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The Continuous Galerkin Method for Fractional Order Viscoelasticity

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Abstract

We consider a fractional order integro-differential equation with a weakly singular convolution kernel. The equation with homogeneous Dirichlet boundary conditions is reformulated as an abstract Cauchy problem, and well-posedness is verified in the context of linear semigroup theory. Then we formulate a continuous Galerkin method for the problem, and we prove stability estimates. These are then used to prove a priori error estimates. The theory is illustrated by a numerical example.

Keywords: finite element, continuous Galerkin, linear viscoelasticity, fractional calculus, weakly singular kernel, stability, a priori error estimate.

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Fardin Saedpanah
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Contents

1 Introduction

For better understanding of the main work that has been done in the presented paper, we bring some basic concepts to see how and why the fractional order differential/integral operators may be used to model viscoelastic materials. Afterwards, we prepare the basic aspects of the linear semigroup theory to make the first part of the paper more understandable for whom may not be familar with the concept. It is assumed that the reader is rather familiar with the finite element methods, especially the continuous Galerkin method.

2 Fractional calculus

Generalization have always been an intereting subject in mathematics. One example is gamma function

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt, \quad x > 0,$$

which interpolates between the factorials. Another one is the fractional differential/integral operators which interpolates between integer order differential/integral operators. In fact analytic continution of the gamma function for $x \leq 0$ plays an important role when we construct the theory of fractional order differential/integral operators from the corresponding integer order operators.

2.1 Fractional differential/integral operators

We recall the Cauchy's formula for repeated integration

$$\begin{aligned} D^{-n} f(x) &= \int_0^x \int_0^{x_{n-1}} \cdots \int_0^{x_1} f(x_0)\, dx_0 \cdots dx_{n-2}\, dx_{n-1} \\ &= \frac{1}{(n-1)!} \int_0^x \frac{f(t)}{(x-t)^{1-n}}\, dt, \quad n = 1, 2, \cdots, \end{aligned}$$

with $D^0 f(x) = f(x)$. Replacing the integer number n with the real number α and the discrete factorial $(n-1)!$ with the continuous gamma function Γ, we obtain the Riemann-Liouville fractional integral

$$D^{-\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x \frac{f(t)}{(x-t)^{1-\alpha}}\, dt, \quad \alpha > 0,$$

where α is the order of integration. Note that the convolution kernel $\frac{1}{\Gamma(\alpha) x^{1-\alpha}}$ is singular but integrable.

The same definition can be used for fractional differentiation of order α by a formal replacement of $-\alpha$ by α ($\alpha \neq 1, 2, 3, \cdots$)

$$D^\alpha f(x) = \frac{1}{\Gamma(-\alpha)} \int_0^x \frac{f(t)}{(x-t)^{1+\alpha}}\, dt, \quad \alpha > 0.$$

The convolution integral above is in general divergent and needs to be interpreted in the sense of its regularization. A convergent expression for the fractional derivative operator is obtained by splitting the derivative operator into an integer order derivative and a fractional integral operator

$$D^\alpha = D^{N-\rho} = D^N D^{-\rho},$$

where N is the integer that satisfies $\alpha < N < \alpha + 1$ and $0 < \rho < 1$. Specializing to $0 < \alpha < 1$, which is the interesting interval in viscoelasticity, we can write the definition of the fractional derivative as

$$D^\alpha f(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \left[\int_0^x \frac{f(t)}{(x-t)^\alpha}\, dt \right]. \tag{2.1}$$

We note that the fractional order differential operator is not a local operator as the integer order differential operator is, i.e., the derivative depends on all function values from its lower limit $t = 0$ up to the evaluation point $t = x$.

In contrast to the term "fractional" the fractional order exponent α can be irrational and even complex. However, in this context we take it to be real.

The text books [27] and [31] are concerened with the definitions and the properties of fractional order differential/integral operators. A survey of the many different applications which have emerged from fractional calculus is given in [29].

3 Fractional order linear viscoelasticity

Linear viscoelasticity in combination with fractional order operators, i.e., the fractional order viscoelastic model, have attracted considerable attention in the last decades. The fractional order viscoelastic model is capable of describing the behavior of many viscoelastic materials.

A perfectly elastic material does not exist since in reality: inelasticity is always present. This inelasticity leads to energy dissipation or damping. Therefore, for a wide class of materials it is not sufficient to use an elastic constitutive model to capture the mechanical behavior. In order to replace extensive experimental tests by numerical simulations there is a need for an accurate material model. Therefore viscoelastic constitutive models have frequently been used to simulate the time dependent behavior of polymeric materials. The classical linear viscoelastic models that use integer order time derivatives in the constitutive laws, require an excessive number of parameters to accurately predict observed material behavior.

Bagley and Torvik [8] used fractional derivatives to construct stress-strain relationships for viscoelastic materials. The advantage of this approach is that very few empirical parameters are required (two elastic constants, one relaxation constant and the fractional order exponent).

When this fractional derivative model of viscoelasticity is incorporated directly into the structural equations a time differential equation of non-integer order higher than two is obtained. One consequence of this is that initial conditions of fractional order higher than one are required. The problems with initial conditions of fractional order have been discussed by Enelund and Olsson [17] and also by Bagley [7] and by Beyer and Kempfle [10]. To avoid the difficulties with fractional order initial conditions some alternative formulations of the fractional derivative viscoelastic model are used in structural modeling. The first form, that we will use, is based on a convolution integral formulation with a singular kernel of Mittag-Leffler type (see [17], [14] and [4]). The second form involves fractional integral operators rather than fractional derivative operators (see [13]). And the third form uses internal variables, see [15], [16] and [1]. The main advantage of these forms is that they lead to well-posed initial value problems.

We recall that a fractional order differential operator is not a local operator, i.e., the derivative depends on the whole history of the function. This increases the complexity of mathematical analysis and the numerical computations of fractional order viscoelastic models.

For extensive overviews, analysis of the fractional order viscoelastic models, the hereditary theory of linear viscoelasticity and the history of linear viscoelasticy the reader is referred to [6], [12], [30] and [1].

3.1 Convolution integral formulation

Let σ_{ij} and u_i denote the usual stress tensor and displacement vector and define the linear strain tensor:

$$\epsilon_{ij} = \frac{1}{2}\Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\Big).$$

With the decompositions

$$s_{ij} = \sigma_{ij} - \tfrac{1}{3}\sigma_{kk}\delta_{ij}, \quad e_{ij} = \epsilon_{ij} - \tfrac{1}{3}\epsilon_{kk}\delta_{ij},$$

we formulate the constitutive equations, see [8],

$$\begin{aligned}
s_{ij}(t) + \tau_1^{\alpha_1} D_t^{\alpha_1} s_{ij}(t) &= 2G_\infty e_{ij}(t) + 2G\tau_1^{\alpha_1} D_t^{\alpha_1} e_{ij}(t), \\
\sigma_{kk}(t) + \tau_2^{\alpha_2} D_t^{\alpha_2} \sigma_{kk}(t) &= 3K_\infty \epsilon_{kk}(t) + 3K\tau_2^{\alpha_2} D_t^{\alpha_2} \epsilon_{kk}(t),
\end{aligned}$$

with initial conditions

$$s_{ij}(0+) = 2Ge_{ij}(0+), \quad \sigma_{kk}(0+) = 3K\epsilon_{kk}(0+),$$

meaning that the initial response follows Hooke's elastic law. Note that we have two relaxation times, $\tau_1,\ \tau_2 > 0$, and fractional orders of differentiation, $\alpha_1,\ \alpha_2 \in (0,1)$, where the fractional order derivative is defined by (2.1).

We solve for σ by means of Laplace transformation, [17]:

$$\begin{aligned}
s_{ij}(t) &= 2G\Big(e_{ij}(t) - \frac{G - G_\infty}{G} \int_0^t f_1(t-s) e_{ij}(s)\, ds\Big), \\
\sigma_{kk}(t) &= 3K\Big(\epsilon_{kk}(t) - \frac{K - K_\infty}{K} \int_0^t f_2(t-s) \epsilon_{kk}(s)\, ds\Big),
\end{aligned}$$

where

$$f_i(t) = -\frac{d}{dt} E_{\alpha_i}\Big(- \Big(\frac{t}{\tau_i}\Big)^{\alpha_i} \Big)$$

and

$$E_\alpha(t) = \sum_{n=0}^{\infty} \frac{t^n}{\Gamma(1 + \alpha n)}$$

is the Mittag-Leffler function [9]. We make the simplifying assumption (synchronous viscoelasticity):

$$\alpha = \alpha_1 = \alpha_2, \quad \tau = \tau_1 = \tau_2, \quad f = f_1 = f_2.$$

Then we may define a parameter γ, a kernel β, and the Lamé constants μ, λ,

$$\gamma = \frac{G - G_\infty}{G} = \frac{K - K_\infty}{K}, \quad \beta(t) = \gamma f(t), \quad \mu = G, \quad \lambda = K - \tfrac{2}{3}G,$$

and the constitutive equations become

$$\sigma_{ij}(t) = \Big(2\mu\epsilon_{ij}(t) + \lambda\epsilon_{kk}(t)\delta_{ij}\Big) - \int_0^t \beta(t-s)\Big(2\mu\epsilon_{ij}(s) + \lambda\epsilon_{kk}(s)\delta_{ij}\Big)\, ds.$$

Note that the viscoelastic part of the model contains only three parameters:

$$0 < \gamma < 1, \quad 0 < \alpha < 1, \quad \tau > 0.$$

The kernel is weakly singular:

$$\beta(t) = -\gamma \frac{d}{dt} E_\alpha\Big(-\Big(\frac{t}{\tau}\Big)^\alpha\Big) = \gamma \frac{\alpha}{\tau}\Big(\frac{t}{\tau}\Big)^{-1+\alpha} E'_\alpha\Big(-\Big(\frac{t}{\tau}\Big)^\alpha\Big) \approx Ct^{-1+\alpha}, \quad t \to 0,$$

and we note the properties

$$\begin{aligned} \beta(t) &\geq 0, \\ \|\beta\|_{L_1(\mathbb{R}^+)} &= \int_0^\infty \beta(t)\,dt = \gamma\big(E_\alpha(0) - E_\alpha(\infty)\big) = \gamma < 1. \end{aligned}$$

Various properties (e.g., regularity and convergence) of the memory kernel function β have been investigated in [17].

The equations of motion now become:

$$\begin{aligned} &\rho u_{i,tt} - \sigma_{ij,j} = f_i, && \text{in } \Omega, \\ &u_i = 0, && \text{on } \Gamma_\mathrm{D}, \\ &\sigma_{ij} n_j = g_i, && \text{on } \Gamma_\mathrm{N}. \end{aligned}$$

Considering also initial values for displacement u and velocity u_t, this can be written as

$$\begin{aligned} &\rho \boldsymbol{u}_{tt}(\boldsymbol{x},t) - \nabla\cdot\boldsymbol{\sigma}_0(\boldsymbol{u};\boldsymbol{x},t) \\ &\quad + \int_0^t \beta(t-s)\nabla\cdot\boldsymbol{\sigma}_1(\boldsymbol{u};\boldsymbol{x},s)\,ds = \boldsymbol{f}(\boldsymbol{x},t) && \text{in } \Omega\times I, \\ &\boldsymbol{u}(\boldsymbol{x},t) = 0 && \text{on } \Gamma_D\times I, \\ &\boldsymbol{\sigma}(\boldsymbol{u};\boldsymbol{x},t)\cdot\boldsymbol{n}(\boldsymbol{x}) = \boldsymbol{g}(\boldsymbol{x},t) && \text{on } \Gamma_N\times I, \\ &\boldsymbol{u}(\boldsymbol{x},0) = \boldsymbol{u}^0(\boldsymbol{x}) && \text{in } \Omega, \\ &\boldsymbol{u}_t(\boldsymbol{x},0) = \boldsymbol{v}^0(\boldsymbol{x}) && \text{in } \Omega, \end{aligned} \tag{3.1}$$

which is equation (1.1) in the appended paper and is a Volterra type integro-differential equation.

We should mention that there are also models with exponential kernels, smooth kernels, which describe polymeric materials, e.g., natural and synthetic rubber. The drawback with this kind of model is that it requires a large number of exponential kernels to describe the behavior of the materials. This is the reason for introducing kernels of Mittag-Leffler type or fractional operators. In [17] and [6] Enelund and Adolfsson have shown that the classical viscoelastic model based on exponential kernels can describe the same viscoelastic behavior as the fractional model if the number of kernels tend to infinity.

4 Semigroups of linear operators

Semigroup theory is the abstract study of first-order ordinary differential equations with values in Banach spaces, driven by linear, but possibly unbounded, operators. This approch provides an elegant alternative to some of the well-posedness theory for evolution equations that is one of the many applications that the theory has in different branches of analysis (such as harmonic analysis, approximation theory and many other subjects). In this section we outline the basics of the theory, without any proof, and present as well the Lumer-Phillips theorem, which will be used in §2 of the the appended article.

Troughout this section we let X denote a real Banach space.

For more complete and advanced details of the theory and its application to partial differential equations one may refer to [28] and [18].

4.1 Definitions and properties

Definition 4.1. A one parameter family $T(t)$, $0 \leq t < \infty$, of bounded linear operators from the Banach space X to X is a *semigroup of bounded linear operator* on X if

$(i)\, T(0) = I, \quad (I \text{ is the identity operator on } X),$

$(ii)\, T(t+s) = T(t)T(s), \quad \text{for every } t,\, s \geq 0 \quad \text{(the semigroup property)}.$

Definition 4.2. The linear operator $\mathcal{A}$ defined by

$$\mathcal{A}x = \lim_{t \searrow 0} \frac{T(t)x - x}{t} = \frac{d^+T(t)x}{dt} \,|_{t=0} \quad \text{for} \quad x \in D(\mathcal{A}),$$

is the *(infinitesimal) generator* of the semigroup $T(t)$, where $D(\mathcal{A})$ is the domain of $\mathcal{A}$ defined by

$$D(\mathcal{A}) = \Big\{ x \in X : \lim_{t \searrow 0} \frac{T(t)x - x}{t} \text{ exists} \Big\}.$$

Definition 4.3. A semigroup $T(t)$, $0 \leq t < \infty$, of bounded linear operators on X is a *strongly continuous* semigroup if

$$\lim_{t \searrow 0} T(t)x = x \quad \forall x \in X.$$

A strongly continuous semigroup of bounded linear operators on X will be called a C_0 semigroup. If moreover $\|T(t)\| \leq 1$ for $t \geq 0$ it is called a C_0 *semigroup of contractions.*

Lemma 4.1. Let the linear operator $\mathcal{A}$ be the generator of a C_0 semigroup $T(t)$. Then for $x \in D(\mathcal{A})$, $T(t)x \in D(\mathcal{A})$ and

$$\frac{d}{dt}T(t)x = \mathcal{A}T(t)x.$$

Definition 4.4. For every $x \in X$ we define the *duality set* $F(x) \subset X^*$ by

$$F(x) = \{x^* : x^* \in X^* \text{ and } \langle x^*, x\rangle = \|x\|^2 = \|x^*\|^2\},$$

where X^* denotes the dual of X. And we note that by Hahn-Banach theorem $F(x) \neq \varnothing$ for every $x \in X$.

Definition 4.5. A linear operator $\mathcal{A}$ is *dissipative* if for every $x \in D(\mathcal{A})$ there is a $x^* \in F(x)$ such that $\langle \mathcal{A}x, x^*\rangle \leq 0$.

Lemma 4.2. Let $\mathcal{A}$ be dissipative with $R(I - \mathcal{A}) = X$. If X is reflexive then $D(\mathcal{A})$ is dense in X, i.e., $\overline{D(\mathcal{A})} = X$.

We use the first part of the following theorem in §2 in the appended paper.

Theorem 4.1. (*Lumer-Phillips*). Let $\mathcal{A}$ be a linear operator with dense domain $D(\mathcal{A})$ in X.

(a) If $\mathcal{A}$ is dissipative and there is a $\lambda > 0$ such that $R(\lambda I - \mathcal{A}) = X$, then $\mathcal{A}$ is the infinitesimal generator of a C_0 semigroup of contractions on X.

(b) If $\mathcal{A}$ is the infinitesimal generator of a C_0 semigroup of contractions on X, then $R(\lambda I - \mathcal{A}) = X$ for all $\lambda > 0$ and $\mathcal{A}$ is dissipative. Moreover, for every $x \in D(\mathcal{A})$ and every $x^* \in F(x)$, $\langle \mathcal{A}x, x^*\rangle \leq 0$.

