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ABSTRACT. We give a local criterion in terms of a residue current for strong holomorphicity of a meromorphic function on an arbitrary puredimensional analytic variety. This generalizes a result by A Tsikh for the case of a reduced complete intersection.

1. INTRODUCTION

Let Z be an analytic variety in a neighborhood of the closed unit ball in $\mathbb{C}^n$. A germ of a holomorphic function ϕ on Z_{reg} is said to be (strongly) holomorphic on Z, $\phi \in \mathcal{O}_Z$, if it is the restriction to Z_{reg} of a holomorphic function Φ defined in the ambient space. A meromorphic function ϕ on Z_{reg} is said to be meromorphic on Z, $\phi \in \mathcal{M}_Z$, if it is the restriction to Z_{reg} of a meromorphic function in the ambient space. This notion is robust and for various alternative definitions, see [10]. On the other hand there are several weaker notions of holomorphicity of ϕ, but all of them imply that ϕ is at least meromorphic.

Suppose that Z is given by a complete intersection, $Z = \{F_1 = \cdots = F_p = 0\}$ and codim $Z = p$, and recall that we then have a well-defined $\bar{\partial}$-closed $(0, p)$-current

$$\mu^F = \bar{\partial}\frac{1}{F_p} \wedge \ldots \wedge \bar{\partial}\frac{1}{F_1},$$

the Coleff-Herrera product, [6], with support on Z. The following criterion was proved by A Tsikh, [16]; see also [10]:

Assume that the Jacobian $dF_1 \wedge \ldots \wedge dF_p$ is non-vanishing on Z_{reg}. A meromorphic function ϕ on Z is (strongly) holomorphic on Z if and only if the current $\phi\mu^F$ is $\bar{\partial}$-closed.

The assumption on the Jacobian implies (and is in fact equivalent to) that the annihilator of μ^F is precisely the sheaf $\mathcal{I}_Z$ of holomorphic functions that vanish on Z. The product $\phi\mu^F$ can be defined as the principal value

(1.1) $$\lim_{\epsilon \to 0} \chi_{|h|>\epsilon}(g/h)\mu^F,$$

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where g/h is a representation of ϕ. For the existence of this limit and the independence of the representation of ϕ, see Section 3. For further reference let us sketch a proof of Tsikh's theorem: If ϕ is strongly holomorphic, then it is represented by a function Φ that is holomorphic in the ambient space, and since μ^F is $\bar{\partial}$-closed it follows that $\phi\mu^F$ is. Conversely, assume that $\phi = g/h$ where g, h are holomorphic in the ambient space (and necessarily) h is generically non-vanising on Z_{reg}. Then formally at least, the assumption implies that

$$g\bar{\partial}\frac{1}{h}\wedge\bar{\partial}\frac{1}{F_p}\wedge\ldots\wedge\bar{\partial}\frac{1}{F_1} = 0,$$

and since also $h, F_1, \ldots, F_p$ form a complete intersection it follows from the duality theorem, [8] and [12], that g is in the ideal generated by $h, F_1, \ldots, F_p$, i.e., $g = \alpha h + \alpha_1 F_1 + \cdots + \alpha_p F_p$. Thus $\phi = g/h = \alpha$ on Z_{reg} so by definition $\phi \in \mathcal{O}_Z$.

Remark 1. One should remark here that it is *not* possible to use the Lelong current $[Z]$; in fact, the meromorphic functions ϕ such that $\phi[Z]$ is $\bar{\partial}$-closed form the wider class ω_Z^0 studied in [10]. □

In this paper we generalize Tsikh's result in two ways. We consider an arbitrary variety Z of pure codimension p, and we consider also the the non-reduced case, i.e., instead of $\mathcal{I}_Z$ we have an arbitrary pure-dimensional coherent ideal sheaf $\mathcal{J}$ with zero variety Z. To formulate our results we first have to discuss an appropriate generalization from [3] of the Coleff-Herrera product above.

In a neighborhood X of the closed unit ball there is a free resolution

$$(1.2) \qquad 0 \to \mathcal{O}(E_N) \xrightarrow{f_N} \ldots \xrightarrow{f_3} \mathcal{O}(E_2) \xrightarrow{f_2} \mathcal{O}(E_1) \xrightarrow{f_1} \mathcal{O}(E_0)$$

of the sheaf $\mathcal{O}/\mathcal{J}$. Here $\mathcal{O}(E_k)$ is the free sheaf associated to the trivial vector bundle E_k over X, and $E_0 \simeq \mathbb{C}$ so that $\mathcal{O}(E_0) \simeq \mathcal{O}$. In [3] we defined, given Hermitian metrics on E_k, a residue current $R = R_p + R_{p+1} + \cdots$ with support on Z, where R_k is a $(0, k)$-current that takes values in $E_k \simeq \text{Hom}(E_0, E_k)$, such that a holomorphic function ϕ is in $\mathcal{J}$ if and only if $R\phi = 0$. For simplicity we think that we have some fixed global frames for E_k and choose the trivial metrics that they induce. In this way we can talk about *the* residue current associated with (1.2).

