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Abstract

The Spectral Problem is to describe possible spectra $\sigma(A_j)$ for an irreducible n-tuple of Hermitian operators s.t. $A_1 + \ldots + A_n$ is a scalar operator. In case when $m_j = |\sigma(A_j)|$ are finite and a rooted tree $\mathrm{T}_{m_1,\ldots,m_n}$ with n branches of lengths $m_1, \ldots, m_n$ is a Dynkin graph the explicit answer to the Spectral Problem was given recently by the authors. In present work we solve the Spectral Problem for all simply laced extended Dynkin graphs, i.e. when $(m_1, \ldots, m_n) \in \{(2,2,2,2), (3,3,3), (4,4,2), (6,3,2)\}$.

KEYWORDS: Dynkin graph, quiver representation, Coxeter functor, Horn's problem, extended Dynkin graph, spectral problem

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Introduction.

1. Let A_1, A_2, …, A_n be Hermitian $m \times m$ matrices with given eigenvalues: $\tau(A_j) = \{\lambda_1(A_j) \geq \lambda_2(A_j) \geq \ldots \geq \lambda_m(A_j)\}$. The well-known classical problem about the spectrum of a sum of two Hermitian matrices (Horn's problem) is to describe possible values of $\tau(A_1), \tau(A_2), \tau(A_3)$ such that $A_1 + A_2 = A_3$. In more symmetric setting one can seek for a connection between $\tau(A_1), \tau(A_2), \ldots, \tau(A_n)$ necessary and sufficient for the existence of Hermitian operators such that $A_1 + A_2 + \ldots + A_n = \gamma I$ for a fixed $\gamma \in \mathbb{R}$.

A recent solution of this problem (see [4, 5] and others) gives a complete description of possible $\tau(A_1), \tau(A_2), \ldots, \tau(A_n)$ in terms of linear inequalities conjectured by A. Horn.

2. A modification of Horn's problem called henceforth the *spectral problem* was considered in [6, 7]. Let A_1, A_2, …, A_n be bounded linear Hermitian operators on a separable Hilbert space H. For an operator X denote by $\sigma(X)$ its spectrum. Given $M_1, M_2, \ldots, M_n$ closed subsets of $\mathbb{R}$ and $\gamma \in \mathbb{R}$ the problem is to determine whether there are Hermitian operators A_1, A_2, …, A_n on H such that $\sigma(A_j) \subseteq M_j$ $(1 \leq j \leq n)$, and $A_1 + A_2 + \ldots + A_n = \gamma I$.

In this work the sets $M_1, M_2, \ldots, M_n$ will be finite. Even for finite M_k it can be very complicated to describe such n-tuples of operators up to unitary equivalence if $|M_k|$ is large enough.

The essential difference with Horn's classical problem is that we do not fix the dimension of H (it may be finite or infinite) and the spectral multiplicities. It seems that the solution of the spectral and strict spectral problems could not be deduced directly from the Horn inequalities, since the number of necessary inequalities increases with m.

3. The Spectral Problem can be stated in terms of $*$-representations of $*$-algebras. Namely, let $\alpha^{(j)} = (\alpha_1^{(j)}, \alpha_2^{(j)}, \ldots, \alpha_{m_j}^{(j)})$ $(1 \leq j \leq n)$ be vectors with positive strictly decreasing coefficients. Put $M_j = \alpha^{(j)}$. Let us consider the associative algebra defined by the following generators and relations (see. [10]):

$$\mathcal{A}_{M_1,\ldots,M_n,\gamma} =$$
$$\mathbb{C}\langle p_1^{(1)}, p_2^{(1)}, \ldots, p_{m_1}^{(1)}, p_1^{(2)}, p_2^{(2)}, \ldots, p_{m_2}^{(2)}, \ldots, p_1^{(n)}, p_2^{(n)}, \ldots, p_{m_n}^{(n)} |$$
$$p_k^{(i)2} = p_k^{(i)}, \sum_{i=1}^{n} \sum_{k=1}^{m_i} \alpha_k^{(i)} p_k^{(i)} = \gamma e,$$
$$\sum_{k=1}^{m_i} p_k^{(i)} = e, p_j^{(i)} p_k^{(i)} = 0, j, k = 1, \ldots, m_i, j \neq k, i = 1, \ldots, n\rangle.$$

Here e is the identity of the algebra. This is a $*$-algebra, if we declare all generators to be self-adjoint.

A $*$-representation π of $\mathcal{A}_{M_1,\ldots,M_n,\gamma}$ determines an n-tuple of non-negative operators $A^{(j)} = \sum_{k=1}^{m_i} \alpha_k^{(i)} P_k^{(i)}$, where each of the families of orthoprojections, $\{P_i = \pi(p_i),\ i = 1, \ldots, k\}$ forms a resolution of the identity and such that $A^{(1)} + \ldots + A^{(n)} = \gamma I$. Moreover, $\sigma(A^{(j)}) \subseteq M_j$. And viceversa, any n-tuple of Hermitian matrices $A^{(1)}$, $A^{(2)}, \ldots,$ $A^{(n)}$ such that $A^{(1)} + \ldots + A^{(n)} = \gamma I$ and $\sigma(A^{(j)}) \subseteq M_j$ determines a representation π of $\mathcal{A}_{M_1,\ldots,M_n,\gamma}$ by taking $\pi(p_k^{(j)})$ to be the spectral projections corresponding to eigenvalues $\alpha_k^{(j)}$ respectively.

So in terms of $*$-representations, the spectral problem is a problem consisting of the following two parts: 1) a description of the set $\Sigma_{m_1,m_2,\dots,m_n}$ of the parameters $\alpha_k^{(j)}$, γ for which there exist $*$-representations of $\mathcal{A}_{M_1,\dots,M_n,\gamma}$. 2) a description of *-representations π of the *-algebra $\mathcal{A}_{M_1,M_2,\dots,M_n,\gamma}$.
4. A natural way to try to solve the spectral problem is to describe all irreducible $*$-representations up to unitary equivalence and then all $*$-representations as sums or direct integrals of irreducible representations. Obviously if there is a $*$-representation of the algebra $\mathcal{A}_{M_1,\dots,M_n\gamma}$ then there is its irreducible $*$-representation. Hence the set $\Sigma_{m_1,m_2,\dots,m_n}$ coincides with the set of parameters for which algebra $\mathcal{A}_{M_1,\dots,M_n,\gamma}$ has at least one irreducible *-representation;

The second part of the spectral problem could be formulated in the following way: find the formulae for the irreducible $*$-representations of the algebra $\mathcal{A}_{M_1,\dots,M_n,\gamma}$ for parameters $(\lambda_k^{(j)},\gamma) \in \Sigma_{m_1,\dots,m_n}$ or at least present an algorithm to construct such representations.
5. A key step in solving spectral problem is to describe the irreducible non-degenerate representations. Let us call a $*$-representation π of the algebra $\mathcal{A}_{M_1,\dots,M_n,\gamma}$ *non-degenerate* if $\pi(p_k^{(j)}) \neq 0$ for all k and j.

Consider the following set: $\Sigma_{m_1,\dots,m_n}^{n.-d.} = \{(\{\lambda_k^{(j)}\},\gamma) |$ *there is a non-degenerate $*$-representation of* $\mathcal{A}_{M_1,\dots,M_n,\gamma}\}$; which depends only on $(m_1, \dots, m_n)$. Every irreducible representation of the algebra $\mathcal{A}_{M_1,M_2,\dots,M_n,\gamma}$ is an irreducible non-degenerate *-representation of an algebra $\mathcal{A}_{\widetilde{M}_1,\dots,\widetilde{M}_n,\gamma}$ for some subsets $\widetilde{M}_j \subset M_j$. Hence $(M_1,\dots,M_n,\gamma) \in \Sigma_{m_1,\dots,m_n}$ if there exist $(\widetilde{M}_1,\dots,\widetilde{M}_n,\gamma) \in \Sigma^{n.-d.}_{|\widetilde{M}_1|,\dots,|\widetilde{M}_n|}$. Thus the description of $\Sigma_{m_1,\dots,m_n}$ follows from the description of $\Sigma^{n.-d.}_{k_1,\dots,k_n}$ where $k_j \leq m_j, 1 \leq j \leq n$.
6. With an integer vector $(m_1,\dots,m_n)$ we will associate a non-oriented star-shape graph G with n branches of the lengths m_1, m_2, $\dots$, m_n stemming from a single root. The graph G and vector $\chi = (\alpha_1^{(1)}, \alpha_2^{(1)}, \dots, \alpha_{m_1}^{(1)}, \alpha_1^{(2)}, \dots, \alpha_{m_2}^{(2)}, \dots, \alpha_1^{(n)}, \dots, \alpha_{m_n}^{(n)}, \gamma)$ completely determine the algebra $\mathcal{A}_{M_1,\dots,M_n,\gamma}$ so we will write $\mathcal{A}_{G,\chi}$ instead of $\mathcal{A}_{M_1,\dots,M_n,\gamma}$.

