CHALMERS | GÖTEBORG UNIVERSITY

PREPRINT 2007:4

Multidimensional Operator Multipliers

K. JUSCHENKO
I. G. TODOROV
L. TUROWSKA

Department of Mathematical Sciences
Division of Mathematics
CHALMERS UNIVERSITY OF TECHNOLOGY
GÖTEBORG UNIVERSITY
Göteborg Sweden 2007

Preprint 2007:4

Multidimensional Operator Multipliers

K. Juschenko, I. G. Todorov and L. Turowska

CHALMERS | GÖTEBORG UNIVERSITY

Department of Mathematical Sciences
Division of Mathematics
Chalmers University of Technology and Göteborg University
SE-412 96 Göteborg, Sweden

Göteborg, January 2007

Preprint 2007:4
ISSN 1652-9715

Matematiska vetenskaper
Göteborg 2007

Multidimensional operator multipliers

K. Juschenko, I. G. Todorov and L. Turowska

Abstract

We introduce multidimensional Schur multipliers and characterise them generalising well known results by Grothendieck and Peller. We define a multidimensional version of the two dimensional operator multipliers studied recently by Kissin and Shulman. The multidimensional operator multipliers are defined as elements of the minimal tensor product of several C*-algebras satisfying certain boundedness conditions. In the case of commutative C*-algebras, the multidimensional operator multipliers reduce to continuous multidimensional Schur multipliers. We show that the multipliers with respect to some given representations of the corresponding C*-algebras do not change if the representations are replaced by approximately equivalent ones. We establish a non-commutative and multidimensional version of the characterisations by Grothendieck and Peller which shows that universal operator multipliers can be obtained as certain weak limits of elements of the algebraic tensor product of the corresponding C*-algebras.

1 Introduction

A bounded function $\varphi : \mathbb{N} \times \mathbb{N} \to \mathbb{C}$ is called a Schur multiplier if $(\varphi(i,j)a_{ij})$ is the matrix of a bounded linear operator on ℓ^2 whenever (a_{ij}) is such. The study of Schur multipliers was initiated by Schur in the early 20th century. A characterisation of these objects was given by A. Grothendieck in his *Résumé* [14], where he showed that Schur multipliers are precisely the functions φ of the form $\varphi(i,j) = \sum_{k=1}^{\infty} a_k(i)b_k(j)$, where $a_k, b_k : \mathbb{N} \to \mathbb{C}$ are such that $\sup_i \sum_{k=1}^{\infty} |a_k(i)|^2 < \infty$ and $\sup_j \sum_{k=1}^{\infty} |b_k(j)|^2 < \infty$. Schur multipliers have had many important applications in Analysis, see e.g. [2], [10] and [23]. One

2000 Mathematics Subject Classification: Primary 46L07; Secondary 47L25
Keywords: multiplier, C*-algebra, multidimensional

of the forms of the celebrated Grothendieck inequality can be given in terms of these objects [23].

One of the most important developments in Analysis in recent years has been "quantisation" [12], starting with the advent of the theory of operator spaces in the 1980's in the work of Blecher, Effros, Haagerup, Paulsen, Pisier, Ruan, Sinclair and many others, and based on Arveson's pioneering work in the 1970's. Operator space (or non-commutative) versions are presently being found for many results in classical Banach space theory [7, 19, 24]. A construction underlying many of the developments in operator space theory is the Haagerup tensor product, as well as its weak counterpart, the extended Haagerup tensor product [8]. Grothendieck's characterisation can be formulated by saying that the set of Schur multipliers coincides with the extended Haagerup tensor product $\ell^\infty \otimes_{eh} \ell^\infty$ of the space ℓ^∞ of all bounded complex sequences, with itself.

Schur multipliers are elements of the commutative von Neumann algebra $\ell^\infty(\mathbb{N}\times\mathbb{N})$, or equivalently of the (von Neumann) tensor product of (the commutative von Neumann algebra) ℓ^∞ with itself. Subsequently, they form a commutative algebra themselves. Their quantisation was initiated by Kissin and Shulman in [18]. Suppose that $\mathcal{A}$ and $\mathcal{B}$ are C*-algebras and π and ρ their representations on H and K, respectively. The Hilbert space tensor product $H \otimes K$ can be naturally identified with the Hilbert space $\mathcal{C}_2(H^{\mathrm{d}}, K)$ of Hilbert-Schmidt operators from the dual H^{d} of H into K. It follows that π and ρ give rise to a representation $\sigma_{\pi,\rho}$ of the minimal tensor product $\mathcal{A}\otimes\mathcal{B}$ of $\mathcal{A}$ and $\mathcal{B}$ on $\mathcal{C}_2(H^{\mathrm{d}}, K)$. Kissin and Shulman call an element $\varphi \in \mathcal{A}\otimes\mathcal{B}$ a π,ρ-multiplier if $\sigma_{\pi,\rho}(\varphi)$ is bounded in the operator norm of $\mathcal{C}_2(H^{\mathrm{d}}, K)$. In [18], they study two sets of problems: the dependence of π,ρ-multipliers on π and ρ and the description of the norm of an operator multiplier. Most of their results are established in the more general setting of symmetrically normed ideals.

Assume that $\mathcal{A}$ and $\mathcal{B}$ are commutative, say $\mathcal{A} = C_0(X)$ and $\mathcal{B} = C_0(Y)$, for some locally compact Hausdorff spaces X and Y, and that the representations π and ρ arise from some spectral measures on X and Y. The notion of a π,ρ-multiplier is in this case closely related to double operator integrals. The theory of these integrals was developed by Birman and Solomyak [3, 4, 5, 6] in connection with various problems of Mathematical Physics and in particular of Perturbation Theory. If $(X,\mathcal{E})$ and $(Y,\mathcal{F})$ are spectral measures on

Hilbert spaces H and K, they defined the double operator integral

$$I_\psi(T) = \int_{X\times Y} \psi(x,y)\, d\mathcal{E}(x) T\, d\mathcal{F}(y)$$

for every bounded measurable function ψ and every operator T from the Hilbert-Schmidt class $\mathcal{C}_2(H,K)$. A function ψ is called a Schur multiplier with respect to $\mathcal{E}$ and $\mathcal{F}$ if I_ψ can be extended to a bounded linear transformer on the space $(\mathcal{B}(H,K), \|\cdot\|_{\mathrm{op}})$ of bounded operators from H to K, i.e., if there exists $C > 0$ such that $\|I_\psi(T)\|_{\mathrm{op}} \le C\|T\|_{\mathrm{op}}$ for all $T \in \mathcal{C}_2(H,K)$. Peller [21] (see also [17]) characterised Schur multipliers with respect to $\mathcal{E}$ and $\mathcal{F}$ in several ways. In particular, he showed that the space of Schur multipliers with respect to $\mathcal{E}$ and $\mathcal{F}$ coincides with the extended Haagerup tensor product $L^\infty(X) \otimes_{eh} L^\infty(Y)$ and the integral projective tensor product $L^\infty(X)\hat{\otimes}_i L^\infty(Y)$.

Several attempts were made to generalise the Birman-Solomyak theory to the case of multiple operator integrals [20, 28, 27]. Such integrals appear, for instance, in the study of differentiability of functions of operators depending on a parameter. A recent definition of multiple operator integrals of Peller's [22] is based on the integral projective tensor product. For some fixed spectral measures $(X_1,\mathcal{E}_1),\dots,(X_n,\mathcal{E}_n)$ on Hilbert spaces $H_1,\dots,H_n$, he defines

$$I_\psi(T_1,\dots,T_{n-1}) = \int_{X_1\times\ldots\times X_n} \psi(x_1,\dots,x_n)\, d\mathcal{E}_1(x_1)T_1\, d\mathcal{E}_2(x_2)\dots T_{n-1}\, d\mathcal{E}_n(x_n),$$

where $\psi \in L^\infty(X_1)\hat{\otimes}_i \dots \hat{\otimes}_i L^\infty(X_n)$ and $T_1,\dots,T_{n-1}$ are bounded linear operators, and shows that

$$\|I_\psi(T_1,\dots,T_{n-1})\|_{\mathrm{op}} \le \|\psi\|_i \|T_1\|_{\mathrm{op}} \dots \|T_{n-1}\|_{\mathrm{op}},$$

where $\|\psi\|_i$ denotes the integral projective tensor norm of ψ. If the spectral measures are multiplicity free and $T_1,\dots,T_{n-1}$ are of Hilbert-Schmidt class and have kernels $f_1,\dots,f_{n-1}$, respectively, then $I_\psi(T_1,\dots,T_{n-1})$ is a Hilbert-Schmidt operator with kernel $S_\psi(f_1,\dots,f_{n-1}) \in L^2(X_1\times X_n)$ equal to

$$\int_{X_2\times\ldots\times X_{n-1}} \psi(x_1,\dots,x_n) f_1(x_1,x_2)\dots f_{n-1}(x_{n-1},x_n)\, d\mathcal{E}_2(x_2)\dots\, d\mathcal{E}_{n-1}(x_{n-1}). \tag{1}$$

This was the starting point for our definition of multidimensional Schur multipliers in Section 3. Let (X_i,μ_i), $i = 1,\dots,n$, be standard σ-finite measure spaces and $\Gamma(X_1,\dots,X_n) = L^2(X_1\times X_2)\odot L^2(X_2\times X_3)\odot\ldots\odot L^2(X_{n-1}\times$

$X_n)$ be the algebraic tensor product of the corresponding L^2-spaces equipped with the projective tensor norm, where each of the L^2-spaces is equipped with its L^2-norm. An element $\psi \in L^\infty(X_1 \times \cdots \times X_n)$ determines a bounded linear map S_ψ from $\Gamma(X_1, \ldots, X_n)$ to $L^2(X_1, X_n)$ given on elementary tensors $f_1 \otimes \ldots \otimes f_n \in \Gamma(X_1, \ldots, X_n)$ by (1) (where the integration is now with respect to μ_i instead of $\mathcal{E}_i$). On the other hand, for any measure spaces (X, μ) and (Y, ν), the space $L^2(X \times Y)$ can be identified with the class of all Hilbert-Schmidt operators from $L^2(X)$ to $L^2(Y)$; to each $f \in L^2(X \times Y)$ there corresponds the operator T_f given by $T_f\xi(y) = \int_X f(x, y)\xi(x)d\mu(x)$, $\xi \in L^2(X)$. Using this identification, one can equip the space $L^2(X \times Y)$ with the opposite operator space structure arising from the inclusion of $L^2(X \times Y)$ into $\mathcal{B}(L^2(X), L^2(Y))$. We further equip $\Gamma(X_1, \ldots, X_n)$ with the Haagerup tensor norm $\|\cdot\|_{\mathrm{h}}$, where the L^2-spaces are given their opposite operator space structure described above, and say that an element $\psi \in L^\infty(X_1 \times \ldots \times X_n)$ is a Schur multiplier (with respect to $\mu_1, \ldots, \mu_n$) if there exists $C > 0$ such that

$$\|S_\psi(\Phi)\|_{\mathrm{op}} \leq C\|\Phi\|_{\mathrm{h}}, \text{ for all } \Phi \in \Gamma(X_1, \ldots, X_n). \tag{2}$$

Using a generalisation of a result of Smith [25] on the complete boundedness of certain bounded bimodule maps to the case of multilinear modular maps, we obtain a characterisation of multidimensional Schur multipliers as the extended Haagerup tensor product $L^\infty(X_1)\otimes_{eh}\ldots\otimes_{eh}L^\infty(X_n)$ (Theorem 3.4). This generalises Grothendieck's and Peller's characterisations in the case $n = 2$. We show that the integral projective tensor product consists of multipliers and, therefore, $L^\infty(X_1)\hat{\otimes}_i \ldots \hat{\otimes}_i L^\infty(X_n) \subset L^\infty(X_1) \otimes_{eh} \ldots \otimes_{eh} L^\infty(X_n)$. The converse inclusion is true in the case $n = 2$ [21] but remains an open problem for $n > 2$.

In Section 4 we consider a non-commutative version of multidimensional multipliers following the Kissin-Shulman approach in the two dimensional case. We replace the functions ψ by elements of the minimal tensor product $\mathcal{A}_1 \otimes \ldots \otimes \mathcal{A}_n$ of some given C*-algebras $\mathcal{A}_1, \ldots, \mathcal{A}_n$ and the measure μ_i by a representation π_i of $\mathcal{A}_i$. We thus obtain a class of operator $\pi_1, \ldots, \pi_n$-multipliers. If each $\mathcal{A}_i$ is a commutative C*-algebra, say $\mathcal{A}_i = C_0(X_i)$ for some locally compact Hausdorff space X_i, and $\pi_i(f)$ is the operator of multiplication by $f \in C_0(X)$ acting on $L^2(X_i, \mu_i)$, then ψ is a $\pi_1, \ldots, \pi_n$-multiplier if and only if ψ is a Schur multiplier with respect to $\mu_1, \ldots, \mu_n$ (Proposition 4.5). As in the two-dimensional case, we show that the set of $\pi_1, \ldots, \pi_n$-multipliers does not change if we replace each π_i by an approximately equivalent representation (Theorem 5.1). A consequence of this result is the fact that the class of continuous (multidimensional) Schur multipliers depends only on the supports of the measures μ_i.

In Section 6 we study universal mutlipliers, i.e., the elements of $\mathcal{A}_1 \otimes \ldots \otimes \mathcal{A}_n$ which are $\pi_1, \ldots, \pi_n$-multipliers for all representations π_i of $\mathcal{A}_i$, $i = 1, \ldots, n$. We characterise such multipliers as the elements of a certain weak completion of the algebraic tensor product $\mathcal{A}_1 \odot \ldots \odot \mathcal{A}_n$ (Theorem 6.6). In the case where the C*-algebras are commutative and $n = 2$ this was proved in [18]; the case of arbitrary C*-algebras was left as a conjecture. Our result may be thought of as a non-commutative and multidimensional version of Grothendieck's and Peller's characterisations of Schur multipliers. The key ingredient in the proof is the observation that a universal multiplier determines a completely bounded multilinear modular map from the Cartesian product of the C*-algebras of compact operators into the C*-algebra of compact operators which allows us to use a result by Christensen and Sinclair [9] providing a description of all such mappings.

Acknowledgements

The authors are grateful to Victor Shulman for stimulating results, questions and discussions. They also thank Aristides Katavolos for providing the reference [16].

The work was partially written when the second author was visiting Chalmers University of Technology in Göteborg, Sweden, supported by funds from Queen's University Belfast. The third author was partially supported by the Swedish Reseach Council, while the first and the third authors were partially supported by Engineering and Physical Sciences Research Council grant EP/D050677/1.

2 Preliminaries

In this section we collect some preliminary notions and results which will be needed in the sequel.

Let H be a Hilbert space. The dual space H^{d} of H is a Hilbert space and there exists an anti-isometry $\partial : H \to H^{\mathrm{d}}$ given by $\partial(x)(y) = (y, x)$, $x, y \in H$. We set $x^{\mathrm{d}} = \partial(x)$.

If H_1 and H_2 are Hilbert spaces, we let $\mathcal{B}(H_1, H_2)$ be the space of all bounded linear operators from H_1 into H_2, and $\|\cdot\|_{\mathrm{op}}$ be the usual operator norm on $\mathcal{B}(H_1, H_2)$. We let $\mathcal{K}(H_1, H_2)$ be the subspace of all compact operators, and $\mathcal{C}_2(H_1, H_2)$ be the subspace of all Hilbert-Schmidt operators, from H_1 into H_2. For each $T \in \mathcal{C}_2(H_1, H_2)$, we denote by $\|T\|_2$ the Hilbert-Schmidt norm of T. The space $\mathcal{C}_2(H_1, H_2)$ is a Hilbert space with respect to the inner product $(T, S) = \mathrm{tr}(TS^*)$, where S^* denotes the adjoint of the operator S. We let $\mathcal{B}(H) = \mathcal{B}(H, H)$, $\mathcal{K}(H) = \mathcal{K}(H, H)$ and $\mathcal{C}_2(H) = \mathcal{C}_2(H, H)$.

If $T \in \mathcal{B}(H_1, H_2)$ we denote by $T^{\mathrm{d}} \in B(H_2^{\mathrm{d}}, H_1^{\mathrm{d}})$ the conjugate of T. We have that $\|T^{\mathrm{d}}\|_{\mathrm{op}} = \|T\|_{\mathrm{op}}$ and $T^{\mathrm{d}} x^{\mathrm{d}} = (T^* x)^{\mathrm{d}}$, whenever $x \in H_2$. Another way of expressing the last identity is

$$T^{\mathrm{d}} = \partial T^* \partial^{-1}. \tag{3}$$

We also have

$$(T^*)^{\mathrm{d}} = (T^{\mathrm{d}})^* \quad \text{and} \quad (\lambda T)^{\mathrm{d}} = \lambda T^{\mathrm{d}}, \quad \lambda \in \mathbb{C}. \tag{4}$$

We let $H_1 \otimes H_2$ be the Hilbert space tensor product of H_1 and H_2. There exists a unitary operator $\theta_{H_1,H_2} : H_1 \otimes H_2 \to \mathcal{C}_2(H_1^{\mathrm{d}}, H_2)$ given on elementary tensors $x \otimes y \in H_1 \otimes H_2$ by

$$\theta_{H_1,H_2}(x \otimes y)(z^{\mathrm{d}}) = (x, z)y, \quad z^{\mathrm{d}} \in H_1^{\mathrm{d}}.$$

If $A \in \mathcal{B}(H_1)$, $B \in \mathcal{B}(H_2)$, $x \in H_1$ and $y \in H_2$, we have that $\theta((A \otimes B)(x \otimes y)) = B\theta(x \otimes y)A^{\mathrm{d}}$, and hence

$$\theta((A \otimes B)\xi) = B\theta(\xi)A^{\mathrm{d}} \quad \text{for all } \xi \in H_1 \otimes H_2. \tag{5}$$

If $\varphi \in \mathcal{B}(H_1 \otimes H_2)$, let $\sigma_{H_1,H_2}(\varphi) \in \mathcal{B}(\mathcal{C}_2(H_1^{\mathrm{d}}, H_2))$ be given by the formula

$$\sigma_{H_1,H_2}(\varphi)(\theta(\xi)) = \theta(\varphi\xi), \quad \xi \in H_1 \otimes H_2.$$

Then σ_{H_1,H_2} implements a unitary equivalence between $\mathcal{B}(H_1 \otimes H_2)$ and $\mathcal{B}(\mathcal{C}_2(H_1^{\mathrm{d}}, H_2))$. An element $\varphi \in \mathcal{B}(H_1 \otimes H_2)$ is called a concrete (operator) multiplier if there exists $C > 0$ such that $\|\sigma_{H_1,H_2}(\varphi)(T)\|_{\mathrm{op}} \leq C\|T\|_{\mathrm{op}}$, for each $T \in \mathcal{C}_2(H_1^{\mathrm{d}}, H_2)$. Suppose that $H_1 = l^2(X)$, $H_2 = l^2(Y)$ for some sets X and Y and φ is the operator on $H_1 \otimes H_2 = \ell^2(X \times Y)$ of multiplication by a function $\phi \in \ell^\infty(X \times Y)$. The concrete operator multipliers of this form are precisely the classical Schur multipliers on $X \times Y$ (see e.g. [23]).

Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be C*-algebras. We denote by $\mathcal{A}_1 \otimes \mathcal{A}_2$ the minimal tensor product of $\mathcal{A}_1$ and $\mathcal{A}_2$. Let $\pi_i : \mathcal{A}_i \to \mathcal{B}(H_i)$ be a representation of $\mathcal{A}_i$, $i = 1, 2$. Then $\pi_1 \otimes \pi_2 : \mathcal{A}_1 \otimes \mathcal{A}_2 \to \mathcal{B}(H_1 \otimes H_2)$, given on elementary tensors by $(\pi_1 \otimes \pi_2)(a \otimes b) = \pi_1(a) \otimes \pi_2(b)$, is a representation of $\mathcal{A}_1 \otimes \mathcal{A}_2$. Let $\sigma_{\pi_1,\pi_2} = \sigma_{H_1,H_2} \circ (\pi_1 \otimes \pi_2)$; clearly, σ_{π_1,π_2} is a representation of $\mathcal{A}_1 \otimes \mathcal{A}_2$ on $\mathcal{C}_2(H_1^{\mathrm{d}}, H_2)$, unitarily equivalent to $\pi_1 \otimes \pi_2$. We moreover have

$$\sigma_{\pi_1,\pi_2}(a \otimes b)(T) = \pi_2(b) T \pi_1(a)^{\mathrm{d}}, \quad a \in \mathcal{A}_1, b \in \mathcal{A}_2, T \in \mathcal{C}_2(H_1^{\mathrm{d}}, H_2).$$

An element $\varphi \in \mathcal{A}_1 \otimes \mathcal{A}_2$ is called a π_1, π_2-multiplier [18] if there exists $C > 0$ such that

$$\|\sigma_{\pi_1,\pi_2}(\varphi)(T)\|_{\mathrm{op}} \leq C \|T\|_{\mathrm{op}}, \quad \text{for each } T \in \mathcal{C}_2(H_1^{\mathrm{d}}, H_2), \tag{6}$$

in other words, if $(\pi_1 \otimes \pi_2)(\varphi)$ is a concrete operator multiplier. The set of all π_1, π_2-multipliers in $\mathcal{A}_1 \otimes \mathcal{A}_2$ is denoted by $\mathbf{M}_{\pi_1,\pi_2}(\mathcal{A}_1, \mathcal{A}_2)$, and the smallest constant C appearing in (6) is denoted by $\|\varphi\|_{\pi_1,\pi_2}$. If φ is a π_1, π_2-multiplier for all representations π_i of $\mathcal{A}_i$, $i = 1, 2$, then φ is called a universal multiplier. The set of all universal multipliers is denoted by $\mathbf{M}(\mathcal{A}_1, \mathcal{A}_2)$; if $\varphi \in \mathbf{M}(\mathcal{A}_1, \mathcal{A}_2)$ we let $\|\varphi\|_{\mathrm{univ}} = \sup_{\pi_1,\pi_2} \|\varphi\|_{\pi_1,\pi_2}$. It is not difficult to see that in this case $\|\varphi\|_{\mathrm{univ}} < \infty$ [18].

We now recall some notions from Operator Space Theory. We refer the reader to [7], [13] and [24] for more details. An operator space is a closed subspace of $\mathcal{B}(H_1, H_2)$, for some Hilbert spaces H_1 and H_2. If $n, m \in \mathbb{N}$, by $M_{n,m}(\mathcal{E})$ we will denote the space of all n by m matrices with entries in $\mathcal{E}$ and let $M_n(\mathcal{E}) = M_{n,n}(\mathcal{E})$. Note that $M_{n,m}(\mathcal{E})$ can be identified in a natural way with a subspace of $\mathcal{B}(H_1^m, H_2^n)$ and hence carries a natural operator norm. If $n = \infty$ or $m = \infty$, we will denote by $M_{n,m}(\mathcal{E})$ the space of all (singly or doubly infinite) matrices with entries in $\mathcal{E}$ which represent a bounded linear operator between the corresponding amplifications of the Hilbert spaces. If $a = (a_{ij}) \in M_{n,m}(\mathcal{E})$, where $a_{ij} \in \mathcal{E}$, we let $a^{\mathrm{d}} = (a_{ij}^{\mathrm{d}})$; thus $a^{\mathrm{d}} \in \mathcal{B}(H_2^{\mathrm{d},m}, H_1^{\mathrm{d},n})$. We also let $a^{\mathrm{t}} = (a_{ji}) \in M_{m,n}(\mathcal{E})$; thus $a^{\mathrm{t}} \in \mathcal{B}(H_1^n, H_2^m)$. We have $\|a^{\mathrm{d}}\|_{\mathrm{op}} = \|a^{\mathrm{t}}\|_{\mathrm{op}}$ and $\|a^{\mathrm{d,t}}\|_{\mathrm{op}} = \|a\|_{\mathrm{op}}$.

If $\mathcal{E}$ and $\mathcal{F}$ are operator spaces, a linear map $\Phi : \mathcal{E} \to \mathcal{F}$ is called completely bounded if the map $\Phi_k : M_k(\mathcal{E}) \to M_k(\mathcal{F})$, given by $\Phi_k((a_{ij})) = (\Phi(a_{ij}))$, is bounded for each $k \in \mathbb{N}$ and $\|\Phi\|_{\mathrm{cb}} \stackrel{def}{=} \sup_k \|\Phi_k\| < \infty$.

