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Abstract

In this paper we consider ideals of a C^*-algebra $C^*(\mathcal{B})$ generated by an operator algebra $\mathcal{B}$. A closed ideal $J \subseteq C^*(\mathcal{B})$ is called K-boundary ideal if the restriction of the quotient map on $\mathcal{B}$ has a completely bounded inverse with cb-norm equal to K^{-1}. For $K = 1$ one gets the notion of boundary ideals introduced by Arveson. We study the properties of K-boundary ideals and characterize them in the case when operator algebra λ-norms itself. Several reformulations of Kadison similarity problem are given. In particular, the affirmative answer to this problem is equivalent to the statement that every bounded homomorphism from $C^*(\mathcal{B})$ onto $\mathcal{B}$ which is projection on $\mathcal{B}$ is completely bounded.

1 Introduction.

An operator algebra is a subalgebra of $B(H)$, the algebra of all bounded operators on a Hilbert space H. The algebra $M_n(B(H))$ of $n \times n$ matrices with entries in $B(H)$ has a norm $\|\cdot\|_n$ via the identification of $M_n(B(H))$ with $B(H^n)$, where H^n is the direct sum of n copies of a Hilbert space H. If $\mathcal{A}$ is a subalgebra of $B(H)$ then $M_n(\mathcal{A})$ inherits a norm $\|\cdot\|_n$ via natural inclusion into $M_n(B(H))$. The norms $\|\cdot\|_n$ are called matrix norms on the operator algebra $\mathcal{A}$. The Blecher-Ruan-Sinclair Theorem [6] abstractly characterize operator algebras in terms of matrix norms.

A linear mapping $\phi : \mathcal{A} \to B(H)$ induces a linear mapping

$$\phi^{(n)} : M_n(\mathcal{A}) \to B(\bigoplus_{i=1}^{n} H)$$

via the formula $\phi^{(n)}((a_{ij})) = (\phi(a_{ij}))$ for every $(a_{ij}) \in M_n(\mathcal{A})$. The map ϕ is called completely bounded if there exists C such that $\|\phi\|_{cb} = \sup_n \|\phi^{(n)}\| < C < \infty$, it is called completely contractive (isometric) if $\phi^{(n)}$ is contractive (corresp. isometric) for every $n \in \mathbb{N}$.

The C^*-envelope of an operator algebra $\mathcal{A}$, denoted by $C^*_e(\mathcal{A})$, is a C^*-algebra generated by $i(\mathcal{A})$ for some completely isometric homomorphism $i : \mathcal{A} \to B(H)$ having the following universal property. For any completely isomorphic homomorphism $\rho : \mathcal{A} \to B(K)$ there exists a unique onto $*$-homomorphism $\pi : C^*(\rho(\mathcal{A})) \to C^*_e(\mathcal{A})$ such that $\pi(\rho(a)) = i(a)$ for every $a \in \mathcal{A}$.

In [1] Arveson defined a noncommutative analog of the Šilov boundary of uniform algebra. It is known that the Šilov boundary of a uniform algebra A is the closure of the Choquet boundary. The (irreducible) boundary representations of A correspond to points of the Choquet boundary. In noncommutative setting let $\mathcal{B}$ be an operator algebra in $B(H)$ and $C^*(\mathcal{B})$ be the C^*-algebra generated by $\mathcal{B}$ in $B(H)$. Then a closed (two-sided) ideal J of $C^*(\mathcal{B})$ is called boundary ideal if the canonical quotient map $q : C^*(\mathcal{B}) \to C^*(\mathcal{B})/J$ is completely isometric on $\mathcal{B}$. The Šilov boundary is a boundary ideal containing all boundary ideals. It can be shown that the C^*-envelope of an operator algebra $\mathcal{B}$ is the quotient of C^*-algebra generated by $\mathcal{B}$ by Šilov boundary whenever the latter exists. Arveson showed the existence of C^*-envelopes for *admissible* operator algebras developing theory of *boundary representations* in [1]. The existence in full generality was shown in [11]. But the question wether there are sufficiently many boundary representations to construct the C^*-envelope was not settled until the works [7] and [2] appeared.

In this paper we study a generalization of boundary ideals. Namely, a closed (two-sided) ideal $J \subset C^*(\mathcal{B})$ will be called K-boundary ideal if a canonical quotient map $q : C^*(\mathcal{B}) \to C^*(\mathcal{B})/J$ restricted to $\mathcal{B}$ has a completely bounded inverse $q|_{\mathcal{B}}^{-1}$ with completely bounded norm equal to K^{-1}. For $K = 1$ we obtain the boundary ideals of [1]. Surprisingly, there is no analog of Šilov boundary for K-boundary ideals for $K \neq 1$. That is in general there is no K-boundary ideal containing every other K-boundary ideal, see Example 3.

