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Abstract

It is widely recognized that when classical optimal strategies are used with parameters estimated from data, the resulting portfolio weights are remarkably volatile and unstable over time. The predominant explanation for this is the difficulty to estimate expected returns accurately. We propose to parameterize an n stock Black-Scholes model as an n factor Arbitrage Pricing Theory model where each factor has the same expected return. Hence the non-unique volatility matrix determines both the covariance matrix and the expected returns. This enables the investor to impose views on the future performance of the assets in the model. We derive an explicit strategy π^* which solves Markowitz' continuous time portfolio problem in our framework. The optimal strategy is to implicitly keep $1/n$ of the wealth invested in stocks in each of the n underlying factors. To illustrate the long-term performance of π^*, we apply it out-of-sample to a large data set. We find that it is stable over time and outperforms all the underlying market assets in terms of Sharpe ratios. Further, π^* had a significantly higher Sharpe ratio than the classical $1/n$ strategy.

Key Words: Black-Scholes model, robust portfolio optimization, equal risk premiums, Markowitz' problem, $1/n$ strategy, ranks.

1 Introduction

The fundamental question of portfolio optimization is natural: How do we trade in the stock market in the best possible way? However, this is not easy to answer. Classical optimal strategies applied with parameters estimated from data are known to give irrational portfolio weights. This is primarily due to the

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difficulty to estimate expected returns with sufficient accuracy, see examples in [2]. This motivates to study how to circumvent this problem.

Several different methods for estimating expected returns have been published which do not rely entirely on statistics. For example, Black and Litterman [2] proposed to estimate the expected returns by combining market equilibrium with subjective investor views. A drawback with this approach is that the investor still has to quantify her beliefs by specifying numbers for the expected returns, admittedly with an uncertainty attached to them. The effects of this action are hard to control.

The Arbitrage Pricing Theory (APT), see [11], is another acclaimed approach. The APT models the discrete time returns of the stocks as a linear combination of independent factors. The APT relies on statistical estimates of the expected returns that are constructed to fit historical data. Hence, it is due to give unstable portfolio weights.

Yet another popular method to estimate expected returns is simply to ignore them. This idea is pursued for example in the classical $1/n$ strategy, which puts $1/n$ of the investor's capital in each of n available assets. However, this strategy does not use the dependence between different stocks. This is a disadvantage, since it is possible to obtain good estimates of, for example, the covariance between stock returns.

Recently, some authors have proposed to let the expected returns depend on ranks. These ranks could, for example, be based on the capital distribution of the market, which is fairly stable over time. For developments of this interesting idea, see [3].

Our goal is to find optimal trading strategies that circumvent the severe problems associated with estimating expected returns. Further, we want to allow for investors to specify their unique market views through the market model in a robust way. To this end, we parameterize the Black-Scholes model as an n factor APT model with no individual error terms, and make the assumption that each factor has the same expected return. Hence, expected returns are determined by the volatility matrix and the expected return of the factors. The non-uniqueness of the volatility matrix allows the investor to impose her views on the market by selecting a volatility matrix which suggests expected returns of the stocks that she believes are reasonable.

Modern portfolio optimization was initialized by Markowitz in [8]. Markowitz measured the risk of a portfolio by the variance of its return. He then formulated a one-period quadratic program where he minimized a portfolio's variance subject to the constraint that the expected return should be greater than some constant. Merton ([9] and [10]) was the first to consider continuous time portfolio optimization. He used dynamic programming and stochastic control to maximize expected utility of the investor's terminal wealth. The first results on continuous time versions of Markowitz problem were published rather recently, see [1], [5], [6], [7], [12], and [13].

We solve Markowitz' continuous time portfolio problem explicitly for our n stock market model. The optimal strategy π^* is to implicitly hold $1/n$ of the wealth invested in stocks in each of the n underlying factors, regardless of how we have chosen expected returns and the dependence between the stocks through the volatility matrix. This is not the same as holding $1/n$ of the wealth in each stock.

We apply π^*, out-of-sample, to two different data sets. For the first data set, we analyze how investor views transforms into expected returns, and how this affects the optimal strategy. For the second data set, the long-term performance of π^* is investigated, when no investor preferences are assumed. We find that π^* is stable over time and outperforms all the underlying market assets in terms of Sharpe ratios. Moreover, we can reject the hypothesis that the classical $1/n$ strategy gives a higher Sharpe ratio than π^* with a very low level of significance.