Let $\mathcal{E}, \mathcal{E}_1, \dots, \mathcal{E}_n$ be operator spaces. We denote by $\mathcal{E}_1 \odot \cdots \odot \mathcal{E}_n$ the algebraic tensor product of $\mathcal{E}_1, \dots, \mathcal{E}_n$. Let $a_k = (a_{ij}^k) \in M_{m_k,m_{k+1}}(\mathcal{E}_k)$, $k =$

$1, \ldots, n$. We denote by

$$a^1 \odot \cdots \odot a^n \in M_{m_1, m_{n+1}}(\mathcal{E}_1 \odot \cdots \odot \mathcal{E}_n)$$

the matrix whose i, j-entry is

$$\sum_{i_2, \ldots, i_n} a^1_{i, i_2} \otimes a^2_{i_2, i_3} \otimes \cdots \otimes a^n_{i_n, j}.$$

Let $\Phi : \mathcal{E}_1 \times \cdots \times \mathcal{E}_n \to \mathcal{E}$ be a multilinear map and

$$\Phi_m : M_m(\mathcal{E}_1) \times M_m(\mathcal{E}_2) \times \cdots \times M_m(\mathcal{E}_n) \to M_m(\mathcal{E}_n)$$

be the multiliear map given by

$$\Phi_m(a^1, \ldots, a^n) = \left(\sum_{i_2, \ldots, i_n} \Phi(a^1_{i, i_2}, a^2_{i_2, i_3}, \ldots, a^n_{i_n, j}) \right)_{i,j}.$$

The map Φ is called completely bounded if there exists $C > 0$ such that for all $m \in \mathbb{N}$ and all elements $a^k \in M_m(\mathcal{E}_k)$, $k = 1, \ldots, n$, we have

$$\|\Phi_m(a^1, \ldots, a^n)\| \leq C \|a^1\| \ldots \|a^n\|.$$

Every completely bounded multilinear map $\Phi : \mathcal{E}_1 \times \cdots \times \mathcal{E}_n \to \mathcal{E}$ gives rise to a completely bounded linear map from the Haagerup tensor product $\mathcal{E}_1 \otimes_{\mathrm{h}} \cdots \otimes_{\mathrm{h}} \mathcal{E}_n$ into $\mathcal{E}$. For details on the Haagerup tensor product we refer the reader to [13].

If $R_1, \ldots, R_{n+1}$ are rings, M_i is an R_i, R_{i+1}-module for each $i = 1, \ldots, n$, and M is an R_1, R_{n+1}-module, a multilinear map $\Phi : M_1 \times \cdots \times M_n \to M$ will be called $R_1, \ldots, R_{n+1}$-modular (or simply modular if $R_1, \ldots, R_{n+1}$ are clear from the context) if

$$\Phi(a_1 m_1 a_2, m_2 a_3, m_3 a_4, \ldots, m_n a_{n+1}) = a_1 \Phi(m_1, a_2 m_2, a_3 m_3, \ldots, a_n m_n) a_{n+1},$$

for all $m_i \in M_i$ $(i = 1, \ldots, n)$ and $a_j \in R_j$ $(j = 1, \ldots, n+1)$. If $R_i = \mathcal{A}_i$ are C*-algebras and $M_i = \mathcal{E}_i$ are operator spaces, we let $\mathcal{B}_{\mathcal{A}_1, \ldots, \mathcal{A}_{n+1}}(\mathcal{E}_1, \ldots, \mathcal{E}_n; \mathcal{E})$ (resp. $CB_{\mathcal{A}_1, \ldots, \mathcal{A}_{n+1}}(\mathcal{E}_1, \ldots, \mathcal{E}_n; \mathcal{E})$) denote the spaces of all bounded (resp. completely bounded) $\mathcal{A}_1, \ldots, \mathcal{A}_{n+1}$-modular maps from $\mathcal{E}_1 \times \cdots \times \mathcal{E}_n$ into $\mathcal{E}$.

3 Multidimensional Schur multipliers

In this section, we define multidimensional Schur multipliers on the direct product of finitely many measure spaces. The main result of the section is Theorem 3.4 which characterises multidimensional Schur multipliers generalising the results of Peller [21] and Spronk [26].

Let (X_i, μ_i), $i = 1, 2, \ldots, n$, be standard σ-finite measure spaces. For notational convenience, integration with respect to μ_i will be denoted by dx_i. Direct products of the form $X_{i_1} \times \cdots \times X_{i_k}$ will be equipped with the corresponding product measure. We equip the space $L^2(X_1 \times X_2)$ with an $L^\infty(X_1), L^\infty(X_2)$-module action by letting $(a\xi b)(x,y) = a(x)\xi(x,y)b(y)$. We will denote by M_a the operator of multiplication by the essentially bounded function a acting on the corresponding L^2-space.

Theorem 3.1 *Let $\varphi \in L^\infty(X_1 \times \cdots \times X_n)$. Then the mapping*

$$S_\varphi : L^2(X_1 \times X_2) \times L^2(X_2 \times X_3) \times \cdots \times L^2(X_{n-1} \times X_n) \to L^2(X_1 \times X_n)$$

where $S_\varphi(f_1, \ldots, f_{n-1})(x_1, x_n)$ is defined as

$$\int_{X_2 \times \cdots \times X_{n-1}} \varphi(x_1, \ldots, x_n) f_1(x_1, x_2) f_2(x_2, x_3) \ldots f_{n-1}(x_{n-1}, x_n) dx_2 \ldots dx_{n-1}$$

is a bounded modular map and $\|S_\varphi\| = \|\varphi\|_\infty$.

Conversely, if

$$S : L^2(X_1 \times X_2) \times L^2(X_2 \times X_3) \times \cdots \times L^2(X_{n-1} \times X_n) \to L^2(X_1 \times X_n)$$

is a bounded modular map then there exists $\varphi \in L^\infty(X_1 \times \cdots \times X_n)$ such that $S = S_\varphi$.

Proof. In the case the variables of the functions appearing in the expressions below are clear from the context, we will omit the corresponding symbols in our notation. Fix $\varphi \in L^\infty(X_1 \times \cdots \times X_n)$ and $f_i \in L^2(X_i \times X_{i+1})$, $i = 1, \ldots, n-1$. We have

$$\|S_\varphi(f_1, \ldots, f_{n-1})\|_2^2 \leq \int_{X_1 \times X_n} \left(\int |\varphi f_1 \ldots f_{n-1}| dx_2 \ldots dx_{n-2} \right)^2 dx_1 dx_n$$

$$\leq \|\varphi\|_\infty^2 \int_{X_1 \times X_n} \left(\int |f_1 \ldots f_{n-1}| dx_2 \ldots dx_{n-2} \right)^2 dx_1 dx_n$$

$$\leq \|\varphi\|_\infty^2 \int_{X_1 \times X_n} (\int_{X_2 \times \cdots \times X_{n-2}} |f_1 \ldots f_{n-3}|$$

$$
\begin{aligned}
&\times \left(\int_{X_{n-1}} |f_{n-2}f_{n-1}|dx_{n-1}\right) dx_2 \dots dx_{n-2})^2 dx_1 dx_n \\
&\leq \|\varphi\|_\infty^2 \int_{X_1\times X_n} (\int_{X_2\times\cdots\times X_{n-2}} |f_1\dots f_{n-3}| \left(\int_{X_{n-1}} |f_{n-2}|^2 dx_{n-1}\right)^{\frac{1}{2}} \\
&\times \left(\int_{X_{n-1}} |f_{n-1}|^2 dx_{n-1}\right)^{\frac{1}{2}} dx_2 \dots dx_{n-2})^2 dx_1 dx_n \\
&= \|\varphi\|_\infty^2 \|f_{n-1}\|_2^2 \int_{X_1} (\int_{X_2\times\cdots\times X_{n-2}} |f_1\dots f_{n-3}| \\
&\times \left(\int_{X_{n-1}} |f_{n-2}|^2 dx_{n-1}\right)^{\frac{1}{2}} dx_2 \dots dx_{n-2})^2 dx_1 \\
&\leq \|\varphi\|_\infty^2 \|f_{n-1}\|_2^2 \int_{X_1} (\int_{X_2\times\cdots\times X_{n-3}} |f_1\dots f_{n-4}| \left(\int_{X_{n-2}} |f_{n-3}|^2 dx_{n-2}\right)^{\frac{1}{2}} \\
&\times \left(\int_{X_{n-2}\times X_{n-1}} |f_{n-2}|^2 dx_{n-2} dx_{n-1}\right)^{\frac{1}{2}} dx_2 \dots dx_{n-2})^2 dx_1 \\
&= \|\varphi\|_\infty^2 \|f_{n-1}\|_2^2 \|f_{n-2}\|_2^2 \int_{X_1} (\int_{X_2\times\cdots\times X_{n-3}} |f_1\dots f_{n-4}| \\
&\times \left(\int_{X_{n-2}} |f_{n-3}|^2 dx_{n-2}\right)^{\frac{1}{2}} dx_2 \dots dx_{n-3})^2 dx_1 \leq \\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
&\leq \|\varphi\|_\infty^2 \|f_{n-1}\|_2^2 \|f_{n-2}\|_2^2 \dots \|f_1\|_2^2.
\end{aligned}
$$

Conversely, let

$$S : L^2(X_1 \times X_2) \times L^2(X_2 \times X_3) \times \cdots \times L^2(X_{n-1} \times X_n) \to L^2(X_1 \times X_n)$$

be a bounded modular map. We first assume that the measures μ_i are finite. Write $K_1 = L^2(X_1 \times X_n)$ and let

$$S_1 : L^2(X_2) \times L^2(X_2) \times L^2(X_3) \times L^2(X_3) \times \cdots \times L^2(X_{n-1}) \times L^2(X_{n-1}) \to K_1$$

be given by

$$S_1(\xi_2, \eta_2, \xi_3, \eta_3, \dots, \xi_{n-1}, \eta_{n-1}) = S(1 \otimes \xi_2, \eta_2 \otimes \xi_3, \dots, \eta_{n-1} \otimes 1)$$

(here and in the sequel we denote by 1 the constant function taking value one). The fact that S is modular implies that

$$S_1(\xi_2 a_2, \eta_2, \xi_3 a_3, \dots, \xi_{n-1} a_{n-1}, \eta_{n-1}) = S_1(\xi_2, a_2\eta_2, \xi_3, \dots, a_{n-1}\eta_{n-1}),$$

whenever $a_i \in L^\infty(X_i)$, $i = 2, \ldots, n-1$. For fixed $\xi_3, \eta_3, \ldots, \xi_{n-1}, \eta_{n-1}$, let $S_2 : L^2(X_2) \times L^2(X_2) \to K_1$ be given by

$$S_2(\xi_2, \eta_2) = S_1(\xi_2, \eta_2, \xi_3, \eta_3, \ldots, \xi_{n-1}, \eta_{n-1}).$$

For $h \in K_1$, let $S_2^h : L^2(X_2) \times L^2(X_2) \to \mathbb{C}$ be defined by $S_2^h(\xi_2, \eta_2) = (S_2(\xi_2, \eta_2), h)$. Clearly,

$$|S_2^h(\xi_2, \eta_2)| \leq \|h\| \|S\| \prod_{i=2}^{n-1} \|\xi_i\| \|\eta_i\|.$$

Hence there exists a bounded operator $T_2^h : L^2(X_2) \to L^2(X_2)$ such that $S_2^h(\xi_2, \eta_2) = (T_2^h \xi_2, \overline{\eta_2})$, for all $\xi_2, \eta_2 \in L^2(X_2)$ and $\|T_2^h\| \leq \|h\| \|S\| \prod_{i=3}^{n-1} \|\xi_i\| \|\eta_i\|$. For each $a \in L^\infty(X_2)$ and $\xi_2, \eta_2 \in L^2(X_2)$ we have that

$$\begin{aligned} (T_2^h M_a \xi_2, \overline{\eta_2}) &= S_2^h(a\xi_2, \eta_2) = S_2^h(\xi_2, a\eta_2) \\ &= (T_2^h \xi_2, \overline{a\eta_2}) = (T_2^h \xi_2, M_{\overline{a}} \overline{\eta_2}) = (M_a T_2^h \xi_2, \overline{\eta_2}). \end{aligned}$$

Thus, there exists $\varphi_2^h \in L^\infty(X_2)$ such that $T_2^h = M_{\varphi_2^h}$. Moreover,

$$\|\varphi_2^h\|_\infty \leq \|h\| \|S\| \prod_{i=3}^{n-1} \|\xi_i\| \|\eta_i\|.$$

For each $f \in L^1(X_2)$, the functional on K_1 given by $h \to \int_{X_2} f(x_2) \varphi_2^h(x_2) dx_2$ is conjugate linear and bounded of norm not exceeding $\|f\|_1 \|S\| \prod_{i=3}^{n-1} \|\xi_i\| \|\eta_i\|$. Hence, there exists $\Phi_2(f) \in K_1$ such that

$$(\Phi_2(f), h) = \int_{X_2} f(x_2) \varphi_2^h(x_2) dx_2,$$

and $\|\Phi_2(f)\|_{K_1} \leq \|f\|_1 \|S\| \prod_{i=3}^{n-1} \|\xi_i\| \|\eta_i\|$. Thus, the mapping $\Phi_2 : L^1(X_2) \to K_1$ is bounded and $\|\Phi_2\| \leq \|S\| \prod_{i=3}^{n-1} \|\xi_i\| \|\eta_i\|$. Since Hilbert spaces possess Radon-Nikodym property, the vector valued Riesz Representation Theorem [11, Theorem 5, p. 63] implies that there exists $\varphi_2 \in L^\infty(X_2, K_1)$ ($L^\infty(X_2, K_1)$ being the space of essentially bounded K_1-valued measurable functions on X_2) such that

$$\Phi_2(f) = \int_{X_2} f(x_2) \varphi_2(x_2) dx_2,$$

where the integral is in Bochner's sense. Moreover,

$$\|\varphi_2\|_{L^\infty(X_2,K_1)} = \operatorname*{esssup}_{x_2\in X_2} \|\varphi_2(x_2)\|_{K_1} = \|\Phi_2\| \le \|S\| \prod_{i=3}^{n-1} \|\xi_i\|\|\eta_i\|.$$

For $\xi_2, \eta_2 \in L^2(X_2)$, we have that $\xi_2\overline{\eta_2} \in L^1(X_2)$ and hence

$$\begin{aligned}(S_2(\xi_2,\eta_2),h) &= (T_2^h\xi_2,\overline{\eta_2}) = \int_{X_2} \varphi_2^h(x_2)\xi_2(x_2)\eta_2(x_2)dx_2\\ &= \left(\int_{X_2} \varphi_2(x_2)\xi_2(x_2)\eta_2(x_2)dx_2, h\right);\end{aligned}$$

in other words,

$$S_2(\xi_2,\eta_2) = \int_{X_2} \varphi_2(x_2)\xi_2(x_2)\eta_2(x_2)dx_2,$$

where the integral is in Bochner's sense.

We consider φ_2 as a function on $X_1 \times X_2 \times X_n$ by letting $\varphi_2(x_1,x_2,x_n) = \varphi_2(x_2)(x_1,x_n)$. Note that φ_2 depends on $\xi_3,\eta_3,\dots,\xi_{n-1},\eta_{n-1}$; we denote this dependence by $\varphi_2 = \varphi_{2,\xi_3,\eta_3,\dots,\xi_{n-1},\eta_{n-1}}$.

Let $K_2 = L^2(X_1 \times X_2 \times X_n)$. We have

$$\begin{aligned}\|\varphi_2\|_{K_2} &= \int_{X_2}\int_{X_1\times X_n} |\varphi_2(x_2)(x_1,x_n)|^2 dx_1dx_ndx_2 = \int_{X_2} \|\varphi_2(x_2)\|_{K_1}^2 dx_2\\ &\le \mu_2(X_2)\|\varphi_2\|_{L^\infty(X_2,K_1)}.\end{aligned}$$

It follows that the mapping $S_3 : L^2(X_3) \times L^2(X_3) \to K_2$ given by

$$S_3(\xi_3,\eta_3) = \varphi_{2,\xi_3,\eta_3,\dots,\xi_{n-1},\eta_{n-1}}$$

is well-defined and

$$\|S_3(\xi_3,\eta_3)\|_{K_2} \le \mu_2(X_2)\|S\| \prod_{i=3}^{n-1} \|\xi_i\|\|\eta_i\|.$$

Hence, S_3 is bounded and $\|S_3\| \le \mu_2(X_2)\|S\| \prod_{i=4}^{n-1} \|\xi_i\|\|\eta_i\|$. An argument similar to the above implies the existence of $\varphi_3 \in L^\infty(X_3,K_2)$ with

$$\|\varphi_3\|_{L^\infty(X_3,K_2)} \le \mu_2(X_2)\|S\| \prod_{i=4}^{n-1} \|\xi_i\|\|\eta_i\|$$

such that

$$S_3(\xi_3, \eta_3) = \int_{X_3} \varphi_3(x_3)\xi_3(x_3)\eta_3(x_3)dx_3,$$

where the integral is in Bochner's sense. We may consider φ_3 as a function on $X_1 \times X_2 \times X_3 \times X_n$ by letting $\varphi_3(x_1, x_2, x_3, x_n) = \varphi_3(x_3)(x_1, x_2, x_n)$. We express the dependence of φ_3 on $\xi_4, \dots, \eta_{n-1}$ by writing $\varphi_3 = \varphi_{3,\xi_4,\dots,\eta_{n-1}}$. We have that

$$S_1(\xi_2, \eta_2, \dots, \xi_{n-1}, \eta_{n-1}) =$$
$$\int_{X_2}\int_{X_3} \varphi_{3,\xi_4,\dots,\eta_{n-1}}(x_1, x_2, x_3, x_n)\xi_2(x_2)\eta_2(x_2)\xi_3(x_3)\eta_3(x_3)dx_3dx_2,$$

where both integrals are in Bochner's sense.

Continuing inductively, we obtain $\varphi \in L^\infty(X_{n-1}, K_{n-2})$, where $K_{n-2} = L^2(X_1 \times \cdots \times X_{n-2} \times X_n)$, such that

$$S_1(\xi_2, \eta_2, \dots, \xi_{n-1}, \eta_{n-1}) =$$
$$\int_{X_2} \dots \int_{X_{n-1}} \varphi(x_1, \dots, x_n)\xi_2\eta_2 \dots \xi_{n-1}\eta_{n-1}dx_{n-1} \dots dx_2,$$

where the integrals are understood in Bochner's sense and φ is viewed as a function on $X_1 \times \cdots \times X_n$ by letting $\varphi(x_1, \dots, x_n) = \varphi(x_{n-1})(x_1, \dots, x_{n-2}, x_n)$.

It is easy to see that if $\psi \in L^1(Y, L^2(Z))$, where Y and Z are finite measure spaces, then $\int_{Y\times Z} |\psi(y)(z)|dydz$ is finite and $\left(\int_Y \psi(y)dy\right)(z) = \int_Y \psi(y)(z)dy$, for almost all $z \in Z$ (the first integral is in Bochner's sense, while the second one is a Lebesgue integral with respect to the variable y). It now follows that the last equality holds when the integrals are interpreted in the sense of Lebesgue.

The modularity of S implies

$$S(a \otimes \xi_2, \eta_2 \otimes \xi_3, \dots, \eta_{n-1} \otimes b) =$$
$$\int_{X_2}\int_{X_3} \dots \int_{X_{n-1}} \varphi(x_1, \dots, x_n)a\xi_2\eta_2 \dots \xi_{n-1}\eta_{n-1}bdx_{n-1} \dots dx_2,$$

for all $a \in L^\infty(X_1)$, $b \in L^\infty(X_n)$ and $\xi_i, \eta_i \in L^2(X_i)$, $i = 2, \dots, n-1$. Letting $a = \chi_{\alpha_1}$, $b = \chi_{\alpha_n}$ and $\xi_i = \eta_i = \chi_{\alpha_i}$, $i = 2, \dots, n-1$, the boundedness of S implies

$$\int_{\alpha_1\times\cdots\times\alpha_n} |\varphi(x_1, \dots, x_n)|dx_1 \dots dx_n \leq \|S\|\mu_1(\alpha_1) \dots \mu_n(\alpha_n).$$

It follows that the mapping

$$f = \sum_{i=1}^{N} \lambda_i \chi_{\alpha_1^i \times \cdots \times \alpha_n^i} \longrightarrow \int_{X_1 \times \cdots \times X_n} \varphi f,$$

where $\{\alpha_1^i \times \cdots \times \alpha_n^i\}$ is a finite family of disjoint Borel rectangles, is a linear functional on a dense subspace of $L^1(X_1 \times \cdots \times X_n)$ of norm not exceeding $\|S\|$. Therefore, $\varphi \in L^\infty(X_1 \times \cdots \times X_n)$ and $\|\varphi\|_\infty \leq \|S\|$.

We have that the mappings S and S_φ coincide on the tuples of the form $a \otimes \xi_2, \eta_2 \otimes \xi_3, \ldots, \eta_{n-1} \otimes b$; by linearity and continuity, they are equal. By the first part of the proof, $\|S\| \leq \|\varphi\|_\infty$ and hence $\|\varphi\|_\infty = \|S\|$.

Now relax the assumption on the finiteness of μ_i, and let X_i^k, $k \in \mathbb{N}$, be a measurable subset of X_i such that $\mu_i(X_i^k) < \infty$, $X_i^k \subseteq X_i^{k+1}$ and $X_i = \cup_{k=1}^\infty X_i^k$, $i = 1, \ldots, n$. For each $k \in \mathbb{N}$, let

$$S_k : L^2(X_1^k \times X_2^k) \times L^2(X_2^k \times X_3^k) \times \cdots \times L^2(X_{n-1}^k \times X_n^k) \to L^2(X_1^k \times X_n^k)$$

be the map given by $S_k(f_1, \ldots, f_{n-1}) = S(\tilde{f}_1, \ldots, \tilde{f}_{n-1})$, where $\tilde{f}_i$ coincides with f_i on X_i^k and is equal to zero on the complement of X_i^k. Since

$$\begin{aligned} S_k(f_1, \ldots, f_{n-1}) &= S(\chi_{X_1^k} \tilde{f}_1, \ldots, \tilde{f}_{n-1} \chi_{X_n^k}) \\ &= \chi_{X_1^k} S(\tilde{f}_1, \ldots, \tilde{f}_{n-1}) \chi_{X_n^k}, \end{aligned}$$

the map S_k is well-defined and $\|S_k\| \leq \|S\|$. Since S_k is obviously $L^\infty(X_n^k)$, $\ldots$, L^∞ (X_1^k)-modular, the above paragraphs imply that there exists $\varphi_k \in L^\infty(X_1^k \times \cdots \times X_n^k)$ such that $S_k = S_{\varphi_k}$, for each $k \in \mathbb{N}$. The space $L^2(X_i^k \times X_{i+1}^k)$ can be considered as a subspace of $L^2(X_i^{k+1} \times X_{i+1}^{k+1})$ in a natural way. We have that the restriction of S_{k+1} to $L^2(X_1^k \times X_2^k) \times L^2(X_2^k \times X_3^k) \times \ldots \times L^2(X_{n-1}^k \times X_n^k)$ coincides with S_k. This implies that the restriction of φ_{k+1} to $X_1^k \times \cdots \times X_n^k$ coincides (almost everywhere) with φ_k. Hence, there exists a function φ defined on $X_1 \times \cdots \times X_n$ which coincides with φ_k on $X_1^k \times \cdots \times X_n^k$, for each $k \in \mathbb{N}$. Since $\|\varphi_k\|_\infty = \|S_k\| \leq \|S\|$, we have that $\|\varphi\|_\infty \leq \|S\|$. We have that S and S_φ coincide on the union of $L^2(X_1^k \times X_2^k) \times L^2(X_2^k \times X_3^k) \times \ldots \times L^2(X_{n-1}^k \times X_n^k)$, $k \in \mathbb{N}$, which is a dense subset of $L^2(X_1 \times X_2) \times L^2(X_2 \times X_3) \times \ldots \times L^2(X_{n-1} \times X_n)$. It follows that $S = S_\varphi$, and by the first part of the proof, $\|S\| = \|\varphi\|_\infty$. $\diamondsuit$

Let (Y_1, ν_1) and (Y_2, ν_2) be measure spaces. A subset $E \subset Y_1 \times Y_2$ is called marginally null [1] if $E \subset A \times Y_2 \cup Y_1 \times B$, $\nu_1(A) = \nu_2(B) = 0$. It is well-known that the projective tensor product $L^2(Y_1)\hat{\otimes}L^2(Y_2)$ can be identified with a space of complex-valued functions, defined marginally almost everywhere on $Y_1 \times Y_2$: the element $\sum_{i=1}^{\infty} f_i \otimes g_i \in L^2(Y_1)\hat{\otimes}L^2(Y_2)$, where $f_i \in L^2(Y_1)$, $g_i \in L^2(Y_2)$ $\sum_{i=1}^{\infty} \|f_i\|^2 < \infty$ and $\sum_{i=1}^{\infty} \|g_i\|^2 < \infty$, is identified with the function h given by $h(x,y) = \sum_{i=1}^{\infty} f_i(x)g_i(y)$ (see e.g. [1]).

Let

$$\Gamma(X_1, \dots, X_n) = L^2(X_1 \times X_2) \odot \cdots \odot L^2(X_{n-1} \times X_n).$$

We identify the elements of $\Gamma(X_1, \dots, X_n)$ with functions on

$$X_1 \times X_2 \times X_2 \times \cdots \times X_{n-1} \times X_{n-1} \times X_n$$

in the obvious fashion. We equip $\Gamma(X_1, \dots, X_n)$ with two norms; one is the projective norm $\| \cdot \|_{2,\wedge}$, where each of the L^2-spaces is equipped with its L^2-norm, and the other is the Haagerup tensor norm $\| \cdot \|_{\mathrm{h}}$, where the L^2-spaces are given their opposite operator space structure arising from the identification of $L^2(X \times Y)$ with the class of Hilbert-Schmidt operators from $L^2(X)$ into $L^2(Y)$ given by

$$(T_f\xi)(y) = \int_X f(x,y)\xi(x)dx, \qquad f \in L^2(X \times Y), \xi \in L^2(X).$$

For a function $\Phi \in \Gamma(X_1, \dots, X_n)$ (of $2n-2$ variables), we write $\tilde{\Phi}$ for the function (of n variables) on $X_1 \times \cdots \times X_n$ given by

$$\tilde{\Phi}(x_1, x_2, \dots, x_{n-1}, x_n) = \Phi(x_1, x_2, x_2, \dots, x_{n-1}, x_{n-1}, x_n). \tag{7}$$

It is easy to see that $\tilde{\Phi}$ is well-defined up to a null set with respect to the product measure on $X_1 \times \cdots \times X_n$.