If $\mathcal{J}$ is Cohen-Macaulay, i.e., each stalk $\mathcal{J}_x$ is a Cohen-Macaulay ideal in $\mathcal{O}_x$, then we may assume that $N = p$ and then $R = R_p$ is $\bar{\partial}$-closed. In general, $f_{k+1}R_{k+1} - \bar{\partial}R_k = 0$ for each k which can be written simply as $\nabla R = 0$ if $\nabla = f - \bar{\partial}$ and $f = \oplus f_k$.

The assumption that $\mathcal{J}$ has pure dimension p means that in each local ring $\mathcal{O}_x$ all the associated primes have codimension p. If Z is reduced, i.e., $Z \sim \mathcal{I}_Z$, then $\mathcal{O}_Z = \mathcal{O}/\mathcal{I}_Z$, and in general we have $\mathcal{O}_Z = \mathcal{O}/\mathcal{J}$.

Let ϕ be a holomorphic in $Z \setminus W$ where W is a subvariety of Z with positive codimension. As in the reduced case we say that ϕ is meromorphic if (locally) it is the restriction to $Z \setminus W$ of a meromorphic function Φ in ambient space; thus Φ and Φ' define the same ϕ if and only if $\Phi - \Phi'$ belongs to $\mathcal{J}$ generically on Z. In Section 3 we give a reasonable definition of ϕR for each meromorphic function ϕ on Z. Here is our main result.

Theorem 1.1. *Suppose that $Z \sim \mathcal{J}$ has pure codimension p and let R be the residue current associated to a resolution of $\mathcal{O}/\mathcal{J}$. Then a meromorphic function ϕ on Z is (strongly) holomorphic if and only if*

$$(1.3) \qquad\qquad \nabla(\phi R) = 0.$$

If $\mathcal{J}$ is Cohen-Macaulay and $N = p$ in (1.2), then $R = R_p$ and so (1.3) means that $\bar{\partial}(\phi R) = 0$.

The reduced case of course corresponds to $\mathcal{J} = \mathcal{I}_Z$.

Remark 2. If $f_1 = (F_1, \ldots, F_p)$ is a complete intersection, one can choose (1.2) as the Koszul complex, and then the residue current is precisely the Coleff-Herrera product μ^F, see, [3]. If $\mathcal{J} = \mathcal{I}_Z$ we thus get back Tsikh's theorem. $\qquad\qquad\square$

As a corollary we obtain the following result due to Malgrange [11] and Spallek [15]. One says that a function ϕ on Z is in $C^k(Z)$ if it is (locally) the restriction to Z of a C^k-function in the ambient space.

Corollary 1.2. *Assume that Z has pure codimension and is reduced. There is a natural number m such that if $\phi \in C^m(Z)$ is holomorphic on Z_{reg} then ϕ is strongly holomorphic on Z.*

Let $\mathcal{J}$ be any ideal sheaf and let (1.2) be a resolution of $\mathcal{O}/\mathcal{J}$. Let Z_k be the analytic set where f_k does not have have optimal rank. These sets Z_k are independent of the choice of resolution, $\subset Z_{k+1} \subset Z_k \subset$, their union is equal to the zero set Z of $\mathcal{J}$, and codim $Z_k \geq k$ for all k. Moreover, if $\mathcal{J}$ has codimension p, then it is pure if and only if codim $Z_k \geq k + 1$ for all $k > p$, and $\mathcal{J}$ is Cohen-Macaulay if and only if $Z_k = \emptyset$ for $k > p$. All these facts are well-known and can be found in, e.g., [9].

It is desirable to express the ideal $\mathcal{J}$ as

$$(1.4) \qquad\qquad \mathcal{J} = \cap_1^\nu \mathrm{ann}\, \mu_\ell,$$

where μ_j are so-called Coleff-Herrera currents, $\mu_j \in \mathcal{CH}_Z$, on Z, see below for a definition. This is because a Coleff-Herrera current is basically a differentiated Lelong current so ϕ annihilating such a current is an elegant intrinsic way to express that certain holomorphic differential operators applied to ϕ vanish on Z. If $\mathcal{J}$ has pure codimension

then it turns out, see, e.g., [1], that $\mathcal{J}$ is equal to the annihilator of the analytic sheaf

$$\mathcal{H}om\,(\mathcal{O}/\mathcal{J},\mathcal{CH}_Z) = \{\mu \in \mathcal{CH}_Z;\ \mathcal{J}\mu = 0\}.$$

This sheaf turns out to be coherent, and therefore there is a finite family of global sections in a neighborhood X of the closed unit ball such that (1.4) holds. One can ask whether there is a criterion for strong holomorphicity expressed in terms of the μ_ℓ.