We present our model in Section 2. Further, we give some examples of procedures for obtaining volatility matrices that imply rates of return with different features. An optimal portfolio for a continuous time version of Markowitz' problem for n stocks is solved explicitly in Section 3. Section 4 contains an empirical study of the optimal strategy.

2 The model

We present in this section the model for the stocks. Further, we discuss how to estimate the volatility matrix, and its relation to the expected returns for different assets.

2.1 The stock price model

For $0 \leq t \leq T < \infty$, we assume as given a complete probability space $(\Omega, \mathcal{F}, P)$ with a filtration $\{\mathcal{F}_t\}_{0 \leq t \leq T}$ satisfying the usual conditions. We take n independent Brownian motions B_i, and define the stocks S_i, $i = 1, ..., n$, to have the dynamics

$$dS_i\,(t) = S_i\,(t) \left(rdt + \sum_{j=1}^{n} \sigma_{i,j} \left[(\mu - r)\, dt + dB_j\,(t) \right] \right), \tag{2.1}$$

for the continuously compounded interest rate $r > 0$, and constant $\mu > r$, where the volatility matrix $\sigma := \{\sigma_{i,j}\}_{i,j=1}^{n}$ is assumed to be non-singular. The stock price processes become

$$S_i\,(t) = S_i\,(0) \exp \left(\left(r - \frac{1}{2} \sum_{j=1}^{n} \sigma_{i,j}^2 \right) t + \sum_{j=1}^{n} \sigma_{i,j} \left[(\mu - r)\, t + B_j\,(t) \right] \right),$$

for $B_1\,(0) = \ldots = B_n\,(0) = 0$. We also equip the market with a risk free bond with dynamics

$$dR\,(t) = rR\,(t)\,dt.$$

This is the classical Black-Scholes model parameterized as an n factor APT model with no individual random error terms. Hence, the stocks S_i depend on n independent *risk factors* $F_j(t) := [(\mu - r)t + B_j(t)]$. In addition, the expected returns are assumed to be equal for each factor. A consequence of this model is that the continuously compounded expected returns are determined by $\mu - r$ and the volatility matrix. We have no additional information on any factor, so equal expected return for all of them is a reasonable assumption.

We discuss now how to choose the volatility matrix.

2.2 The volatility matrix and the rates of return

In mathematical finance, the volatility matrix σ is typically used only to model the covariance between the returns of different stocks. However, a given covariance matrix C does not uniquely define a σ such that $C = \sigma\sigma^T$. The model presented above allows the investor to impose her views on the market by selecting a volatility matrix which suggests expected returns of the stocks that she believes are accurate. Hence expected returns and investor views can be expressed in a manner less sensitive to statistical estimates and guesses. We first describe two basic examples of volatility matrices. It is then shown that all volatility matrices which imply the same C can be written as the Cholesky decomposition of C multiplied by an orthogonal matrix.

Example 2.1 *The Sharpe ratio (SR) of a stock is defined as its yearly expected return in excess of the risk-free return divided by the volatility. The SR are hard to estimate accurately since it requires estimates of expected returns. For the investor, one alternative to setting expected returns is to specify ranks. Assume that the investor has ranked the stocks according to her beliefs for their SR. The stock with the highest presumed SR is assigned rank* 1*, the second highest gets rank* 2*, and so on. We now order the stocks according to their rank, with the stock with rank* n *on line* 1*, the stock with rank* $n-1$ *on line* 2*, and so on. Cholesky decomposition applied to the corresponding ordered covariance matrix gives a clear tendency for stocks with high ranks to have large SR. The reason is that the row sums of the lower triangular volatility matrix will tend to be larger for stocks with high ranks. Hence, the continuously compounded rates of return for these stocks will be larger, and consequently also the yearly expected returns.*

Consider the two stocks S_A*, and* S_B*. The investor ranks* S_A *to have the lowest SR. In this example we take the covariance matrix*

$$C = \begin{pmatrix} 4 & 2 \\ 2 & 5 \end{pmatrix}, \tag{2.2}$$

which is sorted in order of increasing SR. Cholesky decomposition gives then the volatility matrix

$$\sigma = \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}.$$

The continuously compounded rates of return for S_A *and* S_B *are* $r - 2 + 2(\mu - r)$

and $r - \frac{1}{2}\left(1 + 2^2\right) + (1+2)(\mu - r)$, *respectively.*

Another alternative is to assign an economic interpretation to the factors. For example, one can assume that each stock has a unique factor associated to it which represents the uncertainty primarily due to that stock.