Let $\varphi \in L^\infty(X_1 \times \cdots \times X_n)$. We define

$$S_\varphi : (\Gamma(X_1, \dots, X_n), \| \cdot \|_{2,\wedge}) \to (L^2(X_1 \times X_n), \| \cdot \|_2)$$

by

$$S_\varphi(\Phi)(x_1, x_n) = \int_{X_2 \times \cdots \times X_{n-1}} \varphi(x_1, \dots, x_n)\tilde{\Phi}(x_1, \dots, x_n)dx_2 \dots dx_{n-1}.$$

By Theorem 3.1, S_φ is well-defined, bounded and $\|S_\varphi\| = \|\varphi\|_\infty$.

Definition 3.2 *Let $\varphi \in L^\infty(X_1 \times \cdots \times X_n)$. We say that φ is a Schur multiplier (relative to the measure spaces $(X_1, \mu_1), \ldots (X_n, \mu_n)$) if there exists $C > 0$ such that $\|S_\varphi(\Phi)\|_{\mathrm{op}} \leq C\|\Phi\|_{\mathrm{h}}$, for all $\Phi \in \Gamma(X_1, \ldots, X_n)$. The smallest constant C with this property will be denoted by $\|\varphi\|_{\mathrm{m}}$.*

Note that in the case where $n = 2$ and the measure spaces are discrete, the definition above reduces to the definition of the classical Schur multipliers. In the case of arbitrary measure spaces and $n = 2$, we obtain the Schur multipliers studied by Peller [21] (see also [26]).

We will present next a characterisation of the n-dimensional Schur multipliers which generalises Grothendieck's and Peller's characterisations. We will need the following generalisation of a result of Smith [25].

Lemma 3.3 *Let $\mathcal{E}_i \subseteq B(H_i, H_{i+1})$, $i = 1, \ldots, n-1$ and $\mathcal{C} \subseteq B(H_1)$, $\mathcal{D} \subseteq B(H_n)$ be C^*-algebras with cyclic vectors. Assume that $\mathcal{E}_1$ is a right $\mathcal{C}$-module and $\mathcal{E}_n$ is a left $\mathcal{D}$-module. Let $\phi : \mathcal{E}_n \times \cdots \times \mathcal{E}_1 \to B(H_1, H_n)$ be a multilinear $\mathcal{D}, \mathcal{C}$-module map (that is, $\phi(dy, \ldots, xc) = d\phi(y, \ldots, x)c$, whenever $x \in \mathcal{E}_1$, $y \in \mathcal{E}_n$, $c \in \mathcal{C}$ and $d \in \mathcal{D}$) whose linearisation $\mathcal{E}_n \odot \cdots \odot \mathcal{E}_1 \to B(H_1, H_n)$ is bounded in the Haagerup norm. Then ϕ is a completely bounded multilinear map.*

Proof. The proof is a straightforward generalisation of the argument given by Smith [25]. We will denote the linearisation of ϕ defined on $(\mathcal{E}_1 \odot \cdots \odot \mathcal{E}_n, \|\cdot\|_{\mathrm{h}})$ by the same symbol. Assume that $\|\phi\| = 1$. We will show that $\|\phi\|_{\mathrm{cb}} = 1$. Suppose, to the contrary, that $\|\phi\|_{\mathrm{cb}} > 1$. Then there exists $m \in \mathbb{N}$, matrices $(x_{k_i,k_{i+1}}) \in M_m(\mathcal{E}_i)$, $i = 1, \ldots, m$ and column vectors $\xi_0 = (\xi_1, \ldots, \xi_m) \in H_1^m$ and $\eta_0 = (\eta_1, \ldots, \eta_m) \in H_n^m$ of norm strictly less than one such that

$$|(\phi_m([x_{j,k_{n-1}}], \ldots, [x_{k_1,i}])\xi_0, \eta_0)| > 1. \tag{8}$$

If ξ and η are cyclic vectors for $\mathcal{C}$ and $\mathcal{D}$, respectively, we may moreover assume that $\xi_i = a_i\xi$ and $\eta_j = b_j\eta$, for some $a_i \in \mathcal{C}$ and $b_j \in \mathcal{D}$, where $i, j = 1, \ldots, m$. Let $a = \sum_{i=1}^m a_i^* a_i$ and $b = \sum_{j=1}^m b_j^* b_j$. Assume first that a and b are invertible, and let $c_i = a_i a^{-1/2}$, $d_j = b_j b^{-1/2}$, $\tilde{\xi} = a^{1/2}\xi$ and $\tilde{\eta} = b^{1/2}\eta$. Then $\xi_i = c_i\tilde{\xi}$ and $\eta_j = d_j\tilde{\eta}$. The left hand side of (8) becomes

$$\left|\sum_{i,j=1}^m \sum_{k_l=1}^m \left(\phi(x_{j,k_{n-1}}, \ldots, x_{k_1,i},)c_i\tilde{\xi}, d_j\tilde{\eta}\right)\right| =$$

$$\left|\sum_{k_l=1}^{m}\left(\phi\left(\sum_{i=1}^{m} d_j^* x_{jk_{n-1}},\ldots,\sum_{j=1}^{m} x_{k_1,i}c_i\right)\tilde{\xi},\tilde{\eta}\right)\right|. \tag{9}$$

We have that

$$\|\tilde{\xi}\| = (a^{1/2}\xi, a^{1/2}\xi) = (a\xi,\xi) = \sum_{k=1}^{n}\|a_i\xi\|^2 \leq 1,$$

and similarly $\|\tilde{\eta}\| \leq 1$. By assumption, (9) does not exceed the product of the norms of

$$\left(\sum_{j=1}^{m} d_j^* x_{jk_{n-1}}\right)_{k_{n-1}} \in M_{1,m}(\mathcal{E}_n),\ldots\ldots,\left(\sum_{i=1}^{m} x_{k_1,i}c_i\right)_{k_1} \in M_{m,1}(\mathcal{E}_1). \tag{10}$$

But, the first matrix appearing in (10) is equal to the product of $(d_j^*)_j \in M_{1,m}(\mathcal{D})$ and $(x_{jk_{n-1}})_{j,k_{n-1}} \in M_m(\mathcal{E}_n)$. We have

$$\|(d_j^*)_j\| = \left\|\sum_{j=1}^{m} d_j^* d_j\right\| = \|I\| = 1,$$

and hence the norm of the first matrix appearing in (10) does not exceed one. Similarly, the second matrix in (10) is the product of $(x_{k_1 i})_{k_1,i} \in M_{m,m}(\mathcal{E}_1)$ and $(c_i)_i \in M_{m,1}(\mathcal{C})$ and its norm does not exceed one. Hence (9) does not exceed one, a contradiction.

In the case a or b is not invertible, as in [25], one can replace the matrices $(x_{j,k_{n-1}})$, ..., $(x_{k_1,i})$ with $(x_{k,k_{n-1}}) \oplus 0 \in M_{m+1}(\mathcal{E}_n)$, ..., $(x_{k_1,i}) \oplus 0 \in M_{m+1}(\mathcal{E}_1)$, respectively (obviously keeping the same norm), and the vectors $(\xi_1,\ldots,\xi_m)$ and $(\eta_1,\ldots,\eta_m)$ with $(\xi_1,\ldots,\xi_m,\epsilon\xi)$ and $(\eta_1,\ldots,\eta_m,\epsilon\eta)$, respectively, for ϵ small enough so that the norms of these vectors remain less than one. $\diamondsuit$

The main result of this section is the following

Theorem 3.4 *Let $\varphi \in L^\infty(X_1 \times \cdots \times X_n)$. The following are equivalent:*

(i) φ is a Schur multiplier and $\|\varphi\|_{\mathrm{m}} < 1$;

(ii) there exist essentially bounded functions $a_1 : X_1 \to M_{\infty,1}$, $a_n : X_n \to M_{1,\infty}$ and $a_i : X_i \to M_\infty$, $i = 2,\ldots,n-1$, such that, for almost all $x_1,\ldots,x_n$ we have

$$\varphi(x_1,\ldots,x_n) = a_n(x_n)a_{n-1}(x_{n-1})\ldots a_1(x_1) \quad \textit{and} \quad \operatorname*{esssup}_{x_i\in X_i}\prod_{i=1}^{n}\|a_i(x_i)\| < 1.$$

Proof. (i)⇒(ii) Let $\varphi \in L^\infty(X_1 \times \cdots \times X_n)$ be a Schur multiplier with $\|\varphi\|_{\mathrm{m}} < 1$. Then the map S_φ induces a map, denoted in the same way, from $L^2(X_1 \times X_2) \times \cdots \times L^2(X_{n-1} \times X_n)$ into $L^2(X_1 \times X_n)$. Let $H_i = L^2(X_i)$, $\mathcal{D}_i$ be the multiplication masa of $L^\infty(X_i)$, $i = 1, \ldots, n$, and

$$\hat{S}_\varphi : \mathcal{C}_2(H_1, H_2) \times \cdots \times \mathcal{C}_2(H_{n-1}, H_n) \to \mathcal{C}_2(H_1, H_n)$$

be the map defined by $\hat{S}_\varphi(T_{f_1}, \ldots, T_{f_n}) = T_{S_\varphi(f_1,\ldots,f_n)}$. Since φ is a Schur multiplier, the linearisation of the map $\hat{S}_\varphi$ is bounded when the space on the $\mathcal{C}_2$-spaces on the left hand side are given the operator space structure opposite to the natural one, the tensor product is given the Haagerup norm and the space on the right hand side is given its operator norm. If $a_i \in L^\infty(X_i)$, $i = 1, \ldots, n$, then

$$\begin{aligned}
\hat{S}_\varphi(M_{a_1}T_{f_1}, M_{a_2}T_{f_2} \ldots, M_{a_{n-1}}T_{f_n}M_{a_n}) &= \hat{S}_\varphi(T_{a_1f_1}, T_{a_2f_2}, \ldots, T_{a_{n-1}f_na_n}) \\
&= T_{S_\varphi(a_1f_1, a_2f_2, \ldots, a_{n-1}f_na_n)} \\
&= T_{a_1S_\varphi(f_1a_2, f_2a_3, \ldots, f_n)a_n} \\
&= M_{a_1}\hat{S}_\varphi(T_{f_1}M_{a_2}, \ldots, T_{f_n})M_{a_n};
\end{aligned}$$

in other words, $\hat{S}_\varphi$ is modular.

By continuity, the map $\hat{S}_\varphi$ has an extension (denoted in the same way)

$$\hat{S}_\varphi : \mathcal{K}(H_1, H_2) \otimes_{\mathrm{h}} \cdots \otimes_{\mathrm{h}} \mathcal{K}(H_{n-1}, H_n) \to \mathcal{K}(H_1, H_n)$$

to a modular map with norm less than one, where the spaces $\mathcal{K}(H_i, H_{i+1})$ are equipped with the operator space structure opposite to their natural operator space structure. It follows that the map

$$\check{S}_\varphi : \mathcal{K}(H_{n-1}, H_n) \otimes_{\mathrm{h}} \cdots \otimes_{\mathrm{h}} \mathcal{K}(H_1, H_2) \to \mathcal{K}(H_1, H_n)$$

given by

$$\check{S}_\varphi(T_{n-1} \otimes \cdots \otimes T_1) = \hat{S}_\varphi(T_1 \otimes \cdots \otimes T_{n-1})$$

is modular and bounded when the spaces $K(H_i, H_{i+1})$ are given their natural operator space structure. By Lemma 3.3, $\check{S}_\varphi$ is completely bounded. It follows that the second dual

$$\check{S}_\varphi^{**} : \mathcal{B}(H_{n-1}, H_n) \otimes_{\sigma h} \cdots \otimes_{\sigma h} \mathcal{B}(H_1, H_2) \to \mathcal{B}(H_1, H_n)$$

is a weak* continuous map with c.b. norm less than one, which extends the map $\check{S}_\varphi$. (Here $\otimes_{\sigma h}$ denotes the normal Haagerup tensor product, see e.g. [7].)

Denote by $\tilde{S}_\varphi$ the corresponding multilinear map

$$\tilde{S}_\varphi : \mathcal{B}(H_{n-1}, H_n) \times \cdots \times \mathcal{B}(H_1, H_2) \to \mathcal{B}(H_1, H_n).$$

The map $\tilde{S}_\varphi$ is separately weak* continuous and hence modular.

A modification of Corollary 5.9 of [9] now implies that there exist bounded linear operators $V_1 : H_1 \to H_1^\infty$, $V_n : H_n^\infty \to H_n$ and $V_i : H_i^\infty \to H_i^\infty$, $i = 2, \dots, n-1$, such that the entries of V_i belong to $\mathcal{D}_i$ and

$$\tilde{S}_\varphi(T_{n-1}, \dots, T_1) = V_n(T_{n-1} \otimes I)V_{n-1}(T_{n-2} \otimes I) \dots (T_1 \otimes I)V_1.$$

Moreover, the operators V_i can be chosen so that $\prod_{i=1}^n \|V_i\| < 1$. Let $V_1 = (M_{a_1^1}, M_{a_1^2}, \dots)^{\rm t}$, $V_i = (M_{a_i^{kl}})_{k,l}$ and $V_n = (M_{a_n^1}, M_{a_n^2}, \dots)$, for some $a_1 = (a_1^k)_k \in L^\infty(X_1, M_{1,\infty})$, $a_n = (a_n^l)_l \in L^\infty(X_n, M_{1,\infty})$ and $a_i = (a_i^{kl})_{k,l} \in L^\infty(X_i, M_\infty)$, $i = 2, \dots, n-1$. Moreover,

$$\operatorname*{esssup}_{x_i \in X_i} \prod_{i=1}^n \|a_i(x_i)\| = \prod_{i=1}^n \|V_i\| < 1.$$

Let $\xi_i, \eta_i \in H_i$, $i = 1, \dots, n$. Then

$$\begin{aligned}
& \tilde{S}_\varphi(T_{\xi_{n-1}\otimes\eta_n}, \dots, T_{\xi_1\otimes\eta_2})(\eta_1) = V_n(T_{\xi_{n-1}\otimes\eta_n} \otimes I) \dots (T_{\xi_1\otimes\eta_2} \otimes I)V_1(\eta_1) \\
= \ & V_n(T_{\xi_{n-1}\otimes\eta_n} \otimes I) \dots V_2(T_{\xi_1\otimes\eta_2} \otimes I)(M_{a_1^{k_1}}\eta_1)_{k_1} \\
= \ & V_n(T_{\xi_{n-1}\otimes\eta_n} \otimes I) \dots V_2((\int_{X_1} a_1^{k_1}(x_1)\xi_1(x_1)\eta_1(x_1)dx_1)\eta_2)_{\infty,1} \\
= \ & V_n \dots (T_{\xi_2\otimes\eta_3} \otimes I)((\sum_{k_1=1}^\infty \int_{X_1} a_1^{k_1}(x_1)\xi_1(x_1)\eta_1(x_1)dx_1)a_2^{k_2,k_1}\eta_2)_{k_2} \\
= \ & V_n \dots V_3((\sum_{k_1=1}^\infty \int_{X_1\times X_2} a_2^{k_2,k_1}(x_2)a_1^{k_1}(x_1)(\xi_1\eta_1)(x_1)(\xi_2\eta_2)(x_2)dx_1dx_2)\eta_3)_{k_2} \\
= \ & \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
= \ & \sum_{k_n=1}^\infty (\int_{X_1\times\dots\times X_{n-1}} \sum_{k_1,\dots,k_{n-1}=1}^\infty a_{n-1}^{k_{n-1},k_{n-2}}(x_{n-1}) \dots a_1^{k_1}(x_1) \times \\
\times \ & \xi_1(x_1)\eta_1(x_1) \dots \xi_{n-1}(x_{n-1}))dx_1 \dots dx_{n-1})M_{a_n^{k_n}}\eta_n.
\end{aligned}$$

Thus,

$$\begin{aligned}
& \tilde{S}_\varphi(T_{\xi_{n-1}\otimes\eta_n},\dots,T_{\xi_1\otimes\eta_2})(\eta_1)(x_n) \\
= & \ (\int_{X_1\times\cdots\times X_{n-1}} \sum_{k_1,\dots,k_n=1}^{\infty} a_n^{k_n}(x_n)a_{n-1}^{k_{n-1},k_{n-2}}(x_{n-1})\dots a_1^{k_1}(x_1)\times \\
\times & \ \xi_1(x_1)\eta_1(x_1)\dots\xi_{n-1}(x_{n-1})dx_1\dots dx_{n-1})\eta_n(x_n).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
& \tilde{S}_\varphi(T_{\xi_{n-1}\otimes\eta_n},\dots,T_{\xi_1\otimes\eta_2})(\eta_1)(x_n) \\
= & \ T_{S_\varphi(\xi_1\otimes\eta_2,\dots,\xi_{n-1}\otimes\eta_n)}(\eta_1)(x_n) \\
= & \ (\int_{X_1\times\cdots\times X_{n-1}} \varphi(x_1,\dots,x_{n-1},x_n) \\
\times & \ \xi_1(x_1)\eta_1(x_1)\dots\xi_{n-1}(x_{n-1})dx_1\dots dx_{n-1})\eta_n(x_n).
\end{aligned}$$

It follows that

$$\varphi(x_1,\dots,x_n)=a_n(x_n)a_{n-1}(x_{n-1})\dots a_1(x_1),$$

for almost all $x_1,\dots,x_n$.

(ii)$\Rightarrow$(i) Assume that φ is given as in (ii), where $a_1=(a_1^k)_k\in L^\infty(X_1, M_{\infty,1})$, $a_n=(a_n^l)_l\in L^\infty(X_n,M_{1,\infty})$ and $a_i=(a_i^{kl})_{k,l}\in L^\infty(X_n,M_{1,\infty})$, $i=2,\dots,n-1$. Let $V_1:H_1\to H_1^\infty$ be the operator corresponding to the column matrix $V_1=(M_{a_1^k})_k:H_1\to H_1^\infty$, $V_n:H_n^\infty\to H_n$ be the operator corresponding to the row matrix $V_n=(M_{a_n^l})_l$ and $V_i:H_i^\infty\to H_i^\infty$ be the operator corresponding to the matrix $V_i=(M_{a_i^{kl}})_{k,l}$, $i=2,\dots,n-1$. Then $\prod_{i=1}^n\|V_i\|<1$. It follows from the first part of the proof that

$$\tilde{S}_\varphi(T_{\xi_{n-1}\otimes\eta_n},\dots,T_{\xi_1\otimes\eta_2})=V_n(T_{\xi_{n-1}\otimes\eta_n}\otimes I)\dots(T_{\xi_1\otimes\eta_2}\otimes I)V_1,$$

for all $\xi_1\in H_1$, $\eta_n\in H_n$ and $\xi_i,\eta_i\in H_i$, $i=2,\dots,n-1$. Since the operator norm is dominated by the Hilbert-Schmidt norm, we conclude that

$$\tilde{S}_\varphi(T_{f_{n-1}},\dots,T_{f_1})=V_n(T_{f_{n-1}}\otimes I)\dots(T_{f_1}\otimes I)V_1,$$

for all $f_i\in L^2(X_i\times X_{i+1})$, $i=1,\dots,n-1$.

Let

$$F=F_1\odot\cdots\odot F_{n-1}\in L^2(X_1\times X_2)\odot\cdots\odot L^2(X_{n-1}\times X_n),$$

where $F_1 \in M_{1,\infty}(L^2(X_1 \times X_2))$, $F_{n-1} \in M_{\infty,1}(L^2(X_{n-1} \times X_n))$ and $F_i \in M_\infty(L^2(X_i \times X_{i+1})$, $i = 2, \dots, n-2$. Lemma 4.6 implies that

$$T_{S_\varphi(F)} = V_n(T_{F_{n-1}} \otimes I) \dots (T_{F_1} \otimes I)V_1,$$

where $T_{F_i} = (T_{f_i^{lk}})_{k,l}$ whenever $F_i = (f_i^{kl})_{k,l}$. It follows that

$$\|T_{S_\varphi(F)}\|_{\mathrm{op}} \leq \prod_{i=1}^{n-1} \|F_i^t\|_{\mathrm{op}} \prod_{i=1}^{n} \|V_i\|.$$

Taking infimum with respect to all representations of F, we conclude that $\|T_{S_\varphi(F)}\|_{\mathrm{op}} \leq \|F\|_{\mathrm{h}} \prod_{i=1}^n \|V_i\|$ and so $\|\varphi\|_{\mathrm{m}} < 1$. $\diamondsuit$

Remark The space of all functions $\varphi(x_1, \dots, x_n)$ satisfying condition (ii) of Theorem 3.4 is the extended Haagerup tensor product $L^\infty(X_1) \otimes_{eh} L^\infty(X_2) \otimes_{eh} \dots \otimes_{eh} L^\infty(X_n)$.

The next proposition relates our approach with a recent work of Peller [22] on multiple operator integrals. For some fixed spectral measures, Peller defines a multiple operator integral $I_\varphi(T_1, \dots, T_{n-1})$ of a function φ and $n-1$-tuple of operators $(T_1, \dots, T_{n-1})$, and shows that if φ belongs to the integral projective tensor product of the corresponding L^∞-spaces, then $I_\varphi(T_1, \dots, T_{n-1})$ is well-defined and, moreover,

$$\|I_\varphi(T_1, \dots, T_{n-1})\|_{\mathrm{op}} \leq \|\varphi\|_i \|T_1\|_{\mathrm{op}} \dots \|T_{n-1}\|_{\mathrm{op}}.$$

Recall that the integral projective tensor product $L^\infty(X_1) \hat{\otimes}_i \dots \hat{\otimes}_i L^\infty(X_n)$ is the space of all functions φ for which there exists a measure space $(\mathcal{T}, \nu)$ and measurable functions g_i on $X_i \times \mathcal{T}$ such that

$$\varphi(x_1, \dots, x_n) = \int_{\mathcal{T}} g_1(x_1, t) \dots g_n(x_n, t) d\nu(t), \tag{11}$$

for almost all $x_1 \dots, x_n$, where

$$\int_{\mathcal{T}} \|g_1(\cdot, t)\|_\infty \dots \|g_n(\cdot, t)\|_\infty d\nu(t) < \infty.$$

The integral projective norm $\|\varphi\|_i$ of φ is the infimum of the above expressions over all representations of φ of the form (11). It was proved by Peller in [21] that in the case where $n = 2$ the integral projective tensor product

$L^\infty(X_1)\hat{\otimes}_i L^\infty(X_2)$ coinsides with the set of all Schur mulipliers. The next proposition shows that for $n > 2$ the integral projective tensor product consists of multipliers. We do not know whether it coincides with the space of all Schur multipliers.

Proposition 3.5 *Let $\varphi \in L^\infty(X_1)\hat{\otimes}_i \dots \hat{\otimes}_i L^\infty(X_n)$. Then φ is a Schur multiplier and $\|\varphi\|_{\mathrm{m}} \leq \|\varphi\|_i$.*

Proof. Suppose that

$$\varphi(x_1, \dots, x_n) = \int_{\mathcal{T}} g_1(x_1, t) \dots g_n(x_n, t) d\nu(t),$$

for almost all $x_1 \dots, x_n$, where $(\mathcal{T}, \nu)$ is a measure space, g_i is a measurable function on $X_i \times \mathcal{T}$, $i = 1, \dots, n$, such that

$$\int_{\mathcal{T}} \|g_1(\cdot, t)\|_\infty \dots \|g_n(\cdot, t)\|_\infty d\nu(t) < \infty.$$

Let $F = F_1 \odot \dots \odot F_{n-1}$, where $F_1 \in M_{1,k_1}(L^2(X_1 \times X_2))$, $F_{n-1} \in M_{k_{n-2},1}$ $(L^2(X_{n-1} \times X_n))$ and $F_i \in M_{k_{i-1},k_i}(L^2(X_i \times X_{i+1}))$, $i = 2, \dots, n-2$, and $G = \tilde{F}$ (see (7)). We have

$$\begin{aligned}
\|S_\varphi(F)\|_{\mathrm{op}} &= \|\int_{X_2 \times \dots \times X_{n-1}} \varphi G dx_2 \dots dx_{n-1}\|_{\mathrm{op}} \\
&= \|\int_{X_2 \times \dots \times X_{n-1}} \left(\int_{\mathcal{T}} g_1(x_1, t) \dots g_n(x_n, t) dt\right) G dx_2 \dots dx_{n-1}\|_{\mathrm{op}} \\
&= \|\int_{\mathcal{T}} \left(\int_{X_2 \times \dots \times X_{n-1}} g_1(x_1, t) \dots g_n(x_n, t) dx_2 \dots dx_{n-1}\right) G dt\|_{\mathrm{op}} \\
&\leq \|\int_{\mathcal{T}} (\int_{X_2 \times \dots \times X_{n-1}} M_{g_1(\cdot,t)} F_1 (M_{g_2(\cdot,t)} \otimes I)(x_1, x_2)) \odot \dots \\
&\odot \; F_{n-1} M_{g_n(\cdot,t)}(x_{n-1}, x_n) dx_2 \dots dx_{n-1}) dt\|_{\mathrm{op}} \\
&\leq \int_{\mathcal{T}} \|\int_{X_2 \times \dots \times X_{n-1}} M_{g_1(\cdot,t)} F_1 (M_{g_2(\cdot,t)} \otimes I)(x_1, x_2)) \odot \dots \\
&\odot \; F_{n-1} M_{g_n(\cdot,t)}(x_{n-1}, x_n) dx_2 \dots dx_{n-1}\|_{\mathrm{op}} dt \\
&\leq \int_{\mathcal{T}} \|M_{g_1(\cdot,t)}\| \|F_1\|_{\mathrm{op}}^o \|M_{g_2(\cdot,t)}\| \dots \|F_{n-1}\|_{\mathrm{op}}^o \|M_{g_n(\cdot,t)}\| dt \\
&\leq \|\varphi\|_i \|F_1\|_{\mathrm{op}}^o \dots \|F_{n-1}\|_{\mathrm{op}}^o.
\end{aligned}$$

The claim follows by taking infimum over all representations $F = F_1 \odot \cdots \odot F_{n-1}$. $\diamondsuit$

Corollary 3.6 $L^\infty(X_1)\hat{\otimes}_i \ldots \hat{\otimes}_i L^\infty(X_n) \subseteq L^\infty(X_1) \otimes_{eh} \ldots \otimes_{eh} L^\infty(X_n)$.