Henceforth we will denote the set Σ defined in 5 by $\Sigma(G)$ where G is the tree mentioned above. The spectral problem for operators on a Hilbert space can be reformulated in the following way: 1) for a given graph G describe the set $\Sigma(G)$; 2) describe non-degenerate representations $\mathcal{A}_{M_1,\dots,M_n,\gamma}$ up to unitary equivalence.

If the graph G is a Dynkin graph or extended Dynkin graph the problem is greatly simplified. The algebras $\mathcal{A}_{G,\chi}$ associated with Dynkin graphs (resp. extended Dynkin graphs) have a more simple structure than in other cases. In particular, the algebras $\mathcal{A}_{G,\chi}$ are finite dimensional (resp. have polynomial growth) if and only if the associated graph is a Dynkin graph (resp. an extended Dynkin graph) (see [12]).

Irreducible representations of the algebras associated with Dynkin graphs exist only in certain dimensions that are bounded from above (see [9]). In [6, 7, 8] we have given a complete description of $\Sigma(G)$ for all Dynkin graphs G and an algorithm for finding all irreducible representations. In present paper we accomplish the same program for extended Dynkin graphs.
7. To solve the spectral problem for extended Dynkin graphs we will follow the following scheme:

1.) As it was mentioned above it in suffices to describe the sets $(M_1, \dots, M_n, \gamma)$ for which there exist irreducible non-degenerate representations of the algebra $\mathcal{A}_{G,\chi}$. Any such representation is finite dimensional (see [11]).

2.) Using the connection between representation theory of $\mathcal{A}_{G,\chi}$ and locally-scalar representations of the associated graph (see s. 2) we can find generalized dimensions of such representations of the algebra since they correspond to the positive roots of the corresponding root system. The roots can be real or imaginary.

3.) If the generalized dimension is a real root then such representation can be obtained using Coxeter functors (see s. 1) starting from the simplest ones.

4.) If the generalized dimension is an imaginary root then the parameters of the algebra belong to a certain hyperplane (see s.2). In this case the dimension of irreducible representation is the unique minimal imaginary root. Since then the dimension is fixed the solution of the spectral problem could be obtained by direct application of Horn's inequalities.

1 Locally-scalar graph representations and representations of the algebras generated by orthoprojections.

The main tools for our classification are Coxeter functors for locally-scalar graph representations. They allow one to construct all representations starting from the simplest ones which correspond to the vertices of the graph. First we will recall a connection between category of *-representation of algebra $\mathcal{A}_{M_1,\dots,M_n,\gamma}$ associated with the graph G and locally-scalar representations of the graph G. For more details see [6].

The Coxeter functors could be constructed directly on the categories of representations of algebras $\mathcal{A}_{M_1,\dots,M_n,\gamma}$ (see [10]) but the simplest representations of a graph do not correspond to representations of corresponding algebra. This force us to use graph representation terminology and techniques.

Henceforth we will use definitions, notations and results about representations of graphs in the category of Hilbert spaces found in [9].

A graph G consists of a set of vertices G_v, a set of edges G_e and a map ε from G_e into the set of one- and two-element subsets of G_v (the edge is mapped into the set of incident vertices). Henceforth we consider connected finite graphs without cycles (trees). Fix a decomposition of G_v of the form $G_v = \overset{\circ}{G}_v \sqcup \overset{\bullet}{G}_v$ (unique up to permutation) such that for each $\alpha \in G_e$ one of the vertices from $\varepsilon(\alpha)$ belongs to $\overset{\circ}{G}_v$ and the other to $\overset{\bullet}{G}_v$. Vertices in $\overset{\circ}{G}_v$ will be called even, and those in the set $\overset{\bullet}{G}_v$ odd. Let us recall the definition of a representation Π of a graph G in the category of Hilbert spaces $\mathcal{H}$. Let us associate with each vertex $g \in G_v$ a Hilbert space $\Pi(g) = H_g \in \mathrm{Ob}\mathcal{H}$, and with each edge $\gamma \in G_e$ such that $\varepsilon(\gamma) = \{g_1, g_2\}$ a pair of mutually adjoint operators $\Pi(\gamma) = \{\Gamma_{g_1,g_2}, \Gamma_{g_2,g_1}\}$, where $\Gamma_{g_1,g_2} : H_{g_2} \to H_{g_1}$. We now construct a category $\mathrm{Rep}(G, \mathcal{H})$. Its objects are the representations of the graph G in $\mathcal{H}$. A morphism $C : \Pi \to \widetilde{\Pi}$ is a family $\{C_g\}_{g \in G_v}$ of operators $C_g : \Pi(g) \to \widetilde{\Pi}(g)$ such that the following diagrams commute for all edges

$\gamma_{g_2,g_1} \in G_e$:

$$\begin{array}{ccc} H_{g_1} & \xrightarrow{\Gamma_{g_2,g_1}} & H_{g_2} \\ {\scriptstyle C_{g_1}}\big\downarrow & & \big\downarrow{\scriptstyle C_{g_2}} \\ \widetilde{H}_{g_1} & \xrightarrow{\widetilde{\Gamma}_{g_2,g_1}} & \widetilde{H}_{g_2} \end{array}$$

Let M_g be the set of vertices connected with g by an edge. Let us define the operators

$$A_g = \sum_{g' \in M_g} \Gamma_{gg'} \Gamma_{g'g}.$$

A representation Π in $\mathrm{Rep}(G, \mathcal{H})$ will be called *locally-scalar* if all operators A_g are scalar, $A_g = \alpha_g I_{H_g}$. The full subcategory $\mathrm{Rep}(G, \mathcal{H})$, the objects of which are locally-scalar representations, will be denoted by $\mathrm{Rep}\, G$ and called the category of locally-scalar representations of the graph G.

Let us denote by V_G the real vector space consisting of sets $x = (x_g)$ of real numbers x_g, $g \in G_v$. Elements x of V_G we will call G-vectors. A vector $x = (x_g)$ is called positive, $x > 0$, if $x \neq 0$ and $x_g \geq 0$ for all $g \in G_v$. Denote $V_G^+ = \{x \in V_G | x > 0\}$. If Π is a finite dimensional representation of the graph G then the G-vector $(d(g))$, where $d(g) = \dim \Pi(g)$ is called the *dimension* of Π. If $A_g = f(g) I_{H_g}$ then the G-vector $f = (f(g))$ is called the *character* of the locally-scalar representation Π and Π is called the f-representation in this case. The *support* G_v^Π of Π is $\{g \in G_v | \Pi(g) \neq 0\}$. A representation Π is *faithful* if $G_v^\Pi = G_v$. A character of the locally-scalar representation Π is uniquely defined on the support G_v^Π and non-uniquely on its complement. In the general case, denote by $\{f_\Pi\}$ the set of characters of Π. For each vertex $g \in G_v$, denote by σ_g the linear operator on V_G given by the formulae:

$$(\sigma_g x)_{g'} = x_{g'} \text{ if } g' \neq g,$$
$$(\sigma_g x)_g = -x_g + \sum_{g' \in M_g} x_{g'}.$$

The mapping σ_g is called the *reflection* at the vertex g. The composition of all reflections at odd vertices is denoted by $\overset{\bullet}{c}$ (it does not depend on the order of the factors), and at all even vertices by $\overset{\circ}{c}$. A Coxeter transformation is $c = \overset{\circ}{c}\overset{\bullet}{c}$, $c^{-1} = \overset{\bullet}{c}\overset{\circ}{c}$. The transformation $\overset{\bullet}{c}$ ($\overset{\circ}{c}$) is called an odd (even) Coxeter map. Let us adopt the following notations for compositions of the Coxeter maps: $\overset{\bullet}{c}_k = \ldots \overset{\bullet}{c}\overset{\circ}{c}\overset{\bullet}{c}$ (k factors), $\overset{\circ}{c}_k = \ldots \overset{\circ}{c}\overset{\bullet}{c}\overset{\circ}{c}$ (k factors), $k \in \mathbb{N}$.