Example 2.2 *Presume that* S_A *depends as much on what happens to factor* A*, as* S_B *depends on what happens to factor* B*. This is reasonable for companies of about the same size and importance to each other, whether or not they operate on the same market. With the economic interpretation of the factors given above, the volatility matrix should be symmetric. Symmetry can be attained by taking the matrix square root of the covariance matrix.*

For the covariance matrix in Equation (2.2) the square root volatility matrix is

$$\sigma = \begin{pmatrix} 1.940 & 0.485 \\ 0.485 & 2.183 \end{pmatrix}.$$

The stock risk premiums for this method are approximately equal: $1.213\,(\mu - r)$ *for* S_A*, and* $1.193\,(\mu - r)$ *for* S_B*.*

We consider now some standard results from linear algebra. Assume that the positive definite covariance matrix C can be written as $C = VV^T$ for some matrix V. We know by QR factorization that V can be written as $V = LQ$, where L is lower triangular and Q is orthogonal. It follows that $C = VV^T = LQQ^TL^T = LL^T$, regardless of orthogonal Q. But since L is lower triangular, it must be equal to the unique Cholesky decomposition of C. We conclude that all volatility matrices can be written as the Cholesky decomposition multiplied by an orthogonal matrix.

Remark 2.1 *Even though the choice of volatility matrix does not change the covariance matrix, we will see below that it has a crucial impact on the optimal trading strategy. This is due to that the volatility matrix determines the expected returns.*

3 Markowitz' problem in continuous time

We derive in this section an explicit solution to Markowitz' problem in continuous time, given our market model. The optimal strategy is to implicitly keep $1/n$ of the wealth invested in stocks in each factor F_j.

3.1 An explicit solution

Our objective is to solve the continuous time Markowitz problem

$$\begin{aligned} &\min_{\pi \in \mathcal{A}} \left\{ Var\left(W^{\pi}(T)\right)\right\}, \\ &\mathbb{E}\left[W^{\pi}(T)\right] \geq w \exp\left(\lambda T\right). \end{aligned} \tag{3.1}$$

Here w is the initial wealth, λ is the continuously compounded required rate of return, and $W^{\pi}(T)$ is the wealth at the deterministic time T, given that we invest according to the *admissible* strategy $\pi \in \mathcal{A}$. The admissible strategies $\mathcal{A}$ are the set of all $\mathbb{R}^n$-valued stochastic processes that are uniformly bounded and progressively measurable in $\mathcal{F}_t$. We sometimes write W for W^{π} when there is no risk for confusion.

The self-financing wealth process W is defined as

$$W(t) = w + \sum_{i=1}^{n} \int_0^t \frac{\pi_i(s) W(s)}{S_i(s)} dS_i(s) + \int_0^t \frac{\left(1 - \sum_{i=1}^n \pi_i(s)\right) W(s)}{R(s)} dR(s),$$

for all $t \in [0, T]$, where $\pi_i(t) W(t) / S_i(t)$ is the number of shares of stock i which is held at time t. See [4] for a motivating discussion. This gives the wealth dynamics

$$\begin{aligned} dW(t) &= \sum_{i=1}^{n} \pi_i(t) W(t) \left(\sum_{j=1}^{n} \sigma_{i,j} \left[(\mu - r) dt + dB_j(t) \right] \right) + W(t) r dt \\ &= W(t) \left(\sum_{j=1}^{n} p_j(t)(\mu - r) dt + \sum_{j=1}^{n} p_j(t) dB_j(t) + r dt \right), \end{aligned}$$

for the processes $p_j := \sum_{i=1}^n \pi_i \sigma_{i,j}$.

The assumption that π is uniformly bounded implies that the equation for W can be written as

$$W(t) = w \exp\left(\sum_{j=1}^{n} \left[\int_0^t \left[p_j(s)(\mu - r) - \frac{1}{2} p_j^2(s) \right] ds + \int_0^t p_j(s) dB_j(s) + rt \right] \right).$$

To avoid trivial cases, we assume that $\lambda > r$ such that we need to invest in some risky asset to obtain an expected yield larger than $w \exp(\lambda T)$.