In the case where $n = 2$, it follows by Peller's characterisation of Schur multipliers [21] that there is an equality in the inclusion of Corollary 3.6. We do not know whether equality holds in the general case.

We finally point out another interesting open question, namely the one of characterising the class of multipliers defined by using the projective tensor norm instead of the Haagerup tensor norm in (2); equivalently, the class of multipliers obtained after replacing (2) with the weaker condition

$$\|S_\psi(f_1 \otimes \ldots \otimes f_n)\|_{\mathrm{op}} \leq C\|f_1\|_{\mathrm{op}} \ldots \|f_n\|_{\mathrm{op}} \text{ for all } f_i \in L^2(X_i), i = 1, \ldots, n.$$

4 Multidimensional operator multipliers: the definition

In this section we generalise the notion of operator multipliers given by Kissin and Shulman [18] to the multidimensional case.

We recall the mapping $\theta_{K_1,K_2} : K_1 \otimes K_2 \to \mathcal{C}_2(K_1^{\mathrm{d}}, K_2)$, where K_1 and K_2 are Hilbert spaces, which is the unitary operator between the Hilbert spaces $K_1 \otimes K_2$ and $\mathcal{C}_2(K_1^{\mathrm{d}}, K_2)$ given on elementary tensors by

$$\theta_{K_1,K_2}(\xi_1 \otimes \xi_2)(\eta_1^{\mathrm{d}}) = (\xi_1, \eta_1)\xi_2.$$

Note that there is a natural identification of $(K_1 \otimes K_2)^{\mathrm{d}}$ and $K_1^{\mathrm{d}} \otimes K_2^{\mathrm{d}}$. It follows that $\mathcal{C}_2(K_1^{\mathrm{d}}, K_2)^{\mathrm{d}}$ can be identified with $\mathcal{C}_2(K_1, K_2^{\mathrm{d}}) = \mathcal{C}_2((K_1^{\mathrm{d}})^{\mathrm{d}}, K_2^{\mathrm{d}})$; we have that $\theta_{K_1^{\mathrm{d}},K_2^{\mathrm{d}}}(\xi^{\mathrm{d}}) = \theta_{K_1,K_2}(\xi)^{\mathrm{d}}$.

Let $H_1, \ldots, H_n$ be Hilbert spaces and $H = H_1 \otimes \ldots H_n$. For any permutation π of $\{1, \ldots, n\}$, we will identify H with the tensor product $H_{\pi(1)} \otimes \ldots H_{\pi(n)}$ without explicitly mentioning this. The symbol $\xi_{j_1,\ldots,j_k}$ will denote an element of $H_{j_1} \otimes \ldots H_{j_k}$.

We define a Hilbert space $HS(H_1, \ldots, H_n)$, isometrically isomorphic to H. Let $HS(H_1, H_2) = \mathcal{C}_2(H_1^{\mathrm{d}}, H_2)$. In the case where n is even, we let by induction

$$HS(H_1, \ldots, H_n) = \mathcal{C}_2(HS(H_2, H_3)^{\mathrm{d}}, HS(H_1, H_4, \ldots, H_n)),$$

and let

$$\theta_{H_1,\dots,H_n} : H \to HS(H_1,\dots,H_n)$$

be given by

$$\theta_{H_1,\dots,H_n}(\xi_{2,3}\otimes\xi) = \theta_{HS(H_2,H_3),HS(H_1,H_4,\dots,H_n)}(\theta_{H_2,H_3}(\xi_{2,3})\otimes\theta_{H_1,H_4,\dots,H_n}(\xi)),$$

where $\xi \in H_1 \otimes H_4 \otimes \cdots \otimes H_n$. In the case where n is odd, we let

$$HS(H_1,\dots,H_n) = HS(\mathbb{C},H_1,\dots,H_n).$$

If K is a Hilbert space, we will identify $\mathcal{C}_2(\mathbb{C}^{\mathrm{d}},K)$ with K via the map $S \to S(1^{\mathrm{d}})$. Thus, $HS(H_1,\dots,H_n)$ can, in the case of odd n, be defined inductively by letting $HS(H_1) = H_1$ and

$$HS(H_1,\dots,H_n) = \mathcal{C}_2(HS(H_1,H_2)^{\mathrm{d}},HS(H_3,\dots,H_n)).$$

The isomorphism $\theta_{H_1,\dots,H_n}$ is in this case given by

$$\theta_{H_1,\dots,H_n}(\xi) = \theta_{\mathbb{C},H_1,\dots,H_n}(1\otimes\xi).$$

We will usually omit the subscripts and write simply θ, when the corresponding Hilbert spaces are understood.

Lemma 4.1 *(i) Assume n is even. Let $\xi \in H$ be of the form $\xi = \xi_{1,2}\otimes\cdots\otimes\xi_{n-1,n}$. If $\eta_{i,i+1} \in H_i \otimes H_{i+1}$ (i even) then*

$$\theta(\xi)(\theta(\eta^{\mathrm{d}}_{2,3}))(\theta(\eta^{\mathrm{d}}_{4,5}))\dots(\theta(\eta^{\mathrm{d}}_{n-2,n-1})) = \theta(\xi_{n-1,n})\theta(\eta^{\mathrm{d}}_{n-2,n-1})\dots\theta(\eta^{\mathrm{d}}_{2,3})\theta(\xi_{1,2}).$$

(ii) Assume n is odd. Let $\xi \in H$ be of the form $\xi = \xi_1\otimes\xi_{2,3}\cdots\otimes\xi_{n-1,n}$. If $\eta_{i,i+1} \in H_i \otimes H_{i+1}$ (i odd) then

$$\theta(\xi)(\theta(\eta^{\mathrm{d}}_{1,2}))(\theta(\eta^{\mathrm{d}}_{3,4}))\dots(\theta(\eta^{\mathrm{d}}_{n-2,n-1})) = \theta(\xi_{n-1,n})\theta(\eta^{\mathrm{d}}_{n-2,n-1})\dots\theta(\eta^{\mathrm{d}}_{1,2})(\xi_1).$$

Proof. (i) Assume first that $\xi_{i-1,i} = \xi_{i-1}\otimes\xi_i$ and $\eta_{i,i+1} = \eta_i\otimes\eta_{i+1}$ (i even). Fix $\eta^{\mathrm{d}}_1 \in H^{\mathrm{d}}_1$. The image of η^{d}_1 under the operator on the right hand side of the identity in (i) is

$$(\xi_1,\eta_1)(\xi_2,\eta_2)\dots(\xi_{n-1},\eta_{n-1})\xi_n.$$

On the other hand, the image of η_1^{d} under the operator on the left hand side is

$$\begin{aligned}
& (\theta_{H_2,H_3}(\xi_2 \otimes \xi_3), \theta_{H_2,H_3}(\eta_2 \otimes \eta_3)) \\
\times\ & \theta_{H_1,H_4,\dots,H_n}(\xi_1 \otimes \xi_4 \otimes \cdots \otimes \xi_n)(\theta(\eta_{4,5})^{\mathrm{d}}) \dots (\theta(\eta_{n-2,n-1})^{\mathrm{d}})(\eta_1^{\mathrm{d}}) \\
=\ & (\xi_2, \eta_2)(\xi_3, \eta_3) \\
\times\ & \theta_{H_1,H_4,\dots,H_n}(\xi_1 \otimes \xi_4 \otimes \cdots \otimes \xi_n)(\theta(\eta_{4,5})^{\mathrm{d}}) \dots (\theta(\eta_{n-2,n-1})^{\mathrm{d}})(\eta_1^{\mathrm{d}}).
\end{aligned}$$

By induction, (i) holds in the case of elementary tensors.

By linearity, (i) holds for finite sums of elementary tensors. Assume that $\xi_{i-1,i}^{k_{i-1}} \to \xi_{i-1,i}$ and $\eta_{i,i+1}^{k_i} \to \eta_{i,i+1}$ (i even), where $\xi_{i-1,i}^{k_{i-1}}$ and $\eta_{i,i+1}^{k_i}$ are finite sums of elementary tensors. Since the operator norm is dominated by the Hilbert-Schmidt norm, we have that the mapping

$$(S_1, S_2, \dots, S_m) \to S_1 S_2 \dots S_m,$$

defined on the direct product of spaces of the form $\mathcal{C}_2(K_1, K_2)$, is continuous with respect to the product Hilbert-Schmidt norm - topology (on the left) and the operator norm topology (on the right). It follows that

$$\theta(\xi_{n-1,n}^{k_{n-1}})\theta(\eta_{n-2,n-1}^{k_{n-2},\mathrm{d}}) \dots \theta(\eta_{2,3}^{k_2,\mathrm{d}})\theta(\xi_{1,2}^{k_1})$$

converges in the operator norm to

$$\theta(\xi_{n-1,n})\theta(\eta_{n-2,n-1}^{\mathrm{d}}) \dots \theta(\eta_{2,3}^{\mathrm{d}})\theta(\xi_{1,2})$$

as $k_1, \dots, k_{n-1}$ tend to infinity.

On the other hand, since θ is an isometry, we have that

$$\theta(\xi_{1,2}^{k_1} \otimes \cdots \otimes \xi_{n-1,n}^{k_{n-1}}) \longrightarrow \theta(\xi_{1,2} \otimes \cdots \otimes \xi_{n-1,n})$$

in the Hilbert-Schmidt, and hence in the operator, norm as $k_1, k_3, \dots, k_{n-1}$ tend to infinity. Thus,

$$\theta(\xi_{1,2}^{k_1} \otimes \cdots \otimes \xi_{n-1,n}^{k_{n-1}})(\theta(\eta_{2,3}^{k_2,\mathrm{d}})) \longrightarrow \theta(\xi_{1,2} \otimes \cdots \otimes \xi_{n-1,n})(\theta(\eta_{2,3}^{\mathrm{d}}))$$

in the Hilbert-Schmidt, and hence in the operator, norm, as $k_1, k_2, k_3, k_5, \dots, k_{n-1}$ tend to infinity. Continuing inductively, we conclude that

$$\theta(\xi_{1,2}^{k_1} \otimes \cdots \otimes \xi_{n-1,n}^{k_{n-1}})(\theta(\eta_{2,3}^{k_2,\mathrm{d}})) \dots (\theta(\eta_{n-2,n-1}^{k_{n-2},\mathrm{d}}))$$

tends to
$$\theta(\xi_{1,2}\otimes\cdots\otimes\xi_{n-1,n})(\theta(\eta^{\mathrm{d}}_{2,3}))\dots(\theta(\eta^{\mathrm{d}}_{n-2,n-1}))$$
in the operator norm. The identity in (i) now follows.

(ii) By (i),
$$\theta(\xi)(\theta(\eta^{\mathrm{d}}_{1,2}))\dots(\theta(\eta^{\mathrm{d}}_{n-2,n-1}))=\theta(\xi_{n-1,n})\theta(\eta^{\mathrm{d}}_{n-2,n-1})\dots\theta(\eta^{\mathrm{d}}_{1,2})\theta(1\otimes\xi_1)$$
is a Hilbert-Schmidt operator from $\mathbb{C}^{\mathrm{d}}$ into H_n. Since $\theta(1\otimes\xi_1)(1^{\mathrm{d}})=\xi_1$, this operator can be identified with the vector
$$\theta(\xi_{n-1,n})\theta(\eta^{\mathrm{d}}_{n-2,n-1})\dots\theta(\eta^{\mathrm{d}}_{2,3})(\xi_1)\in H_n.$$
$\diamondsuit$

We define a representation $\sigma_{H_1,\dots,H_n}$ of $B(H)$ on $HS(H_1,\dots,H_n)$ by letting
$$\sigma_{H_1,\dots,H_n}(A)\theta(\xi)=\theta(A\xi);$$
clearly, $\sigma_{H_1,\dots,H_n}$ is unitarily equivalent to the identity representation of $B(H)$. If $H_1,\dots,H_n$ are clear from the context we will simply write σ in the place of $\sigma_{H_1,\dots,H_n}$. If $\mathcal{A}_1,\dots,\mathcal{A}_n$ are C*-algebras and $\pi_1,\dots,\pi_n$ corresponding representations on $H_1,\dots,H_n$, we let
$$\sigma_{\pi_1,\dots,\pi_n}=\sigma_{H_1,\dots,H_n}\circ(\pi_1\otimes\cdots\otimes\pi_n)\ ;$$
thus, $\sigma_{\pi_1,\dots,\pi_n}$ is a representation of $\mathcal{A}_1\otimes\cdots\otimes\mathcal{A}_n$ on $HS(H_1,\dots,H_n)$, unitarily equivalent to $\pi_1\otimes\cdots\otimes\pi_n$.

Lemma 4.2 *Let $A_i\in B(H_i)$, $i=1,\dots,n$, and $A=A_1\otimes\cdots\otimes A_n$.*

(i) Assume n is even. Let $\xi_{i-1,i}\in H_{i-1}\otimes H_i$, $\eta_{i,i+1}\in H_i\otimes H_{i+1}$ (i even). If $\xi=\xi_{1,2}\otimes\dots\xi_{n-1,n}$ then
$$\begin{aligned}&\sigma(A)(\theta(\xi))(\theta(\eta^{\mathrm{d}}_{2,3}))\dots(\theta(\eta^{\mathrm{d}}_{n-2,n-1}))\\=\ &A_n\theta(\xi_{n-1,n})A^{\mathrm{d}}_{n-1}\theta(\eta_{n-2,n-1})^{\mathrm{d}}A_{n-2}\dots A_2\theta(\xi_{1,2})A^{\mathrm{d}}_1\\=\ &A_n\theta(\xi)(\theta((A^*_2\otimes A^*_3(\eta_{2,3}))^{\mathrm{d}}))\dots(\theta((A^*_{n-2}\otimes A^*_{n-1}(\eta_{n-2,n-1}))^{\mathrm{d}}))A^{\mathrm{d}}_1.\end{aligned}$$

(ii) Assume n is odd. Let $\xi_1\in H_1$, $\xi_{i-1,i}\in H_{i-1}\otimes H_i$, $\eta_{i,i+1}\in H_i\otimes H_{i+1}$ (i odd). If $\xi=\xi_1\otimes\xi_{2,3}\otimes\dots\xi_{n-1,n}$ then
$$\begin{aligned}&\sigma(A)(\theta(\xi))(\theta(\eta^{\mathrm{d}}_{1,2}))\dots(\theta(\eta^{\mathrm{d}}_{n-2,n-1}))\\=\ &A_n\theta(\xi_{n-1,n})A^{\mathrm{d}}_{n-1}\theta(\eta_{n-2,n-1})^{\mathrm{d}}A_{n-2}\dots A^{\mathrm{d}}_2\theta(\eta^{\mathrm{d}}_{1,2})(A_1\xi_1)\\=\ &A_n\theta(\xi)(\theta((A^*_1\otimes A^*_2(\eta_{1,2}))^{\mathrm{d}}))\dots(\theta((A^*_{n-2}\otimes A^*_{n-1}(\eta_{n-2,n-1}))^{\mathrm{d}})).\end{aligned}$$

Proof. (i) Let first $n = 2$. If $\eta^{\mathrm{d}} \in H_1^{\mathrm{d}}$ and $\xi = \xi_1 \otimes \xi_2$ then

$$\begin{aligned}\sigma(A)(\theta(\xi))(\eta^{\mathrm{d}}) &= \theta(A_1\xi_1 \otimes A_2\xi_2)(\eta^{\mathrm{d}}) = (A_1\xi_1, \eta)A_2\xi_2 \\ &= (\xi_1, A_1^*\eta)A_2\xi_2 = A_2\theta(\xi_1 \otimes \xi_2)((A_1^*\eta)^{\mathrm{d}}) \\ &= A_2\theta(\xi_1 \otimes \xi_2)A_1^{\mathrm{d}}(\eta^{\mathrm{d}}) = A_2\theta(\xi)A_1^{\mathrm{d}}(\eta^{\mathrm{d}}).\end{aligned}$$

It follows by linearity and continuity that $\sigma(A)(\theta(\xi)) = A_2\theta(\xi)A_1^{\mathrm{d}}$, for every $\xi \in H_1 \otimes H_2$. Using Lemma 4.1 (i) we now obtain

$$\begin{aligned}&\sigma(A)(\theta(\xi))(\theta(\eta_{2,3})^{\mathrm{d}}) \dots (\theta(\eta^{\mathrm{d}}_{n-2,n-1})) \\ = \;& \theta((A_1 \otimes \dots A_n)(\xi))(\theta(\eta_{2,3})^{\mathrm{d}}) \dots (\theta(\eta^{\mathrm{d}}_{n-2,n-1})) \\ = \;& \theta(A_{n-1} \otimes A_n(\xi_{n-1,n}))\theta(\eta^{\mathrm{d}}_{n-2,n-1}) \dots \theta(\eta^{\mathrm{d}}_{2,3})\theta(A_1 \otimes A_2(\xi_{1,2})) \\ = \;& A_n\theta(\xi_{n-1,n})A^{\mathrm{d}}_{n-1}\theta(\eta_{n-2,n-1})^{\mathrm{d}}A_{n-2} \dots A_2\theta(\xi_{1,2})A_1^{\mathrm{d}} \\ = \;& A_n\theta(\xi)(\theta((A_2^* \otimes A_3^*(\eta_{2,3}))^{\mathrm{d}})) \dots (\theta((A^*_{n-2} \otimes A^*_{n-1}(\eta_{n-2,n-1}))^{\mathrm{d}}))A_1^{\mathrm{d}}.\end{aligned}$$

(ii) By Lemma 4.1 (ii),

$$\begin{aligned}&\sigma(A)(\theta(\xi))(\theta(\eta_{1,2})^{\mathrm{d}}) \dots (\theta(\eta_{n-2,n-1})^{\mathrm{d}}) \\ = \;& \theta((A_1 \otimes \dots A_n)(\xi))(\theta(\eta_{1,2})^{\mathrm{d}}) \dots (\theta(\eta_{n-2,n-1})^{\mathrm{d}}) \\ = \;& \theta(A_{n-1} \otimes A_n(\xi_{n-1,n}))\theta(\eta^{\mathrm{d}}_{n-2,n-1}) \dots \theta(\eta^{\mathrm{d}}_{1,2})(A_1\xi_1) \\ = \;& A_n\theta(\xi_{n-1,n})A^{\mathrm{d}}_{n-1}\theta(\eta_{n-2,n-1})^{\mathrm{d}}A_{n-2} \dots A_2^{\mathrm{d}}\theta(\eta^{\mathrm{d}}_{1,2})(A_1\xi_1) \\ = \;& A_n\theta(\xi)(\theta((A_1^* \otimes A_2^*(\eta_{1,2}))^{\mathrm{d}})) \dots (\theta((A^*_{n-2} \otimes A^*_{n-1}(\eta_{n-2,n-1}))^{\mathrm{d}})).\end{aligned}$$

$\diamondsuit$

Let $H_1, \dots, H_n$ be Hilbert spaces. If n is even, we let

$$\Gamma(H_1, \dots, H_n) = (H_1 \otimes H_2) \odot (H_2^{\mathrm{d}} \otimes H_3^{\mathrm{d}}) \odot (H_3 \otimes H_4) \odot \dots \odot (H_{n-1} \otimes H_n).$$

If n is odd, we let

$$\Gamma(H_1, \dots, H_n) = (H_1^{\mathrm{d}} \otimes H_2^{\mathrm{d}}) \odot (H_2 \otimes H_3) \odot (H_3^{\mathrm{d}} \otimes H_4^{\mathrm{d}}) \odot \dots \odot (H_{n-1} \otimes H_n).$$

After identifying $\mathbb{C} \otimes H_1$ with H_1, for n odd we have the identification

$$\Gamma(\mathbb{C}, H_1, \dots, H_n) \equiv H_1 \odot \Gamma(H_1, \dots, H_n).$$

Fix $\varphi \in B(H)$. We define a mapping S_φ on $\Gamma(H_1, \dots, H_n)$ taking values in $\mathcal{C}_2(H_1^{\mathrm{d}}, H_n)$ in the case n is even, and in $\mathcal{C}_2(H_1, H_n)$, in the case n is odd. Let first n be even. On elementary tensors

$$\zeta = \xi_{1,2} \otimes \eta^{\mathrm{d}}_{2,3} \otimes \xi_{3,4} \otimes \dots \otimes \xi_{n-1,n} \in \Gamma(H_1, \dots, H_n),$$

we let

$$S_\varphi(\zeta) = \sigma(\varphi)\theta(\xi_{1,2} \otimes \xi_{3,4} \otimes \cdots \otimes \xi_{n-1,n})(\theta(\eta^{\mathrm{d}}_{2,3})) \ldots (\theta(\eta^{\mathrm{d}}_{n-2,n-1}))$$

and extend S_φ on the whole of $\Gamma(H_1, \ldots, H_n)$ by linearity.

Now assume n is odd. Let $\zeta \in \Gamma(H_1, \ldots, H_n)$ and $\xi_1 \in H_1$. Then

$$\xi_1 \otimes \zeta \in H_1 \odot \Gamma(H_1, \ldots, H_n) = \Gamma(\mathbb{C}, H_1, \ldots, H_n).$$

We let $S_\varphi(\zeta)$ be the operator defined on H_1 by

$$S_\varphi(\zeta)(\xi_1) = S_{1\otimes\varphi}(\xi_1 \otimes \zeta).$$

Note that $S_{1\otimes\varphi}(\xi_1 \otimes \zeta)$ is an element of $\mathcal{C}_2(\mathbb{C}^d, H_n)$, which can be identified with H_n in a natural way. In this way, $S_\varphi(\zeta)(\xi_1)$ can be viewed as an element of H_n. We want to show that the operator $S_\varphi(\zeta) : H_1 \to H_n$ belongs to $\mathcal{C}_2(H_1, H_n)$. Clearly, it suffices to show this in the case ζ is an elementary tensor, say

$$\zeta = \eta^{\mathrm{d}}_{1,2} \otimes \xi_{2,3} \otimes \cdots \otimes \xi_{n-1,n}.$$

Fix an orthonormal basis $\{\xi_1^j\}_j$ of H_1. We have

$$\begin{aligned}
& \sum_j \|S_\varphi(\zeta)(\xi_1^j)\|^2 = \sum_j \|S_{1\otimes\varphi}(\xi_1^j \otimes \zeta)\|^2 \\
= & \sum_j \|\sigma(1 \otimes \varphi)\theta((1 \otimes \xi_1^j) \otimes \xi_{2,3} \otimes \cdots \otimes \xi_{n-1,n})(\theta(\eta^{\mathrm{d}}_{1,2})) \ldots (\theta(\eta^{\mathrm{d}}_{n-2,n-1}))\|^2 \\
\leq & \sum_j \|\sigma(1 \otimes \varphi)\theta((1 \otimes \xi_1^j) \otimes \xi_{2,3} \otimes \cdots \otimes \xi_{n-1,n})(\theta(\eta^{\mathrm{d}}_{1,2})) \ldots (\theta(\eta^{\mathrm{d}}_{n-4,n-3}))\|^2_{\mathrm{op}} \\
\times & \|\eta_{n-2,n-1}\|^2 \\
\leq & \sum_j \|\sigma(1 \otimes \varphi)\theta((1 \otimes \xi_1^j) \otimes \xi_{2,3} \otimes \cdots \otimes \xi_{n-1,n})(\theta(\eta^{\mathrm{d}}_{1,2})) \ldots (\theta(\eta^{\mathrm{d}}_{n-4,n-3}))\|^2_2 \\
\times & \|\eta_{n-2,n-1}\|^2 \\
\leq & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}$$

$$\begin{aligned}
\leq & \ \|1 \otimes \varphi\|^2 \|\eta_{1,2}\|^2 \ldots \|\eta_{n-2,n-1}\|^2 \sum_j \|(1 \otimes \xi_1^j) \otimes \xi_{2,3} \otimes \cdots \otimes \xi_{n-1,n}\|^2 \\
= & \ \|1 \otimes \varphi\|^2 \|\eta_{1,2}\|^2 \ldots \|\eta_{n-2,n-1}\|^2 \|\xi_{2,3} \otimes \cdots \otimes \xi_{n-1,n}\|^2 \\
= & \ \|\varphi\|^2 \|\eta_{1,2}\|^2 \ldots \|\eta_{n-2,n-1}\|^2 \|\xi_{2,3}\|^2 \ldots \|\xi_{n-1,n}\|^2,
\end{aligned}$$

hence

$$\|S_\varphi(\zeta)\|_{\mathcal{C}_2(H_1,H_n)} \le \|\varphi\|_{\mathcal{B}(H)}\|\eta_{1,2}\|^2 \dots \|\eta_{n-2,n-1}\|^2\|\xi_{2,3}\|^2 \dots \|\xi_{n-1,n}\|^2. \quad (12)$$

Before proceeding, we identify two norms with which the space $\Gamma(H_1, \dots, H_n)$ can be equipped. The first norm on $\Gamma(H_1,\dots,H_n)$ is the projective tensor norm $\|\cdot\|_{2,\wedge}$, where each of the terms $H_i \otimes H_{i+1}$ (resp. $H_{i-1}^{\rm d} \otimes H_i^{\rm d}$) is given its Hilbert space norm. In order to describe the second norm, note that if K_1 and K_2 are Hilbert spaces then $K_1 \otimes K_2$ can be endowed with an operator space structure by letting

$$\|(\xi_{ij})\| = \|\theta(\xi_{ji})\|_{M_m(\mathcal{B}(K_1^{\rm d},K_2))}, \quad (\xi_{ij}) \in M_m(K_1 \otimes K_2).$$

We write $(K_1 \otimes K_2)^o_{\rm op}$ for this operator space. Note that this is the opposite operator space structure on $\mathcal{C}_2(K_1^{\rm d}, K_2)$, after the identification of $K_1 \otimes K_2$ and $\mathcal{C}_2(K_1^{\rm d}, K_2)$. The norm $\|\cdot\|_{\rm h}$ is the Haagerup norm on $\Gamma(H_1,\dots,H_n)$ when $\Gamma(H_1,\dots,H_n)$ is viewed as the algebraic tensor product of the operator spaces $(H_i \otimes H_{i+1})^o_{\rm op}$ (resp. $(H_{i-1}^{\rm d} \otimes H_i^{\rm d})^o_{\rm op}$). Thus, the norm $\|u\|_{\rm h}$ of a finite sum $u = \sum_i \xi^i_{1,2} \otimes \dots \otimes \xi^i_{n-1,n} \in \Gamma(H_1,\dots,H_n)$ of elementary tensors equals the Haagerup norm of the element $\sum_i \theta(\xi^i_{n-1,n}) \otimes \dots \otimes \theta(\xi^i_{1,2})$.

