta = 1 - e^{(\hat{\mu}+\lambda)(t,r)} \le 1.$$

Proof: It suffices to integrate

$$\partial_r(e^{\mu+\lambda}) = e^{\mu+\lambda}(\mu_r + \lambda_r) = e^{\mu+\lambda} 4\pi r\,(p + \rho),$$

$$\begin{aligned}\partial_r(e^{\hat{\mu}+\lambda}) &= e^{\hat{\mu}+\lambda}(\hat{\mu}_r + \lambda_r) = e^{\hat{\mu}+\lambda}\Big(e^{2\lambda}\frac{m}{r^2} + \Big(4\pi r\rho - \frac{m}{r^2}\Big)e^{2\lambda}\Big)\\ &= 4\pi r\rho\, e^{\hat{\mu}+\lambda} e^{2\lambda},\end{aligned} \tag{4.6}$$

observing that $\lim_{r\to\infty}\hat\mu(t,r) = \lim_{r\to\infty}\lambda(t,r) = \lim_{r\to\infty}\mu(t,r) = 0$. For Vlasov matter, the first relation has been used in [2, Lemma 1]. $\square$

Next we consider outgoing and ingoing radial null geodesics γ^+ and γ^-, respectively.

Lemma 4.3 *Let $\gamma^\pm$ be the solutions to*

$$\frac{d\gamma^\pm}{ds}(s) = \pm\, e^{(\mu-\lambda)(s,\,\gamma^\pm(s))}, \quad \gamma^+(0) = r_0 < r_1 = \gamma^-(0).$$

Then

- *(a) γ^+ is strictly increasing, $s\mapsto m(s,\gamma^+(s))$ is increasing, and the limits $\lim_{s\to\infty}\gamma^+(s)\in\,]r_0,\infty]$ and $\lim_{s\to\infty} m(s,\gamma^+(s))\in[m(0,r_0),M]$ exist.*
- *(b) γ^- is strictly decreasing, $s\mapsto m(s,\gamma^-(s))$ is decreasing, and the limits $\lim_{s\to\infty}\gamma^-(s)\in[0,r_1[$ and $\lim_{s\to\infty} m(s,\gamma^-(s))\in[0,m(0,r_1)]$ exist.*
- *(c) The relation*

$$\frac{d}{ds}(\hat\mu+\lambda)(s,\gamma^\pm(s)) = \Big(\hat\mu_t - 4\pi r\, e^{\mu+\lambda}(j\mp\rho)\Big)\Big|_{(t,\,r)=(s,\gamma^\pm(s))}$$

holds. In particular, if $j\le 0$ and $\rho = j = 0$ along $\gamma^\pm$, then also $\frac{d}{ds}(\hat\mu+\lambda)(s,\gamma^\pm(s))\le 0$.

Proof: Differentiating (4.1) w.r.t. t and using (1.9) implies that $m_t = -4\pi r^2 e^{\mu-\lambda}j$. Since $\rho\ge j$ according to (3.2), this yields

$$\begin{aligned}\frac{d}{ds}\,m(s,\gamma^+(s)) &= m_t(s,\gamma^+(s)) + m_r(s,\gamma^+(s))\frac{d\gamma^+}{ds}(s)\\ &= (-4\pi r^2 e^{\mu-\lambda}j + 4\pi r^2\rho\, e^{\mu-\lambda})\big|_{(t,\,r)=(s,\gamma^+(s))}\ge 0.\end{aligned}$$

Thus part (a) is obtained from $m\le M$. Since $\rho\ge -j$, the proof of (b) is analogous to (a). As to (c), note that by definition of $\hat\mu$, (1.7), and (1.9),

$$\begin{aligned}&\frac{d}{ds}(\hat\mu+\lambda)(s,\gamma^\pm(s))\\ &= \Big(\hat\mu_t + \hat\mu_r\frac{d\gamma^\pm}{ds} + \lambda_t + \lambda_r\frac{d\gamma^\pm}{ds}\Big)\Big|_{(t,\,r)=(s,\gamma^\pm(s))}\\ &= \Big(\hat\mu_t \pm \frac{m}{r^2}\,e^{2\lambda}e^{\mu-\lambda} - 4\pi r\, e^{\mu+\lambda}j \pm \Big(4\pi r\rho - \frac{m}{r^2}\Big)e^{2\lambda}e^{\mu-\lambda}\Big)\Big|_{(t,\,r)=(s,\gamma^\pm(s))}\\ &= \Big(\hat\mu_t - 4\pi r\, e^{\mu+\lambda}(j\mp\rho)\Big)\Big|_{(t,\,r)=(s,\gamma^\pm(s))},\end{aligned}$$

as desired. The last claim follows from Lemma 4.1(d). $\square$

5 Proof of Theorem 3.1

In this section we use the hypotheses stated in Section 3 to prove Theorem 3.1. The proof is short and emphasizes that the crucial mechanism is captured in the (EHC) condition. Our main results which show in particular that the (EHC) condition holds for Vlasov matter are established in the next sections.