4.2 The abstract Cauchy problem

Let $\mathcal{A}$ be a linear operator from $D(\mathcal{A}) \subset X$ into X. Given $x \in X$ the *abstract Cauchy problem* for $\mathcal{A}$ with initial data x consists of finding a solution $u(t)$ to the initial value problem

$$\begin{aligned} &\frac{d}{dt}u(t) = \mathcal{A}u(t) + f(t), \quad t > 0, \\ &u(0) = x, \end{aligned} \tag{4.1}$$

where $f : [0, T) \to X$. And by a solution we mean an X-valued function $u(t)$ such that $u(t)$ is continuous for $t \geq 0$, continuously differentiable and $u(t) \in D(\mathcal{A})$ for $t > 0$ and (4.1) is satisfied. Note that since $u(t) \in D(\mathcal{A})$ for $t > 0$ and u is continuous at $t = 0$, (4.1) cannot have a solution for $x \notin \overline{D(\mathcal{A})}$.

From Lemma 4.1 it is clear that if $\mathcal{A}$ is the (infinitesimal) generator of a C_0 semigroup $T(t)$, the abstract Cauchy problem (4.1) when $f = 0$ has a solution, namely, $u(t) = T(t)x$, for every $x \in D(\mathcal{A})$. So $T(t)$ is called the *operator solution*. It can be proved that the solution is unique.

Definition 4.6. A function u which is differentiable almost everywhere on $[0, T]$ such that $u' \in L_1(0, T; X)$ is called a *strong solution* of (4.1) if $u(0) = x$ and $u'(t) = \mathcal{A}u(t) + f(t)$ a.e. on $[0, T]$.

In the following lemmas we find the sufficient assumptions under which we get a unique strong solution of (4.1).

Lemma 4.3. If $\mathcal{A}$ generates a C_0 semigroup $T(t)$, f is differentiable a.e. on $[0, T]$ and $f' \in L_1(0, T; X)$ then for every $x \in D(\mathcal{A})$ the initial value problem (4.1) has a unique strong solution.

In general, the Lipschitz continuity of f on $[0, T]$ is not sufficient to assure the existence of a strong solution of (4.1) for $x \in D(\mathcal{A})$. However, if X is reflexive, for instance a Hilbert space, and f is Lipschitz continuous on $[0, T]$ that is

$$\|f(t_1) - f(t_2)\| \leq C|t_1 - t_2| \quad \text{for} \quad t_1,\, t_2 \in [0, T],$$

then by a classical result f is differentiable a.e. and $f' \in L_1(0, T; X)$. Therefore Lemma 4.3 implies the following.

Lemma 4.4. Let X be a reflexive Banach space and $\mathcal{A}$ generates a C_0 semigroup $T(t)$ on X. If f is Lipschitz continuous on $[0, T]$ then for every $x \in D(\mathcal{A})$ the initial value problem (4.1) has a unique strong solution u on $[0, T]$ given by the variation of constants formula

$$u(t) = T(t)x + \int_0^t T(t-s)f(s)\,ds.$$

4.3 Application to partial differential equations

One important application of the theory of linear semigroups is analysing partial differential equations, e.g., well-posedness and numerical solution. In order to reformulate a PDE's into a first-order ordinary differential equation, an abstract Cauchy problem, we need to construct suitable spaces and a suitable linear operator $\mathcal{A}$. It should be noticed in the previous sections that an important property for a linear operator $\mathcal{A}$ is to generate a C_0 semigroup (of contractions) of $T(t)$. This is what we have done, inspired by [19], in §2 of the appended paper to prove well-posedness and regularity properties.

To make reading §2 of the paper independent of looking for the theorems in [28], we correspond the important lemmas and theorems in this draft with the main ones in [28] as follows:

here	Lemma 4.2	Theorem 4.1	Lemma 4.3	Lemma 4.4
$\Updownarrow$	$\Updownarrow$	$\Updownarrow$	$\Updownarrow$	$\Updownarrow$
[28]	Theorem 1.4.6	Theorem 1.4.3	Corollary 4.2.10	Corollary 4.2.11

5 Earlier works

A lot of work have been done during the last decades regarding well-posedness of the fractional order linear viscoelasticity and also several methods have been investigated to solve these kinds of models numerically. We try to give just some references to earlier works, but it does not seem to be possible to give a complete list.

Thomée and McLean [25] have proved the existence, uniqueness and regularity of the solution of a reformed model of (3.1) by means of Fourier series. One can also see [11] where Desch and Fašanga have used the context of analytic semigroups.

For more details on semidiscretization in time or space and the relevant methods that have been used, namely discontinuous Galerkin approximation as well as first and second-order backward difference methods in time or continuous Galerkin approximation in space, we refer to, e.g., [5], [2], [3], [21], [34], [26] and [25].

Numerical methods for quasistatic viscoelasticity problems, i.e., $\rho \boldsymbol{u}_{tt} \approx 0$, have been studied, e.g., in [3] and [33] where basically they have used discontinuous Galerkin approximation in time and continuous Galerkin approximation in space.

The drawback of the fractional order viscoelastic models is that the whole strain history must be saved and included in each time step that is due to the non-locality of the fractional order differential operators. The most commonly used algorithms for this integration are based on Lubich convolution quadrature [23] for fractional order operators. One example of the application of this approch to integro-differential equations with a memory term is in [24]. The Lubich convolution quadrature requires uniformly distributed time steps or alternatively logarithmically distributed time steps as outlined in [20]. These are cumbersome restrictions because it is not possible to use adaptivity and goal oriented error estimation. Some efficient numerical algorithms to overcome the mentioned problem of Lubich convolution quadrature can be found in [32] and [22]. Also sparse quadrature as a possible way to overcome the problem with the growing amount of data, that has to be stored and used in each time step, has been studied in [26], [2] and [3]. In this approach variable time steps can be used.

6 Summary of the appended paper

In the appended paper we prove well-posedness of the problem in the context of linear semigroup theory as well as regularity properties. We formulate a continuous Galerkin method of arbitray order q in time and continuous Galerkin approximation of any order p in space. The stability property is investigated and some a priori error estimates, for the linear case $p = q = 1$, that are optimal in $L_\infty(L_2)$ and $L_\infty(H^1)$ are obtained for displacement and velocity, respectively. At the end we illustrate the theory for the linear case $p = q = 1$ by a simple but realistic numerical example. In the presented work we only study the original form of the numerical method and we do not discuss fast or adaptive strategies such as sparse quadrature and adaptive strategy based on a posteriori error estimates, e.g., [25], [2] and [3], to speed up the performance and decreasing the neccessary memory. We postpone this to the forthcoming work.

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THE CONTINUOUS GALERKIN METHOD FOR AN INTEGRO-DIFFERENTIAL EQUATION MODELING DYNAMIC FRACTIONAL ORDER VISCOELASTICITY

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Abstract. We consider a fractional order integro-differential equation with a weakly singular convolution kernel. The equation with homogeneuos Dirichlet boundary conditions is reformulated as an abstract Cauchy problem, and well-posedness is verified in the context of linear semigroup theory. Then we formulate a continuous Galerkin method for the problem, and we prove stability estimates. These are then used to prove a priori error estimates. The theory is illustrated by a numerical example.

1. Introduction

R. L. Bagley and P. J. Torvik [5] have proved that fractional order operators (integrals and derivatives) are very suitable for modeling viscoelastic materials. Basic equations of the viscoelastic dynamic problem, with surface loads, can be written in the strong form,

$$\begin{aligned}
&\rho\ddot{\boldsymbol{u}}(\boldsymbol{x},t) - \nabla\cdot\boldsymbol{\sigma}_0(\boldsymbol{u};\boldsymbol{x},t) \\
&\qquad + \int_0^t \beta(t-s)\nabla\cdot\boldsymbol{\sigma}_1(\boldsymbol{u};\boldsymbol{x},s)\,ds = \boldsymbol{f}(\boldsymbol{x},t) && \text{in } \Omega\times I,\\
&\boldsymbol{u}(\boldsymbol{x},t) = 0 && \text{on } \Gamma_D\times I,\\
&\boldsymbol{\sigma}(\boldsymbol{u};\boldsymbol{x},t)\cdot\boldsymbol{n}(\boldsymbol{x}) = \boldsymbol{g}(\boldsymbol{x},t) && \text{on } \Gamma_N\times I,\\
&\boldsymbol{u}(\boldsymbol{x},0) = \boldsymbol{u}^0(\boldsymbol{x}) && \text{in } \Omega,\\
&\dot{\boldsymbol{u}}(\boldsymbol{x},0) = \boldsymbol{v}^0(\boldsymbol{x}) && \text{in } \Omega,
\end{aligned} \tag{1.1}$$

(through out this text we use '$\cdot$' to denote '$\frac{\partial}{\partial t}$') where $\boldsymbol{u}$ is the displacement vector, ρ is the (constant) mass density, $\boldsymbol{f}$ and $\boldsymbol{g}$ represent, respectively, the volume and surface loads, $\boldsymbol{\sigma}_0$ and $\boldsymbol{\sigma}_1$ are the stresses according to

$$\begin{aligned}
&\boldsymbol{\sigma}(t) = \boldsymbol{\sigma}_0(t) - \int_0^t e(t-s)\boldsymbol{\sigma}_1(s)\,ds, \quad \text{with}\\
&\boldsymbol{\sigma}_0(t) = 2\mu_0\boldsymbol{\epsilon}(t) + \lambda_0\mathrm{tr}\boldsymbol{\epsilon}(t)\mathbf{I}, \quad \boldsymbol{\sigma}_1(t) = 2\mu_1\boldsymbol{\epsilon}(t) + \lambda_1\mathrm{tr}\boldsymbol{\epsilon}(t)\mathbf{I},
\end{aligned} \tag{1.2}$$

where $\lambda_0 > \lambda_1 > 0$ and $\mu_0 > \mu_1 > 0$ are elastic constants of Lamé type, $\boldsymbol{\epsilon}$ is the strain which is defined through the usual linear kinematic relation $\boldsymbol{\epsilon} = \frac{1}{2}(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T)$, and e is the convolution kernel

$$
\begin{aligned}
e(t) &= -\frac{d}{dt}\Big(\mathrm{E}_\alpha(-(t/\tau)^\alpha)\Big) = \frac{\alpha}{\tau}\Big(\frac{t}{\tau}\Big)^{\alpha-1}\dot{\mathrm{E}}_\alpha(-(t/\tau)^\alpha) \\
&\approx Ct^{-1+\alpha}, \ t \to 0.
\end{aligned} \tag{1.3}
$$

Here $\tau > 0$ is the relaxation time and $\mathrm{E}_\alpha(x) = \sum_{k=0}^\infty \frac{x^k}{\Gamma(1+\alpha k)}$ is the Mittag-Leffler function of order $\alpha \in (0,1)$, and γ is introduced to be $\gamma = \frac{\mu_1}{\mu_0} = \frac{\lambda_1}{\lambda_0} < 1$, so we have $\boldsymbol{\sigma}_1 = \gamma \boldsymbol{\sigma}_0$ and we define $\beta(t) = \gamma e(t)$. The convolution term is weakly singular and $\beta \in L_1(0,\infty)$ with $\int_0^\infty \beta(t)\, dt = \gamma$. And we introduce the function

$$
\xi(t) = \gamma - \int_0^t \beta(s) ds = \int_t^\infty \beta(s)\, ds, \tag{1.4}
$$

which is decreasing with $\xi(0) = \gamma$, $\lim_{t\to\infty} \xi(t) = 0$, so that $0 < \xi(t) \le \gamma$.

We let $\Omega \subset \mathbb{R}^d$, $d = 2,3$, be a bounded domain with boundary $\Gamma = \Gamma_D \cup \Gamma_N$ where Γ_D and Γ_N are disjoint and $meas(\Gamma_D) \neq 0$. We introduce the function spaces $H = L_2(\Omega)^d$, $H^{\Gamma_N} = L_2(\Gamma_N)^d$ and $V = \{\boldsymbol{v} \in H^1(\Omega)^d : \boldsymbol{v}\,|_{\Gamma_D} = \boldsymbol{0}\}$. We denote the norms in H and H^{Γ_N} by $\|\cdot\|$ and $\|\cdot\|_{\Gamma_N}$, respectively, and we equip V with the inner product $a(\cdot,\cdot)$ and norm $\|\boldsymbol{v}\|_V^2 = a(\boldsymbol{v},\boldsymbol{v})$. We also define a bilinear form (with the usual summation convention)

$$
a(\boldsymbol{v},\boldsymbol{w}) = \int_\Omega \big(2\mu_0 \epsilon_{ij}(\boldsymbol{v})\epsilon_{ij}(\boldsymbol{w}) + \lambda_0 \epsilon_{ii}(\boldsymbol{v})\epsilon_{jj}(\boldsymbol{w})\big)\, dx, \quad \boldsymbol{v},\boldsymbol{w} \in V, \tag{1.5}
$$

which is coercive on V. Setting $A\boldsymbol{u} = -\nabla\cdot\boldsymbol{\sigma}_0(\boldsymbol{u})$ with $\mathrm{dom}(A) = H^2(\Omega)^d \cap V$ such that $a(\boldsymbol{u},\boldsymbol{v}) = (A\boldsymbol{u},\boldsymbol{v})$ for sufficiently smooth $\boldsymbol{u},\boldsymbol{v} \in V$, we can write the weak form of the equation of motion (1.1) as: Find $\boldsymbol{u}(t) \in V$ such that $\boldsymbol{u}(0) = \boldsymbol{u}^0$, $\dot{\boldsymbol{u}}(0) = \boldsymbol{v}^0$ and,

$$
\begin{aligned}
\rho(\ddot{\boldsymbol{u}}(t),\boldsymbol{v}) + a(\boldsymbol{u}(t),\boldsymbol{v}) - &\int_0^t \beta(t-s) a(\boldsymbol{u}(s),\boldsymbol{v})\, ds \\
&= (\boldsymbol{f}(t),\boldsymbol{v}) + (\boldsymbol{g}(t),\boldsymbol{v})_{\Gamma_N}, \quad \forall \boldsymbol{v} \in V,
\end{aligned} \tag{1.6}
$$

with $(\boldsymbol{g}(t),\boldsymbol{v})_{\Gamma_N} = \int_{\Gamma_N} \boldsymbol{g}(t)\cdot\boldsymbol{v}\, dS$. For more details see [4], [1], [2], [3] and references therein.

We define $\boldsymbol{u}_1 = \boldsymbol{u}$ and $\boldsymbol{u}_2 = \dot{\boldsymbol{u}}$, and henceforth we set $\boldsymbol{f}_2 = \boldsymbol{f}$. Then we can write the weak form (1.6) as: Find $\boldsymbol{u}_1(t)$, $\boldsymbol{u}_2(t) \in V$ such that

$\boldsymbol{u}_1(0) = \boldsymbol{u}^0$, $\boldsymbol{u}_2(0) = \boldsymbol{v}^0$ and,

$$\begin{aligned}
&a\big(\dot{\boldsymbol{u}}_1(t), \boldsymbol{v}_1\big) - a\big(\boldsymbol{u}_2(t), \boldsymbol{v}_1\big) = 0, \\
&\rho(\dot{\boldsymbol{u}}_2(t), \boldsymbol{v}_2) + a(\boldsymbol{u}_1(t), \boldsymbol{v}_2) - \int_0^t \beta(t-s) a(\boldsymbol{u}_1(s), \boldsymbol{v}_2)\, ds \\
&\qquad = (\boldsymbol{f}_2(t), \boldsymbol{v}_2) + (\boldsymbol{g}(t), \boldsymbol{v}_2)_{\Gamma_N}, \quad \forall \boldsymbol{v}_1, \boldsymbol{v}_2 \in V.
\end{aligned} \tag{1.7}$$

In the next section, using (1.6) with $\boldsymbol{g} = \boldsymbol{0}$ $(\Gamma = \Gamma_D)$,we reformulate the problem as an abstract Cauchy problem and prove well-posedness. We also discuss the regularity properties and we obtain some regularity estimates. In §3 we use (1.7) to formulate a continuous Galerkin method based on polynomials of degree at most q in time, and polynomials of degree at most p in space. Then in §4 we show stability estimates for the continuous Galerkin method, and in §5 we use them to prove a priori error estimates, for the linear case $p = q = 1$, that are optimal in $L_\infty(L_2)$ and $L_\infty(H^1)$. Finally, in §6,we illustrate the theory for the linear case by computing the approximated solutions of (1.1) in a simple but realistic numerical example. In this paper we only study the original form of the numerical method and we do not discuss fast or adaptive strategies such as sparse quadrature or adaptive strategy based on a posteriori error estimates, e.g., [8], [1] and [2]. We postpone this to the forthcoming work. We also do not discuss adaptive, fast and oblivious convolution quadrature [10] and [7], to speed up the performance and decreasing the necessary memory.