Remark 4.3 *For each $\varphi \in B(H)$ and each $\zeta \in \Gamma(H_1,\dots,H_n)$, we have*

$$\|S_\varphi(\zeta)\|_2 \le \|\varphi\|_{B(H)}\|\zeta\|_{2,\wedge}.$$

Proof. In the case where n is odd and ζ is an elementary tensor, the inequality coincides with (12). In the case n is even and ζ is an elementary tensor, this is verified similarly. The general case now follows by linearity. $\diamondsuit$

Definition 4.4 *An element $\varphi \in B(H_1 \otimes \dots \otimes H_n)$ is called a concrete (operator) multiplier if there exists $C > 0$ such that*

$$\|S_\varphi(\zeta)\|_{\rm op} \le C\|\zeta\|_{\rm h}, \quad \textit{for each } \zeta \in \Gamma(H_1,\dots,H_n).$$

The smallest such C is denoted by $\|\varphi\|_{\rm m}$.

Let $\mathcal{A}_1,\dots,\mathcal{A}_n$ be C^*-algebras and $\pi_1,\dots,\pi_n$ be corresponding representations on Hilbert spaces $H_1,\dots,H_n$. An element $\varphi \in \mathcal{A}_1 \otimes \dots \otimes \mathcal{A}_n$ is called a $\pi_1,\dots,\pi_n$-multiplier if $(\pi_1 \otimes \dots \otimes \pi_n)(\varphi)$ is a concrete multiplier. We denote the set of all $\pi_1,\dots,\pi_n$-multipliers in $\mathcal{A}_1 \otimes \dots \otimes \mathcal{A}_n$ by $\mathbf{M}_{\pi_1,\dots,\pi_n}(\mathcal{A}_1,\dots \mathcal{A}_n)$. If $\varphi \in \mathbf{M}_{\pi_1,\dots,\pi_n}(\mathcal{A}_1,\dots \mathcal{A}_n)$, we let $\|\varphi\|_{\pi_1,\dots,\pi_n} = \|(\pi_1 \otimes \dots \otimes \pi_n)(\varphi)\|_{\rm m}$.

The element $\varphi \in \mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_n$ is called a universal multiplier if φ is a $\pi_1, \ldots, \pi_n$-multiplier for all representations π_i of $\mathcal{A}_i$, $i = 1, \ldots, n$. We denote by $\mathbf{M}(\mathcal{A}_1, \ldots \mathcal{A}_n)$ the set of all universal multipliers in $\mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_n$.

Remark In the case $n = 2$, Definition 4.4 reduces to the definition of $\mathcal{C}_\infty$-multipliers studied in [18].

Next we show that an element $\varphi \in L^\infty(X_1) \otimes \ldots \otimes L^\infty(X_n) \subset L^\infty(X_1 \times \ldots \times X_n)$ is a Schur multiplier as defined in Section 3 if and only if φ is a $\pi_1, \ldots, \pi_n$-multiplier, where π_i is the canonical representation of $L^\infty(X_i)$ on $L^2(X_i)$ acting by multiplication.

Let $\mathcal{A}$ be a commutative C^*-algebra with maximal ideal space X, acting on a Hilbert space H. It is well-known that, up to unitary equivalence, $H = \oplus_{\gamma\in\Gamma} H_\gamma$, where $H_\gamma = L_2(X, \mu_\gamma)$ is invariant under $\mathcal{A}$ for each $\gamma \in \Gamma$, and an element $f \in \mathcal{A}$ acts as on H_γ by multiplication. Let $j : H \to H$ be given by $\{\xi_\gamma(\lambda)\} \mapsto \{\overline{\xi_\gamma(\lambda)}\}$. Then $V = \partial j$ is a unitary operator from H to H^{d} such that $A^{\mathrm{d}} = VAV^{-1}$ for all $A \in \mathcal{A}$. If K is another Hilbert space then $U(T) = TV$ (resp. $W(S) = V^{-1}S$) is an isometry from $\mathcal{C}_2(H^{\mathrm{d}}, K)$ to $\mathcal{C}_2(H, K)$ (resp. from $\mathcal{C}_2(K, H^{\mathrm{d}})$ to $\mathcal{C}_2(K, H)$).

Let $\mathcal{A}_1, \ldots, \mathcal{A}_n$ be commutative C^*-algebras and let $\pi_1, \ldots, \pi_n$ be corresponding representations on $H_1, \ldots, H_n$. Let $V_i : H_i \to H_i^{\mathrm{d}}$ be unitary operator defined above with the property $\pi_i(a_i)^{\mathrm{d}} = V_i \pi_i(a_i) V_i^{-1}$ for each $a_i \in \mathcal{A}_i$, $i = 1, \ldots, n$. Define $U_{i,k} : \mathcal{C}_2(H_i^{\mathrm{d}}, H_k) \to \mathcal{C}_2(H_i, H_k)$ and $W_{i,k} : \mathcal{C}_2(H_i, H_k^{\mathrm{d}}) \to \mathcal{C}_2(H_i, H_k)$ to be $U_{i,k}(T) = TV_i$ and $W_{i,k}(S) = V_k^{-1}S$. Then for $\varphi = a_1 \otimes \ldots \otimes a_n$ the mapping $S_{(\pi_1\otimes\ldots\otimes\pi_n)(\varphi)}$ can be identified with a mapping $\check{S}_{(\pi_1\otimes\ldots\otimes\pi_n)(\varphi)}$ from $\mathcal{C}_2(H_1, H_2) \odot \mathcal{C}_2(H_2, H_3) \odot \ldots \odot \mathcal{C}_2(H_{n-1}, H_n)$ to $\mathcal{C}_2(H_1, H_n)$ such that

$$\check{S}_{(\pi_1\otimes\ldots\otimes\pi_n)(\varphi)}(R_1 \otimes \ldots \otimes R_{n-1}) = \pi_n(a_n) R_{n-1} \pi_{n-1}(a_{n-1}) R_{n-2} \ldots R_1 \pi_1(a_1)$$

In fact, let $\mathcal{U} = U_{1,2}\theta_{H_1,H_2} \otimes W_{2,3}\theta_{H_2,H_3} \otimes \ldots \otimes U_{n-1,n}\theta_{H_{n-1},H_n}$ if n is even and $\mathcal{U} = W_{1,2}\theta_{H_1,H_2} \otimes U_{2,3}\theta_{H_2,H_3} \otimes \ldots \otimes U_{n-1,n}\theta_{H_{n-1},H_n}$ if n is odd, which maps the space $\Gamma(H_1, H_2 \ldots, H_n)$ to $\mathcal{C}_2(H_1, H_2) \odot \mathcal{C}_2(H_2, H_3) \odot \ldots \odot \mathcal{C}_2(H_{n-1}, H_n)$. Then, in the case where n is even, we have

$$\begin{aligned}
&U_{1,n} S_{\pi_1\otimes\ldots\otimes\pi_n(\varphi)} \mathcal{U}^{-1}(R_1 \otimes \ldots \otimes R_{n-1}) \\
&= U_{1,n}(\pi_n(a_n) U_{n-1,n}^{-1}(R_{n-1}) \pi_{n-1}(a_{n-1})^{\mathrm{d}} W_{n-2,n-1}(R_{n-2}) \ldots \pi_1(a_1)^{\mathrm{d}}) \\
&= \pi_n(a_n) R_{n-1} V_{n-1}^{-1} \pi_{n-1}(a_{n-1})^{\mathrm{d}} V_{n-1} R_{n-2} \ldots R_1 V_1^{-1} \pi_1(a_1)^{\mathrm{d}} V_1 \qquad (13) \\
&= \pi_n(a_n) R_{n-1} \pi_{n-1}(a_{n-1}) R_{n-2} \ldots R_1 \pi_1(a_1) \\
&= \check{S}_{(\pi_1\otimes\ldots\otimes\pi_n)(\varphi)}(R_1 \otimes \ldots \otimes R_{n-1})
\end{aligned}$$

In the case where n is odd one obtains in a similar way that $S_{\pi_1\otimes\ldots\otimes\pi_n(\varphi)}\mathcal{U}^{-1} = \check{S}_{(\pi_1\otimes\ldots\otimes\pi_n)(\varphi)}$.

Let now $\mathcal{A}_i = L^\infty(X_i)$ and let π_i be the representation of $\mathcal{A}_i$ on $L^2(X_i)$ given by $(\pi_i(f)\xi)(x) = f(x)\xi(x)$, $\xi \in L^2(X_i)$, $i = 1,\ldots,n$.

Using (13) and the identification $\psi_{k,l} : f \mapsto T_f$ of $L_2(X_k, X_l)$ with the class of Hilbert-Schmidt operators from $L_2(X_k)$ to $L_2(X_l)$, where

$$(T_f\xi)(y) = \int_{X_k} f(x,y)\xi(x)dx, \quad f \in L_2(X_k \times X_l), \xi \in L^2(X_k), y \in X_l,$$

we obtain for $f_1 \otimes \ldots \otimes f_{n-1} \in \Gamma(X_1,\ldots,X_n)$ and even n

$$\begin{aligned}&\psi_{1,n}^{-1}(\check{S}_{\pi_1\otimes\ldots\otimes\pi_n(\varphi)}(\psi_{1,2}\otimes\ldots\otimes\psi_{n-1,n})(f_1\otimes\ldots\otimes f_{n-1}))(x_1,x_n) \\ &= \int_{X_2\times\ldots\times X_{n-1}} \varphi(x_1,\ldots,x_n)f_1(x_1,x_2)\ldots f_{n-1}(x_{n-1},x_n)dx_2\ldots dx_{n-1} \\ &= S_\varphi(f_1\otimes\ldots\otimes f_{n-1})(x_1,x_n),\end{aligned} \tag{14}$$

Similarly, if n is odd we get

$$\begin{aligned}&\psi_{1,n}^{-1}\check{S}_{\pi_1\otimes\ldots\otimes\pi_n(\varphi)}(\psi_{1,2}\otimes\ldots\otimes\psi_{n-1,n})(f_1\otimes\ldots\otimes f_{n-1}))(x_1,x_n) \\ &= S_\varphi(f_1\otimes\ldots\otimes f_{n-1})(x_1,x_n)\end{aligned} \tag{15}$$

By linearity and continuity we have that (14) and (15) hold for any $\varphi \in L^\infty(X_1)\otimes\ldots\otimes L^\infty(X_n)$ and any $f \in \Gamma(X_1,\ldots,X_n)$. This implies the following

Proposition 4.5 *Let $\varphi \in L^\infty(X_1)\otimes\ldots\otimes L^\infty(X_n)$. Then φ is a Schur multiplier if and only if $\varphi \in \mathbf{M}_{\pi_1,\ldots,\pi_n}(L^\infty(X_1),\ldots,L^\infty(X_n))$.*

Next we want to give a generalisation of Lemma 4.2 for the case where φ is a sum of elementary tensors. Let $V, V_1,\ldots,V_n$ be vector spaces, $L(V_1,V_2)$ be the space of all linear mappings from V_1 into V_2 and $L(V) = L(V,V)$. Recall that if $f : V_1 \to V_2$ is a linear map, we let $f_{k,l} : M_{k,l}(V_1) \to M_{k,l}(V_2)$ be the mapping given by $f_{k,l}((v_{ij})) = (f(v_{ij}))$, for each $(v_{ij}) \in M_{k,l}(V_1)$. For an element $v = (v_{ij}) \in M_{k,l}(V)$ we denote by $v^{\mathrm{t}} = (v_{ji}) \in M_{l,k}(V)$ the transpose of v. Denote by $d : B(K) \to B(K^{\mathrm{d}})$ the mapping sending A to its dual A^{d}. If $A \in M_{k,l}(B(K))$ let $A^{\mathrm{d}} = d_{k,l}(A)$.

We will identify $M_{p,q}(\mathcal{C}_2(K_1,K_2))$ with $\mathcal{C}_2(K_1^q,K_2^p)$. If $\xi \in M_{p,q}(K_1 \otimes K_2)$ then $\theta_{p,q}(\xi) \in M_{p,q}(\mathcal{C}_2(K_1^{\mathrm{d}},K_2))$; using this identification, we will be considering $\theta_{p,q}(\xi)$ as a Hilbert-Schmidt operator from K_1^q to K_2^p. If $A \in B(K_1, K_2)$ then $A \otimes I_k \in B(K_1^k, K_2^k)$ is the k-fold ampliation of A; under the identification $B(K_1^k, K_2^k) = M_k\ (B(K_1,\, K_2))$, the operator $A \otimes I_k$ has a k by k diagonal matrix, whose every diagonal entry is A.

Lemma 4.6 *Let $V_1, \dots, V_n$ be vector spaces, $\mathcal{L}_i \subseteq L(V_i, V_{i+1})$ a subspace, $i = 1, \dots, n-1$, and*

$$S : (L(V_n) \odot L(V_{n-1}) \odot \cdots \odot L(V_1)) \times (\mathcal{L}_{n-1} \odot \cdots \odot \mathcal{L}_1) \to L(V_1, V_n)$$

be a mapping satisfying

$$S(a_n \otimes \cdots \otimes a_1, \lambda_{n-1} \otimes \cdots \otimes \lambda_1) = a_n \lambda_{n-1} a_{n-1} \dots \lambda_1 a_1.$$

Assume that $A_1 \in M_{k_1,1}(L(V_1))$, $A_2 \in M_{k_2,k_1}(L(V_2)), \dots, A_n \in M_{1,k_{n-1}}(L(V_n))$, and that $\Lambda_1 \in M_{l_1,1}(\mathcal{L}_1)$, $\Lambda_2 \in M_{l_2,l_1}(\mathcal{L}_2), \dots,\ \Lambda_{n-1} \in M_{1,l_{n-2}}(\mathcal{L}_{n-1})$. Then

$$S(A_n \odot \cdots \odot A_1, \Lambda_{n-1} \odot \cdots \odot \Lambda_1) = A_n \dots (\Lambda_2 \otimes I_{k_2})(A_2 \otimes I_{l_1})(\Lambda_1 \otimes I_{k_1})A_1.$$

Proof. “A few moments’ thought.” $\diamondsuit$

Lemma 4.7 *Let $A_1 \in M_{1,k_1}(\mathcal{B}(H_1))$, $A_2 \in M_{k_1,k_2}(\mathcal{B}(H_2)), \dots,\ A_n \in M_{k_{n-1},1}$ $(\mathcal{B}(H_n))$ and $\varphi = A_1 \odot A_2 \odot \cdots \odot A_n$.*

(i) Assume n is even. Let $\xi_{1,2} \in M_{1,l_1}(H_1 \otimes H_2)$, $\eta_{2,3} \in M_{l_1,l_2}(H_2^{\mathrm{d}} \otimes H_3^{\mathrm{d}}), \dots,\ \xi_{n-1,n} \in M_{l_{n-2},1}(H_{n-1} \otimes H_n)$ and

$$\zeta = \xi_{1,2} \odot \eta_{2,3} \odot \cdots \odot \xi_{n-1,n} \in \Gamma(H_1, \dots, H_n).$$

Then

$$S_\varphi(\zeta) = A_n^{\mathrm{t}} \dots (A_3^{\mathrm{t,d}} \otimes I_{l_2})(\theta_{l_1,l_2}(\eta_{2,3})^{\mathrm{t}} \otimes I_{k_2})(A_2^{\mathrm{t,}} \otimes I_{l_1})(\theta_{1,l_1}(\xi_{1,2})^{\mathrm{t}} \otimes I_{k_1})A_1^{\mathrm{t,d}}.$$

(ii) Assume n is odd. Let $\eta_{1,2} \in M_{1,l_1}(H_1^{\mathrm{d}} \otimes H_2^{\mathrm{d}})$, $\xi_{2,3} \in M_{l_1,l_2}(H_2 \otimes H_3), \dots,\ \xi_{n-1,n} \in M_{l_{n-2},1}(H_{n-1} \otimes H_n)$ and

$$\zeta = \eta_{1,2} \odot \xi_{2,3} \odot \cdots \odot \xi_{n-1,n} \in \Gamma(H_1, \dots, H_n).$$

Then

$$S_\varphi(\zeta) = A_n^{\mathrm{t}} \dots (A_3^{\mathrm{t}} \otimes I_{l_2})(\theta_{l_1,l_2}(\xi_{2,3})^{\mathrm{t}} \otimes I_{k_2})(A_2^{\mathrm{t,d}} \otimes I_{l_1})(\theta_{1,l_1}(\eta_{1,2})^{\mathrm{t}} \otimes I_{k_1})A_1^{\mathrm{t}}.$$

Proof. Let $f : V_1 \odot \cdots \odot V_n \to V_n \odot \cdots \odot V_1$ be the flip, namely the map given on elementary tensors by $f(v_1 \otimes \cdots \otimes v_n) = v_n \otimes \cdots \otimes v_1$. Note that if $A_1 \in M_{1,k_1}(V_1)$, $A_2 \in M_{k_1,k_2}(V_2), \ldots,$ $A_n \in M_{k_{n-1},1}(V_n)$ then

$$f(A_1 \odot \cdots \odot A_n) = A_n^{\mathrm{t}} \odot \cdots \odot A_1^{\mathrm{t}}.$$

Let

$$D : B(H_1) \odot B(H_2) \odot \cdots \odot B(H_n) \longrightarrow B(H_n) \odot B(H_{n-1}^{\mathrm{d}}) \odot \cdots \odot B(H_1^{\mathrm{d}})$$

be the map

$$D = f \circ (d \otimes \mathrm{id} \otimes d \otimes \cdots \otimes \mathrm{id}).$$

We have that

$$D(A) = A_n^{\mathrm{t}} \odot A_{n-1}^{\mathrm{t,d}} \odot \cdots \odot A_1^{\mathrm{t,d}}.$$

Define a mapping S from

$$(B(H_n) \odot B(H_{n-1}^{\mathrm{d}}) \odot \cdots \odot B(H_1^{\mathrm{d}})) \times (\mathcal{C}_2(H_{n-1}^{\mathrm{d}}, H_n) \odot \cdots \odot \mathcal{C}_2(H_1^{\mathrm{d}}, H_2))$$

into $\mathcal{C}_2(H_1^{\mathrm{d}}, H_n)$ by

$$S(\psi, \zeta') = S_{D^{-1}(\psi)}(\tilde{\theta}^{-1}(\zeta')),$$

where

$$\tilde{\theta} : \Gamma(H_1, \ldots, H_n) \to \mathcal{C}_2(H_{n-1}^{\mathrm{d}}, H_n) \odot \cdots \odot \mathcal{C}_2(H_1^{\mathrm{d}}, H_2)$$

is given on elementary tensors by

$$\tilde{\theta}(\xi_{1,2} \otimes \eta_{2,3} \otimes \cdots \otimes \xi_{n-1,n}) = \theta(\xi_{n-1,n}) \otimes \cdots \otimes \theta(\eta_{2,3}) \otimes \theta(\xi_{1,2}).$$

By Lemma 4.2 (i), the mapping S satisfies the requirements of Lemma 4.6 and

$$S_\varphi(\zeta) = S(A_n^{\mathrm{t}} \odot A_{n-1}^{\mathrm{t,d}} \odot \cdots \odot A_1^{\mathrm{t,d}}, \theta_{l_{n-2},1}(\xi_{n-1,n})^{\mathrm{t}} \odot \cdots \odot \theta_{1,l_1}(\xi_{1,2})^{\mathrm{t}}).$$

The claim now follows from Lemma 4.6.

The proof of (ii) is similar. $\diamondsuit$

5 Multipliers for tensor products of representations

It was proved in [18] that the space of all (π,ρ)-multipliers does not change if the representations π and ρ are replaced by approximately equivalent representations. In this section we will prove a corresponding result for multidimensional multipliers. We first recall the notion of approximate equivalence and approximate suborditation introduced by Voiculescu in [29].

Let π and π' be $*$-representations of a C^*-algebra $\mathcal{A}$ on Hilbert spaces H and H', respectively. We say that π' is *approximately subordinate* to π and write $\pi' \overset{a}{\ll} \pi$ if there is a net $\{U_\lambda\}$ of isometries from H' to H such that

$$\|\pi(a)U_\lambda - U_\lambda\pi'(a)\| \to 0 \text{ for all } a \in \mathcal{A}. \tag{16}$$

The representations π' and π are said to be *approximately equivalent* if the operators U_λ can be chosen to be unitary; in this case we write $\pi' \overset{a}{\sim} \pi$.

For C*-algebras $\mathcal{A}_1,\dots,\mathcal{A}_n$ and corresponding representations $\pi_1,\dots,\pi_n$, we will denote the collection of all $\pi_1,\dots,\pi_n$-multipliers in $\mathcal{A}_1\otimes\dots\otimes\mathcal{A}_n$ simply by $\mathbf{M}_{\pi_1,\dots,\pi_n}$, in case there is no danger of confusion.