Consider the out- and ingoing null geodesics γ^+ and γ^- defined in Lemma 4.3. The claims follow if we can show that these geodesics never intersect. By continuity and monotonicity there exists $T \in]0,\infty]$ such that

$$r_0 \leq \gamma^+(t) < \gamma^-(t) \leq r_1, \quad t \in [0,T[; \tag{5.1}$$

it will be shown that actually $T = \infty$ holds. In view of (2.9) we have initially that $\rho = p = j = 0$ for $r \geq R_1$. The (EHC) condition implies that $j \leq 0$ in D, meaning that the flow of matter is ingoing. Therefore

$$\rho = p = j = 0 \quad \text{and} \quad m = M \quad \text{for} \quad (t,r) \in [0,T[\times[R_1,\infty[. \tag{5.2}$$

By Lemma 4.2, (3.2), the (EHC) condition, and Lemma 4.1(d) for $s \in [0,T[$ and $r \in [\gamma^+(s),\infty[$,

$$\begin{aligned} 1 - e^{(\mu+\lambda)(s,r)} &= \int_r^\infty 4\pi\eta\,(\rho+p)(s,\eta)\,e^{(\mu+\lambda)(s,\eta)}e^{2\lambda(s,\eta)}\,d\eta \\ &\leq 2c_1 \int_r^\infty 4\pi\eta\,|j(s,\eta)|\,e^{(\mu+\lambda)(s,\eta)}e^{2\lambda(s,\eta)}\,d\eta \\ &\leq -2c_1R_1 \int_r^\infty 4\pi j(s,\eta)\,e^{(\mu+\lambda)(s,\eta)}e^{2\lambda(s,\eta)}\,d\eta \\ &= -2c_1R_1\hat{\mu}_t(s,r), \end{aligned}$$

since $j(s,\eta) \neq 0$ implies $\eta \leq R_1$. Thus

$$\hat{\mu}_t(s,r) \leq -\frac{1}{2c_1R_1}\Big(1 - e^{(\mu+\lambda)(s,r)}\Big). \tag{5.3}$$

This in turn implies that

$$
\begin{aligned}
&\hat\mu(t,\gamma^\pm(t)) - \hat\mu(0,\gamma^\pm(0)) \\
&= \int_0^t \frac{d}{ds}\,\hat\mu(s,\gamma^\pm(s))\,ds \\
&= \int_0^t \Big(\hat\mu_t(s,\gamma^\pm(s)) \pm \hat\mu_r(s,\gamma^\pm(s)) e^{(\mu-\lambda)(s,\,\gamma^\pm(s))}\Big)\,ds \\
&\le \int_0^t \Big(-\frac{1}{2c_1R_1}\Big(1-e^{(\mu+\lambda)(s,\,\gamma^\pm(s))}\Big) \pm \frac{m(s,\gamma^\pm(s))}{\gamma^\pm(s)^2}\, e^{(\mu+\lambda)(s,\,\gamma^\pm(s))}\Big)\,ds \\
&\le -\frac{t}{2c_1R_1} + \int_0^t \Big(\frac{1}{2c_1R_1} + \frac{m(s,\gamma^\pm(s))}{\gamma^\pm(s)^2}\Big)\, e^{(\mu+\lambda)(s,\,\gamma^\pm(s))}\,ds. \qquad (5.4)
\end{aligned}
$$

Now for any $r\in[r_0,r_1]$ and $t\in[0,T[$ it follows from $\hat\mu_r\ge 0$ and (4.1) that

$$
\hat\mu(t,r) \le \hat\mu(t,R_1) = -\int_{R_1}^\infty \frac{M\,d\eta}{\eta^2(1-2M/\eta)}. \qquad (5.5)
$$

Using $M=r_1/2$ we get

$$
\hat\mu(t,R_1) = \frac{1}{2}\log\Big(\frac{R_1-r_1}{R_1}\Big),
$$

so that for $r\in[r_0,r_1]$,

$$
e^{\hat\mu(t,r)} \le e^{\hat\mu(t,R_1)} = \sqrt{\frac{R_1-r_1}{R_1}}. \qquad (5.6)
$$

By (3.3) and the properties of the initial matter distribution there is vacuum in the region $\gamma^+(t)\le r\le\gamma^-(t)$. Hence $m(t,r)=M-M_{\rm out}$ and (2.6) imply that

$$
e^{\lambda(t,r)} \le \frac{1}{\sqrt{1-2(M-M_{\rm out})/r_0}} < 3
$$

for $\gamma^+(t)\le r\le\gamma^-(t)$. From Lemma 4.1(b) and (2.8), recalling $\kappa=2c_1$, we obtain in particular that

$$
e^{(\mu+\lambda)(s,\gamma^\pm(s))} \le e^{(\hat\mu+\lambda)(s,\gamma^\pm(s))} < \min\Big\{\frac{1}{2},\frac{r_0^2}{8c_1R_1M}\Big\} =: d.
$$

Thus (5.4) yields

$$\begin{aligned}\hat\mu(t,\gamma^\pm(t))-\hat\mu(0,\gamma^\pm(0)) &\le -\frac{t}{2c_1R_1}+d\int_0^t\Big(\frac{1}{2c_1R_1}+\frac{M}{r_0^2}\Big)\,ds\\ &= -\Big(\frac{1-d}{2c_1R_1}-d\,\frac{M}{r_0^2}\Big)t\\ &\le -\Big(\frac{1}{4c_1R_1}-d\,\frac{M}{r_0^2}\Big)t\\ &\le -\frac{t}{8c_1R_1},\quad t\in[0,T[.\end{aligned}$$

Hence Lemma 4.1(a) leads to the estimate

$$\begin{aligned}|\gamma^\pm(t)-\gamma^\pm(0)| &= \left|\int_0^t e^{(\mu-\lambda)(s,\,\gamma^\pm(s))}\,ds\right| \le \int_0^t e^{\hat\mu(s,\,\gamma^\pm(s))}\,ds\\ &\le e^{\hat\mu(0,\,\gamma^\pm(0))}\int_0^t e^{-\frac{s}{8c_1R_1}}\,ds \le 8c_1R_1\sqrt{\frac{R_1-r_1}{R_1}},\end{aligned}$$

where we used (5.6) in the last inequality. By the third condition in (2.8),

$$\sqrt{\frac{R_1-r_1}{R_1}}<\frac{r_1-r_0}{16c_1R_1},$$

so that

$$|\gamma^\pm(t)-\gamma^\pm(0)|<\frac{r_1-r_0}{2},\quad t\in[0,T[.$$

Since $\gamma^-(0)-\gamma^+(0)=r_1-r_0$, this implies that $\gamma^-(T)-\gamma^+(T)>0$. Hence, if we choose T in (5.1) to be maximal, then $T=\infty$, i.e., γ^+ and γ^- do never intersect. This completes the proof of Theorem 3.1. □

Remark. In the above proof we have obtained the somewhat more explicit information that

$$\lim_{s\to\infty}\gamma^+(s)<\frac{r_0+r_1}{2},\quad m(s,\gamma^+(s))=M-M_{\rm out},\ s\ge 0, \tag{5.7}$$

the latter since all the matter originally to the right of $\gamma^-(s)>\gamma^+(s)$ necessarily stays there.