2. **Existence and uniqueness**

In this section, using the theory of linear operator semigroups, we show that there is a unique solution of (1.6) for $t \geq 0$, when $\boldsymbol{g} = \boldsymbol{0}$ $(\Gamma = \Gamma_D)$, provided the data is regular enough. The techniques are adapted from [6].

We consider the strong form of (1.6), for any fixed $T > 0$, that is

$$\rho\ddot{\boldsymbol{u}}(t) + A\boldsymbol{u}(t) - \int_0^t \beta(t-s) A\boldsymbol{u}(s)\, ds = \boldsymbol{f}(t), \quad 0 < t < T, \tag{2.1}$$

with the initial data

$$\boldsymbol{u}(0) = \boldsymbol{u}^0, \ \dot{\boldsymbol{u}}(0) = \boldsymbol{v}^0. \tag{2.2}$$

We extend $\boldsymbol{u}$ by $\boldsymbol{u}(t) = \boldsymbol{h}(t)$ for $t < 0$ with $\boldsymbol{h}$ to be chosen. Then adding $-\int_{-\infty}^0 \beta(t-s) A\boldsymbol{h}(s)\, ds$ to both sides of (2.1), changing the variables in the convolution terms and defining $\boldsymbol{w}(t,s) = \boldsymbol{u}(t) - \boldsymbol{u}(t-s)$, we get

$$\rho\ddot{\boldsymbol{u}}(t) + \tilde{\gamma} A\boldsymbol{u}(t) + \int_0^\infty \beta(s) A\boldsymbol{w}(t,s)\, ds = \boldsymbol{f}(t) - \int_t^\infty \beta(s) A\boldsymbol{h}(t-s)\, ds, \tag{2.3}$$

where $\tilde{\gamma} = 1 - \gamma = 1 - \int_0^\infty \beta(s)\,ds$.

2.1. **An abstract Cauchy problem.** We choose $\boldsymbol{h}(\cdot) = \boldsymbol{u}^0$ in (2.3), so that

$$\rho\ddot{\boldsymbol{u}}(t) + \tilde{\gamma}A\boldsymbol{u}(t) + \int_0^\infty \beta(s)A\boldsymbol{w}(t,s)\,ds = \tilde{\boldsymbol{f}}(t), \tag{2.4}$$

where, in view of (1.4),

$$\tilde{\boldsymbol{f}}(t) = \boldsymbol{f}(t) - A\boldsymbol{u}^0\xi(t). \tag{2.5}$$

Then we reformulate the equation (2.4) as an abstract Cauchy problem and prove well-posedness.

We set $\boldsymbol{v} = \rho\dot{\boldsymbol{u}}$ and define the Hilbert spaces

$$W = L^2_\beta(0,\infty;V) = \Big\{\boldsymbol{w} : (0,\infty) \to V : \|\boldsymbol{w}\|_W^2 = \rho\int_0^\infty \beta(s)\|\boldsymbol{w}(s)\|_V^2\,ds < \infty\Big\},$$

$$Z = V \times H \times W = \Big\{\boldsymbol{z} = (\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) : \|\boldsymbol{z}\|_Z^2 = \tilde{\gamma}\rho\|\boldsymbol{u}\|_V^2 + \|\boldsymbol{v}\|^2 + \|\boldsymbol{w}\|_W^2 < \infty\Big\}.$$

So the inner products in W and Z are, respectively, $(\cdot,\cdot)_W = \rho\int_0^\infty \beta(s)a(\cdot,\cdot)\,ds$ and $\langle(\cdot,\cdot,\cdot)\,,\,(\cdot,\cdot,\cdot)\rangle_Z = \tilde{\gamma}\rho a(\cdot,\cdot) + (\cdot,\cdot) + (\cdot,\cdot)_W$.

We also define the linear operator $\mathcal{A} : \mathrm{dom}(\mathcal{A}) \to Z$ such that, for $\boldsymbol{z} = (\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})$

$$\mathcal{A}\boldsymbol{z} = \Big(\frac{1}{\rho}\boldsymbol{v}\,,\, -A\Big(\tilde{\gamma}\boldsymbol{u} + \int_0^\infty \beta(s)\boldsymbol{w}(s)\,ds\Big)\,,\, \frac{1}{\rho}\boldsymbol{v} - D\boldsymbol{w}\Big),$$

with

$$\mathrm{dom}(\mathcal{A}) = \Big\{(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) \in Z : \boldsymbol{v} \in V,\, \tilde{\gamma}\boldsymbol{u} + \int_0^\infty \beta(s)\boldsymbol{w}(s)\,ds \in \mathrm{dom}(A),$$
$$D\boldsymbol{w} \in W,\, \boldsymbol{w}(0) = 0\Big\},$$

where

$$D\phi = \frac{\partial}{\partial s}\phi \quad \text{with} \quad \mathrm{dom}(D) = \{\phi \in W : D\phi \in W \text{ and } \phi(0) = 0\}.$$

Then (2.4), with the initial values (2.2), can be written as an abstract Cauchy problem

$$\begin{aligned} &\dot{\boldsymbol{z}}(t) = \mathcal{A}\boldsymbol{z}(t) + F(t), \quad 0 < t < T,\\ &\boldsymbol{z}(0) = \boldsymbol{z}^0, \end{aligned} \tag{2.6}$$

where $F(t) = (0\,,\,\tilde{\boldsymbol{f}}(t)\,,\,0)$ and $\boldsymbol{z}^0 = \big(\boldsymbol{u}^0,\,\boldsymbol{v}^0,\,\boldsymbol{w}^0(\cdot)\big)$ with

$$\boldsymbol{w}^0(\cdot) = \boldsymbol{w}(0,\cdot) = 0, \tag{2.7}$$

since $\boldsymbol{w}(0,s) = \boldsymbol{u}(0) - \boldsymbol{u}(-s) = \boldsymbol{u}(0) - h(-s) = \boldsymbol{u}^0 - \boldsymbol{u}^0 = 0$. We also note that $\boldsymbol{w}(t,0) = \boldsymbol{u}(t) - \boldsymbol{u}(t) = 0$.

A function $\boldsymbol{z}$ which is differentiable a.e. on $[0,T]$ such that $\dot{\boldsymbol{z}} \in L_1([0,T];Z)$ is called a *strong solution* of the initial value problem (2.6) if $\boldsymbol{z}(0) = \boldsymbol{z}^0$, $\boldsymbol{z}(t) \in \mathrm{dom}(\mathcal{A})$, and $\dot{\boldsymbol{z}}(t) = \mathcal{A}\boldsymbol{z}(t) + F(t)$ a.e. on $[0,T]$.

Remark: By a solution of (1.6) we mean a *weak solution* in the way that is defined in (1.6), and by a solution of (2.1), that is often called *strong solution* in the literature, we mean a function $\boldsymbol{u}$ such that

$$\boldsymbol{u}(t) \in \mathrm{dom}(A), \quad \ddot{\boldsymbol{u}}(t) \in H \quad \text{and} \quad A\boldsymbol{u} \in L_1([0,T];H), \tag{2.8}$$

and also satisfies (2.1) a.e. on $[0,T]$ and the initial conditions (2.2). Henceforth, to avoid confusion, we call a weak solution of (1.6) and a strong solution of (2.1) just a *solution* of the relevant problem. We note that a solution $\boldsymbol{u}$ of (2.1) is also a solution of (1.6), when $\Gamma = \Gamma_D$.

Lemma 1. *Let $\boldsymbol{z} = (\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})$ be a strong solution of* (2.6)*. Then $\boldsymbol{u}$ is a solution of* (2.1) *with initial data* (2.2)*.*

Proof. For a given strong solution $\boldsymbol{z}$ of (2.6), considering (2.7), we get $\boldsymbol{u}(0) = \boldsymbol{u}^0$ and $\boldsymbol{v}(0) = \boldsymbol{v}^0$, which are the initial conditions (2.2). Indeed for $\boldsymbol{u} \in V$, $\boldsymbol{v} \in V$ (since $\boldsymbol{z} \in \mathrm{dom}(\mathcal{A})$) and $\boldsymbol{w} \in W$ the components of the strong solution $\boldsymbol{z}$ of (2.6), we have

$$\begin{aligned}
\dot{\boldsymbol{u}}(t) &= \frac{1}{\rho}\boldsymbol{v}, \\
\dot{\boldsymbol{v}}(t) &= -A\Big(\tilde{\gamma}\boldsymbol{u}(t) + \int_0^\infty \beta(s)\boldsymbol{w}(t,s)\,ds\Big) + \tilde{\boldsymbol{f}}(t), \\
\dot{\boldsymbol{w}} &= \frac{1}{\rho}\boldsymbol{v} - D\boldsymbol{w}.
\end{aligned}$$

The first and the third equation with initial value (2.7) imply that $\boldsymbol{w}(t,s) = \boldsymbol{u}(t) - \boldsymbol{u}(t-s)$. This and the fact that (2.4) is obtained from the first two equations, imply that $\boldsymbol{u}$ satisfies (2.1) a.e. on $[0,T]$ by backward calculations from (2.3). From the definition of the operator $\mathcal{A}$ and its domain we deduce (2.8), and this completes the proof. □

Theorem 1. *There is a unique solution $\boldsymbol{u} = \boldsymbol{u}(t)$ of* (2.1)–(2.2) *for all $\boldsymbol{u}^0 \in \mathrm{dom}(A)$ and $\boldsymbol{v}^0 \in V$, if $\boldsymbol{f} : [0,T] \to H$ is Lipschitz continuous. Moreover, we have the regularity estimate*

$$\|\boldsymbol{u}\|_V + \|\dot{\boldsymbol{u}}\| \le C\Big(\|\boldsymbol{u}^0\|_{H^2(\Omega)^d} + \|\boldsymbol{v}^0\| + \int_0^t \|\boldsymbol{f}\|\,ds\Big). \tag{2.9}$$

Proof. For any $\boldsymbol{u}^0 \in \mathrm{dom}(A)$ and $\boldsymbol{v}^0 \in V$, considering (2.7), we have $\boldsymbol{z}^0 = \big(\boldsymbol{u}^0, \boldsymbol{v}^0, \boldsymbol{w}^0(\cdot)\big) \in \mathrm{dom}(\mathcal{A})$. We first show that F in (2.6) is differentiable a.e. on $[0,T]$ and $\dot{F} \in L_1([0,T];Z)$. We then show that the linear operator $\mathcal{A}$ is an infinitesimal generator of a C_0 semigroup $T(t)$ on Z. These prove that there is a unique strong solution of (2.6) by [9], Corollary 4.2.10, and the proof of the first part is complete by Lemma 1.

1. By assumption $\boldsymbol{f}$ is Lipschitz continuous on $[0,T]$. Hence $\boldsymbol{f}$ is differentiable a.e. on $[0,T]$ and $\dot{\boldsymbol{f}} \in L_1([0,T];H)$, since H is a Hilbert space. Since $\dot{\xi}(t) = -\beta(t)$ by (1.4), from (2.5) we get

$$\dot{\tilde{\boldsymbol{f}}}(t) = \dot{\boldsymbol{f}}(t) + A\boldsymbol{u}^0\beta(t),$$

which shows that $\tilde{\boldsymbol{f}}$ is differentiable a.e. on $[0,T]$. Thus F is differentiable a.e. on $[0,T]$ and $\dot{F} \in L_1([0,T];Z)$.

2. We use the Lumer-Philips Theorem [9] to show that $\mathcal{A}$ generates a C_0 semigroup on Z (in fact, $\mathcal{A}$ generates a C_0 semigroup of contractions on Z). To this end we first justify that $\mathcal{A}$ is dissipative. For $\boldsymbol{z} = (\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) \in \mathrm{dom}(\mathcal{A})$ we have

$$\begin{aligned}\langle \mathcal{A}\boldsymbol{z}, \boldsymbol{z}\rangle_Z &= \tilde{\gamma} a(\boldsymbol{u}, \boldsymbol{v}) - \Big(A\big(\tilde{\gamma}\boldsymbol{u} + \int_0^\infty \beta(s)\boldsymbol{w}(s)\big)ds, \boldsymbol{v}\Big) + \big(\frac{1}{\rho}\boldsymbol{v} - D\boldsymbol{w}, \boldsymbol{w}\big)_W \\ &= -\rho \int_0^\infty \beta(s) a(D\boldsymbol{w}(s), \boldsymbol{w}(s))\, ds = -\frac{1}{2}\rho \int_0^\infty \beta(s) D\|\boldsymbol{w}(s)\|_V^2\, ds.\end{aligned}$$

To prove that the last term is non-positive, and hence $\mathcal{A}$ is dissipative, we consider for $\epsilon > 0$,

$$\begin{aligned}\int_\epsilon^\infty \beta(s) D\|\boldsymbol{w}(s)\|_V^2\, ds &= \lim_{M\to\infty} \int_\epsilon^M \beta(s) D\|\boldsymbol{w}(s)\|_V^2\, ds \\ &= \lim_{M\to\infty} \beta(M)\|\boldsymbol{w}(M)\|_V^2 - \beta(\epsilon)\|\boldsymbol{w}(\epsilon)\|_V^2 \\ &\quad - \int_\epsilon^\infty \beta'(s)\|\boldsymbol{w}(s)\|_V^2\, ds \\ &\geq -\beta(\epsilon)\|\boldsymbol{w}(\epsilon)\|_V^2,\end{aligned}$$

using the facts that $\beta'(s) < 0$ and $\lim_{M\to\infty} \beta(M)\|\boldsymbol{w}(M)\|_V^2 = 0$, since $\int_0^\infty \beta(s)\|\boldsymbol{w}(s)\|_V^2\, ds < \infty$. Since $\boldsymbol{w}(\epsilon) = \int_0^\epsilon D\boldsymbol{w}(s)\, ds$, by the Cauchy-Schwarz inequality we have

$$\|\boldsymbol{w}(\epsilon)\|_V^2 \leq \Big(\int_0^\epsilon \|D\boldsymbol{w}(s)\|_V\, ds\Big)^2 \leq \int_0^\epsilon \frac{1}{\beta(s)}\, ds \int_0^\epsilon \beta(s)\|D\boldsymbol{w}(s)\|_V^2\, ds,$$

and consequently we get

$$\int_\epsilon^\infty \beta(s) D\|\boldsymbol{w}(s)\|_V^2\, ds \geq -\int_0^\epsilon \frac{\beta(\epsilon)}{\beta(s)}\, ds \int_0^\epsilon \beta(s)\|D\boldsymbol{w}(s)\|_V^2\, ds.$$

But $\frac{\beta(\epsilon)}{\beta(s)} \leq 1$, which yields $\int_0^\epsilon \frac{\beta(\epsilon)}{\beta(s)}\, ds \leq \int_0^\epsilon ds = \epsilon$, so that

$$\int_\epsilon^\infty \beta(s) D\|\boldsymbol{w}(s)\|_V^2\, ds \geq -\epsilon \int_0^\epsilon \beta(s)\|D\boldsymbol{w}(s)\|_V^2\, ds.$$

Since $(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) \in \mathrm{dom}(\mathcal{A})$ implies $D\boldsymbol{w} \in W$, i.e., $\int_0^\epsilon \beta(s)\|D\boldsymbol{w}(s)\|_V^2\,ds < \infty$. Therefore

$$\langle \mathcal{A}\boldsymbol{z},\boldsymbol{z}\rangle_Z \le -\frac{1}{2}\rho \lim_{\epsilon\searrow 0} \epsilon \int_0^\epsilon \beta(s)\|D\boldsymbol{w}\|_V^2\,ds = 0,$$

and $\mathcal{A}$ is dissipative.