Theorem 1.3. *Assume that $\mathcal{J}$ has pure codimension p and that μ_ℓ, $\ell = 1, \ldots, N$, generate $\mathcal{H}om\,(\mathcal{O}/\mathcal{J},\mathcal{CH}_Z)$. Let ϕ be meromorphic, let A be the smallest variety outside which ϕ is strongly holomorphic, and assume that $\mathrm{codim}\,(A \cap Z_k) \geq k + 2$ for $k > p$. Then ϕ is holomorphic if and only if $\phi\mu_\ell$ are $\bar{\partial}$-closed for all ℓ.*

Remark 3. If for instance $\mathcal{J}$ is Cohen-Macaulay, then Z_k is empty for $k > p$ so the condition is fulfilled for any meromorphic ϕ. If h is holomorphic and generically non-vanishing on Z, then $\bar{\partial}(1/h) \wedge \mu_\ell$ will be Coleff-Herrera currents whose common annihilator is precisely the ideal $(h, \mathcal{J})$, see Theorem 1.4 below. $\square$

The proofs of our main results basically follow the outline of the proof of Tsikh's theorem above. Therefore one is led to discuss products of residue currents of the type above. In case we just have two functions g and h, then there exist analytic continuations so that one can define

$$\bar{\partial}\frac{1}{h} \wedge \bar{\partial}\frac{1}{g} = \frac{\bar{\partial}|h|^{2\lambda} \wedge \bar{\partial}|g|^{2\mu}}{gh}\Big|_{\mu=0}\Big|_{\lambda=0},$$

but the result in general depends on the order of the evaluations. However if g, h form a complete intersection, i.e., $\mathrm{codim}\,(Z^g \cap Z^h) = 2$, then the product is robust; it is independent of the order, and one can also, e.g., put $\lambda = \mu$ before taking $\lambda = 0$. See [14] for various other definitions and the relation to the original definition in [6]. Consider now two general ideal sheaves $\mathcal{I}$ and $\mathcal{J}$, and assume that we have fixed two resolutions $g, \mathcal{O}(E^g)$ and $h, \mathcal{O}(E^h)$ with associated currents R^g and R^h. Following [4] we can define the product $R^h \wedge R^g$ in analogous ways; it is a current that takes values in $E^g \otimes E^h$.

Theorem 1.4. *Assume that $\mathcal{I}$ and $\mathcal{J}$ are ideal sheaves such that*

$$(1.5) \qquad\qquad \mathrm{codim}\,(Z_k^{\mathcal{I}} \cap Z_\ell^{\mathcal{J}}) \geq k + \ell, \quad k, \ell \geq 1.$$

Then the annihilator of $R^h \wedge R^g$ is equal to $(\mathcal{I}, \mathcal{J})$. Moreover, the definition of this current is independent of the order of the limits.

In case both sheaves are Cohen-Macaulay and both resolutions have minimal lengths, then $R^h \wedge R^g$ coincides with the current obtained from the tensor product of the resolutions.

Remark 4. Let $\mathcal{I} = (g_1)$ and $\mathcal{J} = (h_1)$ be complete intersections, and choose the Koszul complexes as resolutions. Then, see [3], R^g and R^h are the Bochner-Martinelli type residues introduced in [13]. Moreover, the tensor product of these resolutions is the Koszul complex generated by (g_1, h_1), and so the last statement in the theorem means that this product coincides with the Bochner-Martinelli residue associated with (g_1, h_1). This fact is proved already in [17]. $\qquad\square$

Corollary 1.5. *Assume that $\mathcal{I}$ has pure codimension and let R be the residue current associated with a resolution. If h is generically nonvanishing on Z, then $(h, \mathcal{I})$ coincides with the annihilator of*

$$\bar{\partial}\frac{1}{h} \wedge R.$$

Remark 5. Theorem 1.4 extends in a quite obvious way, and with essentially the same proof, to any finite number of ideal sheaves. $\qquad\square$

For the proof of our main results we only need Corollory 1.5, but the general result, Theorem 1.4, does not require much more effort, and we provide a proof in Section 4. In the last section we then prove the main results. However first we have to recall some material from [3] and [4].