6 Proof of Theorem 2.2

We first check that the (DEC), (NNP), and (GLO) conditions hold for Vlasov matter. Then we show that there exists a class of initial data such that the corresponding solutions satisfy the (EHC) condition with $c_1 = 3$. Hence Theorem 2.2 will follow from Theorem 3.1.

The characteristic system associated to the Vlasov equation (2.1) is

$$\frac{dR}{ds} = e^{(\mu-\lambda)(s,\,R)}\,\frac{W}{E}, \tag{6.1}$$

$$\frac{dW}{ds} = -\lambda_t(s,R)W - e^{(\mu-\lambda)(s,\,R)}\mu_r(s,R)E + e^{(\mu-\lambda)(s,\,R)}\frac{L}{R^3E}, \tag{6.2}$$

$$\frac{dL}{ds} = 0. \tag{6.3}$$

If $s \mapsto (R,W,L)(s)$ is a solution with data $(R,W,L)(0) = (r,w,L)$, then

$$f(s,R(s),W(s),L) = \mathring{f}(r,w,L)$$

is constant in s. Hence $(R(s),W(s),L) \in \operatorname{supp} f(s)$ iff $(r,w,L) \in \operatorname{supp} \mathring{f}$. Such characteristics will be addressed as characteristics in $\operatorname{supp} f$.

Direct inspection of the definition in (2.3) shows that (NNP) holds for Vlasov matter. It is moreover well-known that the (DEC) condition is satisfied for Vlasov matter; see [1, Sec. 1.4]. Alternatively, we can check (3.2) and (3.3) directly. The latter follows from (6.1) above, whereas the former is a consequence of the expressions for the matter terms given in (2.2), (2.3), and (2.4).

To see that the (GLO) condition holds for any regular initial data set we argue as follows. First of all, a regular initial data launches a local-in-time solution on some time interval $[0,T[$, and the corresponding theorems in [23] or [22] also give a condition under which this local solution can be extended to a global one. In order to see that the local solution can always be extended to the whole outer domain D we first observe that the spherically symmetric Einstein-Vlasov system on D, with (4.1) and (4.2) replacing the usual boundary condition of a regular centre and with (1.10) included, has again a well-posed initial value problem for regular data supported in $]r_0,\infty[$. This can be shown in the same way as for the system on the whole space, the essential point being that no characteristic of the Vlasov equation can enter region D at the boundary $r = \gamma^+(t)$. To the local solution on D we can now apply the arguments from [25] and conclude that the solution exists on all of D. This is possible due to the fact that the estimates in [25] address a situation where matter is bounded away from the centre or is controlled in a neighborhood of the centre so that these estimates can be applied on D. We emphasize that for our present analysis only the behavior of the solution on D plays a role. We have chosen to present our results in the form that we have Vlasov matter also inside $r < \gamma^+(t)$, and this part of the solution may or may not break down, but this is irrelevant for our arguments.

Hence it remains to show that the (EHC) condition holds. To this end we let $0 < r_0 < r_1 < R_1$, $R_0 = (r_1 + R_1)/2$, and $M = r_1/2$. For a parameter $W_- < 0$ to be specified below and regular data $\mathring{f}$ with ADM mass M we formulate the following

General support condition: For all $(r, w, L) \in \operatorname{supp} \mathring{f}$ the following holds:

$$r \in]0, r_0] \cup [R_0, R_1],$$

and if $r \in [R_0, R_1]$ then

$$w \leq W_-$$

and also

$$0 < L < \frac{3L}{\eta}\, \mathring{m}(\eta) + \eta\, \mathring{m}(\eta),\ \ \eta \in [r_0, R_1]. \tag{6.4}$$

We use the notation $\mathring{m}$ when $\rho = \mathring{\rho}$ in (4.2). Furthermore, we abbreviate

$$\Gamma = \Gamma(r_1, R_1) := \sqrt{\frac{R_1 - r_1}{R_1 + r_1}}. \tag{6.5}$$

The following lemma shows that if the support condition holds, then the particles in the outer domain D keep moving inward in a controlled way.

Lemma 6.1 *Let $\mathring{f}$ be regular and satisfy the general support condition for some $W_- < 0$. Then for all $(r, w, L) \in \operatorname{supp} f(t)$ such that $(t, r) \in D$,*

$$w \leq \Gamma(r_1, R_1) W_-.$$

In particular, $j \leq 0$ on D.