Theorem 5.1 *Let $\mathcal{A}_1,\dots,\mathcal{A}_n$ be C^*-algebras and π_i and π_i' be representations of $\mathcal{A}_i$ on Hilbert spaces H_i and H_i', respectively, $i=1,\dots,n$.*

(i) If $\pi_i' \overset{a}{\ll} \pi_i$, $i=1,\dots,n$, then

$$\mathbf{M}_{\pi_1,\dots,\pi_n} \subseteq \mathbf{M}_{\pi_1',\dots,\pi_n'} \text{ and } \|\varphi\|_{\pi_1',\dots,\pi_n'} \le \|\varphi\|_{\pi_1,\dots,\pi_n}, \text{ for } \varphi\in\mathbf{M}_{\pi_1,\dots,\pi_n}.$$

(ii) If $\pi_i' \overset{a}{\sim} \pi_i$, $i=1,\dots,n$, then

$$\mathbf{M}_{\pi_1,\dots,\pi_n} = \mathbf{M}_{\pi_1',\dots,\pi_n'} \text{ and } \|\varphi\|_{\pi_1,\dots,\pi_n} = \|\varphi\|_{\pi_1',\dots,\pi_n'}, \text{ for } \varphi\in\mathbf{M}_{\pi_1,\dots,\pi_n}.$$

Proof. (i) Let first n be even and $\{U_{\lambda_i}\}_{\lambda_i}$ be nets of isometries from H_i' into H_i satisfying

$$\|\pi_i(a_i)U_{\lambda_i} - U_{\lambda_i}\pi_i'(a_i)\| \to 0, \text{ for all } a_i\in\mathcal{A}_i.$$

Set $\pi = \otimes_{i=1}^n\pi_i$, $\pi' = \otimes_{i=1}^n\pi_i'$ and $W_{\lambda_1,\dots,\lambda_n} = U_{\lambda_1}\otimes\ldots\otimes U_{\lambda_n}$. Then $W_{\lambda_1,\dots,\lambda_n}$ are isometries from $\otimes_{i=1}^n H_i'$ to $\otimes_{i=1}^n H_n$ and, for $x\in\mathcal{A}_1\odot\ldots\odot\mathcal{A}_n$, we have

$$\|\pi(x)W_{\lambda_1,\dots,\lambda_n} - W_{\lambda_1,\dots,\lambda_n}\pi'(x)\| \longrightarrow_{(\lambda_1,\dots,\lambda_n)} 0.$$

As $\|W_{\lambda_1,\dots,\lambda_n}\| = 1$ for all $\lambda_1,\dots\lambda_n$, this holds for all $x \in \mathcal{A}_1 \otimes \dots \otimes \mathcal{A}_n$. By Lemma 4.2 (i) we have that, for any $\xi \in \otimes_{i=1}^n H_i$,

$$\begin{aligned} & \theta(W^*_{\lambda_1,\dots,\lambda_n}\xi)(\theta(\eta^{\mathrm{d}}_{2,3}))\dots(\theta(\eta^{\mathrm{d}}_{n-2,n-1})) \\ = \; & U^*_{\lambda_n}\theta(\xi)(\theta((W_{\lambda_2,\lambda_3}\eta_{2,3})^{\mathrm{d}}))\dots(\theta((W_{\lambda_{n-2},\lambda_{n-1}}\eta_{n-2,n-1})^{\mathrm{d}}))(U^*_{\lambda_1})^{\mathrm{d}} \end{aligned}$$

where $W_{\lambda_k,\lambda_{k+1}} = U_{\lambda_k} \otimes U_{\lambda_{k+1}}$. Therefore, if $\zeta = \xi_{1,2} \otimes (\eta_{2,3})^{\mathrm{d}} \otimes \dots \otimes \xi_{n-1,n}$, then

$$\begin{aligned} & S_{W^*_{\lambda_1,\dots,\lambda_n}\pi(\varphi)W_{\lambda_1,\dots,\lambda_n}}(\zeta) = \\ & = U^*_{\lambda_n} S_{\pi(\varphi)}(W_{\lambda_1,\lambda_2}\xi_{1,2} \otimes (W_{\lambda_2,\lambda_3}\eta_{2,3})^{\mathrm{d}} \otimes \dots \otimes W_{\lambda_{n-1},\lambda_n}\xi_{n-1,n})(U^*_{\lambda_1})^{\mathrm{d}}. \end{aligned} \tag{17}$$

Let $\Gamma_{\lambda_1,\dots,\lambda_n} : \Gamma(H'_1,\dots,H'_n) \to \Gamma(H_1,\dots,H_n)$ be the linear operator defined on elementary tensors by

$$\Gamma_{\lambda_1,\dots,\lambda_n}(\xi_{1,2}\otimes\eta^{\mathrm{d}}_{2,3}\otimes\dots\otimes\xi_{n-1,n}) = W_{\lambda_1,\lambda_2}\xi_{1,2}\otimes(W_{\lambda_2,\lambda_3}\eta_{2,3})^{\mathrm{d}}\otimes\dots\otimes W_{\lambda_{n-1},\lambda_n}\xi_{n-1,n}.$$

It follows from (17) and Remark 4.3 that if $\varphi \in \mathbf{M}_{\pi_1,\dots,\pi_n}$ and $\zeta \in \Gamma(H'_1,\dots,H'_n)$ then

$$\begin{aligned} \|S_{\pi'(\varphi)}(\zeta)\|_{\mathrm{op}} & \leq \|S_{W^*_{\lambda_1,\dots,\lambda_n}\pi(\varphi)W_{\lambda_1,\dots,\lambda_n}}(\zeta)\|_{\mathrm{op}} \\ & + \|S_{W^*_{\lambda_1,\dots,\lambda_n}\pi(\varphi)W_{\lambda_1,\dots,\lambda_n}-\pi'(\varphi)}(\zeta)\|_{\mathrm{op}} \\ & \leq \|S_{\pi(\varphi)}(\Gamma_{\lambda_1,\dots,\lambda_n}\zeta)\|_{\mathrm{op}} + \|S_{W^*_{\lambda_1,\dots,\lambda_n}\pi(\varphi)W_{\lambda_1,\dots,\lambda_n}-\pi'(\varphi)}(\zeta)\|_2 \\ & \leq \|\varphi\|_{\pi_1,\dots,\pi_n}\|\Gamma_{\lambda_1,\dots,\lambda_n}\zeta\|_{\mathrm{h}} \\ & + \|W^*_{\lambda_1,\dots,\lambda_n}\pi(\varphi)W_{\lambda_1,\dots,\lambda_n} - \pi'(\varphi)\|_{\mathrm{op}}\|\zeta\|_{2,\wedge}. \end{aligned}$$

Since $\|W^*_{\lambda_1,\dots,\lambda_n}\pi(\varphi)W_{\lambda_1,\dots,\lambda_n} - \pi'(\varphi)\|_{\mathrm{op}} \to 0$, in order to prove that $\varphi \in \mathbf{M}_{\pi'_1,\dots,\pi'_n}$, it suffices to show that $\|\Gamma_{\lambda_1,\dots,\lambda_n}\zeta\|_{\mathrm{h}} \leq \|\zeta\|_{\mathrm{h}}$. If $\xi_{i,i+1} \in H'_i \otimes H'_{i+1}$ then $\theta(W_{\lambda_i,\lambda_{i+1}}\xi_{i,i+1}) = U_{\lambda_{i+1}}\theta(\xi_{i,i+1})U^{\mathrm{d}}_{\lambda_i}$. Let $\zeta \in \Gamma(H'_1,\dots,H'_n)$ be of the form

$$\zeta = \xi_{1,2} \otimes \eta^{\mathrm{d}}_{2,3} \otimes \dots \otimes \xi_{n-1,n}$$

where $\xi_{1,2} \in M_{1,k_2}(H'_1 \otimes H'_2)$, $\eta^{\mathrm{d}}_{2,3} \in M_{k_2,k_3}((H'_2)^{\mathrm{d}} \otimes (H'_3)^{\mathrm{d}})$, $\dots$, and $\xi_{n-1,n} \in M_{k_{n-1},1}(H'_{n-1} \otimes H'_n)$ are such that

$$\|\zeta\|_{\mathrm{h}} = \|\theta_{1,k_2}(\xi_{1,2})^{\mathrm{t}}\|_{\mathrm{op}}\|\theta_{k_2,k_3}(\eta^{\mathrm{d}}_{2,3})^{\mathrm{t}}\|_{\mathrm{op}}\dots\|\theta_{k_{n-1},1}(\xi_{n-1,n})^{\mathrm{t}}\|_{\mathrm{op}}.$$

Then

$$\Gamma_{\lambda_1,\dots,\lambda_n}\zeta = W_{\lambda_1,\lambda_2}\xi_{1,2} \odot (W^{*,\mathrm{d}}_{\lambda_2,\lambda_3} \otimes I_{k_2})\eta^{\mathrm{d}}_{2,3} \odot \dots \odot (W_{\lambda_{n-1},\lambda_n} \otimes I_{k_{n-1}})\xi_{n-1,n}$$

and as

$$\begin{array}{rcl}
\theta_{1,k_2}(W_{\lambda_1,\lambda_2}\xi_{1,2}) & = & U_{\lambda_2}\theta_{1,k_2}(\xi_{1,2})(U^{\rm d}_{\lambda_1}\otimes I_{k_2}),\\
\theta_{k_2,k_3}(((W^*_{\lambda_2,\lambda_3})^{\rm d}\otimes I_{k_2})\eta^{\rm d}_{2,3}) & = & (U^{\rm d}_{\lambda_3}\otimes I_{k_2})\theta_{2,3}(\eta^{\rm d}_{2,3})(U_{\lambda_2}\otimes I_{k_3}),\\
\dots\dots\dots\dots\dots\dots & \dots & \dots\dots\dots\dots\dots\dots\\
\theta_{k_{n-1},1}((W_{\lambda_{n-1},\lambda_n}\otimes I_{k_{n-1}})\xi_{n-1,n}) & = & (U_{\lambda_n}\otimes I_{k_{n-1}})\theta_{k_{n-1},1}(\xi_{n-1,n})U^{\rm d}_{\lambda_{n-1}},
\end{array}$$

we get

$$\begin{array}{rcl}
\|\Gamma_{\lambda_1,\dots,\lambda_n}\zeta\|_{\rm h} & \leq & \|U_{\lambda_2}\otimes I_{k_2}\|_{\rm op}\|\theta_{1,k_2}(\xi_{1,2})^{\rm t}\|_{\rm op}\|U^{\rm d}_{\lambda_1}\|_{\rm op}\dots\\
& \dots & \|\theta_{k_{n-1},1}(\xi_{n-1,n})^{\rm t}\|_{\rm op}\|U^{\rm d}_{\lambda_{n-1}}\otimes I_{k_{n-1}}\|_{\rm op}\\
& = & \|\theta_{1,k_2}(\xi_{1,2})^{\rm t}\|_{\rm op}\dots\|\theta_{k_{n-1},1}(\xi_{n-1,n})^{\rm t}\|_{\rm op} = \|\zeta\|_{\rm h}
\end{array}$$

This completes the proof for the case where n is even. Noe assume that n is odd and let $\Gamma_{\lambda_1,\dots,\lambda_n}:\Gamma(H'_1,\dots,H'_n)\to\Gamma(H_1,\dots,H_n)$ be the linear operator defined on elementary tensors by

$$\Gamma_{\lambda_1,\dots,\lambda_n}(\xi^{\rm d}_{1,2}\otimes\dots\otimes\eta_{n-1,n}) = (W_{\lambda_1,\lambda_2}\xi_{1,2})^{\rm d}\otimes\otimes\dots\otimes W_{\lambda_{n-1},\lambda_n}\eta_{n-1,n}.$$

An estimate similar to the above shows again that $\|\Gamma_{\lambda_1,\dots,\lambda_n}\zeta\|_{\rm h}\leq\|\zeta\|_{\rm h}$.

By the definition of the map $S_{\pi'(\varphi)}$ and the arguments above, we obtain

$$\begin{array}{rcl}
\|S_{\pi'(\varphi)}(\zeta)\|_{\rm op} & \leq & \|S_{W^*_{\lambda_1,\dots,\lambda_n}\pi(\varphi)W_{\lambda_1,\dots,\lambda_n}}(\zeta)\|_{\rm op}\\
& & +\|S_{(W^*_{\lambda_1,\dots,\lambda_n}\pi(\varphi)W_{\lambda_1,\dots,\lambda_n}-\pi'(\varphi))}(\zeta)\|_{\rm op}\\
& = & \sup\limits_{\xi_1\in H'_1,\|\xi_1\|=1}\|S_{1\otimes W^*_{\lambda_1,\dots,\lambda_n}\pi(\varphi)W_{\lambda_1,\dots,\lambda_n}}(\xi_1\otimes\zeta)\|_{H'_n}\\
& & +\|S_{(W^*_{\lambda_1,\dots,\lambda_n}\pi(\varphi)W_{\lambda_1,\dots,\lambda_n}-\pi'(\varphi))}(\zeta)\|_{\rm op}\\
& \leq & \sup\limits_{\xi_1\in H'_1,\|\xi_1\|=1}\|S_{1\otimes\pi(\varphi)}(U_{\lambda_1}\xi_1\otimes\Gamma_{\lambda_1,\dots,\lambda_n}\zeta)\|_{H_n}\\
& & +\|S_{(W^*_{\lambda_1,\dots,\lambda_n}\pi(\varphi)W_{\lambda_1,\dots,\lambda_n}-\pi'(\varphi))}(\zeta)\|_2\\
& \leq & \sup\limits_{\eta_1\in H_1,\|\eta_1\|=1}\|S_{1\otimes\pi(\varphi)}(\eta_1\otimes\Gamma_{\lambda_1,\dots,\lambda_n}\zeta)\|_{H_n}\\
& & +\|W^*_{\lambda_1,\dots,\lambda_n}\pi(\varphi)W_{\lambda_1,\dots,\lambda_n}-\pi'(\varphi)\|_{\rm op}\|\zeta\|_{2,\wedge}\\
& = & \|S_{\pi(\varphi)}(\Gamma_{\lambda_1,\dots,\lambda_n}\zeta)\|_{\rm op}\\
& & +\|W^*_{\lambda_1,\dots,\lambda_n}\pi(\varphi)W_{\lambda_1,\dots,\lambda_n}-\pi'(\varphi)\|_{\rm op}\|\zeta\|_{2,\wedge}\\
& \leq & \|\varphi\|_{\pi_1,\dots,\pi_n}\|\|\Gamma_{\lambda_1,\dots\lambda_n}\zeta\|_{\rm h}\\
& & +\|W^*_{\lambda_1,\dots,\lambda_n}\pi(\varphi)W_{\lambda_1,\dots,\lambda_n}-\pi'(\varphi)\|_{\rm op}\|\zeta\|_{2,\wedge}\\
& \leq & \|\varphi\|_{\pi_1,\dots,\pi_n}\|\|\zeta\|_{\rm h}\\
& & +\|W^*_{\lambda_1,\dots,\lambda_n}\pi(\varphi)W_{\lambda_1,\dots,\lambda_n}-\pi'(\varphi)\|_{\rm op}\|\zeta\|_{2,\wedge}.
\end{array}$$

As $\|W^*_{\lambda_1,\dots,\lambda_n}\pi(\varphi)W_{\lambda_1,\dots,\lambda_n} - \pi'(\varphi)\|_{\rm op} \to 0$ we obtain the desired statement.

(ii) is a direct consequence of (i). $\diamondsuit$

For $T \in B(H)$, set $\mathrm{rank}(T) = \overline{\dim(TH)}$. It was proved in [15, Theorem 5.1] that for $*$-representations π and π' of a C^*-algebra $\mathcal{A}$

$$\pi' \overset{a}{\ll} \pi \iff \mathrm{rank}(\pi'(a)) \leq \mathrm{rank}(\pi(a)) \text{ for each } a \in \mathcal{A}. \tag{18}$$

The next statement is a multidimensional version of [18, Corollary 4.3]. Its proof follows the lines of the proof of the corresponding statement in the two dimensional case and uses Theorem 5.1 instead of [18, Theorem 4.2].

Corollary 5.2 *Let π_i, π_i' be representations of the separable C^*-algebra $\mathcal{A}_i$, $i = 1,\dots,n$. Assume that*

$$\min\{\aleph_0, \mathrm{rank}(\pi_i'(a_i))\} \leq \min\{\aleph_0, \mathrm{rank}(\pi_i(a_i))\}$$

for each $a_i \in \mathcal{A}_i$ and $i = 1,\dots,n$.

Then $\mathbf{M}_{\pi_1,\dots,\pi_n} \subseteq \mathbf{M}_{\pi_1',\dots,\pi_n'}$ and $\|\varphi\|_{\pi_1',\dots,\pi_n'} \leq \|\varphi\|_{\pi_1,\dots,\pi_n}$ for $\varphi \in \mathbf{M}_{\pi_1,\dots,\pi_n}$.

Recall that a $*$-representation π of a C^*-algebra $\mathcal{A}$ has a separating vector if there is a cyclic vector for the commutant $\pi(\mathcal{A})'$.

Lemma 5.3 *Let $\mathcal{H}, H_1,\dots,H_n$ be Hilbert spaces, $\pi_1,\dots,\pi_n$ be representations of the C^*-algebras $\mathcal{A}_1,\dots,\mathcal{A}_n$ on $H_1,\dots,H_n$ and $\pi_i \otimes 1$ be the amplification of π_i on $H_i \otimes \mathcal{H}$, respectively. Assume that π_1 and π_n have separating vectors. Then*

$$\mathbf{M}_{\pi_1,\dots,\pi_n} = \mathbf{M}_{\pi_1\otimes 1,\dots,\pi_n\otimes 1}$$

and the multiplier norms on these spaces coincide.

Proof. We use ideas from the proofs of [25, Theorem 2.1] and Lemma 3.3. For simplicity we assume that $n = 3$ and that $\mathcal{H}$ is separable. Let $\varphi \in \mathbf{M}_{\pi_1,\pi_2,\pi_3}$ with $\|\varphi\|_{\pi_1,\pi_2,\pi_3} = 1$ and set $S = S_{(\pi_1\otimes 1)\otimes(\pi_2\otimes 1)\otimes(\pi_3\otimes 1)(\varphi)}$. The mapping S can be regarded as a mapping on

$$\mathcal{C}_2((H_2 \otimes \mathcal{H})^{\rm d}, H_3 \otimes \mathcal{H}) \odot \mathcal{C}_2((H_1 \otimes \mathcal{H}), (H_2 \otimes \mathcal{H})^{\rm d}) \tag{19}$$

by setting $S(\theta(\xi_{2,3}) \otimes \theta(\eta^{\rm d}_{1,2})) = S(\eta^{\rm d}_{1,2} \otimes \xi_{2,3})$ for $\zeta = \eta^{\rm d}_{1,2} \otimes \xi_{2,3} \in \Gamma(H_1 \otimes \mathcal{H}, H_2 \otimes \mathcal{H}, H_3 \otimes \mathcal{H})$. In what follows the space (19) will be denoted by $HS\Gamma((H_1 \otimes \mathcal{H}, H_2 \otimes \mathcal{H}, H_3 \otimes \mathcal{H})$. Similarly, the mapping $S_{\pi_1\otimes\pi_2\otimes\pi_3(\varphi)}$ can be regarded as a mapping on $HS\Gamma(H_1, H_2, H_3) = \mathcal{C}_2(H_2^{\mathrm{d}}, H_3) \odot \mathcal{C}_2((H_1, H_2^{\mathrm{d}})$. It follows from Lemma 4.7 that $S_{\pi_1\otimes\pi_2\otimes\pi_3(\varphi)}$ is $(\pi_3(\mathcal{A}_3)', (\pi_2(\mathcal{A}_2)')^{\mathrm{d}}, \pi_1(\mathcal{A}_1)')$-modular.

Assume that $\|\varphi\|_{\pi_1\otimes 1,\pi_2\otimes 1,\pi_3\otimes 1} > 1$. Then there exists an element

$$T = (T_1^2, \ldots, T_s^2) \odot (T_1^1, \ldots, T_s^1)^{\mathrm{t}} \in HS\Gamma((H_1 \otimes \mathcal{H}, H_2 \otimes \mathcal{H}, H_3 \otimes \mathcal{H})$$

with

$$\|\sum (T_i^1)^* T_i^1\| \|\sum T_i^2 (T_i^2)^*\| = 1$$

and vectors $\xi_0 \in H_1 \otimes \mathcal{H}$, $\eta_0 \in H_3 \otimes \mathcal{H}$ of norm less than one such that

$$|(S(T)\xi_0, \eta_0)| > 1.$$

Fix a basis $\{f_l\}$ of $\mathcal{H}$ and denote by P_n the projection onto the space generated by the first n vectors in this basis. Then, as

$$(1_{H_3} \otimes P_n)S(T)(1_{H_1} \otimes P_n) \to S(T),$$

weakly, there exists $n \geq 1$ such that

$$|((1_{H_3} \otimes P_n)S(T)(1_{H_1} \otimes P_n)\xi_0, \eta_0)| > 1.$$

Thus we may assume that $\xi_0 \in H_1 \otimes P_n\mathcal{H}$ and $\eta_0 \in H_3 \otimes P_n\mathcal{H}$, say

$$\xi_0 = (\xi_1, \ldots, \xi_n, 0, \ldots), \eta_0 = (\eta_1, \ldots, \eta_n, 0 \ldots).$$

As $\pi_1(\mathcal{A}_1)'$ and $\pi_3(\mathcal{A}_3)'$ have cyclic vectors, say ξ and η respectively, we may assume that $\xi_i = a_i\xi$, $\eta_i = b_i\eta$ for some $a_i \in \pi_1(\mathcal{A}_1)'$ and $b_i \in \pi_3(\mathcal{A}_3)'$. Let $a = \sum a_i^* a_i$, $b = \sum b_i^* b_i$. Assuming first that a, b are invertible we set $\tilde{a}_i = a_i a^{-1/2}$, $\tilde{b}_i = b_i b^{-1/2}$. Then for $\tilde{\xi} = a^{1/2}\xi$, $\tilde{\eta} = b^{1/2}\eta$ we have $\xi_i = \tilde{a}_i\tilde{\xi}$ and $\eta_i = \tilde{b}_i\tilde{\eta}$. We write $T_i^k = (T_{i,k}^{l,m})_{l,m}$, where $T_{i,1}^{l,m} = (1_{H_2^{\mathrm{d}}} \otimes P(f_l^{\mathrm{d}}))T_i^1(1_{H_1} \otimes P(f_m))$, $T_{i,2}^{l,m} = (1_{H_3} \otimes P(f_l))T_i^2(1_{H_2^{\mathrm{d}}} \otimes P(f_m^{\mathrm{d}}))$, where $P(f)$ is the projection onto the one dimensional space generated by f. Using the modularity of $S_{\pi_1\otimes\pi_2\otimes\pi_3(\varphi)}$, we obtain

$$\begin{aligned}
|(S(T)\xi_0, \eta_0)| &= \left|\sum_{i=1}^{s}(S(T_i^2 \otimes T_i^1)\xi_0, \eta_0)\right| \\
&= \left|\sum_{i=1}^{s}\sum_{l,m=1}^{n}\sum_{k=1}^{\infty}(S_{\pi_1\otimes\pi_2\otimes\pi_3(\varphi)}(T_{i,2}^{l,k} \otimes T_{i,1}^{k,m})\tilde{a}_m\tilde{\xi}, \tilde{b}_l\tilde{\eta})\right| \quad (20) \\
&= \left|\sum_{i=1}^{s}\sum_{l,m=1}^{n}\sum_{k=1}^{\infty}(S_{\pi_1\otimes\pi_2\otimes\pi_3(\varphi)}(\tilde{b}_l^* T_{i,2}^{l,k} \otimes T_{i,1}^{k,m}\tilde{a}_m)\tilde{\xi}, \tilde{\eta})\right|
\end{aligned}$$

The next step is to prove that $\sum_{i=1}^{s}\sum_{k=1}^{\infty}\left(\sum_{l=1}^{n}\tilde{b}_l^* T_{i,2}^{l,k}\right)\otimes\left(\sum_{m=1}^{n}T_{i,1}^{k,m}\tilde{a}_m\right)$ belongs to $\mathcal{K}(H_2^{\mathrm{d}}, H_3)\otimes_{\mathrm{h}}\mathcal{K}(H_1, H_2^{\mathrm{d}})$. Observe first that the row operator $R_i = (\sum_{l=1}^{n}\tilde{b}_l^* T_{i,2}^{l,1}, \ldots, \sum_{l=1}^{n}\tilde{b}_l^* T_{i,2}^{l,k}, \ldots)$ is equal to the product of the row operator $\tilde{B} = (\tilde{b}_1, \ldots, \tilde{b}_n, 0, \ldots)$ and the Hilbert-Schmidt operator T_i^2. Set $R = (R_1, \ldots, R_s) = (\tilde{B}T_1^2, \ldots, \tilde{B}T_s^2)$.

As each T_i^2 is the operator norm-limit of operators $T_i^2(1_{H_2^{\mathrm{d}}}\otimes P_k)$ as $k\to\infty$, the operator R_i is the uniform limit of the sequence of truncated operators $R_i^k = (\sum_{l=1}^n \tilde{b}_l^* T_{i,2}^{l,1}, \ldots, \sum_{l=1}^n \tilde{b}_l^* T_{i,2}^{l,k}, 0\ldots)$. Thus

$$RR^* = \sum_{i=1}^{s}\sum_{k=1}^{\infty}\left(\sum_{l=1}^{n}\tilde{b}_l^* T_{i,2}^{l,k}\right)\left(\sum_{l=1}^{n}\tilde{b}_l^* T_{i,2}^{l,k}\right)^*,$$

where the series converges uniformly and the norm

$$\begin{aligned}\|\sum_{i=1}^{s}\sum_{k=1}^{\infty}(\sum_{l=1}^{n}\tilde{b}_l^* T_{i,2}^{l,k})(\sum_{l=1}^{n}\tilde{b}_l^* T_{i,2}^{l,k})^*\| &= \|RR^*\| = \|\sum_{i=1}^{s}R_iR_i^*\| \\ &= \|\tilde{B}(\sum_{i=1}^{s}T_i^2(T_i^2)^*)\tilde{B}^*\| \le \|\tilde{B}\|^2\|\|\sum_{i=1}^{s}T_i^2(T_i^2)^*\| \le 1.\end{aligned}$$

In the same way one shows that the series $\sum_{k=1}^{\infty}(\sum_{m=1}^{n}T_{i,1}^{k,m}\tilde{a}_m)(\sum_{m=1}^{n}T_{i,1}^{k,m}\tilde{a}_m)^*$ converges uniformly and

$$\|\sum_{i=1}^{s}\sum_{k=1}^{\infty}(\sum_{m=1}^{n}T_{i,1}^{k,m}\tilde{a}_m)(\sum_{m=1}^{n}T_{i,1}^{k,m}\tilde{a}_m)^*\| \le 1.$$

Thus $\sum_{i=1}^{s}\sum_{k=1}^{\infty}(\sum_{l=1}^{n}\tilde{b}_l^* T_{i,2}^{l,k})\otimes(\sum_{m=1}^{n}T_{i,1}^{k,m}\tilde{a}_m) \in \mathcal{K}(H_1, H_2^{\mathrm{d}})\otimes_{\mathrm{h}}\mathcal{K}(H_2^{\mathrm{d}}, H_3)$ and

$$\|\sum_{i=1}^{s}\sum_{k=1}^{\infty}(\sum_{l=1}^{n}\tilde{b}_l^* T_{i,2}^{l,k})\otimes(\sum_{m=1}^{n}T_{i,1}^{k,m}\tilde{a}_m)\|_{\mathrm{h}} \le 1.$$

Next $\|\tilde{\xi}\|^2 = (b^{1/2}\xi, b^{1/2}\xi) = (b\xi,\xi) = \sum_i (b_i\xi, b_i\xi) = \|\xi_0\|^2 < 1$. Similarly, $\|\tilde{\eta}\| < 1$. Since $\|\varphi\|_{\pi_1,\pi_2,\pi_3} = 1$, it now follows from (20) that

$$|(S(T)\xi_0,\eta_0)| \le \left\|\sum_{i=1}^{s}\sum_{k=1}^{\infty}\left(\sum_{l=1}^{n}\tilde{b}_l^* T_{i,2}^{l,k}\right)\otimes\left(\sum_{m=1}^{n}T_{i,1}^{k,m}\tilde{a}_m\right)\right\|_{\mathrm{h}}\|\tilde{\xi}\|\|\tilde{\eta}\| \le 1,$$

a contradiction.