As a corollary of [15] and the solution of Halmos problem [17] there exists bounded homomorphism of the disk algebra $A(\mathbb{D})$ into $B(H)$ which is not completely bounded. Adapting this result to the quotient maps in Example 5 we construct an operator algebra $\mathcal{B}$ and an ideal J such that $q|_{\mathcal{B}}^{-1}$ is bounded but not completely bounded.

Further, we prove that if an operator algebra $\mathcal{B}$ λ-norms itself and J is closed ideal of C^*-algebra generated by $\mathcal{B}$ such that the restriction of the quotient map $q : C^*(\mathcal{B}) \to C^*(\mathcal{B})/J$ has continuous inverse $q|_{\mathcal{B}}^{-1}$ then $q|_{\mathcal{B}}^{-1}$ is completely bounded.

In Section 3 we consider the following problem raised by Kadison in 1955. Is any bounded homomorphism π of a C^*-algebra $\mathcal{A}$ into $B(H)$ is similar to a $*$-homomorphism? The similarity above means that there exists an invertible operator $S \in B(H)$ such that $S^{-1}\pi(\cdot)S$ is $*$-homomorphism. Haagerup [10] showed that π is similar to a $*$-homomorphism if and only if it is completely bounded. Obviously, for this problem it is sufficient to consider only faithful homomorphisms. Pitts [19] proved that every bounded faithful homomorphism of a C^*-algebra has a completely bounded inverse. In Section 3 we show that for Kadison's similarity problem it is sufficient to consider bounded faithful homomorphisms that have completely contractive inverses. As a corollary of this result we have that every bounded homomorphism of a C^*-algebra is completely bounded if and only if every bounded homomorphism from $C^*(\mathcal{B})$ onto $\mathcal{B}$ which is projection on $\mathcal{B}$ is completely bounded.

It was proved in [19] that bounded homomorphism π of a C^*-algebra $\mathcal{A}$ is completely bounded if and only if $\pi(\mathcal{A})$ λ-norms itself. Using this result and Theorem 4 we have a reformulation of Kadison's similarity problem in terms of ideals of a C^*-algebra generated by an operator algebra, see Theorem 6.

In this paper all operator algebras are supposed to be unital and for a given operator algebra $\mathcal{A} \subseteq B(H)$ the C^*-algebra generated by $\mathcal{A}$ in $B(H)$ is denoted by $C^*(\mathcal{A})$.

We refer the reader to the books [8], [16] and [17] for precise definitions, basic facts and terminology related to operator algebras, operator spaces and completely bounded maps.

2 K-boundary Ideals.

In this section we define and investigate a generalization of boundary ideals.

Definition 1. *Let $\mathcal{B}$ be a closed subalgebra of a unital C^*-algebra $\mathcal{A}$ such that $\mathcal{B}$ contains the unit and generates $\mathcal{A}$ as a C^*-algebra. A closed ideal J of $\mathcal{A}$ is called a K-boundary ideal if $K > 0$ is the greatest constant having* *the following property*

$$K \cdot \|a\|_{M_n(\mathcal{A})} \leq \|a\|_{M_n(\mathcal{A}/J)} \tag{1}$$

for every $a \in M_n(\mathcal{B})$ and $n \in \mathbb{N}$. In other words the canonical quotient map $q : \mathcal{A} \to \mathcal{A}/J$ restricted to $\mathcal{B}$ is injective and has a completely bounded inverse with completely bounded norm equal to K^{-1}.

If $C^*(\mathcal{B})$ is commutative then every ideal which satisfies inequality (1) with $n = 1$ and $K > 0$ is automatically boundary ideal, i.e. $K = 1$. It follows from the following observation.

Proposition 2. *Let J be closed a ideal of $C^*(\mathcal{B})$ and $0 < K < 1$ be a greatest constant satisfying inequality*

$$K \cdot \|b\|_{C^*(\mathcal{B})} \leq \|b\|_{C^*(\mathcal{B})/J}$$

for every $b \in \mathcal{B}$. Then there exists $b \in \mathcal{B}$ such that $\|b\|_{C^*(\mathcal{B})} = 1$ and $\|b^n\|_{C^*(\mathcal{B})} \to 0$ when $n \to \infty$.

Proof. Straightforward. $\square$

A boundary ideal is called Šilov boundary if it contains every other boundary ideal. In the following example we present an operator algebra such that the C^*-algebra it generates has K-boundary ideals (for some $K < 1$) but does not have a K-boundary ideal that contains all K-boundary ideals.