Proof: Let $[0, T[$ denote the maximal time interval such that for $t < T$

$$w < 0 \text{ for } (r, w, L) \in \operatorname{supp} f(t) \text{ with } (t, r) \in D. \tag{6.6}$$

Since $W_- < 0$, $T > 0$ by continuity. By the definition of j,

$$j(t, r) \leq 0 \text{ for } (t, r) \in D_T := D \cap ([0, T[\times[0, \infty[). \tag{6.7}$$

Let $(R, W, L)(s)$ be a characteristic in $\operatorname{supp} f$. Then

$$\begin{aligned}\frac{d}{ds}(e^{-\lambda}W) &= -\,e^{-\lambda}\Big(W\lambda_t + W\lambda_r\frac{dR}{ds} - \frac{dW}{ds}\Big)\\ &= \frac{4\pi R}{E}\,e^{\mu}(2WEj - W^2\rho - E^2 p) + e^{\mu}\Big(1-\frac{2m}{R}\Big)\frac{L}{R^3E}\\ &\quad + e^{\mu}\frac{m}{R^2}\left(\frac{w^2}{E} - E\right)\\ &= -\,\frac{4\pi^2}{R}\,e^{\mu}\int_{-\infty}^{\infty}\int_0^{\infty}\left[\sqrt{\frac{\tilde E}{E}}\,w - \sqrt{\frac{E}{\tilde E}}\,\tilde w\right]^2 f\,d\tilde L\,d\tilde w\\ &\quad - e^{\mu}\frac{m}{R^2}\left(\frac{1+L/R^2}{E} + \frac{2L}{R^2E}\right) + e^{\mu}\frac{L}{R^3E}\,,\end{aligned}$$

where $E = E(R, W, L)$ and $\tilde E = \tilde E(R, \tilde w, \tilde L)$. Therefore

$$\frac{d}{ds}(e^{-\lambda}W) \le -e^{\mu}\frac{m}{R^2}\left(\frac{1+L/R^2}{E} + \frac{2L}{R^2E}\right) + e^{\mu}\frac{L}{R^3E}.$$

Differentiating (4.1) w.r.t. t and using (1.9) leads to $m_t = -4\pi r^2 e^{\mu-\lambda} j$, which by (6.7) is non-negative on D_T. It follows that $m(s, r) \ge m(0, r) = \mathring{m}(r)$. Thus as long as the characteristic remains in D_T,

$$\begin{aligned}\frac{d}{ds}(e^{-\lambda}W) &\le -e^{\mu}\frac{\mathring{m}(R)}{R^2}\left(\frac{1+L/R^2}{E} + \frac{2L}{R^2E}\right) + e^{\mu}\frac{L}{R^3E}\\ &= e^{\mu}\,\frac{1}{R^3E}\left(L - \frac{3L}{R}\,\mathring{m}(R) - R\,\mathring{m}(R)\right).\end{aligned}$$

Now $R(0) \in [R_0, R_1]$ and $\dot R(s) \le 0$ by (6.1) and (6.6) yields $R_1 \ge R(0) \ge R(s) \ge \gamma^+(s) \ge r_0$. Hence condition (6.4) implies that, as long as the characteristic remains in D_T, $\frac{d}{ds}(e^{-\lambda}W) < 0$, so that

$$W(s) \le e^{\lambda(s,\,R(s)) - \lambda(0,\,R(0))}\,W_-.$$

But $\lambda \ge 0$, so $W_- < 0$ leads to

$$W(s) \le \Big(\min_{r\in[R_0,R_1]} e^{-\lambda(0,\,r)}\Big)\,W_-.$$

In view of (4.1),

$$e^{-\lambda(0,\,r)} \ge \sqrt{1-\frac{2M}{R_0}} = \sqrt{\frac{R_1 - r_1}{R_1 + r_1}},\quad r\in[R_0, R_1],$$

and recalling (6.5) it follows that

$$W(s) \leq \Gamma(r_1, R_1) W_- < 0$$

as long as the characteristic remains in D_T. By the maximality of T in (6.6), $T = \infty$, and the proof is complete. $\square$

In order to specify the initial data set $\mathcal{I}_2$, let r_0, r_1, M, and M_{out} be given as in Section 2 and let R_1 be such that (2.8) holds for $\kappa = 6$. We require that $W_- < 0$ satisfies the estimate

$$\Gamma(r_1, R_1)\, |W_-| \geq 1. \tag{6.8}$$

Then

$$\mathcal{I}_2 := \Big\{ \mathring{f} \mid \mathring{f} \text{ is regular, satisfies (2.9), (2.10), the general support condition,} \\ \text{and for } (r, w, L) \in \operatorname{supp} \mathring{f} \text{ with } r \in [R_0, R_1], \sqrt{L}/r_0 \leq \Gamma\, |W_-| \Big\}. \tag{6.9}$$

Consider now a solution f launched by initial data from this set. Condition (6.8) and Lemma 6.1 imply that

$$|w| \geq \Gamma(r_1, R_1)\, |W_-| \geq 1 \quad \text{on} \quad \operatorname{supp} f \cap D, \tag{6.10}$$

and since L is conserved along characteristics, (6.9) leads to $\sqrt{L}/r \leq \sqrt{L}/r_0 \leq |w|$ for all particles in $\operatorname{supp} f \cap D$. Hence the definition (2.2) of ρ implies that on D,

$$\begin{aligned} \rho(t, r) &\leq \frac{\pi}{r^2} \int_{-\infty}^{\infty} \int_0^{\infty} f\, dL\, dw + \frac{\pi}{r^2} \int_{-\infty}^{\infty} \int_0^{\infty} |w| f\, dL\, dw \\ &\quad + \frac{\pi}{r^2} \int_{-\infty}^{\infty} \int_0^{\infty} \sqrt{L}/r\, f\, dL\, dw \\ &\leq 3\, \frac{\pi}{r^2} \int_{-\infty}^{\infty} \int_0^{\infty} |w| f\, dL\, dw = 3\, |j(t, r)|. \end{aligned} \tag{6.11}$$

Accordingly, $\mathcal{I}_2$ satisfies the (EHC) condition with $c_1 = 3$, and Theorem 2.2 follows from Theorem 3.1. $\square$