If a or b is not invertible, let $\epsilon > 0$ be such that $\hat{\xi}_0 \stackrel{def}{=} (\xi_1, \dots, \xi_n, \epsilon\xi, 0, \dots)$ and $\hat{\eta}_0 \stackrel{def}{=} (\eta_1, \dots, \eta_n, \epsilon\eta, 0, \dots)$ have norm less than one and $|(S(T)\hat{\xi}_0, \hat{\eta}_0)| > 1$. Choose a_i and b_i in the same way as before, and let $a_{n+1} = \epsilon I$, $b_{n+1} = \epsilon I$, $a = \sum_{i=1}^{n+1} a_i^* a_i$ and $b = \sum_{i=1}^{n+1} b_i^* b_i$. Then a and b are invertible and the proof proceeds in the same fashion.

We have proved that $\mathbf{M}_{\pi_1,\dots,\pi_n} \subseteq \mathbf{M}_{\pi_1\otimes 1,\dots,\pi_n\otimes 1}$ and that $\|\cdot\|_{\pi_1\otimes 1,\dots,\pi_n\otimes 1} \leq \|\cdot\|_{\pi_1,\dots,\pi_n}$. The converse inequality is easy to show, and thus the proof is complete. $\diamondsuit$

Corollary 5.4 *Let π_i be a representation of the C*-algebra $\mathcal{A}_i$, $i = 1, \dots, n$. Assume that π_1 and π_n have separating vectors. If*

$$\ker(\pi_i) \subseteq \ker(\pi_i'), \text{ for each } i = 1, \dots, n, \tag{21}$$

then $\mathbf{M}_{\pi_1,\dots,\pi_n} \subseteq \mathbf{M}_{\pi_1',\dots,\pi_n'}$ and $\|\varphi\|_{\pi_1',\dots,\pi_n'} \leq \|\varphi\|_{\pi_1,\dots,\pi_n}$, for each $\varphi \in \mathbf{M}_{\pi_1,\dots,\pi_n}$.

Proof. The proof is similar to that of [18, Corollary 4.8]; we include it for completeness. Let $\mathcal{H}$ be an infinite-dimensional Hilbert space of sufficiently large dimension. Then (21) implies

$$\text{rank}(\pi_i'(a_i)) \leq \text{rank}(\pi_i(a_i) \otimes 1), \text{ for all } a_i \in \mathcal{A}_i.$$

By (18), $\pi_i' \stackrel{a}{\ll} \pi_i \otimes 1$. Applying now Theorem 5.1 and then Lemma 5.3 we obtain the statement. $\diamondsuit$

Using Corollary 5.4 and results from [18] we will now show that if the C*-algebras $\mathcal{A}_i$ are commutative then the space $\mathbf{M}_{\pi_1,\dots,\pi_n}(\mathcal{A}_1, \dots, \mathcal{A}_n)$ of multipliers depends only on the supports of spectral measures corresponding to the representations π_i.

Assume that $\mathcal{A}_i$ is commutative, $i = 1, \dots, n$ and let X_i be the maximal ideal spaces of $\mathcal{A}_i$; then $\mathcal{A}_i \simeq C_0(X_i)$. Let π_i be a representation of $\mathcal{A}_i$ and $\mathcal{E}_{\pi_i}$ be the spectral measure on X_i corresponding to π_i.

It was proved in [18, Lemma 6.2] that if $f \in C_0(X)$ and the representation π of $C_0(X)$ is such that rank $(\pi(f)) < \infty$ then

$$\text{rank } (\pi(f)) = \sum_{x \in S(f, \mathcal{E}_\pi)} \dim(\mathcal{E}_\pi(\{x\})),$$

where $S(f,\mathcal{E}_\pi)=\{x\in\operatorname{supp}\mathcal{E}_\pi : f(x)\neq 0\}$. Thus the condition

$$\operatorname{supp}\mathcal{E}_{\pi'}\subset\operatorname{supp}\mathcal{E}_\pi$$

implies $\ker\pi(f)\subseteq\ker\pi'(f)$. As each representation π of a commutative algebra $C_0(X)$ has a separating vector we have the following

Corollary 5.5 *Let π_i, π_i' be separable representations of the C*-algebra $\mathcal{A}_i=C_0(X_i)$ and $\mathcal{E}_{\pi_i}$ and $\mathcal{E}_{\pi_i'}$ be the corresponding spectral measures ($i=1,\dots,n$). If*

$$\operatorname{supp}\mathcal{E}_{\pi_i'}\subseteq\operatorname{supp}\mathcal{E}_{\pi_i},\ \textit{for each } i=1,\dots,n,$$

then $\mathbf{M}_{\pi_1,\dots,\pi_n}\subseteq\mathbf{M}_{\pi_1',\dots,\pi_n'}$.

Let μ_i be measures on X_i. Let π_i be a representation of $C_0(X_i)$ on $L_2(X_i,\mu_i)$ defined by $(\pi_i(f)h)(x_i)=f(x_i)h(x_i)$. We call $\varphi\in C_0(X_1\times\ldots\times X_n)$ a $(\mu_1,\dots,\mu_n)$-*multiplier* if $\varphi\in\mathbf{M}_{\pi_1,\dots,\pi_n}$ and let $\|\varphi\|_{\mu_1,\dots,\mu_n}=\|\varphi\|_{\pi_1,\dots,\pi_n}$.

By Corollary 5.5, the set of the all $(\mu_1,\dots,\mu_n)$-multipliers depends only on the supports of measures μ_i. The next statement shows the connection between $(\mu_1,\dots,\mu_n)$-multipliers and multidimensional Schur multipliers (with respect to discrete measures).

Corollary 5.6 *Let X_i be locally compact spaces with countable bases and let μ_i be Borel σ-finite measures on X_i with $\operatorname{supp}\mu_i=X_i$. Then $\varphi\in C_0(X_1\times\ldots\times X_n)$ is a $(\mu_1,\dots,\mu_n)$-multiplier iff φ is a Schur multiplier on $X_1\times\ldots\times X_n$. Moreover, in this case $\|\varphi\|_{\mu_1,\dots,\mu_n}=\|S_\varphi\|$.*

Proof. The proof is similar to that of [18, Theorem 6.5]. $\diamondsuit$

6 Universal multipliers

The main goal of this section is to give a full description of the multipliers which do not depend on the choice of the representations of the C*-algebras $\mathcal{A}_1,\mathcal{A}_2,\dots,\mathcal{A}_n$. Recall that an element $\varphi\in\mathcal{A}_1\otimes\ldots\otimes\mathcal{A}_n$ is called a universal multiplier if φ is a $\pi_1,\pi_2,\dots,\pi_n$-multiplier for all representations π_1, π_2,..., π_n of $\mathcal{A}_1$, $\mathcal{A}_2,\dots,\mathcal{A}_n$, respectively. The set of all universal multipliers in $\mathcal{A}_1\otimes\cdots\otimes\mathcal{A}_n$ is denoted by $\mathbf{M}(\mathcal{A}_1,\dots,\mathcal{A}_n)$.

Along with the universal multipliers, we will describe another class of multipliers which we call projective universal multipliers and define as follows. Let $H_1, \ldots, H_n$ be Hilbert spaces. Equip $\Gamma(H_1, \ldots, H_n)$ with the projective tensor norm $\|\cdot\|_\wedge$, where each of the terms $H_i \otimes H_{i+1}$ (resp. $H_{i-1}^{\mathrm{d}} \otimes H_i^{\mathrm{d}}$) is given its operator norm. We call an element $\varphi \in \mathcal{B}(H_1 \otimes \cdots \otimes H_n)$ a concrete projective multiplier if there exists $C > 0$ such that $\|S_\varphi(\zeta)\|_{\mathrm{op}} \leq C\|\zeta\|_\wedge$, for all $\zeta \in \Gamma(H_1, \ldots, H_n)$. If $\mathcal{A}_1, \ldots \mathcal{A}_n$ are C*-algebras, an element $\varphi \in \mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_n$ will be called a **projective universal multiplier** if $(\pi_1 \otimes \cdots \otimes \pi_n)(\varphi)$ is a concrete projective multiplier for all choices of representations $\pi_1, \ldots, \pi_n$ of $\mathcal{A}_1, \ldots, \mathcal{A}_n$, respectively. We denote by $\mathbf{M}^\wedge(\mathcal{A}_1, \ldots, \mathcal{A}_n)$ the set of all projective universal multipliers.

If $\varphi \in \mathbf{M}(\mathcal{A}_1, \ldots, \mathcal{A}_n)$ let

$$\|\varphi\|_{\mathrm{univ}} = \sup_{\pi_1, \pi_2, \ldots, \pi_n} \|\varphi\|_{\pi_1, \pi_2, \ldots, \pi_n}.$$

Note that $\|\varphi\|_{\mathrm{univ}}$ is finite. In fact, assume that there exist representations $\pi_{1,k}, \ldots, \pi_{n,k}$, such that $\|\varphi\|_{\pi_{1,k}, \pi_{2,k}, \ldots, \pi_{n,k}} \to_{k \to \infty} \infty$ and let take $\pi_1 = \bigoplus_k \pi_{1,k}$, $\pi_2 = \bigoplus_k \pi_{2,k}, \ldots, \pi_n = \bigoplus_k \pi_{n,k}$. Then, by Theorem 5.1,

$$\|\varphi\|_{\pi_{1,k}, \pi_{2,k}, \ldots, \pi_{n,k}} \leq \|\varphi\|_{\pi_1, \pi_2, \ldots, \pi_n},$$

for all $k \in \mathbb{N}$, which contradicts the fact that $\varphi \in \mathbf{M}(\mathcal{A}_1, \ldots, \mathcal{A}_n)$.

It is clear that $\mathbf{M}(\mathcal{A}_1, \ldots, \mathcal{A}_n)$ is a subalgebra of $\mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_n$ containing $\mathcal{A}_1 \odot \ldots \odot \mathcal{A}_n$.

Recall that the Haagerup norm on $\mathcal{A}_1 \odot \mathcal{A}_2 \odot \ldots \odot \mathcal{A}_n$ is

$$\|\omega\|_{\mathrm{h}} = \inf\{\|\omega_1\|\|\omega_2\| \ldots \|\omega_n\| : \omega = \omega_1 \odot \omega_2 \odot \ldots \odot \omega_n,$$
$$\omega_1 \in M_{1,i_1}(\mathcal{A}_1), \omega_2 \in M_{i_1,i_2}(\mathcal{A}_2), \ldots, \omega_n \in M_{i_{n-1},1}(\mathcal{A}_n), i_1, i_2, \ldots, i_{n-1} \in \mathbb{N}\}.$$

A modification of the Haagerup norm on the algebraic tensor product of two C^*-algebras was introduced in [18]. Recall the maps $\omega \mapsto \omega^{\mathrm{t}}$ and $\omega \mapsto \omega^{\mathrm{d}}$ on $M_n(\mathcal{A}) = M_n(\mathbb{C}) \otimes \mathcal{A}$ given on elementary tensors by $(a \odot b)^{\mathrm{t}} = a^{\mathrm{t}} \odot b$ and $(a \odot b)^{\mathrm{d}} = a \odot b^{\mathrm{d}}$, here $\mathcal{A}$ is a C^*-subalgebra of $B(H)$ for some Hilbert space H. We set

$$\|\omega\|_{\mathrm{ph}} = \inf\{\prod_{0 \leq i < \frac{n}{2}} \|\omega_{n-2i}^{\mathrm{t}}\|\|\omega_{n-2i-1}\| : \omega = \omega_1 \odot \omega_2 \odot \ldots \odot \omega_n,\ \omega_0 = I,$$
$$\omega_1 \in M_{1,i_1}(\mathcal{A}_1), \omega_2 \in M_{i_1,i_2}(\mathcal{A}_2), \ldots, \omega_n \in M_{i_{n-1},1}(\mathcal{A}_n), i_1, i_2, \ldots, i_{n-1} \in \mathbb{N}\},$$

It is well known that $\|\omega\|_{\min} \leq \|\omega\|_{\mathrm{h}}$ and one can easily prove that $\|\omega\|_{\min} \leq \|\omega\|_{\mathrm{ph}}$.

Lemma 6.1 $\|\omega\|_{\mathrm{univ}} \leq \|\omega\|_{\mathrm{ph}}$ *for all* $\omega \in \mathcal{A}_1 \odot \ldots \odot \mathcal{A}_n$.

Proof. Let π_i be a representation of $\mathcal{A}_i$, $i = 1, \ldots, n$, and let $\omega = \omega_1 \odot \omega_2 \odot \ldots \odot \omega_n$, where $\omega_1 \in M_{1,k_1}(\mathcal{A}_1), \omega_2 \in M_{k_1,k_2}(\mathcal{A}_2), \ldots, \omega_n \in M_{k_{n-1},1}(\mathcal{A}_n)$ for some $k_1, k_2, \ldots, k_{n-1} \in \mathbb{N}$.

Let n be even, $\xi_{1,2} \in M_{1,l_1}(H_1 \otimes H_2)$, $\eta_{2,3} \in M_{l_1,l_2}(H_2^d \otimes H_3^d), \ldots, \xi_{n-1,n} \in M_{l_{n-2},1}(H_{n-1} \otimes H_n)$ and

$$\zeta = \xi_{1,2} \odot \eta_{2,3} \odot \cdots \odot \xi_{n-1,n} \in \Gamma(H_1, \ldots, H_n).$$

By Lemma 4.7,

$$\begin{aligned} S_{\pi_1 \otimes \ldots \otimes \pi_n(\omega)}(\zeta) = (\mathrm{id}_{\,1,k_{n-1}} \otimes \pi_n)(\omega_n^{\mathrm{t}}) \ldots (\theta_{l_1,l_2}(\eta_{2,3})^{\mathrm{t}} \otimes I_{k_2}) \\ \times((\mathrm{id}_{\,k_1,k_2} \otimes \pi_2)(\omega_2^{\mathrm{t}}) \otimes I_{l_1})(\theta_{1,l_1}(\xi_{1,2})^{\mathrm{t}} \otimes I_{k_1})(\mathrm{id}_{\,k_1,1} \otimes \pi_1)(\omega_1^{\mathrm{t}})^{\mathrm{d}}. \end{aligned}$$

Since $\|(\mathrm{id}_{k_{m-1},k_m} \otimes \pi_m)(\omega_m^{\mathrm{t}})^{\mathrm{d}}\| = \|(\mathrm{id}_{k_{m-1},k_m} \otimes \pi_m)(\omega_m)\|$, we have

$$\begin{aligned} \|S_{\pi_1 \otimes \ldots \otimes \pi_n(\omega)}(\zeta)\| &\leq \|\theta_{1,l_1}(\xi_{1,2})^{\mathrm{t}}\| \ldots \|\theta_{l_{n-2},1}(\xi_{n-1,n})^{\mathrm{t}}\| \\ &\times \prod_{0 \leq i < \frac{n}{2}} \|\omega_{n-2i}^{\mathrm{t}}\| \|\omega_{n-2i-1}\| = \|\omega\|_{\mathrm{ph}} \|\zeta\|_{\mathrm{h}}. \end{aligned}$$

Using similar arguments, one can easily see that same inequality holds if n is odd and

$$\zeta = \eta_{1,2} \odot \xi_{2,3} \odot \cdots \odot \xi_{n-1,n} \in \Gamma(H_1, \ldots, H_n),$$

where $\eta_{1,2} \in M_{1,l_1}(H_1^d \otimes H_2^d)$, $\xi_{2,3} \in M_{l_1,l_2}(H_2 \otimes H_3), \ldots,$ $\xi_{n-1,n} \in M_{l_{n-2},1}$ $(H_{n-1} \otimes H_n)$. The proof is complete. $\diamondsuit$

Let $\mathcal{A}_1 \subseteq B(H_1)$, $\mathcal{A}_2 \subseteq B(H_2), \ldots,$ $\mathcal{A}_n \subseteq B(H_n)$ be C*-algebras and $(\mathcal{A}_1 \odot \mathcal{A}_2 \odot \ldots \odot \mathcal{A}_n)^\sharp$ be the linear space of all $\varphi \in \mathcal{A}_1 \otimes \mathcal{A}_2 \otimes \ldots \otimes \mathcal{A}_n$ for which there exists a net $\omega_\nu \in \mathcal{A}_1 \odot \mathcal{A}_2 \odot \ldots \odot \mathcal{A}_n$ weakly converging to φ (as a net of operators in $B(H_1 \otimes H_2 \otimes \ldots \otimes H_n)$) with $\sup_\nu \|\omega_\nu\|_{\mathrm{ph}} < \infty$.

Proposition 6.2 *Let* $\mathcal{A}_i \subseteq B(H_i)$, $i = 1, \ldots, n$, *be C*-algebras. Then* $(\mathcal{A}_1 \odot \cdots \odot \mathcal{A}_n)^\sharp \subseteq \mathbf{M}(\mathcal{A}_1, \ldots, \mathcal{A}_n) \subseteq \mathbf{M}^\wedge(\mathcal{A}_1, \ldots, \mathcal{A}_n)$.

Proof. Since $\|\zeta\|_h \le \|\zeta\|_\wedge$ for all $\zeta \in \Gamma(H_1, \dots, H_n)$ we have $\mathbf{M}(\mathcal{A}_1, \dots, \mathcal{A}_n) \subseteq \mathbf{M}^\wedge(\mathcal{A}_1, \dots, \mathcal{A}_n)$.

Suppose that n is even, for odd n the proof of the proposition is similar. Firstly let us prove that

$$(\mathcal{A}_1 \odot \cdots \odot \mathcal{A}_n)^\sharp \subseteq \mathbf{M}_{\pi_1,\dots,\pi_n}(\mathcal{A}_1, \dots, \mathcal{A}_n),$$

in the case where $\pi_i = \bigoplus_{\lambda_i} \mathrm{id}$ is the sum of λ_i copies of the identity representation. Let $\{\varphi_\nu\} \subseteq \mathcal{A}_1 \odot \dots \odot \mathcal{A}_n$ be a net converging weakly to φ and such that $D = \sup_\nu \|\varphi_\nu\|_{\mathrm{ph}} < \infty$. By Lemma 6.1,

$$\|S_{\pi_1,\dots,\pi_n(\varphi_\nu)}(\zeta)\|_{\mathrm{op}} \le D\|\zeta\|_{\mathrm{h}}$$

for all ν and $\zeta \in \Gamma(H_1, \dots, H_n)$.

To prove that $\|S_{\pi_1\otimes\dots\otimes\pi_n(\varphi_\nu)}(\zeta)\|_{\mathrm{op}} \le D\|\zeta\|_{\mathrm{h}}$, it suffices to show that the net $\{S_{\pi_1\otimes\dots\otimes\pi_n(\varphi_\nu)}(\zeta)\}$ of operators in $B(\widetilde{H}_1, \widetilde{H}_k)$ (where $\widetilde{H}_1 = \bigoplus_{\lambda_1} H_1$ and $\widetilde{H}_n = \bigoplus_{\lambda_n} H_n$), converges weakly to the operator $S_{\pi_1,\dots,\pi_n(\varphi)}(\zeta)$. To this end, it is suffices to prove that

$$(S_{\pi_1\otimes\dots\otimes\pi_n(\varphi_\nu)}(\zeta)x^{\mathrm{d}}, y) \to (S_{\pi_1,\dots,\pi_n(\varphi)}(\zeta)x^{\mathrm{d}}, y)$$

for $x^{\mathrm{d}} \in \widetilde{H}_1^{\mathrm{d}}$ and $y \in \widetilde{H}_n$.

Fix $x^{\mathrm{d}} \in \widetilde{H}_1^{\mathrm{d}}$, $y \in \widetilde{H}_n$, $\zeta = \xi_{1,2} \otimes \eta_{2,3}^{\mathrm{d}} \otimes \dots \otimes \xi_{n-1,n} \in \Gamma(\widetilde{H}_1, \dots, \widetilde{H}_n)$. Then

$$\begin{aligned}
& (S_{\pi_1\otimes\dots\otimes\pi_n(\varphi_\nu)}(\zeta)x^{\mathrm{d}}, y) \\
= \ & (\sigma_{\pi_1\otimes\dots\otimes\pi_n}(\varphi_\nu)\theta(\xi_{1,2} \otimes \dots \otimes \xi_{n-1,n})(\theta(\eta_{2,3}^{\mathrm{d}})) \dots (\theta(\eta_{n-2,n-1}^{\mathrm{d}})), \theta(x \otimes y))_2 \\
= \ & (\sigma_{\pi_1\otimes\dots\otimes\pi_n}(\varphi_\nu)\theta(\xi_{1,2} \otimes \dots \otimes \xi_{n-1,n})(\theta(\eta_{2,3}^{\mathrm{d}})) \dots (\theta(\eta_{n-4,n-3}^{\mathrm{d}})), \\
& \theta(\theta(\eta_{n-2,n-1}) \otimes \theta(x \otimes y)))_2 \\
= \ & (\sigma_{\pi_1\otimes\dots\otimes\pi_n}(\varphi_\nu)\theta(\xi_{1,2} \otimes \dots \otimes \xi_{n-1,n})(\theta(\eta_{2,3}^{\mathrm{d}})) \dots (\theta(\eta_{n-4,n-3}^{\mathrm{d}})), \\
& \theta_{H_1,H_{n-2},H_{n-1},H_n}(x \otimes \eta_{n-2,n-1} \otimes y))_2 \\
= \ & \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
= \ & (\sigma_{\pi_1\otimes\dots\otimes\pi_n}(\varphi_n)\theta(\xi_{1,2} \otimes \dots \otimes \xi_{n-1,n}), \\
& \theta_{H_1,\dots,H_n}(x \otimes \eta_{2,3} \otimes \eta_{4,5} \otimes \dots \otimes \eta_{n-2,n-1} \otimes y))_2
\end{aligned}$$

Since $\|\varphi_\nu\|_{\mathrm{ph}} \le D$ for each ν, we have that $\|\varphi_\nu\|_{\min} \le D$ for each ν. It follows that $\pi_1 \otimes \dots \otimes \pi_n(\varphi_\nu)$ converges weakly to $\pi_1 \otimes \dots \otimes \pi_n(\varphi)$. Since the representation $\pi_1 \otimes \ldots \otimes \pi_n$ is equivalent to the representation $\sigma_{\pi_1,\ldots,\pi_n}$, we have that $\sigma_{\pi_1,\ldots,\pi_n}(\varphi_\nu)$ converges weakly to $\sigma_{\pi_1,\ldots,\pi_n}(\varphi)$. By the previous formuli, $S_{\pi_1\otimes\ldots\otimes\pi_n(\varphi_\nu)}(\zeta)$ converges weakly to $S_{\pi_1\otimes\ldots\otimes\pi_n(\varphi)}(\zeta)$ and hence $\varphi \in \mathbf{M}_{\pi_1,\ldots,\pi_n}(\mathcal{A}_1,\ldots,\mathcal{A}_n)$.

Now let π_1, ..., π_n be representations of $\mathcal{A}_1$, ..., $\mathcal{A}_n$ on H_{π_1}, ..., H_{π_n}. Then

$$\operatorname{rank}(\pi_i(a_i)) \leq \operatorname{rank}\left(\bigoplus_{dim(H_{\pi_i})} \mathrm{id}(a_i)\right)$$

for all $a_i \in \mathcal{A}_i$ and $i = 1,..,n$. By Theorem 5.1 (i),

$$\mathbf{M}_{\bigoplus\limits_{\lambda_1}\mathrm{id},\bigoplus\limits_{\lambda_2}\mathrm{id},\ldots,\bigoplus\limits_{\lambda_k}\mathrm{id}}(\mathcal{A}_1,\ldots,\mathcal{A}_n) \subseteq \mathbf{M}_{\pi_1,\pi_2,\ldots,\pi_k}(\mathcal{A}_1,\mathcal{A}_2,\ldots,\mathcal{A}_n).$$

The proof is complete.