Any real function f on G_v can be identified with a G-vector $f = (f(g))_{g \in G_v}$. If $d(g)$ is the dimension of a locally-scalar graph representation Π, then

$$\overset{\circ}{c}(d)(g) = \begin{cases} -d(g) + \sum_{g' \in M_g} d(g'), & \text{if } g \in \overset{\circ}{G}_v, \\ d(g), & \text{if } g \in \overset{\bullet}{G}_v, \end{cases} \tag{1.1}$$

$$\overset{\bullet}{c}(d)(g) = \begin{cases} -d(g) + \sum_{g' \in M_g} d(g'), & \text{if } g \in \overset{\bullet}{G}_v, \\ d(g), & \text{if } g \in \overset{\circ}{G}_v. \end{cases} \tag{1.2}$$

For $d \in Z_G^+$ and $f \in V_G^+$, consider the full subcategory $\mathrm{Rep}(G, d, f)$ in $\mathrm{Rep}\, G$ (here Z_G^+ is the set of positive integer G-vectors), with the set of objects $Ob\,\mathrm{Rep}(G,d,f) = \{\Pi | \dim \Pi(g) = d(g), f \in \{f_\Pi\}\}$. All representations Π from $\mathrm{Rep}(G,d,f)$ have the same support $X = X_d = G_v^\Pi = \{g \in G_v | d(g) \neq 0\}$. We will consider these categories only if $(d,f) \in S = \{(d,f) \in Z_G^+ \times V_G^+ | d(g) + f(g) > 0, g \in G_v\}$. Let $\overset{\circ}{X} = X \cap \overset{\circ}{G}_v$, $\overset{\bullet}{X} = X \cap \overset{\bullet}{G}_v$. $\mathrm{Rep}_\circ(G,d,f) \subset \mathrm{Rep}(G,d,f)$ ($\mathrm{Rep}_\bullet(G,d,f) \subset \mathrm{Rep}(G,d,f)$) is the full subcategory with objects (Π, f) where $f(g) > 0$ if $g \in \overset{\circ}{X}$ ($f(g) > 0$ if $g \in \overset{\bullet}{X}$). Let $S_0 = \{(d,f) \in S | f(g) > 0 \text{ if } g \in \overset{\circ}{X}_d\}$, $S_\bullet = \{(d,f) \in S | f(g) > 0 \text{ if } g \in \overset{\bullet}{X}_d\}$

Put

$$\overset{\bullet}{c}_d(f)(g) = \overset{\circ}{f}_d(g) = \begin{cases} \overset{\bullet}{c}(f)(g), & \text{if } g \in \overset{\bullet}{X}_d, \\ f(g), & \text{if } g \notin \overset{\bullet}{X}_d, \end{cases} \tag{1.3}$$

$$\overset{\circ}{c}_d(f)(g) = \overset{\bullet}{f}_d(g) = \begin{cases} \overset{\circ}{c}(f)(g), & \text{if } g \in \overset{\circ}{X}_d, \\ f(g), & \text{if } g \notin \overset{\circ}{X}_d. \end{cases} . \tag{1.4}$$

Let us denote

$$\overset{\bullet}{c}{}_d^{(k)}(f) = \dots \overset{\bullet}{c}_{\overset{\circ}{c}_2(d)} \overset{\circ}{c}_{\overset{\circ}{c}(d)} \overset{\bullet}{c}_d(f) \quad (k \text{ factors})$$

$$\overset{\circ}{c}{}_d^{(k)}(f) = \dots \overset{\circ}{c}_{\overset{\bullet}{c}_2(d)} \overset{\bullet}{c}_{\overset{\bullet}{c}(d)} \overset{\circ}{c}_d(f) \quad (k \text{ factors})$$

The even and odd Coxeter reflection functors are defined in [9],

$$\overset{\circ}{F} : \mathrm{Rep}_\circ(G,d,f) \to \mathrm{Rep}_\circ(G, \overset{\circ}{c}(d), \overset{\circ}{f}_d) \quad \text{if } (d,f) \in S_\circ$$

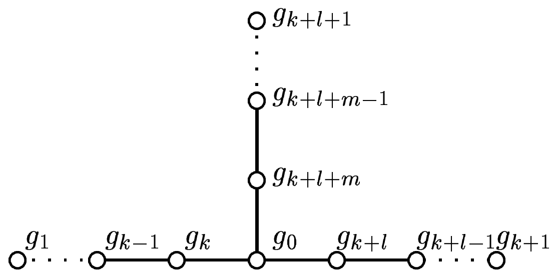
$$\overset{\bullet}{F} : \mathrm{Rep}_\bullet(G,d,f) \to \mathrm{Rep}_\bullet(G, \overset{\bullet}{c}(d), \overset{\bullet}{f}_d) \quad \text{if } (d,f) \in S_\bullet$$

These functors are equivalences of the categories. Let us denote $\overset{\circ}{F}_k(\Pi) = \dots \overset{\circ}{F}\overset{\bullet}{F}\overset{\circ}{F}(\Pi)$ (k factors), $\overset{\bullet}{F}_k(\Pi) = \dots \overset{\bullet}{F}\overset{\circ}{F}\overset{\bullet}{F}(\Pi)$ (k factors), if the compositions exist. Using these functors, an analog of Gabriel's theorem for graphs and their locally-scalar representations has been proven in [9]. In particular, it has been proved that any locally-scalar graph representation decomposes into a direct sum (finite or infinite) of finite dimensional indecomposable representations, and all indecomposable representations can be obtained by odd and even Coxeter reflection functors starting from the simplest representations Π_g of the graph G ($\Pi_g(g) = \mathbb{C}, \Pi_g(g') = 0$ if $g \neq g'$; $g, g' \in G_v$).

2 Representations of algebras generated by projections.

We can assume that each set $M_1, M_2, \dots, M_n$ contains zero (this can be achieved by a translation). Henceforth, $M_j = \alpha^{(j)} \cup \{0\}$. For three operators we will use α, β, δ instead of $\alpha^{(1)}$, $\alpha^{(2)}$, $\alpha^{(3)}$. By χ we will denote the vector $(\alpha_1, \alpha_2, \dots, \alpha_k, \beta_1, \beta_2, \dots, \beta_l, \delta_1, \delta_2, \dots, \delta_m, \gamma)$.

Let us consider a tree G with vertices $\{g_i, i = 0, \dots, k+l+m\}$ and edges $\gamma_{g_i g_j}$.



We will denote the algebra $\mathcal{A}_{M_1,M_2,\ldots,M_n,\gamma}$ as $\mathcal{A}_{G,\chi}$.

Definition 1. *An irreducible finite dimensional $*$-representation π of the algebra $\mathcal{A}_{G,\chi}$ such that*

$$\pi(p_i) \neq 0 \quad (1 \leq i \leq k), \;\; \pi(q_j) \neq 0 \quad (1 \leq j \leq l), \;\; \pi(s_d) \neq 0 \quad (1 \leq d \leq m)$$

and

$$\sum_{i=1}^{k} \pi(p_i) \neq I, \sum_{j=1}^{l} \pi(q_j) \neq I, \sum_{d=1}^{m} \pi(s_d) \neq I$$

will be called non-degenerate. By $\overline{\mathrm{Rep}}\mathcal{A}_{G,\chi}$ we will denote the full subcategory of non-degenerate representations in the category $\mathrm{Rep}\mathcal{A}_{G,\chi}$ of $*$-representations of the $*$-algebra $\mathcal{A}_{G,\chi}$ in the category $\mathcal{H}$ of Hilbert spaces.

Let π be a $*$-representation of $\mathcal{A}_{G,\chi}$ on a Hilbert space H_0. Put $P_i = \pi(p_i)$, $1 \leq i \leq k$, $Q_j = \pi(q_j)$, $1 \leq j \leq l$, $S_t = \pi(s_t)$, $1 \leq t \leq m$. Let $H_{p_i} = \Im\mathrm{m}\, P_i$, $H_{q_j} = \Im\mathrm{m}\, Q_j$, $H_{s_t} = \Im\mathrm{m}\, S_t$. Denote by Γ_{p_i}, Γ_{q_j}, Γ_{s_t} the corresponding natural isometries $H_{p_i} \to H_0$, $H_{q_j} \to H_0$, $H_{s_t} \to H_0$. Then, in particular, $\Gamma_{p_i}^* \Gamma_{p_i} = I_{H_{p_i}}$ is the identity operator on H_{p_i} and $\Gamma_{p_i}\Gamma_{p_i}^* = P_i$. Similar equalities hold for the operators Γ_{q_i} and Γ_{s_i}. Using π we construct a locally-scalar representation Π of the graph G.