2. Some residue theory

The sheaf of Coleff-Herrera currents (or currents of *residual type*) $\mathcal{CH}_Z$ consists of all $\bar{\partial}$-closed $(0, p)$-currents μ with support on Z such that $\bar{\psi}\mu = 0$ for each ψ vanishing on Z, and which in addition fulfills the so-called standard extension property, SEP, see below. Locally, any $\mu \in \mathcal{CH}_Z$ can be realized as a meromorphic differential operator acting on the current of integration $[Z]$ (combined with contractions with holomorphic vector fields), see [5].

In [4] we introduced the sheaf of *hypermeromorphic* currents $\mathcal{HM}$ in X. It is a module over the sheaf $\mathcal{E}_\bullet$ of smooth forms, and closed under $\bar{\partial}$. For any $T \in \mathcal{HM}$ and variety V there exists a restriction $T|_V$ that is in $\mathcal{HM}$ and has support on V, and $T = T|_V$ if and only if T has support on V. Moreover, taking restriction commutes with multiplication by smooth forms. If H is a holomorphic tuple such that $\{H = 0\} = V$, then $\lambda|H|^{2\lambda}T$ has a curren-valued analytuc continuation to $\mathrm{Re}\,\lambda > -\epsilon$ and

$$(2.1) \qquad\qquad T|_{X \setminus V} = |H|^{2\lambda}T\big|_{\lambda = 0}.$$

We say that a current T with support on a variety V has SEP (with respect to V) if $T|_W = 0$ for each $W \subset V$ with positive codimension.

Proposition 2.1. *If $\mu \in \mathcal{HM}$ with bidegree $(*, p)$ has support on a variety V of codimension $k > p$ then $\mu = 0$. Furthermore, if V has codimension p, then the sheaf of Coleff-Herrera currents $\mathcal{CH}_V$ coincides with the subsheaf of $\mathcal{HM}$ of $\bar{\partial}$-closed $(0, p)$-currents with support on V.*

We first briefly recall the construction in [3]. Let

$$(2.2) \qquad 0 \to E_N \xrightarrow{f_N} \ldots \xrightarrow{f_3} E_2 \xrightarrow{f_2} E_1 \xrightarrow{f_1} E_0 \to 0$$

be a generically exact complex of Hermitian vector bundles over X, where $E_0 \simeq \mathbb{C}$ for simplicity, let

$$(2.3) \qquad 0 \to \mathcal{O}(E_N) \xrightarrow{f_N} \ldots \xrightarrow{f_1} \mathcal{O}(E_0)$$

be the corresponding complex of locally free sheaves, and let $\mathcal{J}$ be the ideal sheaf $f_1 \mathcal{O}(E_1) \subset \mathcal{O}$. Then by the Buchsbaum-Eisenbud theorem, see [9], (2.3) is a resolution of $\mathcal{O}/\mathcal{J}$ if and only if $\operatorname{codim} Z_k \geq k$ for all k. Assume that (2.2) is pointwise exact outside the variety Z, and over $X \setminus Z$ let $\sigma_k \colon E_{k-1} \to E_k$ be the minimal inverses of f_k. Then $f\sigma + \sigma f = I$, where I is the identity on $E = \oplus E_k$, $f = \oplus f_k$ and $\sigma = \oplus \sigma_k$. The bundle E has a natural superbundle structure $E = E^+ \oplus E^-$, where $E^+ = \oplus E_{2k}$ and $E^- = \oplus E_{2k+1}$, and f and σ are odd mappings with respect to this structure, see, e.g., [3] for more details.

The operator $\nabla = f - \bar{\partial}$ acts as an odd mapping on $\mathcal{C}^{0,\bullet}(X, E)$, the space of $(0, *)$-currents with values in E, and extends to an odd mapping ∇_{End} on $\mathcal{C}^{0,\bullet}(X, \mathrm{End} E)$, and $\nabla_{\mathrm{End}}^2 = 0$. If

$$u = \sigma + (\bar{\partial}\sigma)\sigma + (\bar{\partial}\sigma)^2\sigma + \cdots$$

then $\nabla_{\mathrm{End}} u = I$ in $X \setminus Z$. One can define a canonical current extension U of u across Z as the analytic continuation to $\lambda = 0$ of $|F|^{2\lambda} u$, where F is any holomorphic function that vanishes on Z; e.g., $F = f_1$ will do if (2.3) is a resolution. In the same way we can define the current

$$R = \bar{\partial}|F|^{2\lambda} \wedge u|_{\lambda=0}$$

with support on Z, and

$$(2.4) \qquad\qquad \nabla_{\mathrm{End}} U = I - R.$$

We have that

$$R = \sum_\ell R^\ell = \sum_{\ell k} R_k^\ell,$$

where R_k^ℓ is a $(0, k - \ell)$-current that takes values in $\mathrm{Hom}\,(E_\ell, E_k)$, i.e.,

$$R_k^\ell \in \mathcal{C}^{0,k-\ell}(X, \mathrm{Hom}\,(E_\ell, E_k)).$$

We also recall from [3] that if (2.3) is a resolution, then $R^\ell = 0$ for all $\ell \geq 1$. From [4] we know that both U^g and R^g are in $\mathcal{HM}$. Thus $R = R^0 = R_p + R_{p+1} + \cdots$.