Example 3. *Consider*

$$\mathcal{B} = \{ \begin{pmatrix} x_1 & x_2 - x_1 \\ 0 & x_2 \end{pmatrix} \oplus x_1 \oplus x_2 \oplus \begin{pmatrix} y_1 & y_2 - y_1 \\ 0 & y_2 \end{pmatrix} \oplus y_1 \oplus y_2 : x_1, x_2, y_1, y_2 \in \mathbb{C}\}.$$

One can easily check that $\mathcal{B}$ is an algebra and the C^*-algebra generated by $\mathcal{B}$ in $M_8(\mathbb{C})$ is $M_2(\mathbb{C}) \oplus \mathbb{C} \oplus \mathbb{C} \oplus M_2(\mathbb{C}) \oplus \mathbb{C} \oplus \mathbb{C}$. Consider the following ideals

$$J_1 = M_2(\mathbb{C}) \oplus 0 \oplus 0 \oplus 0 \oplus \mathbb{C} \oplus \mathbb{C},$$
$$J_2 = 0 \oplus \mathbb{C} \oplus \mathbb{C} \oplus M_2(\mathbb{C}) \oplus 0 \oplus 0.$$

We have

$$\| \begin{pmatrix} x_1 & x_2 - x_1 \\ 0 & x_2 \end{pmatrix} \oplus x_1 \oplus x_2 \oplus \begin{pmatrix} y_1 & y_2 - y_1 \\ 0 & y_2 \end{pmatrix} \oplus y_1 \oplus y_2 \|_{C^*(\mathcal{B})}$$

$$= \max\{\| \begin{pmatrix} x_1 & x_2 - x_1 \\ 0 & x_2 \end{pmatrix} \|, \| \begin{pmatrix} y_1 & y_2 - y_1 \\ 0 & y_2 \end{pmatrix} \|\}$$

$$= \max\{\| \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \|, \| \begin{pmatrix} y_1 & y_2 - y_1 \\ 0 & y_2 \end{pmatrix} \|\}$$

$$\leq C \cdot \max\{\| \begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix} \|, \| \begin{pmatrix} y_1 & y_2 - y_1 \\ 0 & y_2 \end{pmatrix} \|\}$$

$$= C \cdot \| \begin{pmatrix} x_1 & x_2 - x_1 \\ 0 & x_2 \end{pmatrix} \oplus x_1 \oplus x_2 \oplus \begin{pmatrix} y_1 & y_2 - y_1 \\ 0 & y_2 \end{pmatrix} \oplus y_1 \oplus y_2 \|_{C^*(\mathcal{B})/J_1}$$

Let $q : C^*(\mathcal{B}) \to C^*(\mathcal{B})/J_1$ be the canonical quotient map. Using inequalities above we have $\|b\| \leq C \cdot \|q(b)\|$ for every $b \in \mathcal{B}$. Thus the restriction of q to $\mathcal{B}$, $q|_{\mathcal{B}}$, is invertible map into $M_8(\mathbb{C})$ and it is easy to see that $\|q|_{\mathcal{B}}^{-1}\| > 1$. By Smith's theorem [22] the completely bounded norm of $\tau := q|_{\mathcal{B}}^{-1}$ equals to the norm 8-th amplification. Therefore $\|\tau\|_{cb} = \|\tau^{(8)}\| \geq \|\tau\| > 1$ and J_1 is K-boundary ideal with $K = \|\tau\|_{cb}^{-1}$.

Similar arguments prove that J_2 is a K-boundary ideal with the same constant K. There is no K-boundary ideal that contains both ideals since the sum of J_1 and J_2 is the whole $C^*(\mathcal{B})$.

Let us note that inequality (1) with $n = 1$ may fail even when $q|_{\mathcal{B}}$ is injective. The reason is that the image of a Banach subalgebra of a C^*-algebra $\mathcal{A}$ under the quotient map is not necessarily closed. Consider for example the C^*-algebra $C(\overline{\mathbb{D}}_2)$ of continuous function on the disk $\mathbb{D}_2$ of radius 2 and its Banach subalgebra $A(\mathbb{D}_2)$ of analytic functions on $\mathbb{D}_2$ which have continuous extension on $\overline{\mathbb{D}}_2$. Let $\mathbb{D}_1$ be the disk of radius 1 and the same center as $\mathbb{D}_2$. The restriction of functions to $\mathbb{D}_1$ is a $*$-homomorphism $\pi : C(\overline{\mathbb{D}}_2) \to C(\overline{\mathbb{D}}_1)$ and $\pi(A(\mathbb{D}_2))$ is not closed since there are analytic functions on $\mathbb{D}_1$ which are not extendable to the analytic function on $\mathbb{D}_2$. Thus the quotient map $q : C(\overline{\mathbb{D}}_2) \to C(\overline{\mathbb{D}}_2)/J$, where $J = ker(\pi)$, maps $A(\mathbb{D}_2)$ into non-closed subalgebra in $C(\overline{\mathbb{D}}_2)/J$. An example of a Banach operator algebra which is isomorphic to a non-closed self-adjoint operator algebra via contractive isomorphism can be found in [13].