Next we show that $R(I-\mathcal{A}) = Z$. To see this, for an arbitrary $(\boldsymbol{\phi},\boldsymbol{\psi},\boldsymbol{\omega}) \in Z$ we must find $(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) \in \mathrm{dom}(\mathcal{A})$ such that $(I-\mathcal{A})(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) = (\boldsymbol{\phi},\boldsymbol{\psi},\boldsymbol{\omega})$, that is,

$$\begin{aligned}
&\boldsymbol{u} - \frac{1}{\rho}\boldsymbol{v} = \boldsymbol{\phi},\\
&\boldsymbol{v} + A\Big(\tilde{\gamma}\boldsymbol{u} + \int_0^\infty \beta(s)\boldsymbol{w}(s)\,ds\Big) = \boldsymbol{\psi},\\
&\boldsymbol{w} - \frac{1}{\rho}\boldsymbol{v} + D\boldsymbol{w} = \boldsymbol{\omega}.
\end{aligned} \tag{2.10}$$

From the first and third equations and $\boldsymbol{w}(0) = 0$ we get

$$\begin{aligned}
\boldsymbol{v} &= \rho(\boldsymbol{u}-\boldsymbol{\phi}),\\
\boldsymbol{w}(s) &= \int_0^s e^{r-s}\big(\frac{1}{\rho}\boldsymbol{v} + \boldsymbol{\omega}(r)\big)\,dr.
\end{aligned}$$

Substituting these into the second equation of (2.10), we get

$$\rho(\boldsymbol{u}-\boldsymbol{\phi}) + A\Big(\tilde{\gamma}\boldsymbol{u} + \int_0^\infty \beta(s)\int_0^s e^{r-s}\big(\boldsymbol{u}-\boldsymbol{\phi}+\boldsymbol{\omega}(r)\big)\,dr\,ds\Big) = \boldsymbol{\psi},$$

and hence

$$\boldsymbol{u} + \kappa A\boldsymbol{u} = \boldsymbol{\phi} + \frac{1}{\rho}\Big(\boldsymbol{\psi} + \int_0^\infty \beta(s)e^{-s}\int_0^s e^r A\big(\boldsymbol{\phi}-\boldsymbol{\omega}(r)\big)\,dr\,ds\Big), \tag{2.11}$$

where $\kappa = \frac{1}{\rho}\big(1 - \int_0^\infty \beta(s)e^{-s}ds\big)$. Now we need to show that this equation has a solution. We define

$$\Delta = I + \kappa A.$$

Consider the bilinear form

$$(\boldsymbol{u},\boldsymbol{v})_\Delta = (\boldsymbol{u},\boldsymbol{v}) + \kappa a(\boldsymbol{u},\boldsymbol{v}) \quad \text{for } \boldsymbol{u},\boldsymbol{v} \in V,$$

and the linear form

$$L(\boldsymbol{v}) = (\boldsymbol{\phi},\boldsymbol{v}) + \frac{1}{\rho}(\boldsymbol{\psi},\boldsymbol{v}) + \frac{1}{\rho}\int_0^\infty \beta(s)e^{-s}\int_0^s e^r a\big(\boldsymbol{\phi}-\boldsymbol{\omega}(r),\boldsymbol{v}\big)\,dr\,ds.$$

Then for some positive constants C_1, C_2 and C_3 by the boundedness and coercivity of $a(\cdot,\cdot)$

$$
\begin{aligned}
&|(\boldsymbol{u},\boldsymbol{v})_\Delta| \le C_1\|\boldsymbol{u}\|_V\|\boldsymbol{v}\|_V \quad \text{for } \boldsymbol{u},\boldsymbol{v}\in V,\\
&(\boldsymbol{u},\boldsymbol{u})_\Delta \ge C_2\|\boldsymbol{u}\|_V^2 \quad \text{for } \boldsymbol{u}\in V,\\
&|L(\boldsymbol{v})| \le C_3\|\boldsymbol{v}\|_V \quad \text{for } \boldsymbol{v}\in V.
\end{aligned}
$$

Therefore by Riesz representation theorem, there is a unique solution of the problem: find $\boldsymbol{u}\in V$ such that,

$$(\boldsymbol{u},\boldsymbol{v})_\Delta = L(\boldsymbol{v}) \quad \forall \boldsymbol{v}\in V,$$

that implies there is a unique solution of the problem (2.11). Hence $R(I-\mathcal{A}) = Z$.

Since Z is a Hilbert space, it follows from [9], Theorem 1.4.6, that $\overline{\mathrm{dom}(\mathcal{A})} = Z$. So we have verified all the hypotheses of the Lumer-Philips theorem to complete the first part of the proof.

3. Now we have the unique strong solution of (2.6), i.e.,

$$\boldsymbol{z}(t) = T(t)\boldsymbol{z}^0 + \int_0^t T(t-s)F(s)\,ds,$$

and $\|T(t)\|_Z \le 1$, since $\mathcal{A}$ generates a C_0 semigroup of contractions. Therefore we have

$$\|\boldsymbol{z}\|_Z \le \|T(t)\|_Z\|\boldsymbol{z}^0\|_Z + \int_0^t \|T(t-s)F(s)\|_Z\,ds \le \|\boldsymbol{z}^0\|_Z + \int_0^t \|F(s)\|_Z\,ds.$$

Then considering $\boldsymbol{v} = \rho\dot{\boldsymbol{u}}$, $\boldsymbol{z}^0 = (\boldsymbol{u}^0,\,\boldsymbol{v}^0,\,0)$ and $\|F(s)\|_Z = \|\tilde{\boldsymbol{f}}(s)\| = \|\boldsymbol{f}(s) - A\boldsymbol{u}^0\xi(s)\|$, we have

$$
\begin{aligned}
&\Big(\tilde{\gamma}\rho\|\boldsymbol{u}\|_V^2 + \rho^2\|\dot{\boldsymbol{u}}\|^2 + \rho\int_0^\infty \beta(s)\|\boldsymbol{w}(s)\|_V^2\,ds\Big)^{1/2}\\
&\qquad\qquad \le \big(\tilde{\gamma}\rho\|\boldsymbol{u}^0\|_V^2 + \|\boldsymbol{v}^0\|^2\big)^{1/2} + \int_0^t \big(\|\boldsymbol{f}(s)\| + \|\boldsymbol{u}^0\|_{H^2(\Omega)^d}\xi(s)\big)\,ds.
\end{aligned}
$$

Consequently, we have the estimate(2.9) with $C = C(\tilde{\gamma},\rho,T)$. □

Remark: Due to singularity of the kernel β at the origin, $\xi = \xi(t)$ in (1.4) is not Lipschitz continuous. With a smoother kernel β, $\xi = \xi(t)$ would be Lipschitz continuous so that we could get a unique strong solution of (2.6) by [9], Corollary 4.2.11, instead of [9], Corollary 4.2.10, when $\boldsymbol{f}$ is Lipschitz continuous on $[0,T]$.

2.2. **Regularity.** By Theorem 1 there is a unique solution of (1.6), if the data are smooth enough. To find sufficient conditions on the data for more regularity, we assume that the data are smooth enough to justify the following calculations.

We first choose $\boldsymbol{h}(t) = t\boldsymbol{v}^0$ in (2.3), so that

$$\rho\ddot{\boldsymbol{u}}(t) + \tilde{\gamma}A\boldsymbol{u}(t) + \int_0^\infty \beta(s)A\boldsymbol{w}(t,s)\,ds = \breve{\boldsymbol{f}}(t), \tag{2.12}$$

where

$$\breve{\boldsymbol{f}}(t) = \boldsymbol{f}(t) - A\boldsymbol{v}^0 \int_t^\infty (t-s)\beta(s)\,ds. \tag{2.13}$$

Then differentiating the equation (2.12) in time we get

$$\rho\,\dddot{\boldsymbol{u}}(t) + \tilde{\gamma}A\dot{\boldsymbol{u}}(t) + \int_0^\infty \beta(s)A\dot{\boldsymbol{w}}(t,s)\,ds = \dot{\breve{\boldsymbol{f}}}(t), \tag{2.14}$$

which, with an underline instead of one time derivative, can be written as

$$\rho\ddot{\underline{\boldsymbol{u}}}(t) + \tilde{\gamma}A\underline{\boldsymbol{u}}(t) + \int_0^\infty \beta(s)A\underline{\boldsymbol{w}}(t,s)\,ds = \underline{\breve{\boldsymbol{f}}}(t), \tag{2.15}$$

with the initial values

$$\underline{\boldsymbol{u}}(0) = \underline{\boldsymbol{u}}^0 = \boldsymbol{v}^0\,,\;\; \dot{\underline{\boldsymbol{u}}}(0) = \underline{\boldsymbol{v}}^0 = \frac{1}{\rho}\big(\boldsymbol{f}(0) - A\boldsymbol{u}^0\big), \tag{2.16}$$

and

$$\underline{\breve{\boldsymbol{f}}}(t) = \dot{\breve{\boldsymbol{f}}}(t) = \dot{\boldsymbol{f}}(t) - A\boldsymbol{v}^0\xi(t), \tag{2.17}$$

and $\underline{\boldsymbol{w}} = \dot{\boldsymbol{w}}(t,\cdot) = \dot{\boldsymbol{u}}(t) - \dot{\boldsymbol{u}}(t-\cdot) = \underline{\boldsymbol{u}}(t) - \underline{\boldsymbol{u}}(t-\cdot)$, so that $\underline{\boldsymbol{w}}(t,0) = 0$.

Then, in the same way as in §2.1 with $\underline{\boldsymbol{v}} = \rho\dot{\underline{\boldsymbol{u}}}$, we can reformulate (2.15)–(2.16) as the abstract Cauchy problem

$$\begin{aligned} &\dot{\underline{\boldsymbol{z}}}(t) = \mathcal{A}\,\underline{\boldsymbol{z}}(t) + \underline{\breve{F}}(t), \quad 0 < t < T,\\ &\underline{\boldsymbol{z}}(0) = \underline{\boldsymbol{z}}^0, \end{aligned} \tag{2.18}$$

where $\underline{\breve{F}}(t) = (0\,,\, \underline{\breve{\boldsymbol{f}}}(t)\,,\, 0)$ and $\underline{\boldsymbol{z}}^0 = \big(\underline{\boldsymbol{u}}^0,\, \underline{\boldsymbol{v}}^0,\, \underline{\boldsymbol{w}}^0(\cdot)\big)$ with

$$\underline{\boldsymbol{w}}^0(\cdot) = \underline{\boldsymbol{w}}(0,\cdot) = 0, \tag{2.19}$$

since $\underline{\boldsymbol{w}}(0,s) = \underline{\boldsymbol{u}}(0) - \underline{\boldsymbol{u}}(-s) = \underline{\boldsymbol{u}}(0) - \frac{d}{dt}\boldsymbol{h}(t-s)\,|_{t=0} = \boldsymbol{v}^0 - \boldsymbol{v}^0 = 0$.

Lemma 2. *Let $\underline{\boldsymbol{z}} = (\underline{\boldsymbol{u}}, \underline{\boldsymbol{v}}, \underline{\boldsymbol{w}})$ be a strong solution of* (2.18)*. Then $\boldsymbol{u}(t) = \boldsymbol{u}^0 + \int_0^t \underline{\boldsymbol{u}}(s)\,ds$ is a solution of* (2.1) *with initial data* (2.2)*.*

Proof. For a given strong solution $\underline{\boldsymbol{z}}$ of (2.18), considering (2.16) and (2.19), we have

$$\Big(\dot{\boldsymbol{u}}(0),\, \frac{1}{\rho}\big(\boldsymbol{f}(0) - A\boldsymbol{u}(0)\big),\, 0\Big) = \Big(\boldsymbol{v}^0,\, \frac{1}{\rho}\big(\boldsymbol{f}(0) - A\boldsymbol{u}^0\big),\, 0\Big),$$

that gives us the initial data (2.2). Then since $\underline{\dot{\boldsymbol{z}}}(t) = \mathcal{A}\,\underline{\boldsymbol{z}}(t) + \underline{\breve{F}}(t)$ a.e. on $[0, T]$, we have

$$\begin{aligned}
\underline{\dot{\boldsymbol{u}}}(t) &= \frac{1}{\rho}\underline{\boldsymbol{v}}\,,\\
\underline{\dot{\boldsymbol{v}}}(t) &= -A\Big(\tilde{\gamma}\underline{\boldsymbol{u}}(t) + \int_0^\infty \beta(s)\underline{\boldsymbol{w}}(t,s)\,ds\Big) + \underline{\breve{\boldsymbol{f}}}(t),\\
\underline{\dot{\boldsymbol{w}}} &= \frac{1}{\rho}\underline{\boldsymbol{v}} - D\underline{\boldsymbol{w}}.
\end{aligned}$$

The third equation with initial value (2.19) has the unique solution $\underline{\boldsymbol{w}}(t,s) = \underline{\boldsymbol{u}}(t) - \underline{\boldsymbol{u}}(t-s)$ that implies, after integrating with respect to t, $\boldsymbol{w}(t,s) = \boldsymbol{u}(t) - \boldsymbol{u}(t-s)$. And the first two equations give us (2.14) recalling the notation $\underline{\phi} = \dot{\phi}(t)$. Then integrating (2.14), we obtain

$$\begin{aligned}
\rho\ddot{\boldsymbol{u}}(t) - \rho\ddot{\boldsymbol{u}}(0) + \tilde{\gamma}A\boldsymbol{u}(t) - \tilde{\gamma}A\boldsymbol{u}(0) + \int_0^t\int_0^\infty \beta(s)AD_r\boldsymbol{w}(r,s)\,ds\,dr&\\
= \breve{\boldsymbol{f}}(t) - \breve{\boldsymbol{f}}(0).&
\end{aligned}\tag{2.20}$$

From (2.12) for $t = 0$, we have

$$\rho\ddot{\boldsymbol{u}}(0) + \tilde{\gamma}A\boldsymbol{u}(0) + \int_0^\infty \beta(s)A\boldsymbol{w}(0,s)\,ds = \breve{\boldsymbol{f}}(0).\tag{2.21}$$

And the integral in(2.20) is

$$\begin{aligned}
\int_0^t\int_0^\infty \beta(s)AD_r\boldsymbol{w}(r,s)\,ds\,dr &= \int_0^\infty \beta(s)\int_0^t AD_r\boldsymbol{w}(r,s)\,dr\,ds\\
&= \int_0^\infty \beta(s)A\boldsymbol{w}(t,s)\,ds\\
&\quad - \int_0^\infty \beta(s)A\boldsymbol{w}(0,s)\,ds.
\end{aligned}\tag{2.22}$$

Hence (2.20), considering (2.21), (2.22) and $\boldsymbol{w}(t,s) = \boldsymbol{u}(t) - \boldsymbol{u}(t-s)$, gives (2.12), that implies $\boldsymbol{u}$ satisfies (2.1) a.e. on $[0, T]$ by backward calculations from (2.3). Finally from the definition of the operator $\mathcal{A}$ and its domain we deduce (2.8), and this complete the proof. □

In the next theorem we find the circumstances under which, there is a unique solution of (2.1) with more regularity.

Theorem 2. *There is a unique solution* $\boldsymbol{u} = \boldsymbol{u}(t)$ *of* (2.1)–(2.2) *for all* $\boldsymbol{v}^0 \in \mathrm{dom}(A)$, $A\boldsymbol{u}^0 \in V$ *and* $\boldsymbol{f}(0) \in V$, *if* $\dot{\boldsymbol{f}} : [0, T] \to H$ *is Lipschitz continuous. Moreover, we have the regularity estimate*

$$\|\dot{\boldsymbol{u}}\|_V + \|\ddot{\boldsymbol{u}}\| \le C\Big(\|\boldsymbol{u}^0\|_{H^2(\Omega)^d} + \|\boldsymbol{v}^0\|_{H^2(\Omega)^d} + \|\boldsymbol{f}(0)\| + \int_0^t \|\dot{\boldsymbol{f}}\|\,ds\Big).\tag{2.23}$$

Proof. 1. From the assumptions on $\boldsymbol{u}^0$, $\boldsymbol{v}^0$ and $\boldsymbol{f}(0)$ and recalling (2.16), we have $\underline{\boldsymbol{z}}^0 = \big(\underline{\boldsymbol{u}}^0, \underline{\boldsymbol{v}}^0, \underline{\boldsymbol{w}}^0(\cdot)\big) \in \mathrm{dom}(\mathcal{A})$. Also considering

$$\dot{\breve{\underline{\boldsymbol{f}}}}(t) = \ddot{\boldsymbol{f}}(t) + A\boldsymbol{v}^0\beta(t),$$

obtained from (2.17), and the assumptions on $\boldsymbol{v}^0$ and $\dot{\boldsymbol{f}}$, $\breve{\underline{F}}$ is differentiable a.e. on $[0,T]$ and $\dot{\breve{\underline{F}}} \in L_1([0,T];Z)$. Then, since the linear operator $\mathcal{A}$ is an infinitesimal generator of a C_0-semigroup (of contractions) $T(t)$ on Z by the second step of the proof of Theorem 1, there is a unique strong solution of (2.18) by [9], Corollary 4.2.10. Hence the first part of the proof is complete by Lemma 2.