Below we will consider analogues of R and U obtained in a different way. The following proposition is proved precisely as Proposition 2.2 in [3].

Proposition 2.2. *Consider the generically exact complex* (2.2) *and let U and R be any currents such that* (2.4) *holds. If $R^1 = 0$ then* ann $R = \mathcal{J}$. *If $R^\ell = 0$ for all $\ell \geq 1$ then the associated sheaf complex* (2.3) *is exact, i.e., a resolution of $\mathcal{O}/\mathcal{J}$.*

3. Multiplication by meromorphic functions

For any hypermeromorphic current T and holomorphic function h we defined in [4] product $(1/h)T$ as the value at $\lambda = 0$ of $|h|^{2\lambda}T$. It is again a hypermeromorphic current and it is clear that $\alpha(1/h)T = (1/h)\alpha T$ if α is smooth. Notice however, that in general it is *not* true $f(1/fg)T = (1/g)T$.

Lemma 3.1. *Suppose that $Z \sim \mathcal{J}$ has pure codimension p and let R be the residue current associated with a resolution* (1.2). *If h is generically nonvanishing on Z, then $(1/h)R$ has the SEP on Z.*

Proof of Lemma 3.1. Assume that $V \subset Z$ has positive codimension. Then $((1/h)R_p)|_V = 0$ in view of Proposition 2.1. Outside the variety Z_{p+1} we have that $R_{p+1} = \alpha_{p+1}R_p$ where $\alpha_{p+1} = \bar{\partial}\sigma_{p+1}$ is smooth, and hence

$$((1/h)R_{p+1})|_V = ((1/h)\alpha_{p+1}R_p)|_V = $$
$$(\alpha_{p+1}(1/h)R_p)|_V = \alpha_{p+1}((1/h)R_p)|_V = 0.$$

It follows that $((1/h)R_{p+1})|_V$ has support on Z_{p+1} which has codimension $\geq p + 2$, and hence it vanishes by virtue of Proposition 2.1. Now $R_{p+2} = \alpha_{p+2}R_{p+1}$ outside Z_{p+2} that has codimension $\geq p + 3$, and so $(g(1/h)R_{p+2})|_V = 0$ by a similar argument. Continuing in this way the lemma follows. $\square$

Given a meromorphic function ϕ on Z we can define ϕR as $g(1/h)R$ if g/h represents ϕ. Since $(1/h)R$ has the SEP also $g(1/h)R$ has. Since the difference of two representations of ϕ lies in $\mathcal{J}$ outside some $V \subset Z$ of positive codimension and $\mathcal{J}R = 0$, it follows from the SEP that ϕR is well-defined. Moreover, if $\psi \in \mathcal{O}_Z$, it follows that

$$\psi(\phi R) = (\psi\phi)R.$$

One can just as well, with some small extra effort, define ϕR as the limit (1.1), see, e.g., [2].

Remark 6. Let ϕ be holomorphic in $Z \setminus V$, where V has positive codimension and contains Z_{sing}. If ϕ is meromorphic on Z, then we have seen that ϕR has a natural current extension from $X \setminus V$ across V. Also the converse holds. In fact, one can always find a holomorphic form α with values in Hom (E_p, E_0) such that $R_p \cdot \alpha = [Z]$, see [1]. Therefore, if ϕR has an extension across V also $\phi[Z]$ has, and it then follows from [10] that ϕ is meromorphic. $\square$

4. Tensor products of resolutions

Assume that $\mathcal{O}(E_k^g), g_k$ and $\mathcal{O}(E_\ell^h), h_\ell$ are resolutions of $\mathcal{O}/\mathcal{I}$ and $\mathcal{O}/\mathcal{J}$, respectively. We can define a complex (2.3), where

$$(4.1) \qquad\qquad E_k = \bigoplus_{i+j=k} E_i^g \otimes E_j^h,$$

$f = g + h$, or more formally, $f = g \otimes I_{E^h} + I_{E^g} \otimes h$, such that

$$f(\xi \otimes \eta) = g\xi \otimes \eta + (-1)^{\deg \xi}\xi \otimes h\eta.$$

One extends (4.1) to current-valued sections ξ and η and $\deg \xi$ means total degree. It is natural to write $\xi \wedge \eta$ rather than $\xi \otimes \eta$, and of course we can define $\eta \wedge \xi$ as $(-1)^{\deg \xi \deg \eta}\xi \wedge \eta$.