K-boundary ideals can be easily characterized in the case when $\mathcal{B}$ λ-norms itself. Let us recall necessary definitions.

For a given operator algebra $\mathcal{B}$ define norms $|||\cdot|||_n$ on $M_n(\mathcal{B})$ via

$$|||X|||_n = \sup\{\|RXC\| : R \in M_{1,n}(\mathcal{B}), C \in M_{n,1}(\mathcal{B}), \|R\| \leq 1, \|C\| \leq 1\}.$$

Evidently $|||X|||_n \leq \|X\|_n$ for every $X \in M_n(\mathcal{B})$.

An operator algebra $\mathcal{B}$ λ-norms itself, see [20], if there exists $\lambda > 0$ such that

$$\lambda \cdot \|X\|_n \leq |||X|||_n$$

for every $X \in M_n(\mathcal{B})$ and $n \in \mathbb{N}$.

In the following we will need the notion of the maximal enveloping C^*-algebra of an operator algebra. For a given operator algebra $\mathcal{B}$ there exists a C^*-algebra, denoted by $C^*_{max}(\mathcal{B})$, and a completely isometric homomorphism $i : \mathcal{B} \to C^*_{max}(\mathcal{B})$ such that $i(\mathcal{B})$ generates $C^*_{max}(\mathcal{B})$ as a C^*-algebra and has the following universal property (see [3], [5]). If $\pi : \mathcal{B} \to \mathcal{A}$ is a completely contractive homomorphism into a C^*-algebra $\mathcal{A}$ then there exists a unique $*$-homomorphism $\widetilde{\pi} : C^*_{max}(\mathcal{B}) \to \mathcal{A}$ extending π, i.e. $\widetilde{\pi} \circ i = \pi$. Algebra $C^*_{max}(\mathcal{B})$ is called maximal enveloping C^*-algebra of $\mathcal{B}$. The existence follows from the following construction, see [**?**] for details.

Define a semi-norm on the algebraic free product of $\mathcal{B}$ and $\mathcal{B}^*$ by

$$\begin{gathered}\|a\|_{\mathcal{B}*\mathcal{B}^*} = \sup\{\|(\pi * \pi^*)(a)\| : \pi : \mathcal{B} \to B(H) \\ \text{is completely contractive homomorphism}\}\end{gathered}$$

The null-space of this norm is two-sided ideal J. Then $\mathcal{B} * \mathcal{B}^*/J$ is pre-C^*-algebra and $C^*_{max}(\mathcal{B})$ is its completion. Uniqueness of $C^*_{max}(\mathcal{B})$ follows from the universal property.

Theorem 4. *If an operator algebra $\mathcal{B}$ λ-norms itself and $i : \mathcal{B} \to B(H)$ is completely isometric homomorphism then for every ideal $J \subset C^*(i(\mathcal{B}))$ such that the inequality*

$$K' \cdot \|b\|_{C^*(i(\mathcal{B}))} \leq \|b\|_{C^*(i(\mathcal{B}))/J} \tag{2}$$

holds for every $b \in \mathcal{B}$ and some $K' > 0$ we have that J is K-boundary ideal for some $K \geq \lambda K'$.

Proof. Let $X \in M_n(\mathcal{B})$, $R \in M_{1,n}(\mathcal{B})$, $C \in M_{n,1}(\mathcal{B})$ and $\|R\| \leq 1$, $\|C\| \leq 1$.

Since the canonical quotient map is completely contractive we have

$$\begin{aligned}\|RXC\|_{C^*(i(\mathcal{B}))} &= \|\sum_{i,j} R_i X_{ij} C_j\|_{C^*(i(\mathcal{B}))} \\ &\leq \frac{1}{K'} \cdot \|\sum_{i,j} R_i X_{ij} C_j\|_{C^*(i(\mathcal{B}))/J} \\ &= \frac{1}{K'} \cdot \|RXC\|_{C^*(i(\mathcal{B}))/J} \\ &\leq \frac{1}{K'} \cdot \|X\|_{M_n(C^*(i(\mathcal{B}))/J)}.\end{aligned}$$

Taking supremum over all R and C we have

$$\lambda K' \cdot \|X\|_{M_n(C^*(i(\mathcal{B})))} \leq \|X\|_{M_n(C^*(i(\mathcal{B}))/J)}.$$

Thus J is K-boundary ideal for some $K \geq \lambda K'$. $\square$

The example of a semi-simple operator algebra which does not λ-norm itself was given in [19]. In the following we present slightly simplified proof of this result and construct an operator algebra and ideal such that inequality (1) is valid for $n = 1$ but not for all $n \geq 1$.