Let $\Gamma_{ij} : H_j \to H_i$ denote the operator adjoint to $\Gamma_{ji} : H_i \to H_j$, i.e. $\Gamma_{ij} = \Gamma_{ji}^*$. Put

$$\begin{aligned}
\Pi(g_0) &= H^{g_0} = H_0, \\
\Pi(g_k) &= H^{g_k} = H_{p_1} \oplus H_{p_2} \oplus \ldots \oplus H_{p_k}, \\
\Pi(g_{k-1}) &= H^{g_{k-1}} = H_{p_2} \oplus \ldots \oplus H_{p_{k-1}} \oplus H_{p_k}, \\
\Pi(g_{k-2}) &= H^{g_{k-2}} = H_{p_2} \oplus H_{p_3} \oplus \ldots \oplus H_{p_{k-1}}, \\
&\ldots .
\end{aligned}$$

In these equalities the summands are omitted from left and from right in turns. Analogously, we define subspaces $\Pi(g_i)$ for $i = k+1, \ldots, k+l$ and $i = k+l+1, \ldots, k+l+m$. Define the operators $\Gamma_{g_0,g_i} : H^{g_i} \to H^{g_0}$, where $i \in \{k, k+l, k+l+m\}$, by the block-diagonal matrices

$$\begin{aligned}
\Gamma_{g_0,g_k} &= \left[\sqrt{\alpha_1}\Gamma_{p_1} | \sqrt{\alpha_2}\Gamma_{p_2} | \ldots | \sqrt{\alpha_k}\Gamma_{p_k}\right], \\
\Gamma_{g_0,g_{k+l}} &= \left[\sqrt{\beta_1}\Gamma_{q_1} | \sqrt{\beta_2}\Gamma_{q_2} | \ldots | \sqrt{\beta_l}\Gamma_{q_l}\right], \\
\Gamma_{g_0,g_{k+l+m}} &= \left[\sqrt{\delta_1}\Gamma_{s_1} | \sqrt{\delta_2}\Gamma_{s_2} | \ldots | \sqrt{\delta_m}\Gamma_{s_m}\right].
\end{aligned}$$

Now we define the representation Π on the edges γ_{g_0,g_k}, $\gamma_{g_0,g_{k+l}}$, $\gamma_{g_0,g_{k+l+m}}$ by the rule

$$\Pi(\gamma_{g_0,g_k}) = \{\Gamma_{g_0,g_k}, \Gamma_{g_k,g_0}\},$$
$$\Pi(\gamma_{g_0,g_{k+l}}) = \{\Gamma_{g_0,g_{k+l}}, \Gamma_{g_{k+l},g_0}\},$$
$$\Pi(\gamma_{g_0,g_{k+l+m}}) = \{\Gamma_{g_0,g_{k+l+m}}, \Gamma_{g_{k+l+m},g_0}\}.$$

It is easy to see that

$$\Gamma_{g_0,g_k}\Gamma_{g_k,g_0} + \Gamma_{g_0,g_{k+l}}\Gamma_{g_{k+l},g_0} + \Gamma_{g_0,g_{k+l+m}}\Gamma_{g_{k+l+m},g_0} = \gamma I_{H^{g_0}}.$$

Let $\mathcal{O}_{H,0}$ denote the operators from the zero space to H, and $\mathcal{O}_{0,H}$ denote the zero operator from H into the zero subspace. For the operators $\Gamma_{g_j,g_i} : H^{g_i} \to H^{g_j}$ with $i, j \neq 0$, put

$$\Gamma_{g_{k-1},g_k} = \mathcal{O}_{0,H_{p_1}} \oplus \sqrt{\alpha_1 - \alpha_2} I_{H_{p_2}} \oplus \ldots \oplus \sqrt{\alpha_1 - \alpha_k} I_{H_{p_k}}, \tag{2.1}$$
$$\Gamma_{g_{k-1},g_{k-2}} = \sqrt{\alpha_2 - \alpha_k} I_{H_{p_2}} \oplus \ldots \oplus \sqrt{\alpha_{k-1} - \alpha_k} I_{H_{p_{k-1}}} \oplus \mathcal{O}_{H_{p_k},0}, \tag{2.2}$$
$$\Gamma_{g_{k-3},g_{k-2}} = \mathcal{O}_{0,H_{p_2}} \oplus \sqrt{\alpha_2 - \alpha_3} I_{H_{p_3}} \oplus \ldots \oplus \sqrt{\alpha_2 - \alpha_{k-1}} I_{H_{p_{k-1}}}, \tag{2.3}$$
$$\ldots$$

The corresponding operators for the rest of the edges of G can be constructed analogously. One can check that the operators Γ_{g_i,g_j}, where $\Gamma_{g_i,g_j} = \Gamma^*_{g_i,g_j}$, define a locally-scalar representation of the graph G with the following character f: $f(g_0) = \gamma$ and

$$\begin{array}{lll}
f(g_k) = \alpha_1, & f(g_{k+l}) = \beta_1, & f(g_{k+l+m}) = \delta_1, \\
f(g_{k-1}) = \alpha_1 - \alpha_k, & f(g_{k+l-1}) = \beta_1 - \beta_l, & f(g_{k+l+m-1}) = \delta_1 - \delta_m, \\
f(g_{k-2}) = \alpha_2 - \alpha_k, & f(g_{k+l-2}) = \beta_2 - \beta_l, & f(g_{k+l+m-2}) = \delta_2 - \delta_m, \\
f(g_{k-3}) = \alpha_2 - \alpha_{k-1}, & f(g_{k+l-3}) = \beta_2 - \beta_{l-1}, & f(g_{k+l+m-3}) = \delta_2 - \delta_{m-1}, \\
f(g_{k-4}) = \alpha_3 - \alpha_{k-1}, & f(g_{k+l-4}) = \beta_3 - \beta_{l-1}, & f(g_{k+l+m-4}) = \delta_3 - \delta_{m-1}, \\
\ldots & \ldots & \ldots
\end{array}$$

And vice versa, if a locally-scalar representation of the graph G with the character $f(g_i) = x_i \in \mathbb{R}^*$ corresponds to a non-degenerate representation of $\mathcal{A}_{G,\chi}$, then one can check that

$$\alpha_1 = x_k,$$
$$\alpha_k = x_k - x_{k-1}, \alpha_2 = x_k - x_{k-1} + x_{k-2},$$
$$\alpha_{k-1} = x_k - x_{k-1} + x_{k-2} - x_{k-3},$$
$$\alpha_3 = x_k - x_{k-1} + x_{k-2} - x_{k-3} + x_{k-4},$$
$$\ldots.$$

Here $x_j = 0$ if $j \leq 0$. Analogously one can find β_j and δ_t. We will denote Π by $\Phi(\pi)$.

Let π and $\widetilde{\pi}$ be non-degenerate representations of the algebra $\mathcal{P}_{\alpha,\beta,\delta,\gamma}$ and C_0 an intertwining operator for these representations; this is a morphism from π to $\widetilde{\pi}$ in the category $\operatorname{Rep} G$), $C_0 : H_0 \to \widetilde{H}_0$, $C_0\pi = \widetilde{\pi}C_0$. Put

$$\begin{gathered}
C_{p_i} = \widetilde{\Gamma}^*_{p_i} C_0 \Gamma_{p_i}, C_{p_i} : H_{p_i} \to \widetilde{H}_{p_i},\ 1 \le i \le k, \\
C_{q_j} = \widetilde{\Gamma}^*_{q_j} C_0 \Gamma_{q_j}, C_{q_j} : H_{q_j} \to \widetilde{H}_{q_j},\ k+1 \le j \le k+l, \\
C_{s_t} = \widetilde{\Gamma}^*_{s_t} C_0 \Gamma_{s_t}, C_{s_t} : H_{s_t} \to \widetilde{H}_{s_t}.\ k+l+1 \le t \le k+l+m, \\
\dots
\end{gathered}$$

Put

$$\begin{gathered}
C^{(g_0)} = C_0 : H^{(g_0)} \to \widetilde{H}^{(g_0)}, \\
C^{(g_k)} = C_{p_1} \oplus C_{p_2} \oplus \ldots \oplus C_{p_k} : H^{(g_k)} \to \widetilde{H}^{(g_k)}, \\
C^{(g_{k-1})} = C_{p_2} \oplus \ldots \oplus C_{p_{k-1}} \oplus C_{p_k} : H^{(g_{k-1})} \to \widetilde{H}^{(g_{k-1})}, \\
C^{(g_{k-2})} = C_{p_2} \oplus C_{p_3} \oplus \ldots \oplus C_{p_{k-1}} : H^{(g_{k-2})} \to \widetilde{H}^{(g_{k-2})}, \\
\dots
\end{gathered}$$

Analogously one can construct the operators $C^{(g_i)}$ for $i \in \{k+l, \ldots, k+l+m\}$. It is routine to check that the operators $\{C^{(g_i)}\}_{0\le i\le k+l+m}$ intertwine the representations $\Pi = \Phi(\pi)$ and $\widetilde{\Pi} = \Phi(\widetilde{\pi})$. Put $\Phi(C_0) = \{C^{(g_i)}_{0\le i\le k+l+m}\}$. Thus we have defined a functor $\Phi : \overline{\operatorname{Rep}}\mathcal{A}_{G,\chi} \to \operatorname{Rep}G$, see [7]. Moreover, the functor Φ is univalent and full. Let $\widetilde{\operatorname{Rep}}(G,d,f)$ be the full subcategory of irreducible representations of $\operatorname{Rep}(G,d,f)$. $\Pi \in \operatorname{Ob}\widetilde{\operatorname{Rep}}(G,d,f)$, $f(g_i) = x_i \in \mathbb{R}^+, d(g_i) = d_i \in \mathbb{N}_0$, where f is the character of Π, d its dimension. It easy to verify that the representation Π is isomorphic (unitary equivalent) to an irreducible representation from the image of the functor Φ if and only if

$$0 < x_1 < x_2 < \ldots < x_k; 0 < x_{k+1} < x_{k+2} < \ldots < x_{k+l}; \tag{2.4}$$
$$0 < x_{k+l+1} < x_{k+l+2} < \ldots < x_{k+l+m}; \tag{2.5}$$
$$0 < d_1 < d_2 < \ldots < d_k < d_0; 0 < d_{k+1} < d_{k+2} < \ldots < \tag{2.6}$$
$$d_{k+l} < d_0; 0 < d_{k+l+1} < d_{k+l+2} < \ldots < d_{k+l+m} < d_0. \tag{2.7}$$