We briefly show that the set $\mathcal{I}_2$ is far from empty. Therefore fix $0 < r_0 < r_1 < R_0 < R_1$, $M = r_1/2$, and $0 < M_{\text{out}} < M$ such that $R_0 = (r_1 + R_1)/2$, (2.6), and (2.8) are satisfied. Let $0 \leq f_1 \in C^1$ have r-support in $[r_0 - \delta, r_0]$ for some $0 < \delta < r_0/9$, and let $0 \leq f_2 \in C^1$ have r-support in $[R_0, R_1]$. Fix the compact w-support of f_2 in $]-\infty, W_-]$ with $W_- < 0$ such that (6.8) holds, and fix its L-support in $[0, L_2]$ so that

$$\frac{\sqrt{L_2}}{r_0} \leq \Gamma(r_1, R_1)\,|W_-|$$

and

$$L < (M - M_{\rm out})\Big(\frac{3L}{\eta} + \eta\Big), \quad L \in [0, L_2], \quad \eta \in [r_0, R_1].$$

Now take $\overset{\circ}{f} = Af_1 + Bf_2$, where $A > 0$ and $B > 0$ are chosen such that (2.9) and (2.10) are satisfied. Note that $\overset{\circ}{m}(\eta) \geq M - M_{\rm out}$ for $\eta \in [r_0, R_1]$, whence (6.4) holds as well; thus the general support condition if verified. It remains to check (4.3). If $r \in]0, r_0 - \delta]$, then $\overset{\circ}{m}(r) = 0$. If $r \in [r_0 - \delta, R_0]$, then $\overset{\circ}{m}(r) \leq M - M_{\rm out}$ yields in view of (2.6),

$$\frac{2\overset{\circ}{m}}{r} \leq \frac{2(M - M_{\rm out})}{r_0 - \delta} < 1.$$

If $r \in [R_0, \infty[$, then

$$\frac{2\overset{\circ}{m}}{r} \leq \frac{2M}{R_0} < 1,$$

since $2M = r_1 < R_0$. Hence $\overset{\circ}{f}$ is regular and has all the properties that are required in the definition of $\mathcal{I}_2$.

Remark. The set $\mathcal{I}_2$ has "non-empty interior", in the sense that sufficiently small perturbations of initial data in the "interior" of this set belong to $\mathcal{I}_2$ as well, provided that the support is changed very little and M is left invariant. This is due to the fact that the various parameters entering into the definition of $\mathcal{I}_2$ are defined in terms of inequalities and hence can be varied.

7 Proof of Theorem 2.1

The set up is closely related to the set up in the proof of Theorem 2.2. As we saw above, the (DEC), (NNP), and (GLO) conditions are satisfied for Vlasov matter, and we will again construct an initial data set such that the (EHC) condition holds with $c_1 = 3$. However, since this result relies on condition (2.7) instead of (2.8), we cannot simply invoke Theorem 3.1 after the (EHC) condition has been verified; instead an additional step needs to be added to the proof. For this new argument a slightly stronger condition on the momentum variable w needs to be imposed on supp $\overset{\circ}{f}$. We now require that $W_- < 0$ satisfies

$$\Gamma(r_1, R_1)^2 |W_-|^2 \geq \frac{10}{d}, \tag{7.1}$$

where

$$d := \min\left\{\frac{1}{2}, \frac{r_0}{12R_1}, \frac{r_1 - r_0}{300R_1}\right\}.$$

Then

$$\mathcal{I}_1 := \Big\{\mathring{f} \mid \mathring{f} \text{ is regular, satisfies (2.9), (2.10), the general support condition,}$$
$$\text{and for } (r,w,L) \in \operatorname{supp}\mathring{f} \text{ with } r \in [R_0, R_1], \sqrt{L}/r_0 \le 1.\Big\} \qquad (7.2)$$

The same construction as at the end of the previous section shows that this set is not empty, and the same remark as at the end of the previous section applies.

Let f be a solution launched by initial data from $\mathcal{I}_1$. It is clear from these conditions that Lemma 6.1 applies, and since $10/d \ge 1$, it follows that (6.10) holds as well. Thus the argument leading to $\rho \le 3|j|$ on D in the proof of Theorem 2.2 applies again. Hence, the (EHC) condition is satisfied with $c_1 = 3$.

Consider the expression

$$\rho(s,r) - p(s,r) = \frac{\pi}{r^2}\int_{-\infty}^{\infty}\int_0^{\infty}\left(E - \frac{w^2}{E}\right) f(s,r,w,L)\, dL\, dw.$$

Since $E^2 \ge w^2 \ge \Gamma^2(r_1,R_1)\, W_-^2$ by Lemma 6.1, we get for $r \in [\gamma^+(s), R_1]$ from $\sqrt{L}/r_0 \le 1$,

$$E - \frac{w^2}{E} = \frac{1}{E}\,(E^2 - w^2) = \frac{1}{E}\left(1 + \frac{L}{r^2}\right) \le \frac{2}{E} \le \frac{2}{\Gamma^2\, W_-^2}\, E =: c_0 E, \qquad (7.3)$$

so that

$$\rho(s,r) - p(s,r) \le c_0 \rho(s,r). \qquad (7.4)$$

After this preparation, we again show that the out- and ingoing null geodesics γ^+ and γ^- do not intersect. We choose $T \in]0,\infty[$ such that (5.1) holds. In this case we cannot rely on the smallness of $e^{\hat\mu}$ as in the proof of Theorem 3.1, so we need to control the evolution also when $e^{\hat\mu}$ is not small. For this part the estimate (7.4) is essential.