For the optimal strategy π^*, we must have that

$$\mathbb{E}\left[W^{\pi^*}(T) \right] = w \exp(\lambda T). \tag{3.2}$$

To see this, consider a strategy π with $\mathbb{E}[W^{\pi}(T)] > w \exp(\lambda T)$. We know that

$$\begin{aligned} Var(W^{\pi}(T)) &= \mathbb{E}[Var(W^{\pi}(T) | p)] \\ &= w^2 \mathbb{E}\left[\exp\left(2 \left((\mu - r) \sum_{j=1}^{n} \int_0^T p_j(t) dt + rT \right) \right) \right. \\ &\qquad \left. \left(\exp\left(\sum_{j=1}^{n} \int_0^T p_j^2(t) dt \right) - 1 \right) \right]. \end{aligned}$$

It is a necessary condition for an optimal strategy that $\sum_{j=1}^{n} p_j(t) \geq 0$ for all $(t, \omega) \in [0, T] \times \Omega$. The reason is that whenever this condition is violated, exchanging π for the strategy to put all the money in the risk free asset will both increase expected return and lower the variance. Hence, the strategy $\alpha\pi$, for any $\alpha \in (0, 1)$ such that Equation (3.1) holds, has lower variance than π by the definition of the p_j.

We consider now the deterministic and constant process

$$\tilde{p}_1(t) = ... = \tilde{p}_n(t) = \frac{1}{n}\frac{\lambda - r}{\mu - r} =: \tilde{p}, \tag{3.3}$$

and associated strategy $\tilde{\pi}$. Note that $\mathbb{E}\left[W^{\tilde{\pi}}(T)\right] = w\exp(\lambda T)$. This implies, for any strategy $\pi \in \mathcal{A}$ which satisfies Equation (3.2), that

$$\begin{aligned} Var(W^{\pi}(T)) &= \mathbb{E}\left[Var(W^{\pi}(T)|p)\right] \\ &= w^2\exp(2\lambda T)\left(\exp\left(n\tilde{p}^2 T\right) - 1\right) \\ &\quad \times \mathbb{E}\left[\frac{\exp\left(2\left((\mu - r)\sum_{j=1}^{n}\int_0^T p_j(t)\,dt + rT\right)\right)}{\exp(2\lambda T)}\right. \\ &\quad \left.\frac{\left(\exp\left(\sum_{j=1}^{n}\int_0^T p_j^2(t)\,dt\right) - 1\right)}{\left(\exp\left(n\tilde{p}^2 T\right) - 1\right)}\right]. \end{aligned}$$

Set $I_p := \frac{1}{nT}\sum_{j=1}^{n}\int_0^T p_j(t)\,dt$. We can use Jensen's inequality to see that

$$\exp\left(\sum_{j=1}^{n}\int_0^T p_j^2(t)\,dt\right) \geq \exp\left(nI_p^2 T\right),$$

for all $\omega \in \Omega$, so

$$\begin{aligned} Var(W^{\pi}(T)) \geq\ & w^2\exp(2\lambda T)\left(\exp\left(n\tilde{p}^2 T\right) - 1\right) \\ & \times \mathbb{E}\left[\frac{\exp(2((\mu - r)nI_p + r)T)\left(\exp\left(nI_p^2 T\right) - 1\right)}{\exp(2\lambda T)\left(\exp\left(n\tilde{p}^2 T\right) - 1\right)}\right]. \end{aligned}$$

We have assumed that

$$\mathbb{E}\left[\frac{\exp(((\mu - r)nI_p + r)T)}{\exp(\lambda T)}\right] = 1.$$

We see now that

$$\frac{\exp(((\mu - r)nI_p + r)T)\left(\exp\left(nI_p^2 T\right) - 1\right)}{\exp(\lambda T)\left(\exp\left(n\tilde{p}^2 T\right) - 1\right)} > 1,$$

for $I_p > \tilde{p}$, and

$$\frac{\exp\left(\left(\left(\mu-r\right)nI_p+r\right)T\right)\left(\exp\left(nI_p^2T\right)-1\right)}{\exp\left(\lambda T\right)\left(\exp\left(n\tilde{p}^2T\right)-1\right)}<1,$$