(All matrices of the representation of the graph G, except for Γ_{g_0,g_k}, Γ_{g_k,g_0}, $\Gamma_{g_0,g_{k+l}}$, Γ_{g_{k+l},g_0}, $\Gamma_{g_0,g_{k+l+m}}$, Γ_{g_{k+l+m},g_0}, can be brought to the "canonical" form (2.1) by admissible transformations. Then the rest of the matrices will naturally be partitioned into blocks, which gives the matrices $\Gamma_{p_i}, \Gamma_{q_i}, \Gamma_{s_i}$, and thus the projections P_i, Q_i, R_i). An irreducible representation Π of the graph G satisfying conditions (2.4)–(2.7) will be called *non-degenerate*. Let

$$\begin{gathered}
\dim H_{p_i} = n_i, 1 \le i \le k; \\
\dim H_{q_j} = n_{k+j}, 1 \le j \le l; \\
\dim H_{s_t} = n_{k+l+t}, 1 \le t \le m; \\
\dim H_0 = n_0.
\end{gathered}$$

The vector $n = (n_0, n_1, \ldots, n_{k+l+m})$ is called the *generalized dimension* of the representation π of the algebra $\mathcal{A}_{G,\chi}$. Let $\Pi = \Phi(\pi)$ for a non-degenerate representation of the algebra $\mathcal{A}_{G,\chi}$, $d = (d_1, \ldots, d_{k+l+m}, d_0)$ be the dimension of Π. It is easy to see that

$$\begin{aligned}
n_1 + n_2 + \ldots + n_k &= d_k, && (2.8)\\
n_2 + \ldots + n_{k-1} + n_k &= d_{k-1}, && (2.9)\\
n_2 + \ldots + n_{k-1} &= d_{k-2}, && (2.10)\\
n_3 + \ldots + n_{k-2} + n_{k-1} &= d_{k-3}, && (2.11)\\
&\ldots
\end{aligned}$$

Thus

$$\begin{aligned}
n_1 &= d_k - d_{k-1}, && (2.12)\\
n_k &= d_{k-1} - d_{k-2}, && (2.13)\\
n_2 &= d_{k-2} - d_{k-3}, && (2.14)\\
&\ldots
\end{aligned}$$

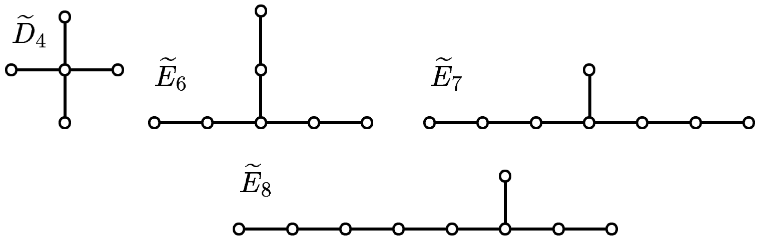
Analogously one can find n_{k+1}, …, n_{k+l} from $d_{k+1}, \ldots, d_{k+l}$ and n_{k+l+1}, …, n_{k+l+m} from d_{k+l+1}, …, d_{k+l+m}

Denote by $\overline{\mathrm{Rep}}G$ the full subcategory in $\mathrm{Rep}\, G$ of non-degenerate locally-scalar representations of the graph G. As a corollary of the previous arguments we obtain the following theorem.

Theorem 1. *Let $\mathcal{A}_{G,\chi}$ be associated with a graph G. The functor Φ is an equivalence between the category $\overline{\mathrm{Rep}\mathcal{A}_{G,\chi}}$ of non-degenerate *-representations of the algebra $\mathcal{A}_{G,\chi}$ and the category $\overline{\mathrm{Rep}}G$ of non-degenerate locally-scalar representations of the graph G.*

Let us define the Coxeter functors for the $*$-algebras $\mathcal{A}_{G,\chi}$, by putting $\overset{\circ}{\Psi} = \Phi^{-1}\overset{\circ}{F}\Phi$ and $\overset{\bullet}{\Psi} = \Phi^{-1}\overset{\bullet}{F}\Phi$.

Utilizing the results of [9] we will present a description of $*$-representation of the $*$-algebras $\mathcal{A}_{G,\chi}$ and hence a complete solution of the Spectral Problem for the simply laced extended Dynkin graphs which are listed below.



3 Spectral problem for algebras associated with extended Dynkin graphs.

Let us recall a few facts about root systems associated with extended Dynkin diagrams. Let G be a simple connected graph. Then its *Tits form* is

$$q(\alpha) = \sum_{i \in G_v} \alpha_i^2 - \frac{1}{2} \sum_{\beta \in G_e, \{i,j\} = \epsilon(\beta)} \alpha_i \alpha_j \quad (\alpha \in V_G).$$

The *symmetric bilinear form* is $(\alpha,\beta) = q(\alpha+\beta) - q(\alpha) - q(\beta)$. Vector $\alpha \in V_G$ is called sincere if each component is non-zero.

It is well known that for Dynkin graphs (and only for them) bilinear form $(\cdot,\cdot)$ is positive definite. The form is positive semi-definite for extended Dynkin graphs. And in the letter case $\operatorname{Rad} q = \{v|q(v) = 0\}$ is equal to $\mathbb{Z}\delta$ where δ is a minimal imaginary root. For other graphs (which are neither Dynkin nor extended Dynkin) there are vectors $\alpha \geq 0$ such that $q(\alpha) < 0$ and $(\alpha, \epsilon_j) \leq 0$ for all j.

For an extended Dynkin graph G a vertex j is called *extending* if $\delta_j = 1$. The graph obtained by deleting extending vertex is the corresponding Dynkin graph. The set of *roots* is $\Delta = \{\alpha \in V_G | \alpha_i \in \mathbb{Z} \text{ for all } i \in G_v, \alpha \neq 0, q(\alpha) \leq 0\}$. A root α is *real* if $q(\alpha) = 1$ and *imaginary* if $q(\alpha) = 0$. Every root is either positive or negative, i.e. all coordinates are simultaneously non-negative or non-positive.

For our classification purposes we will need the following fact (see [2]): for an extended Dynkin graph the set $\Delta \cup \{0\}/\mathbb{Z}\delta$ is finite. Moreover, if e is an extending vertex then the set $\Delta_f = \{\alpha \in \Delta \cup \{0\} | \alpha_e = 0\}$ is a complete set of representatives of the cosets from $\Delta \cup \{0\}/\mathbb{Z}\delta$. If α is a root then $\alpha + \delta$ is again a root. We call a coset $\alpha + \delta\mathbb{Z}$ the δ-series and a coset $\alpha + 2\delta\mathbb{Z}$ the 2δ-series. If α is a root then its images under the action of the group generated by $\overset{\circ}{c}$ and $\overset{\bullet}{c}$ will be called a Coxeter series or C-series for short. It turns out that each C-series decomposes into a finite number of δ-series or 2δ-series of roots.

Note that to find formulae of the locally-scalar representations of a given extended Dynkin graph we need to consider two principally different cases: the case when the vector of generalized dimension is a real root and the case when it is an imaginary root. In the letter case the vector of parameters χ must satisfy (in order for the representations to exist) a certain linear relation which is obtained by taking traces from the both sides of the equation $A_1 + \ldots + A_n = \gamma I$. Hence χ must belong to a ceratin hyperplane h_G which depends only on the graph G. A simple calculation yields that for extended Dynkin graphs $\widetilde{D}_4$, $\widetilde{E}_6$, $\widetilde{E}_7$, $\widetilde{E}_8$ these hyperplanes are the following:

$\widetilde{D}_4$	$\widetilde{E}_6$	$\widetilde{E}_7$	$\widetilde{E}_8$
$\alpha_1 + \beta_1 + \delta_1 + \eta_1 = 2\gamma$	$\alpha_1 + \alpha_2 + \beta_1 + \beta_2 + \delta_1 + \delta_2 = 3\gamma$	$\alpha_1 + \alpha_2 + \alpha_3 + \beta_1 + \beta_2 + \beta_3 + 2\delta_1 = 4\gamma$	$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + 2(\beta_1 + \beta_2) + 3\delta_1 = 6\gamma$

It is know (see [12]) that in case $\chi \in h_G$ the dimension of any irreducible representation is bounded (by 2 for $\widetilde{D}_4$, by 3 for $\widetilde{E}_6$, by 4 for $\widetilde{E}_7$ and by 6 for $\widetilde{E}_8$). Thus in case $\chi \in h_G$ we can describe the set of admissible parameters χ using Horn's inequalities. In case $\chi \notin h_g$ the dimension of any irreducible locally-scalar representation is a real root. In what follows we will relay on the following result due to V.Ostrovskij [11]

Theorem 2. *Let π be an irreducible *-representation of the algebra $\mathcal{A}_{G,\chi}$ associated with extended Dynkin graph G and $\widehat{\pi}$ corresponding representation of the graph G. Then either generalized dimension d of $\widehat{\pi}$ is a singular root or vector-parameter $\chi \in h_G$.*

Hence we will solve the spectral problem if we describe the parameters χ for which there are locally-scalar representations with vector of generalized dimension being real singular roots and, separately, parameters belonging to the hyperplane h_G for which there exist representations of the algebra.