2. We have the unique strong solution of (2.18), i.e.

$$\underline{\boldsymbol{z}}(t) = T(t)\underline{\boldsymbol{z}}^0 + \int_0^t T(t-s)\breve{\underline{F}}(s)\,ds,$$

with $\|T(t)\|_Z \leq 1$, since $\mathcal{A}$ generates a C_0-semigroup of contractions. Then we have

$$\|\underline{\boldsymbol{z}}\|_Z \leq \|\underline{\boldsymbol{z}}^0\|_Z + \int_0^t \|\breve{\underline{F}}(s)\|_Z\,ds.$$

Therefore considering $\underline{\boldsymbol{v}} = \rho\dot{\underline{\boldsymbol{u}}}$, $\underline{\boldsymbol{z}}^0 = (\underline{\boldsymbol{u}}^0, \underline{\boldsymbol{v}}^0, 0)$ and $\|\breve{\underline{F}}(s)\|_Z = \|\breve{\underline{\boldsymbol{f}}}(s)\| = \|\dot{\boldsymbol{f}}(s) - A\boldsymbol{v}^0\xi(s)\|$, we have

$$\begin{aligned}
&\Big(\tilde{\gamma}\rho\|\dot{\boldsymbol{u}}\|_V^2 + \rho^2\|\ddot{\boldsymbol{u}}\|^2 + \rho\int_0^\infty \beta(s)\|\dot{\boldsymbol{w}}(s)\|_V^2\,ds\Big)^{1/2} \\
&\qquad \leq \big(\tilde{\gamma}\rho\|\boldsymbol{v}^0\|_V^2 + \frac{1}{\rho^2}\|\boldsymbol{f}(0) - A\boldsymbol{u}^0\|^2\big)^{1/2} + \int_0^t \big(\|\dot{\boldsymbol{f}}(s)\| + \|\boldsymbol{v}^0\|_{H^2(\Omega)^d}\xi(s)\big)\,ds.
\end{aligned}$$

Consequently, for some $C = C(\tilde{\gamma}, \rho, T)$, we get (2.23). □

2.3. **Higher regularity.** We want to generalize the procedure in §2.1 and §2.2 to obtain higher regularity. To this end we choose, for $n \geq 2$,

$$h(t) = \frac{t^n}{n!}\boldsymbol{u}_n(0),$$

in (2.3) to get

$$\rho\ddot{\boldsymbol{u}}(t) + \tilde{\gamma}A\boldsymbol{u}(t) + \int_0^\infty \beta(s)A\boldsymbol{w}(t,s)\,ds = \boldsymbol{f}(t) - A\boldsymbol{u}_n(0)\int_t^\infty \frac{(t-s)^n}{n!}\beta(s)\,ds,$$

where by $\phi_n(t)$ we mean $\frac{\partial^n}{\partial t^n}\phi(t)$, considering the trivial cases $\phi_0 = \phi$ and $\phi_1 = \dot{\phi}$. The cases $n = 0$ and $n = 1$ have already been discussed in the previous sections.

Then differentiating n times with respect to t, we have

$$\rho\ddot{\boldsymbol{u}}_n(t) + \tilde{\gamma}A\boldsymbol{u}_n(t) + \int_0^\infty \beta(s)A\boldsymbol{w}_n(t,s)\,ds = \boldsymbol{f}_n(t) - A\boldsymbol{u}_n(0)\xi(t), \tag{2.24}$$

with initial values

$$\boldsymbol{u}_n(0) = \boldsymbol{u}_n^0 \quad \text{and} \quad \boldsymbol{u}_{n+1}(0) = \dot{\boldsymbol{u}}_n(0) = \boldsymbol{v}_n^0, \tag{2.25}$$

where

$$\boldsymbol{u}_m(0) = \frac{1}{\rho}\big(\boldsymbol{f}_{m-2}(0) - A\boldsymbol{u}_{m-2}(0)\big), \quad \text{for} \quad 2 \le m \le n+1,$$

is obtained from (2.1) and (2.2) by recursion, and implies that for $n = 2k\,(k = 1, 2, \cdots)$

$$\boldsymbol{u}_n(0) = \frac{A^k}{\rho^k}\boldsymbol{u}^0 + \sum_{j=1}^{k}(-1)^{j-1}\frac{A^{j-1}}{\rho^j}\boldsymbol{f}_{2k-2j}(0), \tag{2.26}$$

and for $n = 2k+1\,(k = 1, 2, \cdots)$

$$\boldsymbol{u}_n(0) = \frac{A^k}{\rho^k}\boldsymbol{v}^0 + \sum_{j=1}^{k}(-1)^{j-1}\frac{A^{j-1}}{\rho^j}\boldsymbol{f}_{2k-2j+1}(0). \tag{2.27}$$

Similar to §2.1 and §2.2 we can reformulate (2.24)–(2.25) as an abstract Cauchy problem

$$\begin{aligned} &\dot{\boldsymbol{z}}_n(t) = \mathcal{A}\boldsymbol{z}_n(t) + F_n(t), \quad 0 < t < T,\\ &\boldsymbol{z}_n(0) = \boldsymbol{z}_n^0, \end{aligned} \tag{2.28}$$

where $F_n(t) = (0\,,\, \boldsymbol{f}_n(t) - A\boldsymbol{u}_n(0)\xi(t)\,,\, 0)$ and $\boldsymbol{z}_n^0 = \big(\boldsymbol{u}_n^0, \boldsymbol{v}_n^0, \boldsymbol{w}_n^0(\cdot)\big)$ with

$$\boldsymbol{w}_n^0(\cdot) = \boldsymbol{w}_n(0,\cdot) = 0, \tag{2.29}$$

since $\boldsymbol{w}_n(0,s) = \boldsymbol{u}_n(0) - \boldsymbol{u}_n(-s) = \boldsymbol{u}_n(0) - \frac{\partial^n}{\partial t^n}h(t-s)\,|_{t=0} = \boldsymbol{u}_n^0 - \boldsymbol{u}_n^0 = 0$. We also note that $\boldsymbol{w}_n(t,0) = \boldsymbol{u}_n(t) - \boldsymbol{u}_n(t) = 0$.

Theorem 3. *Let $\frac{\partial^n}{\partial t^n}\boldsymbol{f} = \boldsymbol{f}_n : [0,T] \to H$ be Lipschitz continuous. Recalling* $\mathrm{dom}(A) = H^2(\Omega)^d \cap V$*, we also assume that:*
for $n = 2k\,(k = 1, 2, \cdots)$

$$\begin{aligned} &A^k\boldsymbol{u}^0 \in \mathrm{dom}(A), \quad A^k\boldsymbol{v}^0 \in V,\\ &\boldsymbol{f}_{2k-2j}(0) \in H^{2j}(\Omega)^d \cap V, \quad \boldsymbol{f}_{2k-2j+1}(0) \in H^{2j-1}(\Omega)^d \cap V,\\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{for } j = 1, \cdots, k, \end{aligned} \tag{2.30}$$

and for $n = 2k+1\,(k = 1, 2, \cdots)$

$$\begin{aligned} &A^k\boldsymbol{v}^0 \in \mathrm{dom}(A), \quad A^k\boldsymbol{u}^0 \in V,\\ &\boldsymbol{f}_{2k-2j+1}(0) \in H^{2j}(\Omega)^d \cap V, \qquad \text{for } j = 1, \cdots, k,\\ &\boldsymbol{f}_{2k-2j+2}(0) \in H^{2j-1}(\Omega)^d \cap V, \qquad \text{for } j = 1, \cdots, k+1, \end{aligned} \tag{2.31}$$

where $H^m(\Omega)^d$ is the standard Sobolev space with the standard norm denoted by $\|\cdot\|_m$.

Then there is a unique solution of (2.1)–(2.2)*. Moreover, for some $C = C(\tilde{\gamma}, \rho, T)$:*

$$\begin{aligned}\|\boldsymbol{u}_n\|_V + \|\boldsymbol{u}_{n+1}\| \le C\bigg(&\|\boldsymbol{u}^0\|_{2k+2} + \|\boldsymbol{v}^0\|_{2k} + \sum_{j=1}^{k}\|\boldsymbol{f}_{2k-2j}(0)\|_{2j} \\ &+ \sum_{j=1}^{k}\|\boldsymbol{f}_{2k-2j+1}(0)\|_{2j-2} + \int_0^t \|\boldsymbol{f}_n(s)\|\,ds\bigg),\end{aligned} \tag{2.32}$$

for $n = 2k\,(k = 1, 2, \cdots)$, and

$$\begin{aligned}\|\boldsymbol{u}_n\|_V + \|\boldsymbol{u}_{n+1}\| \le C\bigg(&\|\boldsymbol{u}^0\|_{2k+2} + \|\boldsymbol{v}^0\|_{2k+2} + \sum_{j=1}^{k}\|\boldsymbol{f}_{2k-2j+1}(0)\|_{2j} \\ &+ \sum_{j=1}^{k+1}\|\boldsymbol{f}_{2k-2j+2}(0)\|_{2j-2} + \int_0^t \|\boldsymbol{f}_n(s)\|\,ds\bigg),\end{aligned} \tag{2.33}$$

for $n = 2k+1\,(k = 1, 2, \cdots)$.

Proof. 1. The assumptions in Theorem 1 and Theorem 2 are fulfilled from the given hypothesis here, respectively, for even and odd n. Therefore existence and uniqueness of the solution $\boldsymbol{u}$ of (2.1)–(2.2) is proved.

2. To prove the regularity estimates (2.32) and (2.33) we need to find a strong solution of (2.28). To this end, we note that $\boldsymbol{z}_n^0 = \big(\boldsymbol{u}_n^0, \boldsymbol{v}_n^0, \boldsymbol{w}_n^0(\cdot)\big) = \big(\boldsymbol{u}_n^0, \boldsymbol{v}_n^0, 0\big) \in \mathrm{dom}(\mathcal{A})$ by (2.29) and the fact that $\boldsymbol{u}_n^0 \in \mathrm{dom}(A)$ and $\boldsymbol{v}^0 \in V$, since assumptions (2.30) and (2.31) and recalling (2.26) and (2.27).

The next step is to show that $F_n = (0,\, \boldsymbol{f}_n - A\boldsymbol{u}_n^0\xi(t),\, 0)$ is differentiable a.e. on $[0,T]$ and $F_n \in L_1([0,T];Z)$. Since, by assumption, $\boldsymbol{f}_n$ is Lipschitz continuous into Hilbert space H, $\boldsymbol{f}_n$ is differentiable a.e. on $[0,T]$ and $\boldsymbol{f}_n \in L_1([0,T];H)$. Indeed $\xi(t)$ is differentiable a.e. on $[0,T]$ and $\dot{\xi}(t) = -\beta(t) \in L_1([0,T])$ from (1.4), and we also have $A\boldsymbol{u}_n^0 \in H$ by assumptions (2.30) and (2.31) and considering (2.26) and (2.27). These give us the desired fact about F_n.

Finally, considering the fact that the linear operator $\mathcal{A}$ is an infinitesimal generator of a C_0-semigroup (of contractions) $T(t)$ on Z by Theorem 1, there is a unique strong solution $\boldsymbol{z}_n(t) = (\boldsymbol{u}_n(t), \boldsymbol{v}_n(t), \boldsymbol{w}_n(t,\cdot)) \in \mathrm{dom}(\mathcal{A})$ of (2.28), by [9], Corollary 4.2.10, so that

$$\boldsymbol{z}_n(t) = T(t)\boldsymbol{z}_n^0 + \int_0^t T(t-s)F_n(s)\,ds,$$

with $\|T(t)\|_Z \leq 1$. Then taking Z-norm $\|\cdot\|_Z$ of both sides the equation and recalling $\boldsymbol{v}_n = \rho\dot{\boldsymbol{u}}_n$, $\boldsymbol{z}_n^0 = \big(\boldsymbol{u}_n^0,\, \boldsymbol{v}_n^0,\, 0\big)$ and $\|F_n(s)\|_Z = \|\boldsymbol{f}_n(s) - A\boldsymbol{u}_n^0\xi(s)\|$, we have

$$\Big(\tilde{\gamma}\rho\|\boldsymbol{u}_n\|_V^2 + \|\boldsymbol{v}_n\|^2 + \rho\int_0^\infty \beta(s)\|\boldsymbol{w}_n(s)\|_V^2\,ds\Big)^{1/2}$$
$$\leq \big(\tilde{\gamma}\rho\|\boldsymbol{u}_n^0\|_V^2 + \|\boldsymbol{v}_n^0\|^2\big)^{1/2} + \int_0^t \big(\|\boldsymbol{f}_n(s)\| + \|A\boldsymbol{u}_n^0\|\xi(s)\big)\,ds.$$

Hence for some $C = C(\tilde{\gamma},\, \rho,\, T)$ we get

$$\|\boldsymbol{u}_n\|_V + \|\boldsymbol{u}_{n+1}\| \leq C\Big(\|\boldsymbol{u}_n^0\|_2 + \|\boldsymbol{v}_n^0\| + \int_0^t \|\boldsymbol{f}_n(s)\|\,ds\Big),$$

that implies the desired estimates (2.32) and (2.33), considering (2.25), (2.26) and (2.27) and the assumptions (2.30) and (2.31). □

Remark: Inspired by Lemma 1 and Lemma 2, one may prove (by induction) that for any strong solution $\boldsymbol{z}_n = \big(\boldsymbol{u}_n,\, \boldsymbol{v}_n,\, \boldsymbol{w}_n(\cdot)\big)$, $\boldsymbol{u}$ is a solution of (2.1)–(2.2). This, of course, gives an alternative to prove existence and uniqueness of $\boldsymbol{u}$ in Theorem 3.

3. **The continuous Galerkin method**

Recalling the function spaces $H = L_2(\Omega)^d$, $H^{\Gamma_N} = L_2(\Gamma_N)^d$ and $V = \{\boldsymbol{v} \in H^1(\Omega)^d : \boldsymbol{v}|_{\Gamma_D} = 0\}$ $(d = 2, 3)$, we provide some definitions which will be used in the forthcoming discussions.

Let $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n < \cdots < t_N = T$ be a partition of the time interval $I = [0, T]$. To each time subinterval $I_n = (t_{n-1}, t_n)$ of length $k_n = t_n - t_{n-1}$, associate a triangulation $\mathcal{T}_{h,n}$ of Ω with meshsize function h_n defined by

$$h_n(x) = \mathrm{diam}(K)\,, \quad \text{where } K \in \mathcal{T}_{h,n} \text{ and } x \in K, \tag{3.1}$$

for all $x \in \Omega$, and for $p \geq 1$ corresponding finite element space $V_{h,n}^{(p)}$ of vector-valued continuous piecewise polynomials in Ω of degree at most p, that vanish on Γ_D (This requires that the mesh is adjusted to fit Γ_D.). We also define the spaces, for $q \geq 0$,

$$W^{(q,p)} = \Big\{\boldsymbol{w} : \boldsymbol{w}|_{\Omega\times I_n} = \boldsymbol{w}^n \in W_n^{(q)},\, n = 1, \dots, N\Big\}\,,$$

where, with $\mathbb{P}_q^d$ the set of all vector-valued polynomials of degree at most q,

$$W_n^{(q,p)} = \Big\{\boldsymbol{w} : \boldsymbol{w}(x, \cdot) \in \mathbb{P}_q^d(I_n),\, \boldsymbol{w}(\cdot, t) \in V_{h,n}^{(p)},\, (x, t) \in \Omega \times I_n,\, \Big\}\,.$$

Note that $\boldsymbol{w} \in W^{(q,p)}$ may be discontinuous at $t = t_n$, and $w \in W^{(0,p)}$ is piecewise constant in time. In the sequel we write $W^{(q)} = W^{(q,p)}$ for short.