for $I_p < \tilde{p}$. This gives

$$Var\left(W^{\pi}\left(T\right)\right)\geq w^2\exp\left(2\lambda T\right)\left(\exp\left(n\tilde{p}^2T\right)-1\right),$$

with equality only for $\pi = \tilde{\pi}$. Hence, for sufficiently high bounds on the admissible strategies, the strategy π^* that solves

$$\begin{aligned}\pi_1\sigma_{1,1}+...+\pi_n\sigma_{n,1} &= \frac{1}{n}\frac{\lambda-r}{\mu-r} \\ &\vdots \\ \pi_1\sigma_{1,n}+...+\pi_n\sigma_{n,n} &= \frac{1}{n}\frac{\lambda-r}{\mu-r},\end{aligned}\tag{3.4}$$

for all $t \in [0,T]$ minimizes the variance of the terminal wealth $W(T)$ subject to the growth constraint in Equation (3.1). The equation for W^{π^*} becomes

$$\begin{aligned}W^{\pi^*}\left(t\right) &= w\exp\left(\lambda t-\left(\frac{\lambda-r}{\mu-r}\right)^2\frac{t}{2n}+\frac{1}{n}\frac{\lambda-r}{\mu-r}\sum_{j=1}^{n}B_j\left(t\right)\right) \\ &=_d w\exp\left(\left(\lambda-\frac{1}{2n}\left(\frac{\lambda-r}{\mu-r}\right)^2\right)t+\frac{1}{\sqrt{n}}\frac{\lambda-r}{\mu-r}B\left(t\right)\right)\end{aligned}$$

for a Brownian motion B, where "$=_d$" denotes equality in distribution.

Remark 3.1 *The effect on the wealth process W^{π^*} from increasing the number of stocks n is illustrated in Figure 3.1. The figure shows that the higher expected return the investor requires, the more she will have to risk. Nonetheless, the risk will decrease as the number of stocks n increases. Note that W^{π^*} is strictly positive with probability* 1, *so the investor does not risk bankruptcy.*

Remark 3.2 *There are interesting connections between π^* and Merton's classical portfolio problems. For example, assume that we want to find a strategy $\tilde{\pi}$ that maximizes expected utility of terminal wealth $\mathbb{E}\left[U\left(W^{\pi}\left(T\right)\right)\right]$. Merton's $\hat{\pi}$ for logarithmic utility $U(w) = \log(w)$ in our model becomes then*

$$\hat{\pi}=\left(\mu-r\right)\left(\sigma\sigma^T\right)^{-1}\sigma\mathbf{1}=\left(\mu-r\right)\left(\sigma^T\right)^{-1}\sigma^{-1}\sigma\mathbf{1}=\left(\mu-r\right)\left(\sigma^T\right)^{-1}\mathbf{1}.$$

This $\hat{\pi}$ is of the same form as the optimal strategy π^* for our continuous time Markowitz' problem.

Remark 3.3 *The optimal strategy π^* has the advantage that the investor can apply it without estimating the parameters μ and λ: Assume that the investor*

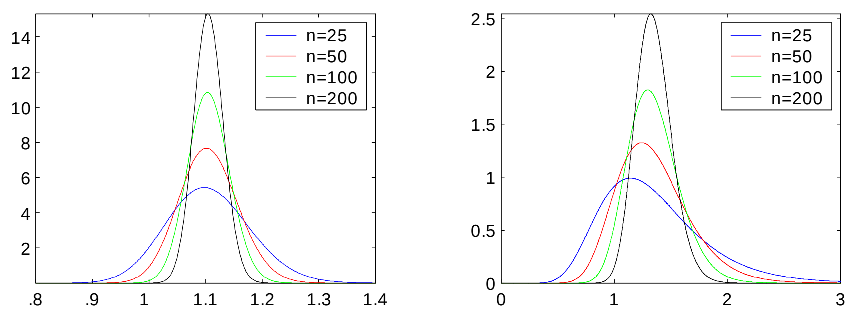

Figure 3.1: *Left:* Distributions of optimal wealth at time $T = 1$ for different n with parameters $w = 1$, $\lambda = 0.1$, $\mu = 0.2$, and $r = 0.03$. *Right:* Distributions of optimal wealth at time $T = 1$ for different n with parameters $w = 1$, $\lambda = 0.3$, $\mu = 0.2$, and $r = 0.03$.

has chosen a fraction of her wealth to invest in risky assets. This choice, together with Equations (3.4), determines $(\lambda - r) / (\mu - r)$.