In the next sections we will do the following: we know how to construct all irreducible locally-scalar representations of Dynkin graphs with the aid of Coxeter reflection functors starting from the simplest ones. In particular, we can find their dimensions and characters [9]. Next we single out non-generate representations and apply the equivalence functor Φ, see Theorem 1.

For a vector $v = (v_0, \ldots, v_n)$ and $0 \le s \le n$ we will write $v \ge_s 0$ if $v_j > 0$ for all $j \ne s$ and $v_s = 0$. If $\Pi \in \mathrm{Rep}(G, w, \xi)$ is a locally-scalar representation of a graph G with w being a singular root then $e_j = \overset{\bullet}{c}_k(w)$ or $e_j = \overset{\circ}{c}_k(w)$ for some positive integer k and a coordinate vector e_j. Thus there is a locally-scalar representation $\Pi' \in \mathrm{Rep}(G, e_j, \xi')$ such that applying corresponding sequence of Coxeter functors $\ldots \overset{\circ}{F}\overset{\bullet}{F}$ or $\ldots \overset{\bullet}{F}\overset{\circ}{F}$ to Π' we obtain Π and hence w belong to C-orbit of e_j and $\overset{\bullet}{c}_d^{(k)}(\xi) = \xi'$ or $\overset{\circ}{c}_d^{(k)}(\xi) = \xi'$. Thus the necessary and sufficient conditions on ξ for the representation Π to exist can be written as

$$\overset{\bullet}{c}_d^{(k)}(\xi) \ge_j 0, \quad \overset{\bullet}{c}_k(w) = e_j, \tag{3.1}$$

or

$$\overset{\circ}{c}_d^{(k)}(\xi) \ge_j 0, \quad \overset{\circ}{c}_k(w) = e_j. \tag{3.2}$$

Let C_j denote the C-orbit of e_j. It can be checked by direct computations that for each extended Dynkin graph every C-orbit is a union of finite number of δ-series or 2δ-series of roots, i.e. $C = (v_0 + \epsilon\delta\mathbb{Z}) \cup \ldots \cup (v_m + \epsilon\delta\mathbb{Z})$ where $\epsilon \in \{1, 2\}$ and $\overset{\circ}{c}(v_{2r}) = v_{2r+1}$, $\overset{\bullet}{c}(v_{2r-1}) = v_{2r}$ (or $\overset{\bullet}{c}(v_{2r}) = v_{2r+1}$, $\overset{\circ}{c}(v_{2r-1}) = v_{2r}$). We have presented this finite sequences $(v_0, \ldots, v_m)$ in the tables at the end of the paper. Elements of C-series can be written then as $w_k = v_{k \mod (m+1)} + \epsilon[\frac{k}{m+1}]\delta$ where $[x]$ denote the integer part of x.

Let t be minimal such that w_t is non-degenerate then w_l is also non-degenerate for all $l > t$. We will denote by $D_{j,t}$ the matrix which transform the character of a locally-scalar graph representation with dimension w_t to the one with dimension e_j, i.e. $D_{j,k}(x_1, x_2, \ldots, x_n, x_0)^T = (x_1', x_2', \ldots, x_7', x_0')^T$ where $(x_1', x_2', \ldots, x_7', x_0')$ obtained from $(x_1, x_2, \ldots, x_7, x_0)$ by applying the corresponding sequence of Coxeter maps that transform v_k to e_j.

By Theorem 1 there is an equivalence functor Φ which assigns to every representation $\pi \in \mathcal{A}_{G,\chi}$ of generalized dimension $(l_1, \ldots, l_n)$ a unique locally-scalar representation of graph G with a character $(x_1, \ldots, x_n, x_0)$ and dimension $(v_1, \ldots, v_n, v_0)$. Let M_f denote the transition matrix which transform the vector χ to $(x_1, \ldots, x_n, x_0)$. Let M_d be the transition matrix which transforms generalized dimension $(v_1, \ldots, v_n, v_0)$ of a graph representation to generalized dimension $(l_1, \ldots, l_n)$ of the corresponding algebra representation. Then generalized dimension l of an irreducible representation $\pi\,\mathrm{Rep}\,\mathcal{A}_{G,\chi}$ is of the form $M_d w$ where w in non-degenerate root of root system associated with graph G and conditions (3.1), (3.2) give the following necessary and sufficient conditions on χ of existence of representation in dimension l

$$\overset{\bullet}{c}_d^{(k)} M_f \chi \ge_j 0,$$

or

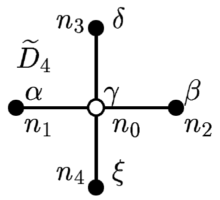
$$\overset{\circ}{c}{}_d^{(k)} M_f \chi \geq_j 0.$$

Let us note that there are no irreducible $*$-representations of the algebra $\mathcal{A}_{G,\chi}$ of generalized dimension $M_d w$ where w is a real regular root. For $\chi \notin h_G$ this follows from Theorem 2. If $\chi \in h_G$ and $\pi \in \overline{\mathrm{Rep}}(\mathcal{A}_{G,\chi})$ of dimension $M_d w$ then since $w + k\delta$ belong to C-orbit of w for arbitrarily large k we obtain by applying Coxeter functors that there exists an irreducible representation of $\mathcal{A}_{G,\chi}$ in arbitrarily large dimension. This contradicts the fact that $\mathcal{A}_{G,\chi}$ is PI-algebra for every $\chi \in h_G$ and thus generalized dimension of π can not correspond to a real regular root.

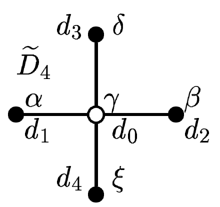
The explicit results of the communications of C-orbits, matrices M_f, M_d, etc. are gathered in the tables at the end of the paper. In the following sections we present explicit answer to the Spectral problem for all extended Dynkin graphs.

4 Representations of $\mathcal{A}_{\widetilde{D}_4,\chi}$.

The parameters $\chi = (\alpha, \beta, \xi, \delta, \gamma)$ of the algebra $\mathcal{A}_{\widetilde{D}_4,\chi}$ and the vector of generalized dimension $n = (n_1, \ldots, n_4, n_0)$ will be plotted on the associated graph according to the following picture:



The category of non-degenerate $*$-representations of $\mathcal{A}_{\widetilde{D}_4,\chi}$ is equivalent to the category of non-degenerate locally-scalar representations of the graph $\widetilde{D}_4$ with the character and generalized dimension given on the following picture:



Obviously the transition matrix M_f such that $M_f(\chi) = (x_1, x_2, \ldots, x_4, x_0)$ and the transition matrix M_d such that $M_d(d_1, \ldots, d_4, d_0)^T = (n_1, \ldots, n_4, n_0)^T$ are identity matrices.

For $\Pi \in \mathrm{Rep}(G, d, f)$ with sincere d the Coxeter map $\overset{\bullet}{C}\overset{\circ}{C}$ transform character $x = (x_1, \ldots, x_4, x_0)$ by multiplying from the left the vector-column x on the matrix $M_c = \begin{pmatrix} -1 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & 1 \\ -1 & -1 & -1 & -1 & 3 \end{pmatrix}$.