We fix $t_*^{\pm} \in [0, T[$ by requiring that

$$e^{(\hat\mu+\lambda)(s,\,\gamma^{\pm}(s))} > d \text{ for } s \in [0, t_*^{\pm}[, \quad e^{(\hat\mu+\lambda)(s,\,\gamma^{\pm}(s))} \le d \text{ for } s \in [t_*^{\pm}, T[.$$

First we note that $t_*^{\pm}$ is well-defined, since by Lemma 4.3(c),

$$\frac{d}{ds}\, e^{(\hat\mu+\lambda)(s,\,\gamma^{\pm}(s))} \le 0. \qquad (7.5)$$

Step 1: Consider $s \in [0, t_*^\pm]$; if $t_*^\pm = 0$, then this step is omitted. For $\eta \geq \gamma^\pm(s)$,

$$d \leq e^{(\hat{\mu}+\lambda)(s,\,\gamma^\pm(s))} \leq e^{(\hat{\mu}+\lambda)(s,\,\eta)},$$

since $(\hat{\mu}+\lambda)_r = 4\pi r \rho\, e^{2\lambda} \geq 0$ by (4.6). Hence Lemma 4.1(c) and (7.4) yield

$$\begin{aligned}
(\mu-\lambda)(s,\gamma^\pm(s)) &= 2\hat{\mu}(s,\gamma^\pm(s)) + \int_{\gamma^\pm(s)}^\infty 4\pi\eta\,(\rho-p)(s,\eta)\,e^{2\lambda(s,\,\eta)}\,d\eta \\
&\leq 2\hat{\mu}(s,\gamma^\pm(s)) + \frac{c_0}{d}\int_{\gamma^\pm(s)}^\infty 4\pi\eta\,\rho(s,\eta)\,e^{(\hat{\mu}+\lambda)(s,\,\eta)}e^{2\lambda(s,\,\eta)}\,d\eta \\
&\leq 2\hat{\mu}(s,\gamma^\pm(s)) + \frac{c_0}{d},
\end{aligned}$$

where for the last estimate Lemma 4.2 has been used.

Now we make the following observation: There is at least one characteristic $(\bar{R},\bar{W},\bar{L})(s)$ with $\bar{R}(0) \in [R_0,R_1]$, which does not leave the strip $[r_1,R_1]$ during the finite time interval $[0,T]$. In fact, if at time $t=T$ all characteristics had left the strip $[r_1,R_1]$ (and thus had entered the region $r<r_1$), then $m(T,r_1)=M$. From (4.1) and $2M=r_1$ it would follow that $\lambda(T,r_1)=\infty$. However, this contradicts the (GLO) condition which holds for Vlasov matter.

Since $\gamma^\pm(s) \leq r_1 \leq \bar{R}(s)$, and since $\hat{\mu}_r \geq 0$, we thus obtain in view of Lemma 4.1(a) that

$$\begin{aligned}
(\mu-\lambda)(s,\gamma^\pm(s)) &\leq 2\hat{\mu}(s,\gamma^\pm(s)) + \frac{c_0}{d} \leq 2\hat{\mu}(s,\bar{R}(s)) + \frac{c_0}{d} \\
&\leq (\mu-\lambda)(s,\bar{R}(s)) + \frac{c_0}{d}, \quad s \in [0,t_*^\pm].
\end{aligned}$$

Next note that $|W| \geq 1$ by (6.10), and hence due to (6.1) and observing $\bar{R}^2 \geq r_0^2 \geq L$,

$$|\dot{\bar{R}}| = \frac{|W|}{E}\,e^{\mu-\lambda} \geq \frac{|W|}{\sqrt{2+W^2}}\,e^{\mu-\lambda} \geq \frac{1}{2}\,e^{\mu-\lambda}.$$

Therefore for all $t \in [0,t_*^\pm]$ the estimate

$$\begin{aligned}
|\gamma^\pm(t)-\gamma^\pm(0)| &= \left|\int_0^t \pm\, e^{(\mu-\lambda)(s,\,\gamma^\pm(s))}\,ds\right| \leq e^{\frac{c_0}{d}}\int_0^t e^{(\mu-\lambda)(s,\,\bar{R}(s))}\,ds \\
&\leq -2e^{\frac{c_0}{d}}\int_0^t \dot{\bar{R}}(s)\,ds = 2e^{\frac{c_0}{d}}\,(\bar{R}(0)-\bar{R}(t)) \\
&\leq 2e^{\frac{c_0}{d}}\,(R_1-r_1) \qquad (7.6)
\end{aligned}$$

is obtained. *Step 2:* Let $t \in [t_*^\pm, T[$; if $t_*^\pm = T$, then this step is omitted. The arguments here are basically the ones presented in Section 5. The computation leading to (5.4) is almost identical, and

$$\hat{\mu}(t,\gamma^\pm(t)) - \hat{\mu}(t_*^\pm,\gamma^\pm(t_*^\pm)) \\ \leq -\frac{t-t_*^\pm}{2c_1R_1} + \int_{t_*^\pm}^t \left(\frac{1}{2c_1R_1} + \frac{m(s,\gamma^\pm(s))}{\gamma^\pm(s)^2}\right) e^{(\mu+\lambda)(s,\gamma^\pm(s))}\, ds \quad (7.7)$$

for $c_1 = 3$. By Lemma 4.1(b), $e^{(\mu+\lambda)(s,\gamma^\pm(s))} \leq e^{(\hat{\mu}+\lambda)(s,\gamma^\pm(s))} \leq d$. Next we use the facts that $m/r < 1/2$, $\gamma^\pm(s) \geq r_0$, and the definition of d to obtain the estimate

$$\begin{aligned} \hat{\mu}(t,\gamma^\pm(t)) - \hat{\mu}(t_*^\pm,\gamma^\pm(t_*^\pm)) &\leq -\frac{1}{2c_1R_1}(t-t_*^\pm) + d\int_{t_*^\pm}^t \left(\frac{1}{2c_1R_1} + \frac{1}{2r_0}\right) ds \\ &= -\left(\frac{1-d}{2c_1R_1} - d\,\frac{1}{2r_0}\right)(t-t_*^\pm) \\ &\leq -\left(\frac{1}{4c_1R_1} - d\,\frac{1}{2r_0}\right)(t-t_*^\pm) \\ &\leq -\frac{1}{8c_1R_1}(t-t_*^\pm), \quad t \in [t_*^\pm, T[. \end{aligned}$$