4 An empirical study of π^*

We investigate the empirical performance of π^* by analyzing two different data sets. For the first data set, we examine how investor preferences are translated into expected returns, and the effect these expected returns have on the optimal strategy. For the second data set, we analyze the long-term efficiency of π^* when no investor views are assumed. Throughout this section, we set $r = 0.05$. Further, we assume that the investor is fully invested in the stock market at all times.

4.1 The strategy π^* with ranks

The first data set comprises three different stocks traded at OMX - The Nordic Exchange. The stocks are Ericsson B, Hennes & Mauritz B, and Volvo B. The data is from the time period 2002-07-01 to 2007-01-01. The covariance matrix is estimated from a window of 18 months of data, and it is updated each month. We have ranked the stocks with regards to their presumed SR; Ericsson B: rank 1; Hennes & Mauritz B: rank 2; Volvo B: rank 3. We find the volatility matrix by applying Cholesky decomposition as in Example 2.1. The optimal strategy π^* is applied out-of-sample, with daily adjustments of the portfolio weights. It can be seen from Figure 4.1 that the optimal strategy is stable over time, and that the portfolio weights are positive for every stock the entire time period. The wealth process associated with π^* is presented in Figure 4.2.

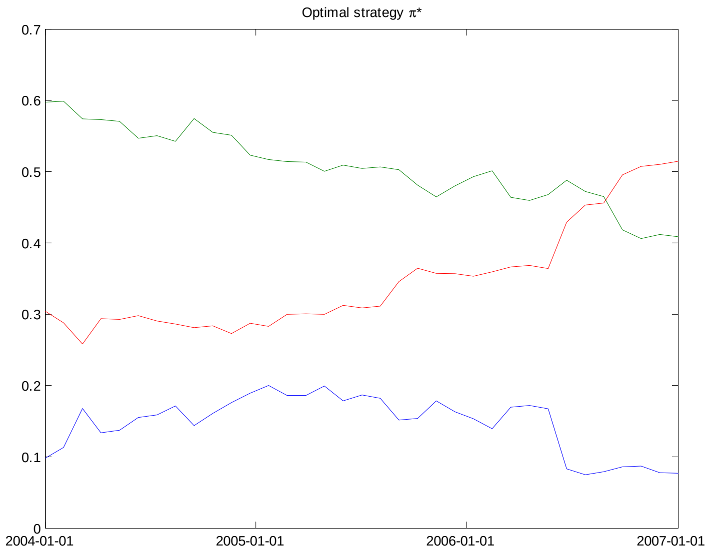

Figure 4.1: Blue, green, and red lines denote optimal fractions of wealth, based on ranks, to be held in Volvo B, Hennes & Mauritz B, and Ericsson B, respectively.

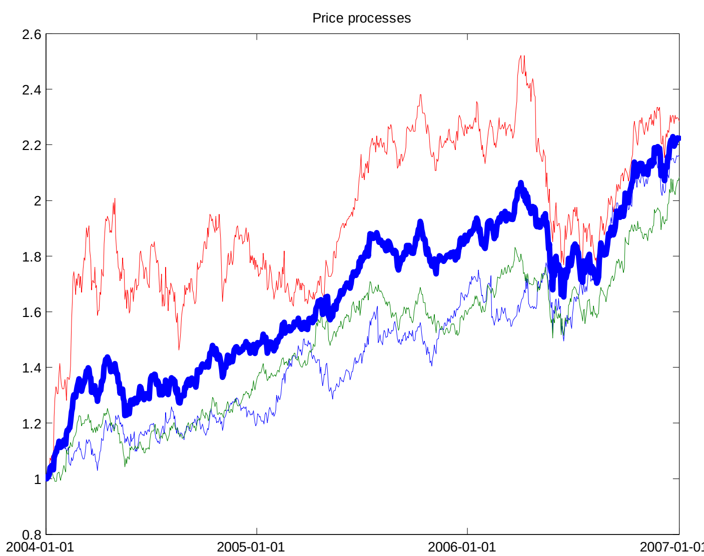

Figure 4.2: *Colored lines:* Prices for Volvo B, Hennes & Mauritz B, and Ericsson B. *Blue thick line:* Wealth process for the optimal strategy π^* based on ranks for the three stocks.