It is easy to check that the Jordan form of M_c is

$$J = -1 \oplus -1 \oplus -1 \oplus J_2(1),$$

where

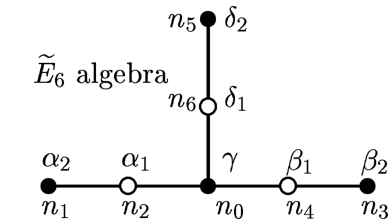
$$J_2(1) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Theorem 3. *Let $d_k^{(t)} = v_{k \bmod m_t}^{(t)} + [\frac{k}{m_t}]\delta$ for $k \geq k_t$ where $v_s^{(t)}$ is the s-th vector in the set C_t from the table 8, $m_t = |C_t|$ and $d_1 = 5$, $d_0 = 2$.*

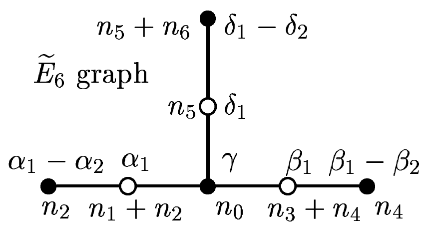
If the vector $\chi \notin h_{\widetilde{D}_4}$, i.e χ does not satisfy $\alpha_1 + \beta_1 + \delta_1 + \eta_1 = 2\gamma$ then the algebra $\mathcal{A}_{\widetilde{D}_4,\chi}$ has an irreducible non-degenerate representation in a generalized dimensions v if and only if for some $t \in \{0,1\}$ and some $k \geq d_t$, $v = M_d d_k^{(t)}$ and $\mathrm{A}_{t,k}\chi \geq_t 0$ where the matrix $\mathrm{A}_{t,k}$ is taken from table 8. Such representation is unique. If χ satisfies $\alpha_1 + \beta_1 + \delta_1 + \eta_1 = 2\gamma$ then irreducible non-degenerate representations of $\mathcal{A}_{\widetilde{D}_4,\chi}$ may exist only in the generalized dimension $\delta_{alg}(\widetilde{D}_4)$. They exist if and only if χ satisfies conditions $H_{\widetilde{D}_4}$.

5 Representations of $\mathcal{A}_{\widetilde{E}_6,\chi}$.

The parameters $\chi = (\alpha_1, \alpha_2, \beta_1, \beta_2, \delta_1, \delta_2, \gamma)$ of the algebra $\mathcal{A}_{\widetilde{E}_6,\chi}$ and the vector of generalized dimension $d = (d_1, \ldots, d_6, d_0)$ will be plotted on the associated graph according to the following picture:



The category of non-degenerate $*$-representations of $\mathcal{A}_{\widetilde{E}_6,\chi}$ is equivalent to the category of non-degenerate locally-scalar representations of the graph $\widetilde{E}_6$ with the character and generalized dimension given on the following picture:



It is easy to see that transition matrix M_f such that $M_f(\chi) = (x_1, x_2, \ldots, x_6, x_0)$ is block-diagonal

$$M_f = T_{6,1} \oplus T_{6,1} \oplus T_{6,1} \oplus 1,$$

where

$$T_{6,1} = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}.$$

The transition matrix M_d such that $M_d(d_1,\ldots,d_6,d_0)^T = (n_1,\ldots,n_6,n_0)^T$ is also block-diagonal

$$M_d^{-1} = T_{6,2} \oplus T_{6,2} \oplus T_{6,2} \oplus 1,$$

where

$$T_{6,2} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

For $\Pi \in \mathrm{Rep}(G,d,f)$ with sincere d the Coxeter map $\overset{\bullet}{C}\overset{\circ}{C}$ transform character $x = (x_1,\ldots,x_7,x_0)$ by multiplying from the left the vector-column x on the matrix $M_c = SJS^{-1}$ where the Jordan form

$$J = -1 \oplus J_2(1) \oplus z \oplus z \oplus \overline{z} \oplus \overline{z},$$

where

$$J_2(1) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

and $z = 1/2(-1 - i\sqrt{3})$.

Theorem 4. *Let for $k \in \{0,1,2\}$ $d_k^{(t)} = v_{k \bmod m_t}^{(t)} + [\frac{k}{m_t}]\delta$ for $k \geq k_t$ where $v_s^{(t)}$ is the s-th vector in the set C_t from the table 9, $m_t = |C_t|$ and $d_0 = 4$, $d_1 = 14$, $d_2 = 7$.*

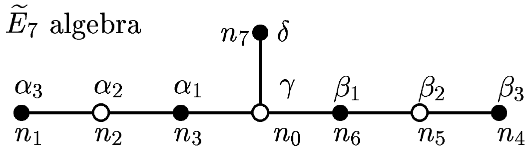
If the vector $\chi \notin h_{\widetilde{E}_6}$, i.e χ does not satisfy

$$\alpha_1 + \alpha_2 + \beta_1 + \beta_2 + \delta_1 + \delta_2 = 3\gamma \tag{5.1}$$

then the algebra $\mathcal{A}_{\widetilde{E}_6,\chi}$ has an irreducible non-degenerate representation in a generalized dimensions v if and only if for some $t \in \{0,1,2\}$ and some $k \geq d_t$ $v = \tau M_d d_k^{(t)}$ where transposition $\tau \in \{(1,3),(1,5),(2,4),(2,6)\}$ permutes the coordinates of the vector $M_d d_k^{(t)}$ and $\mathrm{A_{t,k}}\tau\chi \geq_{\mathrm{t}} 0$ where the matrix $\mathrm{A}_{t,k}$ is taken from table 9. Such representation is unique. If χ satisfies (5.1) then irreducible non-degenerate representations of $\mathcal{A}_{\widetilde{E}_6,\chi}$ may exist only in the generalized dimension $\delta_{alg}(E_6)$. They exist if and only if χ satisfies conditions $H_{\widetilde{E}_6}$.

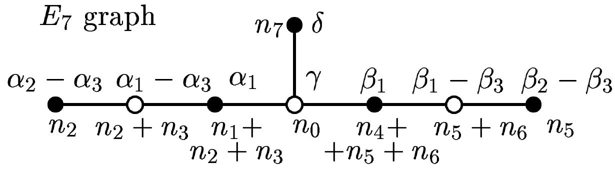
6 Representations of $\mathcal{A}_{\widetilde{E}_7,\chi}$.

The parameters $\chi = (\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2,\beta_3,\delta,\gamma)$ of the algebra $\mathcal{A}_{\widetilde{E}_7,\chi}$ and the vector of generalized dimension $d = (d_0,d_1,\ldots,d_7)$ will be plotted on the associated graph according to the following picture:



The category of non-degenerate $*$-representations of $\mathcal{A}_{\widetilde{E}_7,\chi}$ is equivalent to the category of non-degenerate locally-scalar representations of the graph $\widetilde{E}_7$ with the character and generalized dimension given on the following picture:

$\widetilde{E}_7$ graph



It is easy to see that transition matrix M_f such that $M_f(\chi) = (x_1, x_2, \ldots, x_7, x_0)$ is block-diagonal

$$M_f = \operatorname{diag}(T_1, T_1, 1, 1),$$

where

$$T_1 = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix}.$$

The transition matrix M_d such that $M_d(d_0, d_1, \ldots, d_7)^T = (n_0, n_1, \ldots, n_7)^T$ is also block-diagonal

$$M_d^{-1} = \operatorname{diag}(T_2, T_2, 1),$$

where

$$T_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

For $\Pi \in \operatorname{Rep}(G, d, f)$ with sincere d the Coxeter map $\overset{\bullet}{C}\overset{\circ}{C}$ transform character $x = (x_1, \ldots, x_7, x_0)$ by multiplying from the left the vector-column x on the matrix $M_c = SJS^{-1}$ where the Jordan form

$$J = \operatorname{diag}(-1, -1, J_2(1), -i, i, 1/2(-1 - i\sqrt{3}), 1/2(-1 + i\sqrt{3})),$$

where

$$J_2(1) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Theorem 5. *Let for $k \in \{0, 1, 2, 3, 7, 8\}$ $d_k^{(t)} = v^{(t)}_{k \bmod m_t} + \epsilon_t[\frac{k}{m_t}]\delta$ for $k \geq k_t$ where $v_s^{(t)}$ is the s-th vector in the set C_t from the table 10, $m_t = |C_t|$ and $d_0 = 4$, $d_1 = 27$, $d_2 = 14$, $d_3 = 9$, $d_7 = 11$, $d_8 = 6$, $\epsilon_j = 1$ if $j \notin \{2, 5\}$ and $\epsilon_2 = \epsilon_5 = 2$.*

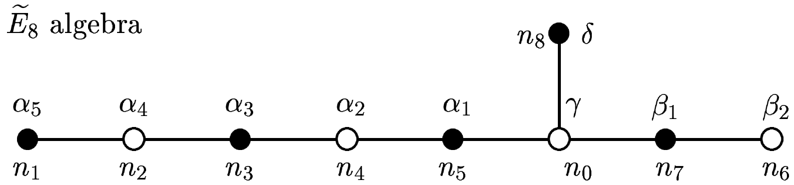
If the vector $\chi \notin h_{\widetilde{E}_7}$, i.e χ does not satisfy

$$\alpha_1 + \alpha_2 + \alpha_3 + \beta_1 + \beta_2 + \beta_3 + 2\delta_1 = 4\gamma \tag{6.1}$$