The Ritz (elliptic) and orthogonal projections $\mathcal{R}_{h,n} : V \to V_{h,n}^{(p)}$, $\mathcal{P}_{h,n} : H \to V_{h,n}^{(p)}$ and $\mathcal{P}_{k,n} : L_2(I_n)^d \to \mathbb{P}_{q-1}^d(I_n)$ are defined, respectively, by

$$
(3.2)\qquad
\begin{aligned}
&a(\mathcal{R}_{h,n}\boldsymbol{v} - \boldsymbol{v}, \boldsymbol{\chi}) = 0, \quad \forall \boldsymbol{v} \in V \text{ and } \boldsymbol{\chi} \in V_{h,n}^{(p)}\,,\\
&(\mathcal{P}_{h,n}\boldsymbol{v} - \boldsymbol{v}, \boldsymbol{\chi}) = 0, \quad \forall \boldsymbol{v} \in H \text{ and } \boldsymbol{\chi} \in V_{h,n}^{(p)}\,,\\
&\int_{I_n} (\mathcal{P}_{k,n}\boldsymbol{v} - \boldsymbol{v}) \cdot \boldsymbol{\psi}\, dt = 0, \quad \forall \boldsymbol{v} \in L_2(I_n)^d \text{ and } \boldsymbol{\psi} \in \mathbb{P}_{q-1}^d(I_n)\,.
\end{aligned}
$$

Correspondingly, we define $\mathcal{R}_h\boldsymbol{v}$ and $\mathcal{P}_h\boldsymbol{v}$ for $t \in I_n\,(n = 1, \cdots, N)$, by $(\mathcal{R}_h\boldsymbol{v})(t) = \mathcal{R}_{h,n}\boldsymbol{v}(t)$ and $(\mathcal{P}_h\boldsymbol{v})(t) = \mathcal{P}_{h,n}\boldsymbol{v}(t)$, and also $\mathcal{P}_k\boldsymbol{v} \in \prod_{n=1}^N \mathbb{P}_{q-1}^d(I_n)$ by $\mathcal{P}_k\boldsymbol{v} = \mathcal{P}_{k,n}(\boldsymbol{v}|_{I_n})$ on I_n. We also define the orthogonal projections, $R_n : L_2(I_n, V) \to W_n^{(q-1)}$, $P_n : L_2(I_n, H) \to W_n^{(q-1)}$ and $P_n^{\Gamma_N} : L_2(I_n, H^{\Gamma_N}) \to W_n^{(q-1)}$, such that

$$
(3.3)\qquad
\begin{aligned}
&\int_{I_n} a(R_n\boldsymbol{u} - \boldsymbol{u}, \boldsymbol{\psi})\, dt = 0, \quad \forall \boldsymbol{\psi} \in W_n^{(q-1)}\,,\ \boldsymbol{u} \in L_2(I_n, V)\,,\\
&\int_{I_n} (P_n\boldsymbol{u} - \boldsymbol{u}, \boldsymbol{\psi})\, dt = 0, \quad \forall \boldsymbol{\psi} \in W_n^{(q-1)}\,,\ \boldsymbol{u} \in L_2(I_n, H)\,,\\
&\int_{I_n} (P_n^{\Gamma_N}\boldsymbol{u}, \boldsymbol{\psi})\, dt = \int_{I_n} (\boldsymbol{u}, \boldsymbol{\psi})_{\Gamma_N}\, dt, \quad \forall \boldsymbol{\psi} \in W_n^{(q-1)}\,,\ \boldsymbol{u} \in L_2(I_n, H^{\Gamma_N})\,.
\end{aligned}
$$

Correspondingly, we define $R : L_2(I, V) \to W^{(q-1)}$, $P : L_2(I, H) \to W^{(q-1)}$ and $P^{\Gamma_N} : L_2(I, H^{\Gamma_N}) \to W^{(q-1)}$ in the obvious way.

One can easily show that

$$
(3.4)\qquad R = \mathcal{R}_h\mathcal{P}_k = \mathcal{P}_k\mathcal{R}_h\,, \quad P = \mathcal{P}_h\mathcal{P}_k = \mathcal{P}_k\mathcal{P}_h\,,
$$

and $\forall \boldsymbol{u} \in W_n^{(q)}\,,\ \boldsymbol{v} \in W_n^{(q-1)}$,

$$
(3.5)\qquad \int_{I_n} (\boldsymbol{u}, \boldsymbol{v})\, dt = \int_{I_n} (\mathcal{P}_{k,n}\boldsymbol{u}, \boldsymbol{v})\, dt,
$$

$$
(3.6)\qquad \int_{I_n} a(\boldsymbol{u}, \boldsymbol{v})\, dt = \int_{I_n} a(\mathcal{P}_{k,n}\boldsymbol{u}, \boldsymbol{v})\, dt.
$$

We introduce the linear operator $A_{h,n,r} : V_{h,r}^{(p)} \to V_{h,n}^{(p)}$ by

$$
a(\boldsymbol{v}_r, \boldsymbol{w}_n) = (A_{h,n,r}\boldsymbol{v}_r, \boldsymbol{w}_n) \qquad \forall \boldsymbol{v}_r \in V_{h,r}^{(p)},\ \boldsymbol{w}_n \in V_{h,n}^{(p)}.
$$

We set $A_{h,n} = A_{h,n,n}$, with discrete norms

$$
|\boldsymbol{v}_n|_{h,l} = \|A_{h,n}^{l/2}\boldsymbol{v}_n\| = \sqrt{(\boldsymbol{v}_n, A_{h,n}^l\boldsymbol{v}_n)}\,, \quad \boldsymbol{v}_n \in V_{h,n}^{(p)} \text{ and } l \in \mathbb{R}\,,
$$

and A_h so that $A_h\boldsymbol{v} = A_{h,n}\boldsymbol{v}$ for $\boldsymbol{v} \in V_{h,n}^{(p)}$. For later use in our error analysis we note that

$$\mathcal{P}_h A = A_h \mathcal{R}_h.$$

We define the bilinear form $B : \mathcal{W} \times \mathcal{W} \to \mathbb{R}$, and the linear forms $L, \hat{L} : \mathcal{W} \to \mathbb{R}$ by

$$\begin{aligned}
B(\boldsymbol{u},\boldsymbol{v}) &= \sum_{n=1}^{N} \int_{I_n} -a(\boldsymbol{u}_2,\boldsymbol{v}_1) + a(\dot{\boldsymbol{u}}_1,\boldsymbol{v}_1) + \rho(\dot{\boldsymbol{u}}_2,\boldsymbol{v}_2) + a(\boldsymbol{u}_1,\boldsymbol{v}_2)\,dt \\
&\quad - \sum_{n=1}^{N} \int_{I_n} \int_0^t \beta(t-s) a\big(\boldsymbol{u}_1(s),\boldsymbol{v}_2(t)\big) ds\,dt, \\
L(\boldsymbol{w}) &= \sum_{n=1}^{N} \int_{I_n} (\boldsymbol{f}_2,\boldsymbol{w}_2) + (\boldsymbol{g},\boldsymbol{w}_2)_{\Gamma_N}\,dt, \\
\hat{L}(\boldsymbol{w}) &= \sum_{n=1}^{N} \int_{I_n} a(\boldsymbol{f}_1,\boldsymbol{w}_1) + (\boldsymbol{f}_2,\boldsymbol{w}_2) + (\boldsymbol{g},\boldsymbol{w}_2)_{\Gamma_N}\,dt,
\end{aligned}$$

where $\mathcal{W}$ is the space of pairs of vector-valued functions $\boldsymbol{u} = (\boldsymbol{u}_1,\, \boldsymbol{u}_2)$ that are piecewise smooth with respect to the temporal mesh. We may note that $(W^{(q)})^2 \subset \mathcal{W}$ for $q \geq 0$.

The continuous Galerkin method of degree (q,p) is based on the variational formulation (1.7) and reads: Find $U = (U_1, U_2) \in \big(W^{(q)}\big)^2$ such that, for $n = 1, \cdots, N$,

$$\begin{aligned}
&\int_{I_n} a(\dot{U}_1, V_1) - a(U_2, V_1)\,dt = 0, \\
&\int_{I_n} \Big(\rho(\dot{U}_2, V_2) + a(U_1, V_2) - \int_0^t \beta(t-s) a\big(U_1(s), V_2(t)\big) ds\Big)\,dt \\
&\qquad = \int_{I_n} (\boldsymbol{f}_2, V_2)\,dt + \int_{I_n} (\boldsymbol{g}, V_2)_{\Gamma_N} dt, \quad \forall (V_1, V_2) \in \Big(W_n^{(q-1)}\Big)^2, \\
&U_{1,n-1}^{+} = \mathcal{R}_{h,n} U_{1,n-1}^{-}\,, \quad U_{2,n-1}^{+} = \mathcal{P}_{h,n} U_{2,n-1}^{-}\,,
\end{aligned} \tag{3.7}$$

where $U_{1,0}^{-} = \boldsymbol{u}^0$, $U_{2,0}^{-} = \boldsymbol{v}^0$. Then $U \in \big(W^{(q)}\big)^2$, which was defined in (3.7), satisfies:

$$\begin{aligned}
&B(U, \mathcal{P}_k V) = L(\mathcal{P}_k V), \qquad \forall\, V \in \Big(W^{(q)}\Big)^2, \\
&U_{1,n-1}^{+} = \mathcal{R}_{h,n} U_{1,n-1}^{-}\,, \quad U_{2,n-1}^{+} = \mathcal{P}_{h,n} U_{2,n-1}^{-}\,, \\
&U_{1,0}^{-} = \boldsymbol{u}^0, \quad U_{2,0}^{-} = \boldsymbol{v}^0,
\end{aligned}$$

where $\mathcal{P}_k V = (\mathcal{P}_k V_1, \mathcal{P}_k V_2)$.

Since the variational form (1.7) can be written as: Find $\boldsymbol{u} \in \mathcal{W}$ such that

$$B(\boldsymbol{u}, \boldsymbol{v}) = L(\boldsymbol{v}), \quad \forall\, \boldsymbol{v} \in \mathcal{W},$$

we may, for later reference, note the Galerkin orthogonality

$$B(U - \boldsymbol{u}, \mathcal{P}_k V) = 0, \quad \forall\, V \in \Big(W^{(q)}\Big)^2. \tag{3.8}$$

For $q = p = 1$, considering the fact that functions in $W_n^{(0,p)}$ are constant with respect to time, we can write (3.7) as

$$\begin{aligned}
&A_{h,n}(U_{1,n}^- - U_{1,n-1}^+) - \frac{k_n}{2} A_{h,n}(U_{2,n}^- + U_{2,n-1}^+) = 0,\\
&A_{h,n}\Big((\frac{k_n}{2} - \gamma\omega_{n,n}^-) U_{1,n}^- + (\frac{k_n}{2} - \gamma\omega_{n,n-1}^+) U_{1,n-1}^+ \Big) + \rho\, (U_{2,n}^- - U_{2,n-1}^+)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad = H_n + b_n,
\end{aligned}$$

where

$$\begin{aligned}
b_n &= k_n (P_n \boldsymbol{f}_2 + P_n^{\Gamma_N} \boldsymbol{g}),\\
H_n &= \gamma \sum_{r=1}^{n-1} k_r A_{h,n,r} (\omega_{n,r}^- + \omega_{n,r-1}^+),\\
\omega_{n,r}^- &= \int_{I_n} \int_{t_{r-1}}^{t_r \wedge t} \beta(t-s) \psi_r^-(s)\, ds\, dt\,, \qquad t_r \wedge t = \min(t_r, t),\\
\omega_{n,r-1}^+ &= \int_{I_n} \int_{t_{r-1}}^{t_r \wedge t} \beta(t-s) \psi_{r-1}^+(s)\, ds\, dt,
\end{aligned}$$

and ψ_n^-, ψ_{n-1}^+ are the linear Lagrange basis functions on I_n, so that, for $i = 1, 2$,

$$U_i(x,t)\,|_{\Omega\times I_n} = \psi_{n-1}^+(t) U_{1,n-1}^+(x) + \psi_n^-(t) U_{1,n}^-(x).$$

If we do not change the mesh, or just refine the mesh from a time step to the next, that is $V_{h,n-1}^{(p)} \subset V_{h,n}^{(p)}$, then $\mathcal{R}_{h,n}$ and $\mathcal{P}_{h,n}$ reduce to the identity, i.e., $U_{i,n} = U_{i,n}^- = U_{i,n}^+$, $(n = 0, 1, \dots, N\,,\ i = 1, 2)$.

From now on, we assume that $V_{h,n-1}^{(p)} \subset V_{h,n}^{(p)}\,,\ n = 2, \dots, N$. So we have defined the initial values of the discrete form to be $U_1(\cdot, 0) = \boldsymbol{u}_h^0 = \mathcal{R}_{h,1}\boldsymbol{u}^0$ and $U_2(\cdot, 0) = \boldsymbol{v}_h^0 = \mathcal{P}_{h,1}\boldsymbol{v}^0$. In this case U in continuous with respect to t.

We also consider a modified problem by adding an extra load function, say $\boldsymbol{f}_1 = \boldsymbol{f}_1(t)$, to the first equation of (3.7). This kind of problem will occur in our error analysis below. Then the continuous Galerkin method of order (q,p) is: Find $U \in \big(W^{(q)}\big)^2$ such that, for $n=1,\cdots,N$,

$$
\begin{aligned}
&\int_{I_n} a(\dot U_1, V_1) - a(U_2, V_1)\,dt = \int_{I_n} a(\boldsymbol{f}_1, V_1)\,dt,\\
&\int_{I_n}\Big(\rho(\dot U_2, V_2) + a(U_1, V_2) - \int_0^t \beta(t-s)a\big(U_1(s), V_2(t)\big)ds\Big)\,dt\\
&\qquad = \int_{I_n}(\boldsymbol{f}_2, V_2)\,dt + \int_{I_n}(\boldsymbol{g}, V_2)_{\Gamma_N}\,dt\,, \quad \forall (V_1,V_2)\in \Big(W_n^{(q-1)}\Big)^2,\\
&U_1,\,U_2 \ \text{ continuous at } t_{n-1},\\
&U_1(\cdot,0) = \boldsymbol{u}_h^0 = \mathcal{R}_{h,1}\boldsymbol{u}^0\,,\quad U_2(\cdot,0) = \boldsymbol{v}_h^0 = \mathcal{P}_{h,1}\boldsymbol{v}^0\,.
\end{aligned}
\tag{3.9}
$$

Then U satisfies:

$$
\begin{aligned}
&B(U, \mathcal{P}_k V) = \hat L(\mathcal{P}_k V), \quad \forall\, V \in \Big(W^{(q)}\Big)^2,\\
&U_1,\,U_2 \ \text{ continuous at } t_{n-1},\\
&U_1(.,0) = \boldsymbol{u}_h^0\,,\quad U_2(.,0) = \boldsymbol{v}_h^0\,.
\end{aligned}
\tag{3.10}
$$

4. **Stability estimates**

In the next theorem we prove an energy identity for problem (3.9) which will be used for proving the error estimates in the next section.

Theorem 4. *Let $U = (U_1, U_2)$ be the solution of (3.9). Then for any $l \in \mathbb{R}\,,\ T > 0$, we have the equality*

$$
\begin{aligned}
\rho|U_{2,N}|_{h,l}^2 &+ \tilde\xi(T)|U_{1,N}|_{h,l+1}^2 + \int_0^T \beta|U_1|_{h,l+1}^2\,dt\\
&+ \int_0^T\int_0^t \beta(t-s)D_t|W_1(t,s)|_{h,l+1}^2\,ds\,dt\\
&= \rho|\boldsymbol{v}_h^0|_{h,l}^2 + |\boldsymbol{u}_h^0|_{h,l+1}^2\\
&\quad + 2\int_0^T (P\boldsymbol{f}_2, A_h^l U_2)\,dt + 2\int_0^T (P^{\Gamma_N}\boldsymbol{g}, A_h^l U_2)\,dt\\
&\quad + 2\int_0^T \tilde\xi a(R\boldsymbol{f}_1, A_h^l U_1)\,dt\\
&\quad + 2\int_0^T\int_0^t \beta(t-s)a\big(R\boldsymbol{f}_1(t), A_h^l W_1(t,s)\big)\,ds\,dt,
\end{aligned}
\tag{4.1}
$$

where $W_1(t,s) = U_1(t) - U_1(s)$ and, recalling (1.4),

$$
\tilde\xi(t) = \xi(t) + 1 - \gamma,
\tag{4.2}
$$

with $0 < 1-\gamma < \tilde\xi(t) \le 1$. All terms on the left hand side are non-negative.

Proof. Throughout the proof we take into account that U_i $(i = 1, 2)$ are continuous, hence piecewise differentiable, so that $\dot{U}_i$ exist a.e. in $[0, T]$. We organize our proof in 6 steps.