Taking $H = h_1$ and following, e.g., the proof of Proposition 2.1 in [4] we can define hypermeromorphic currents

$$U^h \wedge R^g = |H|^{2\lambda}u^h \wedge R^g|_{\lambda=0}, \quad R^h \wedge R^g = \bar{\partial}|H|^{2\lambda}u^h \wedge R^g|_{\lambda=0}.$$

We let

$$U = I^h \wedge U^g + U^h \wedge R^g, \quad R = R^h \wedge R^g.$$

Analogously with $G = g_1$ we define

$$R^g \wedge U^h = \bar{\partial}|G|^{2\lambda} \wedge u^g \wedge U^h, \quad R^g \wedge R^h = \bar{\partial}|G|^{2\lambda} \wedge u^g \wedge R^h.$$

Lemma 4.1. *With the assumptions in Theorem 1.4 we have that*

$$(4.2) \qquad\qquad \nabla_{\text{End}}U = I - R$$

and

$$(4.3) \qquad\qquad U^h \wedge R^g = R^g \wedge U^h.$$

Proof. We first claim that

$$(4.4) \qquad\qquad |H|^{2\lambda}R^g|_{\lambda=0} = R^g.$$

This is equivalent to saying that the restriction $R^g|_{Z^h}$ of R^g to $Z^h = \{H = 0\}$ vanishes. To see this, first notice that $R_1^g|_{Z^h} = 0$ since it is a hypermeromorphic $(0,1)$-current with support on $Z_1^g \cap Z_1^h$ which by assumption has codimension at least 2. Outside Z_2^g we can multiply with α_2^g and get that $R_2^g|_{Z^h} = 0$ there. Thus $R_2^g|_{Z^h} = 0$ has support on $Z_2^g \cap Z_1^h$ and again it must vanish for degree reasons. Continuing in this way (4.4) follows. Now, since $\nabla_{\text{End}}R^g = 0$, for $\lambda >> 0$ we have

$$\nabla_{\text{End}}\big(|H|^{2\lambda}u^h \wedge R^g\big) = |H|^{2\lambda}I^h \wedge R^g - \bar{\partial}|H|^{2\lambda} \wedge u^h \wedge R^g,$$

and in view of (4.4) therefore

$$(4.5) \qquad\qquad \nabla_{\text{End}}(U^h \wedge R^g) = I^h \wedge R^g - R^h \wedge R^g.$$

From this (4.2) follows. In the same way

$$(4.6) \qquad\qquad \nabla_{\text{End}}(R^g \wedge U^h) = R^g \wedge I^h - R^g \wedge R^h.$$

Notice that

$$(4.7) \qquad\qquad (U^h)_1^0 \wedge (R^g)_1^0 - (R^g)_1^0 \wedge (U^h)_1^0 = 0$$

outside Z_1^h since $(U^h)_1^0$ is smooth there. On the other hand, both terms have support on Z_1^g. Thus the left hand side is a hypermeromorphic $(0,1)$-current with support on $Z_1^g \cap Z_1^h$. Since this set has codimension at least 2, the current must vanish identically. Outside $Z_{\ell+1}^h$ we have

$$(4.8) \qquad \alpha_{\ell+1}^h(U_\ell^h \wedge R^g) = U_{\ell+1}^h \wedge R^g, \quad \alpha_{\ell+1}^h(R^g \wedge U_\ell^h) = R^g \wedge U_{\ell+1}^h$$

where $\alpha_{\ell+1}^h$ is smooth. Similarly, outside Z_{k+1}^g we have

$$(4.9) \qquad \alpha_{k+1}^g(U^h \wedge R_k^g) = U^h \wedge R_{k+1}^g, \quad \alpha_{k+1}^g(R_k^g \wedge U^h) = R_{k+1}^g \wedge U^h.$$

Now assume that we have proved that

$$(4.10) \qquad\qquad (U^h)_\ell^0 \wedge (R^g)_1^0 - (R^g)_k^0 \wedge (U^h)_1^0 = 0$$

when $\ell + k < m$ and suppose that $\ell + k = m$. If say $\ell = 1$ then (4.10) holds at least outside Z_1^h. Moreover, in view of (4.9) it holds outside Z_m^g by the induction hypothesis. Thus the current in (4.7) has support on $Z_1^h \cap Z_{m-1}^g$. If $\ell > 1$, for similar reasons, it follows that (4.10) holds outside $Z_\ell^h \cap Z_k^g$. Since the current has bidegree $(0, m-1)$ and $\mathrm{codim}\,(Z_\ell^h \cap Z_k^g) \geq \ell + k = m$ it must vanish. Thus we have proved that

$$(U^h)^r \wedge (R^g)^s = (R^g)^r \wedge (U^h)^s$$

for $r = s = 0$. The general case follows in the same way. $\qquad\square$

Along the same lines one can verify that

$$(4.11) \qquad\qquad U^h \wedge R^g = |H|^{2\lambda} u^h \wedge \bar{\partial}|G|^{2\lambda} \wedge u^g\big|_{\lambda=0}.$$