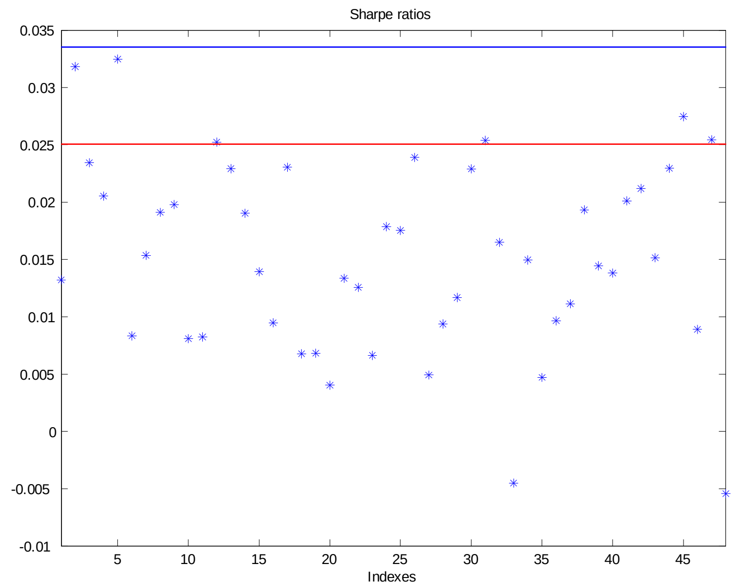

Figure 4.3: Blue and red lines represent the Sharpe ratio for π^* and the classical $1/n$ strategy, respectively. Blue stars are the Sharpe ratios for the individual industry portfolios.

4.2 The strategy π^* without investor views

The second data set consists of 48 value weighted industry portfolios, which we treat as stocks, consisting of each stock traded at NYSE, AMEX, and NASDAQ. The data is from the time period 1963-07-01 to 2005-12-30. The covariance matrix is estimated from a five-year window of data, with the Black Monday of 1987 removed, and it is updated each month. The Black Monday is not removed from the return data. It is reasonable to assume that the industry portfolios are approximately equally important to each other. Also, the investor has no preferences regarding any assets. Hence, we apply the matrix square root to the covariance matrix to get the volatility matrix, see Example 2.2. The optimal strategy π^* is applied out-of-sample, with daily adjustments of the portfolio weights. For this data set, the strategy π^* outperformed the underlying market assets in terms of Sharpe ratios. Further, π^* obtained 29% more wealth than the classical $1/n$ strategy, and with 16% lower volatility. Consequently, Memmel's corrected Jobson & Korkie test of the hypothesis that the classical $1/n$ strategy gives a higher Sharpe ratio than π^* had a p value smaller than 10^{-6}, see Figure 4.3. Figure 4.4 shows the evolvement of the estimated strategies π_i^*, which are quite stable over time. The industry portfolio with the largest average fraction of wealth invested in it is the Paper index.

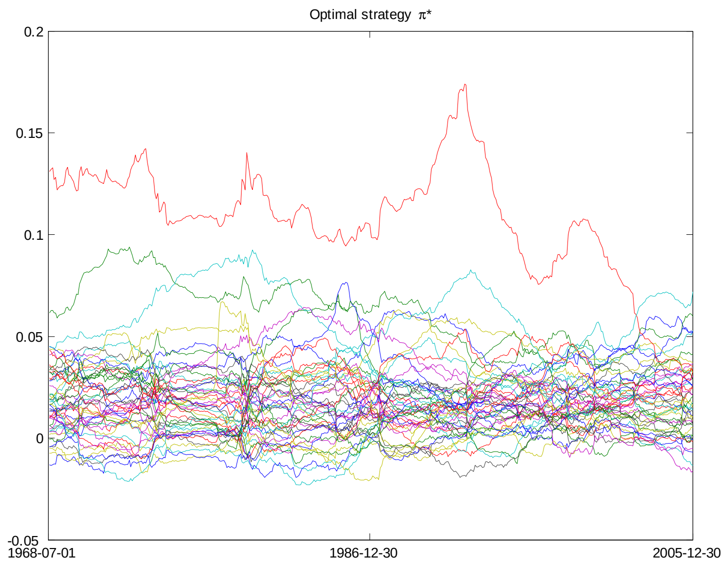

Figure 4.4: The optimal strategy π^* for the 48 industry portfolios.

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