1. Expressing U_2 in terms of U_1. For any $n = 1, \cdots, N$ the first equation of (3.9) may be written as,

$$\int_{I_n} a(U_2, V_1)\, dt = \int_{I_n} a(\dot{U}_1 - R_n \boldsymbol{f}_1, V_1)\, dt, \quad \forall\, V_1 \in W_n^{(q-1)}.$$

Then by (3.6)

$$\int_{I_n} a(\mathcal{P}_{k,n} U_2, V_1)\, dt = \int_{I_n} a(\dot{U}_1 - R_n \boldsymbol{f}_1, V_1)\, dt, \quad \forall\, V_1 \in W_n^{(q-1)}.$$

Therefore, we get

$$\mathcal{P}_k U_2(t) = \dot{U}_1(t) - R\boldsymbol{f}_1(t), \quad t \in I. \tag{4.3}$$

2. Recalling the definitions of the orthogonal projections P and P^{Γ_N} in (3.3) and the functions W_1 and $\tilde{\xi}$, we can write the second equation of (3.9) in the form

$$\begin{aligned} \int_0^T \Big(\rho(\dot{U}_2, V_2) + \tilde{\xi}(t) a(U_1, V_2) + \int_0^t \beta(t-s) a\big(W_1(t,s), V_2(t)\big)\, ds \Big)\, dt \\ = \int_0^T \Big((P_n \boldsymbol{f}_2, V_2) + (P_n^{\Gamma_N} \boldsymbol{g}, V_2) \Big)\, dt, \quad \forall V_2 \in W^{(q-1)}. \end{aligned}$$

Then choosing $V_2 = A_h^l \mathcal{P}_k U_2$ gives us

$$\begin{aligned} \int_0^T \rho(\dot{U}_2, A_h^l \mathcal{P}_k U_2)\, dt + \int_0^T \tilde{\xi}(t) a(U_1, A_h^l \mathcal{P}_k U_2)\, dt \\ + \int_0^T \int_0^t \beta(t-s) a\big(W_1(t,s), A_h^l \mathcal{P}_k U_2(t)\big)\, ds\, dt \\ = \int_0^T \Big((P\boldsymbol{f}_2, A_h^l U_2) + (P^{\Gamma_N} \boldsymbol{g}, A_h^l U_2) \Big)\, dt. \end{aligned} \tag{4.4}$$

There are three terms in the left hand side of the above equation.

3. Using (3.5) and $\dot{U}_2(t) \in W^{(q-1)}$, we can write the first term of the left hand side of (4.4) as

$$\begin{aligned} \int_0^T \rho\big(\dot{U}_2, A_h^l \mathcal{P}_k U_2\big)\, dt &= \rho \int_0^T \big(\dot{U}_2, A_h^l U_2\big)\, dt = \frac{\rho}{2} \int_0^T D_t |U_2|_{h,l}^2\, dt \\ &= \frac{\rho}{2} \sum_{n=1}^N \Big(|U_{2,n}|_{h,l}^2 - |U_{2,n-1}|_{h,l}^2 \Big) = \frac{\rho}{2} \Big(|U_{2,N}|_{h,l}^2 - |\boldsymbol{v}_h^0|_{h,l}^2 \Big), \end{aligned}$$

where in the last equality we have used the continuity of U_2 in time, due to the assumption $V_{h,n-1}^{(p)} \subset V_{h,n}^{(p)}$.

4. With (4.3), we can write the second term as

$$\int_0^T \tilde{\xi} a\big(U_1, A_h^l \mathcal{P}_k U_2\big)\, dt = \frac{1}{2} \sum_{n=1}^{N} \int_{I_n} \tilde{\xi} D_t |U_1|_{h,l+1}^2 \, dt - \int_0^T \tilde{\xi} a(U_1, A_h^l R \boldsymbol{f}_1)\, dt.$$

Then we integrate by parts in the first term of the right hand side and use the facts that $\dot{\tilde{\xi}}(t) = -\beta(t)$ and $\tilde{\xi}(0) = 1$, to get

$$\begin{aligned}
\int_0^T \tilde{\xi} a\big(U_1, A_h^l \mathcal{P}_k U_2\big)\, dt &= \frac{1}{2} \sum_{n=1}^{N} \Big(\tilde{\xi}(t_n) |U_{1,n}|_{h,l+1}^2 - \tilde{\xi}(t_{n-1}) |U_{1,n-1}|_{h,l+1}^2 \Big) \\
&\quad - \frac{1}{2} \sum_{n=1}^{N} \int_{I_n} \dot{\tilde{\xi}} |U_1(t)|_{h,l+1}^2 \, dt - \int_0^T \tilde{\xi} a(U_1, A_h^l R \boldsymbol{f}_1)\, dt \\
&= \frac{1}{2} \Big(\tilde{\xi}(T) |U_{1,N}|_{h,l+1}^2 - |\boldsymbol{u}_h^0|_{h,l+1}^2 \Big) \\
&\quad + \frac{1}{2} \int_0^T \beta |U_1(t)|_{h,l+1}^2 \, dt - \int_0^T \tilde{\xi} a(R \boldsymbol{f}_1, A_h^l U_1)\, dt\,,
\end{aligned}$$

where again we used the continuity of U_1.

5. Consider now the third term in the left hand side of (4.4). Using (4.3) and the fact that $\dot{W}_1(t) = \dot{U}_1(t)$ we have

$$\begin{aligned}
\int_0^T \int_0^t & \beta(t-s) a\big(W_1(t,s), A_h^l \mathcal{P}_k U_2\big)\, ds\, dt \\
&= \frac{1}{2} \int_0^T \int_0^t \beta(t-s) D_t |W_1(t,s)|_{h,l+1}^2 ds\, dt \\
&\quad - \int_0^T \int_0^t \beta(t-s) a\Big(A_h^l W_1(t,s), R \boldsymbol{f}_1(t)\Big)\, ds\, dt.
\end{aligned}$$

The first term of the right hand side is non-negative. To prove this, for a fixed mesh, take $\epsilon \in (0,t)$. Then

$$
\begin{aligned}
&\int_0^T \int_0^{t-\epsilon} \beta(t-s) D_t |W_1(t,s)|_{h,l+1}^2 \, ds \, dt \\
&\qquad = \int_0^{T-\epsilon} \int_{s+\epsilon}^T \beta(t-s) D_t |W_1(t,s)|_{h,l+1}^2 dt \, ds \\
&\qquad = \int_0^{T-\epsilon} \beta(T-S) |W_1(T,s)|_{h,l+1}^2 \, ds \\
&\qquad\quad - \int_0^{T-\epsilon} \beta(\epsilon) |W_1(s+\epsilon,s)|_{h,l+1}^2 \, ds \\
&\qquad\quad - \int_0^{T-\epsilon} \int_{s+\epsilon}^T \dot\beta(t-s) |W_1(t,s)|_{h,l+1}^2 dt \, ds \\
&\qquad \geq -\beta(\epsilon) \int_0^{T-\epsilon} |W_1(s+\epsilon,s)|_{h,l+1}^2 \, ds \,,
\end{aligned} \tag{4.5}
$$

where we changed the order of the integrals in the first equation, we integrated by parts in the next one , and we considered the facts that $\dot\beta(t) \leq 0$ and $\beta(t) \geq 0$ for the last inequality. Then using $W_1(s+\epsilon, s) = \int_s^{s+\epsilon} D_t W_1(t,s) \, dt$ and the Cauchy-Schwarz inequality we get

$$
\begin{aligned}
|W_1(s+\epsilon,s)|_{h,l+1}^2 &\leq \Big(\int_s^{s+\epsilon} |D_t W_1(t,s)|_{h,l+1} \, dt \Big)^2 \\
&\leq \int_s^{s+\epsilon} \frac{dt}{\beta(t-s)} \int_s^{s+\epsilon} \beta(t-s) |D_t W_1(t,s)|_{h,l+1}^2 \, dt.
\end{aligned}
$$

So (4.5) can be written as

$$
\begin{aligned}
&\int_0^T \int_0^{t-\epsilon} \beta(t-s) D_t |W_1(t,s)|_{h,l+1}^2 \, ds \, dt \\
&\qquad \geq - \int_0^{T-\epsilon} \Big(\int_s^{s+\epsilon} \frac{\beta(\epsilon)}{\beta(t-s)} \, dt \int_s^{s+\epsilon} \beta(t-s) |D_t W_1(t,s)|_{h,l+1}^2 \, dt \Big) \, ds \\
&\qquad \geq -\epsilon \int_0^{T-\epsilon} \int_s^{s+\epsilon} \beta(t-s) |D_t W_1(t,s)|_{h,l+1}^2 \, dt \, ds,
\end{aligned}
$$

since $\frac{\beta(\epsilon)}{\beta(t-s)} \leq 1$. Obviously, for a fixed mesh, the last integral in the right hand side is bounded by

$$
\int_0^T \int_s^T \beta(t-s) |D_t W_1(t,s)|_{h,l+1}^2 \, dt \, ds < \infty.
$$

Therefore, letting $\epsilon \to 0$ we get

$$\int_0^T \int_0^t \beta(t-s) D_t |W_1(t,s)|_{h,l+1}^2 \, ds \, dt \geq 0 \, .$$

6. Putting the results from steps 3, 4 and 5 into (4.4) completes the proof.

Remark: In [4] the auxiliary function $\boldsymbol{w}(t,s) = \boldsymbol{u}(t) - \boldsymbol{u}(t-s)$, was used the same as in our §2, to obtain stability estimates for the spatially semidiscrete finite element method. This does not work here because $U_1(t) - U_1(t-s)$ does not belong to $W^{(q)}$ if the temporal mesh is non-uniform.

From now on we specialize to the case $p = q = 1$. We use (4.1) to obtain a stability estimate to be used in the error analysis. To this end, from (4.1) with $\boldsymbol{g} = \boldsymbol{0}$, we have

$$\begin{aligned}
\rho |U_{2,N}|_{h,l}^2 + \tilde{\xi}(T) |U_{1,N}|_{h,l+1}^2 \leq \rho |\boldsymbol{v}_h^0|_{h,l}^2 + |\boldsymbol{u}_h^0|_{h,l+1}^2 + 2 \int_0^T (A_h^l P \boldsymbol{f}_2, U_2) \, dt \\
+ 2 \int_0^T a(A_h^l R \boldsymbol{f}_1, U_1) \, dt \\
+ 2 \int_0^T \int_0^t \beta(t-s) a\big(A_h^l R \boldsymbol{f}_1(t), W_1(t,s)\big) \, ds \, dt.
\end{aligned}$$

Therefore using (3.4), $1 - \gamma < \tilde{\xi}(t) \leq 1$ and $\int_0^t \beta(s) \, ds \leq \gamma$, we get

$$\begin{aligned}
&\rho |U_{2,N}|_{h,l}^2 + (1-\gamma) |U_{1,N}|_{h,l+1}^2 \\
&\quad \leq \rho |\boldsymbol{v}_h^0|_{h,l}^2 + |\boldsymbol{u}_h^0|_{h,l+1}^2 + 2 \int_0^T (A_h^{l/2} \mathcal{P}_k \mathcal{P}_h \boldsymbol{f}_2, A_h^{l/2} U_2) \, dt \\
&\qquad + 2 \int_0^T a(A_h^{l/2} \mathcal{P}_k \mathcal{R}_h \boldsymbol{f}_1, A_h^{l/2} U_1) \, dt \\
&\qquad + 2 \int_0^T \int_0^t \beta(t-s) a\big(A_h^{l/2} \mathcal{P}_k \mathcal{R}_h \boldsymbol{f}_1(t), A_h^{l/2} W_1(t,s)\big) \, ds \, dt \\
&\quad \leq \rho |\boldsymbol{v}_h^0|_{h,l}^2 + |\boldsymbol{u}_h^0|_{h,l+1}^2 + 2 \int_0^T |\mathcal{P}_k \mathcal{P}_h \boldsymbol{f}_2|_{h,l} |U_2|_{h,l} \, dt \\
&\qquad + 2 \int_0^T |\mathcal{P}_k \mathcal{R}_h \boldsymbol{f}_1|_{h,l+1} |U_1|_{h,l+1} \, dt \\
&\qquad + 2\gamma \int_0^T |\mathcal{P}_k \mathcal{R}_h \boldsymbol{f}_1(t)|_{h,l+1} \max_{0 \leq s \leq T} |W_1(t,s)|_{h,l+1} \, dt \\
&\quad \leq \rho |\boldsymbol{v}_h^0|_{h,l}^2 + |\boldsymbol{u}_h^0|_{h,l+1}^2 + \frac{1}{2} \rho \max_{[0,T]} |U_2|_{h,l}^2 + C \Big(\int_0^T |\mathcal{P}_k \mathcal{P}_h \boldsymbol{f}_2|_{h,l} \, dt \Big)^2 \\
&\qquad + \frac{1}{2} (1-\gamma) \max_{[0,T]} |U_1|_{h,l+1}^2 + C \Big(\int_0^T |\mathcal{P}_k \mathcal{R}_h \boldsymbol{f}_1|_{h,l+1} \, dt \Big)^2
\end{aligned}$$

where $C = C(\rho, \gamma)$. Using that, for $q = 1$, we have

$$\max_{[0,T]} |U_i| \le \max_{[0,T]} |U_{i,n}|,$$

and

$$\int_0^T |\mathcal{P}_k \boldsymbol{f}|\, dt \le \int_0^T |\boldsymbol{f}|\, dt,$$

and that the above inequality holds for arbitrary N, we conclude in a standard way

$$(4.6)\qquad \begin{aligned} &|U_{2,N}|_{h,l} + |U_{1,N}|_{h,l+1} \\ &\quad \le C\bigg(|\boldsymbol{v}_h^0|_{h,l} + |\boldsymbol{u}_h^0|_{h,l+1} + \int_0^T \Big(|\mathcal{R}_h \boldsymbol{f}_1|_{h,l+1} + |\mathcal{P}_h \boldsymbol{f}_2|_{h,l} \Big)\, dt \bigg), \end{aligned}$$

with $C = C(\rho,\, \gamma)$.

5. A priori error estimates

To simplify the notation we denote the Sobolev norms $\|\cdot\|_{H^i(\Omega)}$ by $\|\cdot\|_i$. We define the standard interpolant $I_k \boldsymbol{v} \in W^{(1)}$ by

$$(5.1)\qquad I_k \boldsymbol{v}(t_n) = \boldsymbol{v}(t_n), \quad n = 0,\, 1,\, \cdots,\, N.$$

By standard arguments in approximation theory we see that, for $q = 0,\, 1$,

$$(5.2)\qquad \int_0^T \|I_k \boldsymbol{v} - \boldsymbol{v}\|_i \, dt \le C k^{q+1} \int_0^T \|D_t^{q+1} \boldsymbol{v}\|_i \, dt, \quad \text{for } i = 0,\, 1,\, 2,$$

where

$$k = \max_{1 \le n \le N} k_n.$$

We assume the elliptic regularity estimate $\|\boldsymbol{v}\|_2 \le C \|A\boldsymbol{v}\|,\ \forall\, \boldsymbol{v} \in \mathrm{dom}(A)$, so that the following error estimates for the Ritz projection (3.2), hold true

$$(5.3)\qquad \|\mathcal{R}_h v - v\| \le C h^s \|v\|_s\,, \quad \forall \boldsymbol{v} \in H^s \cap V, \quad s = 1,\, 2.$$

Theorem 5. *Assume $q = p = 1$ and let $\boldsymbol{u}$ and U be the solutions of* (1.7) *and* (3.9). *Then, with $\boldsymbol{e} = U - \boldsymbol{u}$ and $C = C(\rho, \gamma)$, we have*

$$\begin{aligned}
\|\boldsymbol{e}_{2,N}\| &\leq Ch^2\Big(\|\boldsymbol{v}^0\|_2 + \|\boldsymbol{u}_{2,N}\|_2 + \int_0^T \|\dot{\boldsymbol{u}}_2\|_2\,dt\Big)\\
&\quad + Ck^2\int_0^T \big(|D_t^2\boldsymbol{u}_2|_1 + \|D_t^2\boldsymbol{u}_1\|_2\big)\,dt,\\
|\boldsymbol{e}_{1,N}|_1 &\leq Ch\Big(\|\boldsymbol{u}_{1,N}\|_2 + \|\boldsymbol{v}^0\|_1 + \int_0^T \|\dot{\boldsymbol{u}}_2\|_1\,dt\Big)\\
&\quad + Ck^2\int_0^T \big(|D_t^2\boldsymbol{u}_2|_1 + \|D_t^2\boldsymbol{u}_1\|_2\big)\,dt,\\
\|\boldsymbol{e}_{1,N}\| &\leq Ch^2\Big(\|\boldsymbol{u}_{1,N}\|_2 + \int_0^T \|\boldsymbol{u}_2\|_2\,dt\Big)\\
&\quad + Ck^2\int_0^T \big(\|D_t^2\boldsymbol{u}_2\| + |D_t^2\boldsymbol{u}_1|_1\big)\,dt.
\end{aligned}$$