Proof of Theorem 1.4. Applying ∇_{End} to (4.3) we get by (4.5) and (4.6) that

$$(I^h - R^h) \wedge R^g = R^g \wedge (I^h - R^h)$$

and thus

$$(4.12) \qquad\qquad R^h \wedge R^g = R^g \wedge R^h.$$

Using (4.11) it also follows that

$$R^h \wedge R^g = \bar{\partial}|H|^{2\lambda} \wedge u^h \wedge \bar{\partial}|G|^{2\lambda} \wedge u^g|_{\lambda=0}.$$

Since $(R^g)^s = 0$ for $s \geq 1$ we have that

$$R = \sum_{s,r \geq 0} (R^h)^r \wedge (R^g)^s = \sum_{r \geq 0} (R^h)^m \wedge (R^g)^0.$$

However, in view (4.12) we thus have that

$$R = (R^h)^0 \wedge (R^g)^0 = R^0$$

i.e., $R^m = 0$ for $m \geq 1$. From Proposition 2.2 we now conclude that $\mathcal{O}(E), f$ is a resolution and $\mathrm{ann}\, R = (\mathcal{I}, \mathcal{J})$.

Finally, assume that $\mathcal{I}$ and $\mathcal{J}$ are Cohen-Macaulay sheafs and the resolutions $\mathcal{O}(E^g), g$ and $\mathcal{O}(E^h), h$ have minimal lengths $\mathrm{codim}\,\mathcal{I}$ and

codim $\mathcal{J}$, respectively. Then the product resolution $\mathcal{O}(E), f$ has (minimal) length $p = \operatorname{codim} \mathcal{I} + \operatorname{codim} \mathcal{J}$. Let U^f, R^f denote the currents associated with this complex. Then R^f as well as $R^h \wedge R^g$ are $\bar{\partial}$-closed hypermeromorphic currents of bidegree $(0, p)$ with support on $Z = Z^g \cap Z^h$ which has codimension p, and hence they are Coleff-Herrera currents, according to Proposition 2.1. Moreover,

$$\nabla_{\mathrm{End}}(U^f - U^h \wedge U^g) = R^f - R^h \wedge R^g.$$

It follows from [1] (analogously as for a complete intersection in [2]) that $R^f - R^h \wedge R^g$ must vanish. □

Remark 7. If $\mathcal{O}(E^g), g$ and $\mathcal{O}(E^h), h$ are resolutions one can verify (without residue calculus) that product complex is a resolution as well if and only if (1.5) holds. Since this should be well-known we just sketch an argument: It is not too hard to see that (for each fixed point x)

$$(4.13) \qquad H^m(E^h \otimes E^g) = \otimes_{\ell+k=m} H^\ell(E^h) \otimes H^k(E^g).$$

For instance, one can choose Hermitian metrics on E^g and E^h. If h^* and h^* and $f^* = g^* + h^*$ are the induced adjoint mappings and $\Delta^f = ff^* + f^*f$, etc, then $\Delta^f = \Delta^g + \Delta^h$. As usual each class in $H^m(E^h \otimes E^g)$ has a unique harmonic representative

$$v = \sum_{\ell+k=m} \xi_\ell \wedge \eta_k.$$

However, it is easily verified that $\Delta^f v = 0$ if and only if $\Delta^g \xi_\ell = 0 = \Delta^h \eta_k$ for all ℓ, k. Thus (4.13) follows.

   Let $Z_k^{\mathcal{I}}$ and $Z_\ell^{\mathcal{J}}$ be the varieties associated to the sheaves $\mathcal{I}$ and $\mathcal{J}$. Since $\mathcal{O}(E^g), g$ is exact, it follows that $H^k(E^g) = 0$ at a given point x if and only if $x \notin Z_k^{\mathcal{I}}$ and similarly for E^h. In view of (4.13), therefore $H^m(E) \neq 0$ at x if and only if

$$x \in \cup_{\ell+k=m} Z_k^{\mathcal{I}} \cap Z_\ell^{\mathcal{J}}.$$

Thus codim $Z_m \geq m$ for all m if and only if (1.5) holds, and in view of the Buchsbaum-Eisenbud theorem, see [9], therefore $\mathcal{O}(E), f$ is a resolution if and only if (1.5) holds. □

5. Proofs of the main results

   We begin with

Proof of Theorem 1.1. If ϕ is strongly holomorphic, then it is represented by a function Φ that is holomorphic in a neighborhood of Z. Thus $\nabla(\phi R) = \nabla(\Phi R) = \Phi \nabla R = 0$.