Example 5. *An operator $T \in B(H)$ is called polynomially bounded operator if there exists a bounded homomorphism $u_T : A(\mathbb{D}) \to B(H)$ such that $u_T(p) = p(T)$ for every polynomial p. An operator T is called completely polynomially bounded if u_T is completely bounded. In [15] it was shown that T is completely polynomially bounded if and only if T is similar to a contraction. There exists a polynomially bounded operator which is not similar to a contraction, see [17]. Thus there is $T \in B(H)$ such that u_T is bounded but not completely bounded homomorphism. Since T is polynomially bounded we have that $\sigma(T) \subseteq \mathbb{D}$. Let U be unitary operator such that $\sigma(U) = \mathbb{T}$. Then $u_{T\oplus U}$ is bounded but not completely bounded and $\mathbb{T} \subseteq \sigma(T \oplus U)$. Thus $\|u_{T\oplus U}(f)\| \geq \|f\|$ for every $f \in \mathbb{A}(\mathbb{D})$ and $\mathcal{B} = u_{T\oplus U}(\mathbb{A}(\mathbb{D}))$ is a Banach algebra. Since $u^{-1}_{T\oplus U} : \mathcal{B} \to \mathbb{A}(\mathbb{D})$ acts into commutative C^*-algebra we have $\|u^{-1}_{T\oplus U}\|_{cb} = \|u^{-1}_{T\oplus U}\| \leq 1$. Let $i : \mathcal{B} \to C^*_{max}(\mathcal{B})$ be embedding of $\mathcal{B}$ into its maximal enveloping C^*-algebra and let $\tau : C^*_{max}(\mathcal{B}) \to C(\overline{\mathbb{D}})$ be $*$-homomorphism extending $u^{-1}_{T\oplus U} \circ i^{-1} : i(\mathcal{B}) \to \mathbb{A}(\mathbb{D})$. Since $\tau(C^*_{max}(\mathcal{B}))$ is a C^*-algebra generated by $\mathbb{A}(\mathbb{D})$ we have that τ is surjective. Consider a canonical quotient map*

$$q : C^*_{max}(\mathcal{B}) \to C^*_{max}(\mathcal{B})/\ker(\tau) \simeq C(\overline{\mathbb{D}}).$$

Then there is $\widetilde{K} > 0$ *such that*

$$\widetilde{K}\|b\|_{C^*_{max}(\mathcal{B})} \leq \|b\|_{C^*_{max}(\mathcal{B})/\ker(\tau)}.$$

Since $q|_{\mathcal{B}}^{-1} = i \circ u_{T\oplus U}$ *is not completely bounded we have that* $\ker(\tau)$ *is not a* K*-boundary ideal. Thus Theorem 4 implies that* $u_{T\oplus U}(A(\mathbb{D}))$ *does not* λ*-norm itself.*

3 K-boundary Ideals and the Similarity Problem.

Let $\mathcal{A}$ be a C^*-algebra and $\pi : \mathcal{A} \to B(H)$ be a bounded homomorphism. It was shown in [19] that $\pi(\mathcal{A})$ is a Banach algebra. Moreover, if π is injective then it has a completely bounded inverse, see [19], [13]. Define $\breve{\pi} : \mathcal{A} \to B(H \oplus H)$ by the following rule:

$$\breve{\pi}(a) = \pi(a) \oplus \pi(a^*)^*.$$

Evidently π is completely bounded iff $\breve{\pi}$ is completely bounded. Thus by theorem of Haagerup (see [10]) we have that π is similar to $*$-homomorphism iff $\breve{\pi}$ is such. A simple proof of this fact which does not use the Haagerup's results can be found in [21] (see also [14]).

Let J be a unitary operator. A homomorphism $\pi : \mathcal{A} \to B(H)$ is called J-symmetric if $\pi(a^*) = J\pi(a)^*J^*$ for every $a \in \mathcal{A}$.

Let $J : B(H\oplus H) \to B(H\oplus H)$ be unitary operator defined by $J(x\oplus y) = y \oplus x$. Then $\breve{\pi}(a^*) = \pi(a^*) \oplus \pi(a)^* = J\pi(a)^*J^*$ and $\breve{\pi}$ is J-symmetric.