Hence by Lemma 4.1(a),

$$\begin{aligned} |\gamma^\pm(t) - \gamma^\pm(t_*^\pm)| &= \left| \int_{t_*^\pm}^t e^{(\mu-\lambda)(s,\gamma^\pm(s))}\, ds \right| \leq \int_{t_*^\pm}^t e^{\hat{\mu}(s,\gamma^\pm(s))}\, ds \\ &\leq e^{\hat{\mu}(t_*^\pm,\gamma^\pm(t_*^\pm))} \int_{t_*^\pm}^t e^{-\frac{(s-t_*^\pm)}{8c_1R_1}}\, ds \\ &\leq e^{(\hat{\mu}+\lambda)(t_*^\pm,\gamma^\pm(t_*^\pm))} \int_{t_*^\pm}^\infty e^{-\frac{(s-t_*^\pm)}{8c_1R_1}}\, ds \leq 8c_1R_1 d. \quad (7.8) \end{aligned}$$

Adding the contributions (7.6) from Step 1 and (7.8) from Step 2, the final estimate

$$|\gamma^\pm(t) - \gamma^\pm(0)| \leq 2e^{c_0/d}(R_1 - r_1) + 8c_1R_1 d$$

is obtained for all $t \in [0, T[$. From (7.3) and (7.1) we have $c_0/d \leq 1/5$. The third condition on d together with (2.7) thus imply that

$$|\gamma^\pm(t) - \gamma^\pm(0)| < \frac{r_1 - r_0}{2}.$$

As in the proof of Theorem 3.1 we conclude that γ^+ and γ^- do not intersect, completing the proof of Theorem 2.1. □

Remarks. (a) The sharper estimates stated in (5.7) clearly hold also in this case.
(b) The solution must necessarily enter the regime of Step 2, more precisely,

$$\lim_{s\to\infty} e^{(\hat\mu+\lambda)(s,\gamma^\pm(s))} = 0$$

for both null geodesics. Otherwise, the monotonicity implied by Eqn. (7.5) yields a positive constant $c > 0$ such that $e^{(\hat\mu+\lambda)(s,\gamma^\pm(s))} > c$ for all time, and hence,

$$|\dot\gamma^\pm| = e^{\mu-\lambda} = e^{\hat\mu+\lambda}e^{\check\mu-2\lambda} > ce^{\check\mu-2\lambda}.$$

Since no matter can cross the two null geodesics,

$$\begin{aligned}(\check\mu - 2\lambda)(s,r) &= \int_r^\infty 4\pi\eta(2\rho - p)e^{2\lambda}d\eta + 2\hat\mu(s,r)\\ &\geq 2\hat\mu(s,r) = -2\int_r^\infty \frac{\mathring{m}(r_0)}{\eta^2}\frac{1}{1-2\mathring{m}(r_0)/\eta}\,d\eta\\ &= \ln\frac{r-2\mathring{m}(r_0)}{r}\end{aligned}$$

for $r = \gamma^\pm(s)$. If we insert this into the estimate for $\dot\gamma^\pm$ it follows that this quantity is bounded from below by a positive constant which contradics the finite limits of $\gamma^\pm(s)$ as $s\to\infty$.

8 Light refreshments to a black hole

In this section we are going to show that for the case of Vlasov matter initial data can be arranged such that for large times the solution behaves exactly as in Theorems 2.1 or 2.2, but some of the matter which initially is in the exterior region D is swallowed by the null geodesic γ^+. We carry out the argument, which essentially works by continuous dependence on initial data, for Theorem 2.1.

Let $\mathring{f} \in \mathcal{I}_1$ be such that in Eqn. (7.1) the inequality is strict. The corresponding local solution of the spherically symmetric Einstein-Vlasov system on the whole space exists on some time interval $[T,0]$ with $T<0$. By choosing T sufficiently close to 0 we can make sure that $r_- = \gamma^+(T) > 0$ and that the matter which at time $t=0$ is in the strip $[R_0,R_1]$ is moving inward on the time interval $[T,0]$. Clearly $r_- < r_0$. The desired initial data is to be constructed from the state $f(T)$.

If at time T there is matter in the strip $[r_-,r_0]$ it can only have come out of the region $r<\gamma^+(t)$ when moving backward in time so that this matter is swallowed by $\gamma^+(t)$ when we move forward in time during $[T, 0]$. Hence $f(T)$ provides initial data of the desired sort.

If there is no matter in the strip $[r_-, r_0]$ at time T, i.e., no matter came out of the inner region when moving backward in time, we construct new regular initial data as follows. We take a small part of matter inside $]0, r_-[$— of the order ϵ with respect to the L^∞ norm of f—and place it in the strip $]r_-, (r_- + r_0)/2[$ in such a way that firstly the quasi-local mass $m(T, r_0)$ is unchanged, secondly all the matter in the strip $]r_-, (r_- + r_0)/2[$ is moving inward, i.e., has $w < 0$, and thirdly the condition on L in the general support condition holds for the matter outside r_-. This modification of $f(T)$ can be carried out in such a way that the condition (2.5) is preserved, and it yields initial data of the desired sort. To see this let $\tilde{f}$ denote the corresponding solution. Using continuous dependence of the solution on the initial data, cf. [22], we can, by making ϵ small, make sure that this perturbed solution exists on the time interval $[T, 0]$, and the radial null geodesic $\tilde{\gamma}^+$ which starts at r_- at time T is as close to r_0 at time $t = 0$ as we wish, in particular, $\tilde{r}_0 = \tilde{\gamma}^+(0) > (r_- + r_0)/2$. Since Lemma 6.1 applies, all the matter outside $\tilde{\gamma}^+$ is moving inward during the time interval $[T, 0]$, and the matter starting in the strip $]r_-, (r_- + r_0)/2[$ at time T is swallowed by $\tilde{\gamma}^+$ during the time interval $[T, 0]$.