   Now assume that $\nabla(\phi R) = 0$ and ϕ is represented by g/h. Then by Leibniz' rule, see [4], we have that

$$0 = \nabla(g(1/h)R) = -g\bar{\partial}\frac{1}{h} \wedge R.$$

This means that g annihilates the current $\bar{\partial}(1/h)\wedge R$, and by Corollary 1.5 therefore $g = \alpha h + \psi$, where $\psi \in \mathcal{J}$. It follows that $g/h - \alpha$ belongs to $\mathcal{J}$ outside the zero set of h, and hence by definition ϕ is strongly holomorphic. $\qquad\square$

Since ϕR is a well-defined, we also have a well-defined current

$$\bar{\partial}\phi\wedge R = -\nabla(\phi R) = g\bar{\partial}\frac{1}{h}\wedge R = \bar{\partial}|h|^{2\lambda}\wedge\frac{g}{h}R|_{\lambda=0}.$$

Proof of Corollary 1.2. First assume that ϕ is (strongly) smooth and holomorphic on Z_{reg}. It is well-known that each weakly holomorphic function on Z (i.e., ϕ holomorphic on Z_{reg} and locally bounded at Z_{sing}) is meromorphic, see, e.g., [7]. Therefore, we have a priori two definitions of ϕR; either as multiplication of smooth function times R or as multiplication by the meromorphic function ϕ. However, they coincide on Z_{reg} and by the SEP therefore they coincide even across Z_{sing}. Therefore also the two possible definitions of $\nabla(\phi R) = -\bar{\partial}\phi\wedge R$ coincide. Since ϕ is holomorphic on Z_{reg} it follows that $\bar{\partial}\phi\wedge R$ has support on Z_{sing}. On the other hand,

$$(\bar{\partial}\phi\wedge R)|_{Z_{sing}} = \bar{\partial}\phi\wedge R_{Z_{sing}} = 0$$

by the SEP, and hence $\nabla(\phi R) = -\bar{\partial}\phi\wedge R = 0$. Now the corollary follows from Theorem 1.1 with $m = \infty$. However, a careful inspection of all arguments reveals that only a finite number of derivatives (not depending on ϕ) come into play. We omit the details. $\qquad\square$

Proof of Theorem 1.3. The hypothesis means that $0 = \bar{\partial}(\phi\mu)$ for all $\mu \in \mathcal{H}om\,(\mathcal{O}/\mathcal{J},\mathcal{CH}_Z)$. It is proved in [1] that each current μ in $\mathcal{H}om\,(\mathcal{O}/\mathcal{J},\mathcal{CH}_Z)$ can be written $\mu = \xi R_p$ for some $\xi \in \mathcal{O}(E^*)$ such that $f_{p+1}^*\xi = 0$ and conversely for each such ξ the current $\mu = \xi R_p$ is in $\mathcal{H}om\,(\mathcal{O}/\mathcal{J},\mathcal{CH}_Z)$. Here f_k^* are the induced mapping(s) on the dual complex $\mathcal{O}(E_k^*)$. Thus

$$0 = \bar{\partial}\phi\wedge\xi R_p$$

for each such ξ. At a given stalk outside Z_{p+1}, the ideal $\mathcal{J}_x$ is Cohen-Macaulay, so if we choose a minimal resolution $\mathcal{O}(\tilde{E}), \tilde{f}$ there it will have length p. If $\tilde{R}_p$ denotes the resulting (germ of a) residue current, then the hypothesis implies that

$$0 = \bar{\partial}\phi\wedge\tilde{R}_p$$

since then trivially $\tilde{f}_{p+1}^*\xi = 0$ for each $\xi \in \mathcal{O}(\tilde{E}_p^*)$. However, $R_p = \alpha\tilde{R}_p$, where α is smooth, see [3]. It follows that $\bar{\partial}\phi\wedge R_p$ vanishes outside Z_{p+1}. Since $R_{p+1} = \alpha_{p+1}R_p$ outside Z_{p+1} it follows that also $\bar{\partial}\phi\wedge R_{p+1}$ has support on Z_{p+1}. However, it is clear that $\bar{\partial}\phi\wedge R$ must have support on A. Using the hypothesis $\mathrm{codim}\,(A\cap Z_k) \geq k+2$ for $k > p$, it follows by induction that $\bar{\partial}\phi\wedge R = 0$. Thus ϕ is strongly holomorphic according to Theorem 1.1. $\qquad\square$

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