Assume that n is even. Then the mapping $S_{\mathrm{id}\otimes\ldots\otimes\mathrm{id}(\varphi)}$ acting on $\Gamma(H_1, \ldots, H_n) = (H_1 \otimes H_2) \odot (H_2^{\mathrm{d}} \otimes H_3^{\mathrm{d}}) \odot \ldots \odot (H_{n-1} \otimes H_n)$ can be regarded as a mapping on the algebraic tensor product

$$HS(H_{n-1}, H_n) \odot HS(H_{n-2}, H_{n-1})^{\mathrm{d}} \odot \ldots \odot HS(H_1, H_2) \tag{22}$$

of the corresponding spaces of Hilbert-Schmidt operators by letting

$$S_\varphi(\theta(\xi_{n-1,n}) \otimes \theta(\eta_{n-2,n-1})^{\mathrm{d}} \otimes \theta(\xi_{n-3,n-2}) \otimes \ldots \otimes \theta(\xi_{1,2})) = S_\varphi(\zeta),$$

where $\zeta = \xi_{1,2} \otimes \eta_{2,3}^{\mathrm{d}} \otimes \xi_{3,4} \otimes \ldots \otimes \xi_{n-1,n}$. Denote the space (22) by $HS\Gamma$ $(H_1,\ldots,H_n)$. If φ is an elementary tensor then Lemma 4.7 (i) shows that $S_{\mathrm{id}\otimes\ldots\otimes\mathrm{id}(\varphi)}$ is $\mathcal{A}_n', (\mathcal{A}_{n-1}^{\mathrm{d}})', \ldots, \mathcal{A}_2', (\mathcal{A}_1{}^{\mathrm{d}})'$-modular. It follows by continuity that $S_{\mathrm{id}\otimes\ldots\otimes\mathrm{id}(\varphi)}$ is $\mathcal{A}_n', (\mathcal{A}_{n-1}^{\mathrm{d}})', \ldots, \mathcal{A}_2', (\mathcal{A}_1{}^{\mathrm{d}})'$-modular for every $\varphi \in \mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_n$. If moreover $\varphi \in \mathbf{M}_{\mathrm{id},\ldots,\mathrm{id}}(\mathcal{A}_1,\ldots,\mathcal{A}_n)$ then $S_{\mathrm{id}\otimes\ldots\otimes\mathrm{id}(\varphi)}$ can be extended to a bounded mapping (denoted in the same way) from the algebraic tensor product

$$\mathcal{K}(H_{n-1}^{\mathrm{d}}, H_n) \odot \mathcal{K}(H_{n-2}^{\mathrm{d}}, H_{n-1})^{\mathrm{d}} \odot \cdots \odot \mathcal{K}(H_1^{\mathrm{d}}, H_2)$$

into $\mathcal{K}(H_1^{\mathrm{d}}, H_n)$. By continuity, this extension is also $\mathcal{A}_{n-1}^{\mathrm{d}})', \ldots, \mathcal{A}_2', (\mathcal{A}_1{}^{\mathrm{d}})'$-modular.

Similarly, if n is odd and $\varphi \in \mathbf{M}_{\mathrm{id},\ldots,\mathrm{id}}(\mathcal{A}_1,\ldots,\mathcal{A}_n)$ then $S_{\mathrm{id}\otimes\ldots\otimes\mathrm{id}(\varphi)}$ can be regarded as a multilinear $\mathcal{A}_n', (\mathcal{A}_{n-1}{}^{\mathrm{d}})', \ldots, (\mathcal{A}_2{}^{\mathrm{d}})', \mathcal{A}_1'$-modular map from

$$\mathcal{K}(H_{n-1}^{\mathrm{d}}, H_n) \odot \mathcal{K}(H_{n-2}^{\mathrm{d}}, H_{n-1})^{\mathrm{d}} \odot \cdots \odot \mathcal{K}(H_1, H_2^{\mathrm{d}})$$

into $\mathcal{K}(H_1^{\mathrm{d}}, H_n)$. Denote by $\mathbf{M}^{cb}_{\mathrm{id},\ldots,\mathrm{id}}(\mathcal{A}_1,\ldots,\mathcal{A}_n)$ the set of all $(\mathrm{id},\ldots,\mathrm{id})$-multipliers for which the mapping $S_{\mathrm{id}\otimes\ldots\otimes\mathrm{id}(\varphi)}$ is completely bounded.

Proposition 6.3 *Let $\mathcal{A}_i \subseteq B(H_i)$, $i = 1, \ldots, n$, be von Neumann algebras. Then $\mathbf{M}^{cb}_{\mathrm{id},\ldots,\mathrm{id}}(\mathcal{A}_1, \ldots, \mathcal{A}_n) \subseteq (\mathcal{A}_1 \odot \cdots \odot \mathcal{A}_n)^\sharp$.*

Proof. We will prove the inclusion in the case n is even; the case of odd n is similar. For notational simplicity we assume that H_i is separable, $1 = 1, .., n$.

Let $\varphi \in \mathbf{M}^{cb}_{\mathrm{id},\ldots,\mathrm{id}}(\mathcal{A}_1, \ldots, \mathcal{A}_n)$. Then $S_{\mathrm{id} \otimes \mathrm{id} \otimes \ldots \otimes \mathrm{id}(\varphi)}$ is a multilinear $\mathcal{A}'_n$, $(\mathcal{A}^{\mathrm{d}}_{n-1})'$, …, $\mathcal{A}'_2$, $(\mathcal{A}_1{}^{\mathrm{d}})'$-modular mapping on

$$\mathcal{K}(H^{\mathrm{d}}_{n-1}, H_n) \times \mathcal{K}(H^{\mathrm{d}}_{n-2}, H_{n-1})^{\mathrm{d}} \times \ldots \times \mathcal{K}(H^{\mathrm{d}}_1, H_2)$$

taking values in $\mathcal{K}(H^{\mathrm{d}}_1, H_n)$. Let $H^\infty = H \otimes l^2$ and I_∞ be the identity operator on l^2.

Let $\zeta = \theta(\xi_{n-1,n}) \otimes \theta(\eta_{n-2,n-1})^{\mathrm{d}} \otimes \ldots \otimes \theta(\xi_{1,2}) \in HS\Gamma(H_1, \ldots, H_n)$. It follows from [9] that there exist bounded linear operators $A_1 : H^{\mathrm{d}}_1 \to (H^{\mathrm{d}}_1)^\infty$, $A_j : H^\infty_j \to H^\infty_j$, if j is even, $A_j : (H^{\mathrm{d}}_j)^\infty \to (H^{\mathrm{d}}_j)^\infty$ if j is odd ($j = 2, \ldots, n-1$) and $A_n : H^\infty_n \to H_n$ such that the entries of A_j with respect to the corresponding direct sum decomposition belong to $\mathcal{A}''_j = \mathcal{A}_j$ for even j and to $(\mathcal{A}^{\mathrm{d}}_j)'' = \mathcal{A}^{\mathrm{d}}_j$ for odd j,

$$S_{\mathrm{id} \otimes \ldots \otimes \mathrm{id}(\varphi)}(\zeta) = A_n(\theta(\xi_{n-1,n}) \otimes I_\infty)A_{n-1}(\theta(\eta_{n-2,n-1})^{\mathrm{d}} \otimes I_\infty)A_{n-2} \ldots A_1,$$

for all $\zeta \in HS\Gamma(H_1, \ldots, H_n)$ and

$$\|S_{\mathrm{id} \otimes \ldots \otimes \mathrm{id}(\varphi)}\|_{cb} = \prod_{1 \le i \le n} \|A_i\|.$$

Let $P_{m,\nu} = (p^m_{ij})^\infty_{i,j=1}$ be the projection with $p^m_{ij} \in B(H_m)$ (resp. $p^m_{ij} \in B(H^{\mathrm{d}}_m)$), $p^m_{ii} = I_{H_m}$ (resp. $p^m_{ii} = I_{H^{\mathrm{d}}_m}$) if m is even (resp. if m is odd) and $1 \le i \le \nu$, and $p^m_{ij} = 0$ otherwise.

Set $\varphi_\nu = A^{\mathrm{d,t}}_1 P^{\mathrm{d}}_{1,\nu} \odot P_{2,\nu} A_2 P_{2,\nu} \odot P_{3,\nu} A^{\mathrm{d}}_3 P_{3,\nu} \ldots \odot P_{n,\nu} A_n$. Clearly, $\|\varphi_\nu\|_{\mathrm{ph}} \le \prod_{1 \le i \le n} \|A_i\|$ for each ν; it hence suffices to prove that $\{\varphi_\nu\}$ converges weakly to φ.

As $S_{\mathrm{id} \otimes \ldots \otimes \mathrm{id}(\varphi_\nu)}(\zeta)$ equals

$$A_n P_{n,\nu}(\theta(\xi_{n-1,n}) \otimes I_\infty) P_{n-1,\nu} A_{n-1} P_{n-1,\nu}(\theta(\eta_{n-2,n-1})^{\mathrm{d}} \otimes I_\infty) \ldots P_{1,\nu} A_1$$

and $P_{l,\nu}$ converges strongly to I_{H_l}, we have that $S_{\mathrm{id} \otimes \ldots \otimes \mathrm{id}(\varphi_\nu)}(\zeta)$ converges weakly to $S_{\mathrm{id} \otimes \ldots \otimes \mathrm{id}(\varphi)}(\zeta)$. By the proof of Proposition 6.2, if $x^{\mathrm{d}} \in H^{\mathrm{d}}_1$ and $y \in H_n$ then $(S_{\mathrm{id} \otimes \ldots \otimes \mathrm{id}(\varphi_\nu)}(\zeta) x^{\mathrm{d}}, y)$ equals

$$(\sigma_{\mathrm{id} \otimes \ldots \otimes \mathrm{id}}(\varphi_\nu) \theta(\xi_{1,2} \otimes \ldots \otimes \xi_{k-1,k}), \theta(x \otimes \eta_{2,3} \otimes \ldots \otimes \eta_{k-2,k-1} \otimes y))_2.$$

Thus $\sigma_{\text{id},\dots,\text{id}}(\varphi_\nu)$ converges weakly to $\sigma_{\text{id},\dots,\text{id}}(\varphi)$ on $\theta(H_1 \odot \dots \odot H_n)$. On the other hand, $\|\varphi_\nu\|_{\min} \leq \|\varphi_\nu\|_{\text{ph}}$ and hence $\{\|\varphi_\nu\|_{\min}\}_\nu$ is bounded. Since $\theta(H_1 \odot \dots \odot H_n)$ is dense in $HS(H_1, \dots, H_n)$, we conclude that $\sigma_{\text{id}\otimes\dots\otimes\text{id}}(\varphi_\nu)$ converges weakly to $\sigma_{\text{id},\dots,\text{id}}(\varphi)$. Thus $\varphi \in (\mathcal{A}_1 \odot \dots \odot \mathcal{A}_n)^\sharp$ and so $\mathbf{M}^{cb}_{\text{id},\dots,\text{id}}(\mathcal{A}_1, \dots, \mathcal{A}_n) \subseteq (\mathcal{A}_1 \odot \dots \odot \mathcal{A}_n)^\sharp$. $\diamondsuit$

Proposition 6.4 *Let $\mathcal{A}_i \subseteq B(H_i)$, $i = 1, \dots, n$, be C*-algebras. Then $\mathbf{M}^\wedge(\mathcal{A}_1, \dots, \mathcal{A}_n) \subseteq \mathbf{M}^{cb}_{\text{id},\dots,\text{id}}(\mathcal{A}_1, \dots, \mathcal{A}_n)$.*

Proof. Let $\varphi \in \mathbf{M}^\wedge(\mathcal{A}_1, \dots, \mathcal{A}_n)$. Then there exists a constant $D > 0$ such that

$$\|\sigma_{\pi_1,\dots,\pi_n}(\varphi)(\zeta)\|_{\text{op}} \leq D\|\zeta\|_\wedge$$

for all $\zeta \in \Gamma(H_1, \dots, H_n)$ and all representations $\pi_1, \dots, \pi_n$ of $\mathcal{A}_1, \dots, \mathcal{A}_n$, respectively.

Let $k \in \mathbb{N}$. The space $HS\Gamma(H_1^k, \dots, H_n^k)$ is naturally isomorphic to

$$M_k(HS(H_{n-1}, H_n)) \odot M_k(HS(H_{n-2}, H_{n-1})^{\text{d}}) \odot \dots \odot M_k(HS(H_1, H_2)), \tag{23}$$

and thus the mapping $S_{(\text{id}\otimes 1_k)\otimes\dots\otimes(\text{id}\otimes 1_k)(\varphi)}$ is well-defined on the space (23). One can easily check that

$$S^{(k)}_{\text{id}\otimes\dots\otimes\text{id}(\varphi)}(\Xi_{n-1} \odot \dots \odot \Xi_1) = S_{(\text{id}\otimes 1_k)\otimes\dots\otimes(\text{id}\otimes 1_k)(\varphi)}(\Xi_{n-1} \otimes \dots \otimes \Xi_1), \tag{24}$$

where $\Xi_i \in M_k(HS(H_i, H_{i+1}))$ (resp. $\Xi_i \in M_k(HS(H_i, H_{i+1})^{\text{d}})$) if i is even (resp, if i is odd) and $\Xi_i \in M_k(HS(H_i, H_{i+1})^{\text{d}})$ (resp. $\Xi_i \in M_k(HS(H_i, H_{i+1})))$ if i is odd (resp, if i is even). If the matrices Ξ_i are of arbitrary sizes such that the product $\Xi_{n-1} \odot \dots \odot \Xi_1$ is well defined then they may be considered as square matrices, all of the same size, by complementing with zeros, and identity (24) will still hold. It follows that

$$\|S^{(k)}_{\text{id}\otimes\dots\otimes\text{id}(\varphi)}(\Xi_1 \odot \dots \odot \Xi_{n-1})\|_{\text{op}} \leq D \prod_{1 \leq i \leq n-1} \|\Xi_i\|_{\text{op}}, \text{ for all } \Xi_1, \dots \Xi_{n-1},$$

and hence the mapping $S_{\text{id}\otimes\dots\otimes\text{id}(\varphi)}$ is completely bounded and φ is an $(\text{id}, \dots, \text{id})$-multiplier. $\diamondsuit$

Theorem 6.5 *Let $\mathcal{A}_i \subseteq B(H_i)$, $i = 1, \ldots, n$, be C^*-algebras. Then* $\mathbf{M}(\mathcal{A}_1, \ldots, \mathcal{A}_n) = \mathbf{M}^\wedge(\mathcal{A}_1, \ldots, \mathcal{A}_n) = (\mathcal{A}_1 \odot \cdots \odot \mathcal{A}_n)^\sharp$.

Proof. By Propositions 6.2, 6.3 and 6.4,

$$\mathbf{M}^{cb}_{\mathrm{id},\ldots,\mathrm{id}}(\mathcal{A}_1'', \ldots, \mathcal{A}_n'') = (\mathcal{A}_1'' \odot \ldots \odot \mathcal{A}_n'')^\sharp.$$

Evidently,

$$\mathbf{M}^{cb}_{\mathrm{id},\ldots,\mathrm{id}}(\mathcal{A}_1, \ldots, \mathcal{A}_n) \subseteq \mathbf{M}^{cb}_{\mathrm{id},\ldots,\mathrm{id}}(\mathcal{A}_1'', \ldots, \mathcal{A}_n'') \cap (\mathcal{A}_1 \otimes \ldots \otimes \mathcal{A}_n).$$

Applying Propositions 6.2, 6.3 and 6.4, we obtain

$$\begin{aligned}
(\mathcal{A}_1 \odot \ldots \odot \mathcal{A}_n)^\sharp &\subseteq \mathbf{M}(\mathcal{A}_1, \ldots, \mathcal{A}_n)\\
&\subseteq \mathbf{M}^\wedge(\mathcal{A}_1, \ldots, \mathcal{A}_n)\\
&\subseteq \mathbf{M}^{cb}_{\mathrm{id},\ldots,\mathrm{id}}(\mathcal{A}_1, \ldots, \mathcal{A}_n)\\
&\subseteq \mathbf{M}^{cb}_{\mathrm{id},\ldots,\mathrm{id}}(\mathcal{A}_1'', \ldots, \mathcal{A}_n'') \cap (\mathcal{A}_1 \otimes \ldots \otimes \mathcal{A}_n)\\
&= (\mathcal{A}_1'' \odot \ldots \odot \mathcal{A}_n'')^\sharp \cap (\mathcal{A}_1 \otimes \ldots \otimes \mathcal{A}_n).
\end{aligned}$$

It hence suffices to show that

$$(\mathcal{A}_1'' \odot \ldots \odot \mathcal{A}_n'')^\sharp \cap \mathcal{A}_1 \otimes \ldots \otimes \mathcal{A}_n \subseteq (\mathcal{A}_1 \odot \ldots \odot \mathcal{A}_n)^\sharp.$$

Let $\varphi \in (\mathcal{A}_1'' \odot \ldots \odot \mathcal{A}_n'')^\sharp \cap \mathcal{A}_1 \otimes \ldots \otimes \mathcal{A}_n$. Then there exists a net $\{\varphi_\nu\}_{\nu \in J} \subseteq \mathcal{A}_1'' \odot \ldots \odot \mathcal{A}_n''$ such that $\sup_\nu \|\varphi_\nu\| < \infty$ and $\varphi = \text{w-}\lim_{\nu \in J} \varphi_\nu$. Write $\varphi_\nu = A_{1,\nu} \odot \ldots \odot A_{k,\nu}$, where $A_{1,\nu} \in M_{1,i_1}(\mathcal{A}_1'')$, $A_{2,\nu} \in M_{i_1,i_2}(\mathcal{A}_2''), \ldots, A_{n,\nu} \in M_{i_n,1}(\mathcal{A}_n'')$.

By Kaplansky density theorem for J^*-algebras ([16]) for each pair (m, ν) there exists a net $\{A_{m,\nu,\tau(m)}\}_{\tau(m)} \subset M_{i_{m-1},i_m}(\mathcal{A}_m)$ converging weakly to $A_{m,\nu}$ and such that $||A_{m,\nu,\tau(m)}|| \leq ||A_{m,\nu}||$ for all $\tau(m)$. Thus if $A_{\nu,\tau} = A_{1,\nu,\tau(1)} \odot A_{2,\nu,\tau(2)} \odot \ldots \odot A_{n,\nu,\tau(n)}$, where $\tau = (\tau(1), \ldots, \tau(n)\}$, then the net $\{A_{\nu,\tau}\}_\tau$ converges weakly to φ_ν and $||A_{\nu,\tau}|| \leq ||\varphi_\nu||$.

The convergence of the net $\{\varphi_\nu\}_{\nu \in J}$ to φ in weak operator topology implies that for every neighborhood U of 0 there exists $\nu(U)$ such that for every $\lambda \in J$ with $\lambda \geq \nu(U)$, we have that $\varphi_\lambda - \varphi \in U$. The convergence of $\{A_{\nu,\tau}\}_\tau$ to φ_ν implies the existence of $T(\nu(U), U)$ such that for every $\tau \geq T(\nu(U), U)$, we have that $A_{\nu(U),\tau} - \varphi_{\nu(U)} \in U$. Consider the net $A_U = A_{\nu(U),T(\nu(U),U)}$ indexed by the set of neighborhoods of 0 directed by inclusion. It is easy to check that A_U converges weakly to φ. The proof is complete. $\diamondsuit$

Denote by $(\mathcal{A}_1 \odot \ldots \odot \mathcal{A}_n)^\sim$ the set of all $\varphi \in \mathcal{A}_1 \otimes \ldots \otimes \mathcal{A}_n$ for which there exists a net $\{\varphi_\nu\} \subseteq \mathcal{A}_1 \odot \cdots \odot \mathcal{A}_n$, such that $\sup_\nu \|\varphi_\nu\|_{\rm ph} < \infty$ and if π_i is an irreducible representation of $\mathcal{A}_i$, $i = 1, \ldots, n$, then $\{(\pi_1 \otimes \ldots \otimes \pi_n)(\varphi_\nu)\}$ converges weakly to $(\pi_1 \otimes \ldots \otimes \pi_n)(\varphi)$. In [18] it was shown that $\mathbf{M}(\mathcal{A}, \mathcal{B}) = (\mathcal{A} \odot \mathcal{B})^\sim$ for commutative C^*-algebras $\mathcal{A}$ and $\mathcal{B}$, and the question of whether equality holds for arbitrary C*-algebras was posed. As a corollary of Theorem 6.5, we have the following description of universal multipliers.

Theorem 6.6 *Let $\mathcal{A}_i$, $i = 1, \ldots, n$, be C*-algebras. Then*

$$\mathbf{M}(\mathcal{A}_1, \ldots, \mathcal{A}_n) = \mathbf{M}^\wedge(\mathcal{A}_1, \ldots, \mathcal{A}_n) = (\mathcal{A}_1 \odot \ldots \odot \mathcal{A}_n)^\sim.$$

Proof. Let $\pi_1 = \bigoplus_{\pi \in IrrRep(\mathcal{A}_1)} \pi, \ldots, \pi_k = \bigoplus_{\pi \in IrrRep(\mathcal{A}_k)} \pi$, where $IrrRep(\mathcal{A}_i)$ is the set all irreducible representations of $\mathcal{A}_i$. Then

$$\begin{aligned} \mathbf{M}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) &= (\pi_1 \otimes \ldots \otimes \pi_n)^{-1}(\pi_1(\mathcal{A}_1) \odot \ldots \odot \pi_k(\mathcal{A}_n))^\sharp \\ &\subseteq (\mathcal{A}_1 \odot \ldots \odot \mathcal{A}_n)^\sim. \end{aligned}$$

Using arguments similar to the ones from the proof of Proposition 6.2, one can show that

$$(\mathcal{A}_1 \odot \ldots \odot \mathcal{A}_n)^\sim \subseteq \mathbf{M}(\mathcal{A}_1, \ldots, \mathcal{A}_n),$$

which together with Theorem 6.5 gives the statement of the theorem. $\diamondsuit$

References

[1] W.B. Arveson, *Operator Algebras and Invariant Subspaces*, Annals of Mathematics 100 (1974), 433-532

[2] C. Badea and V.I. Paulsen, *Schur multipliers and operator valued Foguel-Hankel operators*, Indiana Univ. Math. J. 50 (2001), no. 4, 1509-1522

[3] M.S. Birman and M.Z. Solomyak, *Stieltjes double-integral operators. II*, (Russian) Prob. Mat.Fiz. 2 (1967), 26-60

[4] M.S. Birman and M.Z. Solomyak, *Stieltjes double-integral operators, III (Passage to the limit under the integral sign)*, (Russian) Prob. Mat. Fiz. No 6 (1973), 27-53

[5] M.S. Birman and M.Z. Solomyak, *Operator Integration, perturbations and commutators*, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) Issled. Linein. Oper. Teorii Funktsii. 17, 170 (1989), 34-66

[6] M.S. Birman and M.Z. Solomyak, *Double operator integrals in a Hilbert space*, Integral Equations Operator Theory 47 (2003), no. 2, 131–168

[7] D.P. Blecher and C. Le Merdy, *Operator algebras and their modules – an operator space approach*, Oxford University Press, 2004

[8] D.P. Blecher and R. Smith, *The dual of the Haagrup tensor product*, J. London Math. Soc. (2) 4 (1992), 126-144

[9] E. Christensen and A.M. Sinclair, *Representations of completely bounded multilinear operators*, J. Funct. Anal. 72 (1987), 151-181

[10] K.R. Davidson and V.I. Pauslen, *Polynomially bounded operators*, J. Reine Angew. Math. 487 (1997), 153-170

[11] J. Diestel and J.J. Uhl, Jr., *Vector measures*, American Mathematical Society, Providence, 1977

[12] E.G. Effros, *Advances in quantized functional analysis*, Proceedings of the International Congress of Mathematicians (1987), 906-916

[13] E. Effros and Z-j. Ruan, *Operator Spaces*, Clarendon Press, Oxford, 2000

[14] A. Grothendieck, *Resume de la theorie metrique des produits tensoriels topologiques*, Boll. Soc. Mat. Sao-Paulo 8 (1956), 1-79

[15] D.W. Hadwin, *Nonseparable approximate equivalence*, Trans. of Amer. Math. Soc. 266 (1981), no 1, 203-231

[16] L.A. Harris, *A generalization of C^*-algebras*, Proc. London Math. Soc. (3) 42 (1981), no. 2, 331–361

[17] F. Hiai and H. Kosaki, *"Means of Hilbert Space Operators", Lecture Notes in Mathematics*, Vol 1820, Springer-Verlag, New York, Heidelberg, Berlin, 2003

[18] E. Kissin and V.S. Shulman, *Operator multipliers*, Pacific J. Math. 227 (2006), no. 1, 109-141

[19] V. Paulsen, *Completely bounded maps and operator algebras*, Cambridge University Press, 2002

[20] B.S. Pavlov, *Multidimensional operator integrals*, Problems of Math. Anal., No. 2: Linear Operators and Operator Equations (Russian), pp. 99–122. Izdat. Leningrad. Univ., Leningrad, 1969

[21] V.V. Peller, *Hankel operators in the perturbation theory of unitary and selfadjoint operators*, Funktsional. Anal. i Prilozhen. 19 (1985), no. 2, 37–51, 96

[22] V.V. Peller, *Multiple operator integrals and higher operator derivatives*, J. Funct. Anal. 233 (2006), no. 2, 515–544

[23] G.Pisier, *"Similarity Problems and Completely Bounded Maps", Lecture Notes in Mathematics,* Vol 1618, Springer-Verlag, Berlin, New York, 2001

[24] G. Pisier, *Introduction to Operator Space Theory*, Cambridge University Press, 2003

[25] R. R. Smith, *Completely bounded module maps and the Haagerup tensor product*, J. Funct. Anal. 102 (1991), 156-175

[26] N. Spronk, *Measurable Schur multipliers and completely bounded multipliers of the Fourier algebras*, Proc. London Math. Soc. (3) 89 (2004), no. 1, 161–192

[27] M.Z. Solomjak and V.V. Stenkin, *A certain class of multiple operator Stieltjes integrals. (Russian)*, Problems of Math. Anal., no. 2: Linear Operators and Operator Equations (Russian), pp. 122–134. Izdat. Leningrad. Univ., Leningrad, 1969

[28] V.V. Stenkin, *Multiple operator integrals. (Russian)*, Izv. Vysh. Uchebn. Zaved. Matematika. 4 (79) (1977), 102-115. English translation: Soviet Math. (Iz.VUZ) 21:4 (1977), 88-99

[29] D.Voiculescu *A non-commutative Weyl-von Neumann theorem*, Rev. Roumaine math. Pures Appl. 21 (1976), 97-113