Theorem 6. *If* $\pi : \mathcal{A} \to B(H)$ *is bounded injective homomorphism then* $\breve{\pi}$ *has a completely contractive inverse.*

Proof. Let $\mathcal{B} = \breve{\pi}(\mathcal{A})$ and $r(a)$ denotes the spectral radius of $a \in \mathcal{A}$. Since $\mathcal{B}$ is a Banach algebra and isomorphism preserves the spectrum we have $\sigma_{M_n(\mathcal{A})}(a) = \sigma_{M_n(\breve{\pi}(\mathcal{A}))}(\breve{\pi}^{(n)}(a))$ and

$$\begin{aligned}
\|a\|^2 &= \|a^*a\| = r(a^*a) \\
&= r(\breve{\pi}^{(n)}(a^*a)) \leq \|\breve{\pi}^{(n)}(a^*a)\| \\
&\leq \|\breve{\pi}^{(n)}(a^*)\| \cdot \|\breve{\pi}^{(n)}(a)\| \\
&= \|(J \otimes I_n)\breve{\pi}^{(n)}(a)^*(J \otimes I_n)\| \cdot \|\breve{\pi}^{(n)}(a)\| \\
&\leq \|\breve{\pi}^{(n)}(a)\|^2,
\end{aligned}$$

for every $a \in M_n(\mathcal{A})$. Thus the inverse homomorphism $\breve{\pi}^{-1} : \mathcal{B} \to \mathcal{A}$ is completely contractive. $\square$

Assume that $\breve{\pi}$ is as in Theorem 6. Let $i : \mathcal{B} \to C^*_{max}(\mathcal{B})$ be the canonical inclusion of $\mathcal{B}$ into its maximal enveloping C^*-algebra. Replacing $\breve{\pi}$ by $i \circ \breve{\pi}$, which does not effect completely boundedness of $\breve{\pi}$, we have that $\breve{\pi}(\mathcal{A})$ generate its maximal enveloping C^*-algebra. Therefore by universal property of the maximal enveloping C^*-algebra $\breve{\pi}^{-1}$ can be extended to a $*$-homomorphism $\rho : C^*_{max}(\mathcal{B}) \to \mathcal{A}$. Now we have $C^*_{max}(\mathcal{B})/ker(\rho) \simeq \mathcal{A}$ and

$$\widetilde{K} \cdot \|b\|_{C^*_{max}(\mathcal{B})} \leq \|b\|_{C^*_{max}(\mathcal{B})/ker(\rho)} \leq \|b\|_{C^*_{max}(\mathcal{B})} \tag{3}$$

for every $b \in \mathcal{B}$ and some $\widetilde{K} > 0$.

Note that $\breve{\pi}$ is completely bounded if and only if $ker(\rho)$ is a K-boundary ideal for some $K > 0$.

Proposition 7. *The kernel of ρ is the ideal J generated by $\{\breve{\pi}(a) - \breve{\pi}(a^*)^* : a \in \mathcal{A}\}$. If a closed ideal J' satisfies (3) for some $\widetilde{K} > 0$ and $J \subseteq J'$ then $J = J'$.*

Proof. Since ρ is a $*$-homomorphism $\rho(\breve{\pi}(a) - \breve{\pi}(a^*)^*) = 0$ and $J \subseteq ker(\rho)$.

Let us prove the converse inclusion. Let $q : C^*_{max}(\mathcal{B}) \to C^*_{max}(\mathcal{B})/J$ be a canonical quotient map. Since $\mathcal{B}$ is isomorphic to a C^*-algebra $\mathcal{A}$ we have that $\mathcal{B}$ is semisimple Banach algebra. The image $\mathcal{C} = q \circ \breve{\pi}(\mathcal{A})$ of bounded homomorphism $q \circ \breve{\pi}$ of C^*-algebra $\mathcal{A}$ is Banach algebra. By Johnson's theorem on uniqueness of norm topology on a semisimple Banach algebra (see [12]) the restriction of q to $\mathcal{B}$ is bicontinuous isomorphism of $\mathcal{B}$ and $\mathcal{C}$.