Proof. We set

$$\boldsymbol{e} = \boldsymbol{\theta} + \boldsymbol{\eta} + \boldsymbol{\rho} = (U - \pi\boldsymbol{u}) + (\pi\boldsymbol{u} - J\boldsymbol{u}) + (J\boldsymbol{u} - \boldsymbol{u}), \tag{5.4}$$

for some suitable operators π and J which will be specified in terms of the time interpolant I_k in (5.1) and projectors $\mathcal{R}_h$ and $\mathcal{P}_h$ in (3.2), so that $\pi\boldsymbol{u} \in W^{(1)}$ and $\boldsymbol{\eta}$ and $\boldsymbol{\rho}$ will correspond to the temporal and spatial errors, respectively. Due to (5.2)–(5.3) we just need to estimate $\boldsymbol{\theta}$. To this end, using the Galerkin orthogonality (3.8) and the definition of $\boldsymbol{\theta}$, we get

(5.5)
$$\begin{aligned}
B(\boldsymbol{\theta}, \mathcal{P}_k V) &= -B(\boldsymbol{\eta}, \mathcal{P}_k V) - B(\boldsymbol{\rho}, \mathcal{P}_k V)\\
&= \int_0^T a(\boldsymbol{\eta}_2, \mathcal{P}_k V_1) - a(\dot{\boldsymbol{\eta}}_1, \mathcal{P}_k V_1) - \rho(\dot{\boldsymbol{\eta}}_2, \mathcal{P}_k V_2) - a(\boldsymbol{\eta}_1, \mathcal{P}_k V_2)\,dt\\
&\quad + \int_0^T\int_0^t \beta(t-s)a\big(\boldsymbol{\eta}_1(s), \mathcal{P}_k V_2(t)\big)\,ds\,dt\\
&\quad + \int_0^T a(\boldsymbol{\rho}_2, \mathcal{P}_k V_1) - a(\dot{\boldsymbol{\rho}}_1, \mathcal{P}_k V_1) - \rho(\dot{\boldsymbol{\rho}}_2, \mathcal{P}_k V_2) - a(\boldsymbol{\rho}_1, \mathcal{P}_k V_2)\,dt\\
&\quad + \int_0^T\int_0^t \beta(t-s)a\big(\boldsymbol{\rho}_1(s), \mathcal{P}_k V_2(t)\big)\,ds\,dt\\
&= \sum_{j=1}^{10} E_j, \quad \forall\, V \in \big(W^{(1)}\big)^2.
\end{aligned}$$

We consider two different choices of the operators π and J. In order to prove the first two error estimates we set, for $i = 1, 2$,

$$\boldsymbol{\theta}_i = U_i - I_k\mathcal{R}_h\boldsymbol{u}_i, \quad \boldsymbol{\eta}_i = (I_k - I)\mathcal{R}_h\boldsymbol{u}_i, \quad \boldsymbol{\rho}_i = (\mathcal{R}_h - I)\boldsymbol{u}_i.$$

Integrating by parts in E_2 and E_3 with respect to time and using (5.1) we have for both cases

$$E_2 = E_3 = 0. \tag{5.6}$$

Moreover, by the definitions of $\boldsymbol{\eta}$ and $\boldsymbol{\rho}$, we have

$$E_6 = E_7 = E_9 = E_{10} = 0.$$

Therefore,

$$\begin{aligned} B(\boldsymbol{\theta}, \mathcal{P}_kV) &= \int_0^T a(\boldsymbol{\eta}_2, \mathcal{P}_kV_1)\,dt \\ &\quad + \int_0^T \Big(a\big(-\boldsymbol{\eta}_1 + \int_0^t \beta(t-s)\boldsymbol{\eta}_1(s)ds, \mathcal{P}_kV_2\big) - \rho\big(\dot{\boldsymbol{\rho}}_2, \mathcal{P}_kV_2\big)\Big)\,dt \\ &= \hat{L}(\mathcal{P}_kV), \quad \forall\, V \in \big(W^{(1)}\big)^2, \end{aligned}$$

which is of the form (3.10) with $\boldsymbol{f}_1 = \boldsymbol{\eta}_2$, $\boldsymbol{f}_2 = A_h\big(-\boldsymbol{\eta}_1 + \int_0^t \beta(t - s)\boldsymbol{\eta}_1(s)\,ds\big) - \rho\dot{\boldsymbol{\rho}}_2$ and $\boldsymbol{g} = \boldsymbol{0}$.

Applying the stability inequality (4.6) with $l = 0$, and considering the fact that $|\cdot|_{0,h} = \|\cdot\|$ and $|\cdot|_{h,1} = |\cdot|_1$, we have

$$\begin{aligned} \|\boldsymbol{\theta}_{2,N}\| &+ |\boldsymbol{\theta}_{1,N}|_1 \\ &\leq C\Big(\|\boldsymbol{\theta}_2(0)\| + |\boldsymbol{\theta}_1(0)|_1\Big) + C\int_0^T |\mathcal{R}_h\boldsymbol{\eta}_2|_1\,dt \\ &\quad + C\int_0^T \Big(\|\mathcal{P}_hA_h\boldsymbol{\eta}_1\| + \Big\|\mathcal{P}_hA_h\int_0^t \beta(t-s)\boldsymbol{\eta}_1(s)\,ds\Big\| + \rho\|\mathcal{P}_h\dot{\boldsymbol{\rho}}_2\|\Big)\,dt, \end{aligned}$$

where $\boldsymbol{\theta}_1(0) = 0$, since $U_1(0) = \mathcal{R}_{h,1}\boldsymbol{u}^0$. Since $|\mathcal{R}_h\boldsymbol{v}|_1 \leq |\boldsymbol{v}|_1$, $\|\mathcal{P}_h\boldsymbol{v}\| \leq \|\boldsymbol{v}\|$, $\forall \boldsymbol{v} \in V$ and $\mathcal{R}_hA_h = \mathcal{P}_hA$, we have

$$\begin{aligned} |\mathcal{R}_h\boldsymbol{\eta}_2|_1 &= |(I_k - I)\mathcal{R}_h\boldsymbol{u}_2|_1 \leq |(I_k - I)\boldsymbol{u}_2|_1, \\ \|\mathcal{P}_hA_h\boldsymbol{\eta}_1\| &= \|A_h\boldsymbol{\eta}_1\| = \|(I_k - I)A_h\mathcal{R}_h\boldsymbol{u}_1\| = \|(I_k - I)\mathcal{P}_hA\boldsymbol{u}_1\| \\ &\leq \|(I_k - I)A\boldsymbol{u}_1\| \leq C\|(I_k - I)\boldsymbol{u}_1\|_2, \end{aligned}$$

and

$$\begin{aligned}\int_0^T \Big\| \mathcal{P}_h A_h \int_0^t \beta(t-s)\boldsymbol{\eta}_1(s)\,ds \Big\|\,dt &\leq \int_0^T \Big\| A_h \int_0^t \beta(t-s)\boldsymbol{\eta}_1(s)\,ds \Big\|\,dt \\ &\leq C\int_0^T\int_0^t \beta(t-s)\|(I_k-I)\boldsymbol{u}_1(s)\|_2\,ds\,dt \\ &\leq C\int_0^T \beta(t)dt\int_0^T \|(I_k-I)\boldsymbol{u}_1\|_2\,dt \\ &\leq C\gamma\int_0^T \|(I_k-I)\boldsymbol{u}_1\|_2\,dt.\end{aligned}$$

Therefore by $\boldsymbol{\theta}=\boldsymbol{e}-\boldsymbol{\eta}-\boldsymbol{\rho}$, $\boldsymbol{\eta}(t_n)=0$ and $\boldsymbol{\theta}_1(0)=0$, we get

$$\begin{aligned}\|\boldsymbol{e}_{2,N}\| &\leq \|\boldsymbol{\rho}_{2,N}\| + C\boldsymbol{\theta}_2(0) \\ &\quad + C\int_0^T \Big(|(I_k-I)\boldsymbol{u}_2|_1 + \|(I_k-I)\boldsymbol{u}_1\|_2 + \|(\mathcal{R}_h-I)\dot{\boldsymbol{u}}_2\|\Big)\,dt, \\ |\boldsymbol{e}_{1,N}|_1 &\leq |\boldsymbol{\rho}_{1,N}|_1 + C\boldsymbol{\theta}_2(0) \\ &\quad + C\int_0^T \Big(|(I_k-I)\boldsymbol{u}_2|_1 + \|(I_k-I)\boldsymbol{u}_1\|_2 + \|(\mathcal{R}_h-I)\dot{\boldsymbol{u}}_2\|\Big)\,dt,\end{aligned}$$

wich implies the first two estimates by (5.2) and (5.3).

Finally, we choose

$$\begin{aligned}\boldsymbol{\theta}_1 &= U_1 - I_k\mathcal{R}_h\boldsymbol{u}_1, \quad \boldsymbol{\eta}_1 = (I_k-I)\mathcal{R}_h\boldsymbol{u}_1, \quad \boldsymbol{\rho}_1 = (\mathcal{R}_h-I)\boldsymbol{u}_1, \\ \boldsymbol{\theta}_2 &= U_2 - I_k\mathcal{P}_h\boldsymbol{u}_2, \quad \boldsymbol{\eta}_2 = (I_k-I)\mathcal{P}_h\boldsymbol{u}_2, \quad \boldsymbol{\rho}_2 = (\mathcal{P}_h-I)\boldsymbol{u}_2.\end{aligned}$$

By the definitions of $\mathcal{R}_h$ and $\mathcal{P}_h$ in (3.2), this implies

$$E_7 = E_8 = E_9 = E_{10} = 0,$$

and we still have (5.6). Therefore, (5.5) becomes

$$\begin{aligned}B(\boldsymbol{\theta},\mathcal{P}_kV) &= \int_0^T a(\boldsymbol{\eta}_2+\boldsymbol{\rho}_2,\mathcal{P}_kV_1)\,dt \\ &\quad + \int_0^T a\big(-\boldsymbol{\eta}_1 + \int_0^t \beta(t-s)\boldsymbol{\eta}_1(s)ds, \mathcal{P}_kV_2\big)\,dt \\ &= \hat{L}(\mathcal{P}_kV), \quad \forall\, V\in \big(W^{(1)}\big)^2,\end{aligned}$$

which is of the form (3.10) with $\boldsymbol{f}_1 = \boldsymbol{\eta}_2+\boldsymbol{\rho}_2$, $\boldsymbol{f}_2 = A_h\big(-\boldsymbol{\eta}_1+\int_0^t\beta(t-s)\boldsymbol{\eta}_1(s)\,ds\big)$ and $\boldsymbol{g}=\boldsymbol{0}$.

Again applying the stability inequality (4.6), this time with $l=-1$, and using $|\cdot|_{h,0}=\|\cdot\|$, we have

$$\begin{aligned}\|\boldsymbol{\theta}_{1,N}\| &\le C\int_0^T\Big(\|\mathcal{R}_h\boldsymbol{\eta}_2\|+\|\mathcal{R}_h\boldsymbol{\rho}_2\|\Big)\,dt\\ &\quad+C\int_0^T\Big(|\mathcal{P}_hA_h\boldsymbol{\eta}_1|_{h,-1}+|\mathcal{P}_hA_h\int_0^t\beta(t-s)\boldsymbol{\eta}_1(s)ds|_{h,-1}\Big)\,dt,\end{aligned}$$

where we used that $\boldsymbol{\theta}(0)=0$, since $U_1(0)=\mathcal{R}_{h,1}\boldsymbol{u}^0$ and $U_2(0)=\mathcal{P}_{h,1}\boldsymbol{v}^0$. Then, since

$$\begin{aligned}&\|\mathcal{R}_h\boldsymbol{\eta}_2\|=\|(I_k-I)\mathcal{P}_h\boldsymbol{u}_2\|\le\|(I_k-I)\boldsymbol{u}_2\|,\\ &\|\mathcal{R}_h\boldsymbol{\rho}_2\|=\|\mathcal{P}_h(I-\mathcal{R}_h)\boldsymbol{u}_2\|\le\|(\mathcal{R}_h-I)\boldsymbol{u}_2\|,\\ &|\mathcal{P}_hA_h\boldsymbol{\eta}_1|_{h,-1}=|A_h\mathcal{R}_h(I_k-I)\boldsymbol{u}_1|_{h,-1}=|\mathcal{R}_h(I_k-I)\boldsymbol{u}_1|_{h,1}\le|(I_k-I)\boldsymbol{u}_1|_1\,,\end{aligned}$$

and

$$\begin{aligned}\int_0^T|\mathcal{P}_hA_h\int_0^t\beta(t-s)\boldsymbol{\eta}_1(s)\,ds|_{h,-1}\,dt &\le\int_0^T\int_0^t\beta(t-s)|(I_k-I)\boldsymbol{u}_1(s)|_1\,ds\,dt\\ &\le\gamma\int_0^T|(I_k-I)\boldsymbol{u}_1|_1\,dt,\end{aligned}$$

we conclude

$$\|\boldsymbol{e}_{1,N}\|\le\|\boldsymbol{\rho}_{1,N}\|+C\int_0^T\Big(\|(I_k-I)\boldsymbol{u}_2\|+\|(\mathcal{R}_h-I)\boldsymbol{u}_2\|+|(I_k-I)\boldsymbol{u}_1|_1\Big)\,dt,$$

which implies the last estimate by (5.2) and (5.3). □

6. Numerical example

In this section we demonstrate the numerical method by solving a simple but realistic example for a two dimensional structure, see Figure 1 (a), using piecewise linear polynomials, i.e., $q=p=1$.

We consider the initial conditions: $\boldsymbol{u}(x,0)=\mathbf{0}\,m$, $\dot{\boldsymbol{u}}(x,0)=\mathbf{0}\,m/s$, the boundary conditions: $\boldsymbol{u}=\mathbf{0}$ at $x=0$, $\boldsymbol{g}=(0,-1)\,Pa$ at $x=1.5$ and zero on the rest of the boundary. The volume load is assumed to be $\boldsymbol{f}=\mathbf{0}\,N/m^3$. And the model parameters are: $\gamma=0.5$, $\tau=0.25$, $\nu=0.3$, $E=5MPa$ and $\rho=7000\,kg/m^3$. The deformed mesh at $t/\tau=9$ for $\alpha=1/2$ is displayed in Figure 1 (b), with the displacement magnified by the factor 10^5, and the computed vertical displacement at the point $(1.5,1.5)$ for different α are shown in Figure 2.

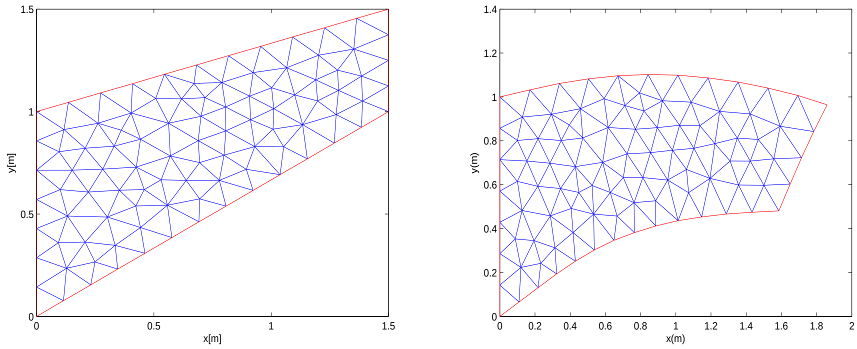

FIGURE 1. (a) Undeformed mesh. (b) Deformed mesh at $t/\tau = 9$ for $\alpha = 1/2$.

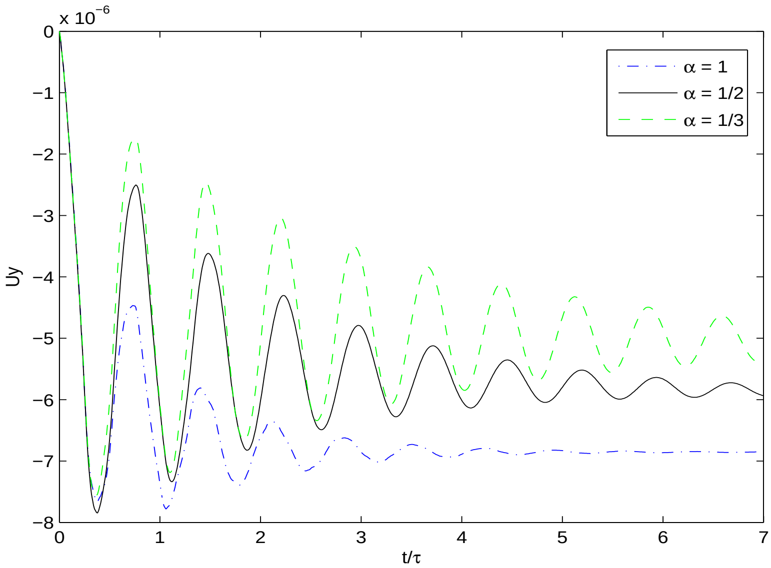

FIGURE 2. Vertical displacement for different α.

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