It remains to show that the behavior for $t > 0$ of the solution $\tilde{f}$ on $\tilde{D}$ is qualitatively the same as the one for f. Between the matter originally in the strip $[R_0, R_1]$ and the matter starting in $]0, (r_- + r_0)/2[$ at time T there is a vacuum region where $m_t = -4\pi e^{\mu-\lambda} j = 0$ so that $m(t, r_0)$ is unchanged during $[T, 0]$. Since the motion of the matter originally in the strip $[R_0, R_1]$ is affected by the matter further in only through the quasi-local mass m, the motion of this matter is completely unchanged during the time interval $[T, 0]$. This means that $\tilde{f}(t = 0) = \mathring{f}$ on $[r_0, \infty[$. Since $\tilde{r}_0$ is as close to r_0 as we wish and the matter outside is the same as in the unperturbed case, $\tilde{f}(t = 0)$ belongs to the set $\mathcal{I}_1$, and we are done.

Acknowledgement : The authors are grateful for discussions with A. Rendall, who also suggested the title of what is now Section 8.

References

[1] H. Andréasson, The Einstein-Vlasov System/Kinetic Theory, *Living Rev. Relativity* **8** (2005).

[2] H. Andréasson, On global existence for the spherically symmetric Einstein-Vlasov system in Schwarzschild coordinates, *Indiana Univ. Math. J.* **56**, 523–552 (2007).

[3] H. Andréasson, M. Kunze, G. Rein, Global existence for the spherically symmetric Einstein-Vlasov system with outgoing matter, to appear in *Comm. Partial Differential Eqns.*

[4] H. Andréasson, G. Rein, A numerical investigation of the stability of steady states and critical phenomena for the spherically symmetric Einstein-Vlasov system, *Class. Quantum Gravity* **23**, 3659–3677 (2006).

[5] J. Binney, S. Tremaine, *Galactic Dynamics*, Princeton University Press 1987.

[6] D. Christodoulou, Violation of cosmic censorship in the gravitational collapse of a dust cloud, *Comm. Math. Phys.* **93**, 171–195 (1984).

[7] D. Christodoulou, A mathematical theory of gravitational collapse, *Comm. Math. Phys.* **109**, 613–647 (1987).

[8] D. Christodoulou, The formation of black holes and singularities in spherically symmetric gravitational collapse, *Comm. Pure Appl. Math.* **44**, 339–373 (1991).

[9] D. Christodoulou, Examples of naked singularity formation in the gravitational collapse of a scalar field, *Ann. of Math.* (2) **140**, 607-653 (1994).

[10] D. Christodoulou, The instability of naked singularities in the gravitational collapse of a scalar field, *Ann. of Math.* (2) **149**, 183-217 (1999).

[11] D. Christodoulou, On the global initial value problem and the issue of singularities, *Class. Quantum Gravity* **16**, A23–A35 (1999).

[12] M. Dafermos, Spherically symmetric spacetimes with a trapped surface, *Class. Quantum Gravity* **22**, 2221–2232 (2005).

[13] M. Dafermos, A. D. Rendall, An extension principle for the Einstein-Vlasov system in spherical symmetry, *Ann. Henri Poincaré* **6**, 1137–1155 (2005).

[14] C. Gundlach, Critical Phenomena in Gravitational Collapse, *Living Rev. Relativity* **2** (1999).

[15] S. Hawking, G. F. R. Ellis, *The Large Scale Structure of Space-time,* Cambridge University Press 1975.

[16] P.-L. Lions, B. Perthame, Propagation of moments and regularity for the 3-dimensional Vlasov-Poisson system, *Invent. Math.* **105**, 415–430 (1991).

[17] I. Olabarrieta, M. W. Choptuik, Critical phenomena at the threshold of black hole formation for collisionless matter in spherical symmetry, *Phys. Rev. D.* **65**, 024007 (2002).

[18] J. R. Oppenheimer, H. Snyder, On continued gravitational contraction, *Phys. Rev.* **56**, 455–459 (1939).

[19] R. Penrose, Gravitational collapse and space-time singularities, *Phys. Rev. Lett.* **14**, 57–59 (1965).

[20] K. Pfaffelmoser, Global classical solutions of the Vlasov-Poisson system in three dimensions for general initial data, *J. Differential Equations* **95**, 281–303 (1992).

[21] E. Poisson, *A Relativist's Toolkit; The Mathematics of Black Hole Mechanics,* Cambridge University Press 2004.

[22] G. Rein, *The Vlasov-Einstein System with Surface Symmetry*, Habilitationsschrift, München 1995.

[23] G. Rein, A. D. Rendall, Global existence of solutions of the spherically symmetric Vlasov-Einstein system with small initial data, *Comm. Math. Phys.* **150**, 561–583 (1992). Erratum: *Comm. Math. Phys.* **176**, 475–478 (1996).

[24] G. Rein, A. D. Rendall, The Newtonian limit of the spherically symmetric Vlasov-Einstein system, *Comm. Math. Phys.* **150**, 585–591 (1992).

[25] G. Rein, A. D. Rendall, J. Schaeffer, A regularity theorem for solutions of the spherically symmetric Vlasov-Einstein system, *Comm. Math. Phys.* **168**, 467–478 (1995).

[26] G. Rein, A. D. Rendall, J. Schaeffer, Critical collapse of collisionless matter—a numerical investigation, *Phys. Rev. D* **58**, 044007 (1998).

[27] A. D. Rendall, The Newtonian limit for asymptotically flat solutions of the Vlasov-Einstein system. *Comm. Math. Phys.* **163**, 89–112 (1994).

[28] A. D. Rendall, Cosmic censorship and the Vlasov equation, *Class. Quantum Gravity* **9**, L99–L104 (1992).

[29] A. D. Rendall, Theorems on existence and global dynamics for the Einstein equations, *Living Rev. Relativity* **6** (2005).

[30] R. Wald, *General Relativity*, Chicago University Press 1984.