Assume that there exists $x \in ker(\rho) \setminus J$. Since $\mathcal{B}$ generates $C^*_{max}(\mathcal{B})$ as a C^*-algebra we have that x is a uniform limit of polynomials $P_k(b_1, \ldots, b_{n_k}, b_1^*, \ldots, b_{n_k}^*)$, where $b_i \in \mathcal{B}$. Thus

$$q(P_k(b_1, \ldots, b_{n_k}, b_1^*, \ldots, b_{n_k}^*)) = P_k(q(b_1), \ldots, q(b_{n_k}), q(b_1)^*, \ldots, q(b_{n_k})^*)$$

converge uniformly in $C^*_{max}(\mathcal{B})/J$. Clearly $q(b_j)^* = q(\widetilde{b_j})$ for some $\widetilde{b_j} \in \mathcal{B}$ and the elements

$$P_k(q(b_1), \ldots, q(b_{n_k}), q(\widetilde{b}_1), \ldots, q(\widetilde{b}_{n_k})) = q(P_k(b_1, \ldots, b_{n_k}, \widetilde{b}_1, \ldots, \widetilde{b}_{n_k}))$$

converge in $\mathcal{C}$. Since $q : \mathcal{B} \to \mathcal{C}$ is bicontinuous $P_k(b_1, \ldots, b_{n_k}, \widetilde{b}_1, \ldots, \widetilde{b}_{n_k}) \in \mathcal{B}$ converge to some element $y \in \mathcal{B}$. Clearly $x - y \in J$. Hence $y \in \mathcal{B} \setminus J$. Since

$J \subseteq ker(\rho)$ and $x \in ker(\rho)$ we have $y \in ker(\rho)$ which is a contradiction. Thus $J = ker(\rho)$.

To prove the second statement of the theorem consider the canonical quotient map $q_{J'} : C^*_{max}(\mathcal{B}) \to C^*_{max}(\mathcal{B})/J'$ and let $x \in J' \backslash J$. Then x is a uniform limit of polynomials $P_k(b_1, \ldots, b_{n_k}, b_1^*, \ldots, b_{n_k}^*)$ and there are $\widetilde{b}_j \in \mathcal{B}$ such that $q(b_j^*) = q(\widetilde{b}_j)$. Then $q_{J'}(P_k(b_1, \ldots, b_{n_k}, b_1^*, \ldots, b_{n_k}^*)) = q_{J'}(P_k(b_1, \ldots, b_{n_k}, \widetilde{b}_1, \ldots, \widetilde{b}_{n_k}))$ uniformly converges to 0. Thus $P_k(b_1, \ldots, b_{n_k}, \widetilde{b}_1, \ldots, \widetilde{b}_{n_k})$ converges to 0 in $\mathcal{B}$. Since

$$q(P_k(b_1, \ldots, b_{n_k}, b_1^*, \ldots, b_{n_k}^*)) = q(P_k(b_1, \ldots, b_{n_k}, \widetilde{b}_1, \ldots, \widetilde{b}_{n_k}))$$

converges to $q(x)$ and we have $q(x) = 0$. □

Another way to make $\breve{\pi}^{-1}$ extendable to a $*$-homomorphism from C^*-algebra generated by $\mathcal{B} = \breve{\pi}(\mathcal{A})$ into $\mathcal{A}$ is the following. Since $\|a\| \le \|\breve{\pi}(a)\|$ the embedding $\mathcal{B}$ into $\mathcal{B} \oplus \mathcal{A}$ via $i : \breve{\pi}(a) \mapsto \breve{\pi}(a) \oplus a$ is completely isometric isomorphism. Let $\tau = (i \circ \breve{\pi})^{-1}$. Then $\tau(\breve{\pi}(a) \oplus a) = a$ and τ has a contractive extension, $\widetilde{\tau}$, to the C^*-algebra generated by $i(\mathcal{B})$, such that $\widetilde{\tau}(a_1 \oplus a_2) = a_2$ for every $a_1 \oplus a_2 \in C^*(i(\mathcal{B}))$. Since $\widetilde{\tau}$ is unital and contractive we have that $\widetilde{\tau}$ is a $*$-homomorphism.

Now we can summarize our observations in several reformulations of the Kadison's similarity problem.

Theorem 8. *The following are equivalent:*

(i) Kadison's conjecture has affirmative answer,

(ii) for every operator algebra $\mathcal{B}$ and every bounded homomorphism $\rho : C^*(\mathcal{B}) \to \mathcal{B}$, such that $\rho(b) = b$ for every $b \in \mathcal{B}$, ρ is completely bounded.

(iii) if $\mathcal{B}$ is an operator algebra and $\rho : C^*_{max}(\mathcal{B}) \to \mathcal{B}$ is bounded homomorphism such that $\rho(b) = b$ for every $b \in \mathcal{B}$ and the restriction of ρ to $\mathcal{B}^*$ is completely isometric then ρ is completely bounded.

(iv) if an operator algebra $\mathcal{B}$ is isomorphic to a C^*-algebra and $J \subset C^*(\mathcal{B})$ is a closed ideal such that

$$C \cdot \|b\|_{C^*(\mathcal{B})} \le \|b\|_{C^*(\mathcal{B})/J}$$

for every $b \in \mathcal{B}$ and some $C > 0$, then J is a K-boundary ideal.

Proof. Evidently (i)⇒(ii) and (ii)⇒(iii).

(iii)⇒(i). Let $\pi : \mathcal{A} \to B(H)$ be bounded injective homomorphism from C^*-algebra $\mathcal{A}$. By Theorem 6 and considerations preceeding Theorem 8 we have bounded injective J-symmetric homomorphism $\breve{\pi} : \mathcal{A} \to B(H \oplus H)$ and completely isometric homomorphism $i : \breve{\pi}(\mathcal{A}) \to C^*_{max}(\breve{\pi}(\mathcal{A}))$. Let $\rho = i \circ \breve{\pi}$ and $\mathcal{B} = \rho(\mathcal{A})$. Then $\|\rho\|_{cb} = \|\pi\|_{cb}$ and $\rho^{-1} : \mathcal{B} \to \mathcal{A}$ extends to $*$-homomorphism $\tau : C^*_{max}(\mathcal{B}) \to \mathcal{A}$. Thus we have bounded homomorphism

$$\rho \circ \tau : C^*_{max}(\mathcal{B}) \to \mathcal{B}$$

such that $\rho \circ \tau(b) = b$ for every $b \in \mathcal{B}$. Consider the restriction of $\rho \circ \tau$ to B^*. Let $(b_{ij})_{i,j} \in M_n(\mathcal{B}^*)$ then $b_{ij} = \rho(a_{ij})^*$ for some $a_{ij} \in \mathcal{A}$. Since $\breve{\pi}$ is J-symmetric we have

$$\begin{aligned}
\|\rho \circ \tau|^{(n)}_{B^*}((\rho(a_{ij})^*)_{i,j})\| &= \|(\rho \circ \tau(\rho(a_{ij})^*))_{i,j}\| \\
&= \|(\rho(\tau(\rho(a_{ij}))^*))_{i,j}\| \\
&= \|(\rho(a^*_{ij}))_{i,j}\| = \|(\breve{\pi}(a^*_{ij}))_{i,j}\| \\
&= \|(J \otimes I_n)(\breve{\pi}(a_{ij})^*)_{i,j}(J \otimes I_n)\| \\
&= \|(\breve{\pi}(a_{ij})^*)_{i,j}\| = \|(\breve{\pi}(a_{ji}))_{i,j}\| \\
&= \|(\rho(a_{ji}))_{i,j}\| = \|(\rho(a_{ij})^*)_{i,j}\|.
\end{aligned}$$

Thus $\rho \circ \tau|_{B^*}$ is a complete isometry. By (iii) $\rho \circ \tau$ is completely bounded. Then $\mathcal{B}$ is similar to some C^*-algebra $\mathcal{C}$, i.e. there exists $S \in B(K)$ such that $\mathcal{B} = S\mathcal{C}S^{-1}$. Since $AdS \circ \rho : \mathcal{A} \to \mathcal{C}$ is bounded isomorphism between two C^*-algebras by Gardner's theorem [9] we have that $AdS \circ \rho$ is similar to $*$-homomorphism which proves similarity of ρ to $*$-homomorphism.

(iv)⇒(ii). Since $ker(\rho)$ is ideal of $C^*(B)$ and satisfies conditions of (iv) we have that $ker(\rho)$ is a K-boundary ideal. Therefore ρ is completely bounded.

(i)⇒(iv). In [19] it was proved that if Kadison conjecture has affirmative answer then every Banach operator algebra which is isomorphic to a C^*-algebra λ-norms itself for some $\lambda > 0$. By Proposition 4 we have that ideal of (iv) is a K-boundary ideal. □

Remark 9. *By example 5 we have that in condition (iv) of Theorem 8 it is not enough to require $\mathcal{B}$ be isomorphic to a semi-simple Banach algebra.*

Question. Note that ρ in Theorem 8 (ii) is $\mathcal{B}$-bimodule map. Let $\mathcal{C}$ and $\mathcal{D}$ be C^*-algebras. In [23] Smith proved that if $\tau : \mathcal{C} \to B(H)$ is bounded $\mathcal{D}$-bimodule map and $\mathcal{D}$ has cyclic vector then τ is completely bounded and

$\|\tau\|_{cb} = \|\tau\|$. Is Smith's theorem true if $\mathcal{D}$ is an operator algebra with cyclic vector?

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