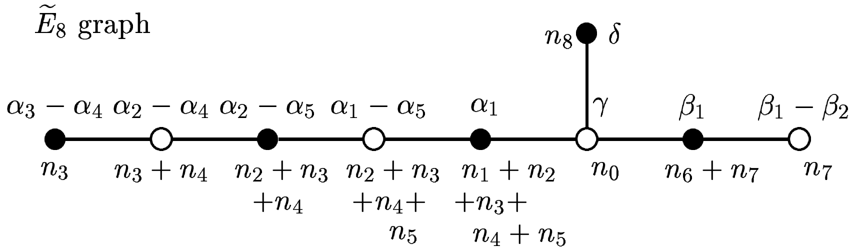
then the algebra $\mathcal{A}_{\widetilde{E}_7,\chi}$ has an irreducible non-degenerate representation in a generalized dimensions v if and only if for some $t \in \{0, 1, 2\}$ and some $k \geq d_t$ $v = \tau M_d d_k^{(t)}$ where transposition $\tau \in \{(1, 4), (2, 5), (3, 6)\}$ permutes the coordinates of the vector $M_d d_k^{(t)}$ and $\mathrm{A}_{t,k}\tau\chi \geq_t 0$ where the matrix $\mathrm{A}_{t,k}$ is taken from table 10. Such representation is unique. If χ satisfies (6.1) then irreducible non-degenerate representations of $\mathcal{A}_{\widetilde{E}_7,\chi}$ may exist only in the generalized dimension $\delta_{alg}(\widetilde{E}_7)$. They exist if and only if χ satisfies conditions $H_{\widetilde{E}_7}$.

7 Representations of $\mathcal{A}_{\widetilde{E}_8,\chi}$.

The parameters $\chi = (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \beta_1, \beta_2, \delta, \gamma)$ of the algebra $\mathcal{A}_{\widetilde{E}_8,\chi}$ and the vector of generalized dimension $d = (d_1, \dots, d_8, d_0)$ will be plotted on the associated graph according to the following picture:



The category of non-degenerate $*$-representations of $\mathcal{A}_{\widetilde{E}_8,\chi}$ is equivalent to the category of non-degenerate locally-scalar representations of the graph $\widetilde{E}_8$ with the character and generalized dimension given on the following picture:



It is easy to see that transition matrix M_f such that $M_f(\chi) = (x_1, x_2, \dots, x_8, x_0)$ is block-diagonal

$$M_f = \operatorname{diag}(T_2, T_3, 1, 1),$$

where

$$T_2 = \begin{pmatrix} 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}, T_3 = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}.$$

The transition matrix M_d such that $M_d(d_1, \dots, d_8, d_0)^T = (n_1, \dots, n_8, n_0)^T$ is also block-diagonal

$$M_d^{-1} = \operatorname{diag}(T_4, T_5, 1, 1),$$

where

$$T_4 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}, T_5 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

For $\Pi \in \operatorname{Rep}(G, d, f)$ with sincere d the Coxeter map $\overset{\bullet}{C}\overset{\circ}{C}$ transform character $x = (x_1, \dots, x_7, x_0)$ by multiplying from the left the vector-column x on the matrix $M_c = SJS^{-1}$ where the Jordan form

$$J = \operatorname{diag}(-1, J_2(1), 1/2(-1 - i\sqrt{3}), 1/2(-1 + i\sqrt{3}), \zeta, \zeta^2, \zeta^3, \zeta^4),$$

where

$$J_2(1) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

and ζ is a prime root of unity of degree 5

Theorem 6. *Let for $k \in \{0, 1, \ldots, 8\}$ $d_k^{(t)} = v_{k \bmod m_t}^{(t)} + \epsilon_t[\frac{k}{m_t}]\delta$ for $k \geq k_t$ where $v_s^{(t)}$ is the s-th vector in the set C_t from the table 10, $m_t = |C_t|$ and $(k_0, \ldots, k_8) = (10, 65, 34, 23, 14, 13, 28, 17, 19)$, $\epsilon_j = 1$ if $j \notin \{4, 7\}$ and $\epsilon_4 = \epsilon_7 = 2$.*

If the vector $\chi \notin h_{\widetilde{E}_8}$, i.e χ does not satisfy

$$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + 2(\beta_1 + \beta_2) + 3\delta_1 = 6\gamma \tag{7.1}$$

then the algebra $\mathcal{A}_{\widetilde{E}_8,\chi}$ has an irreducible non-degenerate representation in a generalized dimensions v if and only if for some $t \in \{0, 1, \ldots, 8\}$ and some $k \geq d_t$ $v = M_d d_k^{(t)}$ and $\mathrm{A}_{\mathrm{t,k}}\chi \geq_{\mathrm{t}} 0$ where the matrix $\mathrm{A}_{t,k}$ is taken from table 11. Such representation is unique. If χ satisfies (7.1) then irreducible non-degenerate representations of $\mathcal{A}_{\widetilde{E}_8,\chi}$ may exist only in the generalized dimension $\delta_{alg}(\widetilde{E}_8)$. They exist if and only if χ satisfies conditions $H_{\widetilde{E}_8}$.

In the following tables coordinates of the root vectors $(v_1, \ldots, v_n, v_0)$ correspond to the enumeration of vertices shown on the pictures in sections 4-7. We will omit the parentheses and commas in the vectors to shorten notations. If a coordinate is not a decimal digit we will take it in parentheses.

8 Table. C-orbits and matrices $\mathrm{A}_{i,j}$ for $\widetilde{D}_4$.

1. Δ consists of 25 δ-orbits.
2. Δ is a union of 5 C-series consisting of singular roots and 6 δ-series of regular roots from the set

$$\begin{aligned} \pm \ &\{10011, 10101, 11001, 00111,\\ &01011, 01101\} \end{aligned}$$

and one δ-series of imaginary roots.
3. $C_1 : 10000, 10001, 01111, 01112$. $\overset{\circ}{C}v_0 = v_1, \ldots, \overset{\bullet}{C}v_3 = v_0 + \delta$.
4. $C_0 : 00001, 11111$. $\overset{\bullet}{C}v_0 = v_1, \overset{\circ}{C}v_1 = v_0 + \delta$.
5.

$$\mathrm{A}_{1,\mathrm{k}} = D_{1,5}\begin{cases} C^{s-3}\overset{\bullet}{C}M_f & \text{if } k = 2s \\ C^{s-3}M_f & \text{if } k = 2s-1 \end{cases}, \qquad \mathrm{A}_{0,\mathrm{k}} = D_{0,2}\begin{cases} C^{s-1}M_f & \text{if } k = 2s \\ C^{s-1}\overset{\bullet}{C}M_f & \text{if } k = 2s+1 \end{cases};$$

$$D_{1,5} = \begin{pmatrix} -2 & -1 & -1 & -1 & 3 \\ -1 & 0 & -1 & -1 & 2 \\ -1 & -1 & 0 & -1 & 2 \\ -1 & -1 & -1 & 0 & 2 \\ -3 & -1 & -1 & -1 & 4 \end{pmatrix}, \qquad D_{0,2} = \begin{pmatrix} -1 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & 1 \\ -1 & -1 & -1 & -1 & 3 \end{pmatrix};$$

6. $\delta_{alg}(\widetilde{D}_4) = (1,1,1,1;2)$.
Hyperplane conditions $H_{\widetilde{D}_4}$:

$$\alpha+\beta+\xi>\gamma, \alpha+\xi+\delta>\gamma, \alpha+\beta+\delta>\gamma, \beta+\xi+\delta>\gamma.$$

9 Table. C-orbits and matrices $\mathrm{A}_{i,j}$ for $\widetilde{E}_6$.

1. Δ consists of 73 δ-orbits.
2. Δ is a union of 7 C-series consisting of singular roots and 11 δ-series of regular roots from the set

$$\pm \quad \{0001111, 0011011, 0100111, 0101012, 0111001, \\ 0111122, 0112112\}$$

and one δ-series of imaginary roots.
3. C_1 :

$$1000000, 1100000, 0100001, 0001011, 0011111, 0111111, \\ 1101012, 1201012, 1211112, 1112122, 0112123, 0212123.$$

$\overset{\circ}{C}v_0 = v_1, \dots, \overset{\bullet}{C}v_{11} = v_0 + \delta$.
4. C_2 :

$$0100000, 1100001, 1101011, 0111112, 0112122, 1112123.$$

$\overset{\bullet}{C}v_0 = v_1, \dots, \overset{\circ}{C}v_5 = v_0 + \delta$.
5. C_0 :

$$0000001, 0101011, 1111112, 1212122.$$

$\overset{\circ}{C}v_0 = v_1, \dots, \overset{\bullet}{C}v_3 = v_0 + \delta$
6.

$$\mathrm{A}_{1,\mathrm{k}} = D_{1,14}\begin{cases} C^{s-7}M_f & \text{if } k=2s \\ C^{s-7}\overset{\bullet}{C}M_f & \text{if } k=2s+1 \end{cases}, \qquad \mathrm{A}_{2,\mathrm{k}} = D_{2,7}\begin{cases} C^{4-s}\overset{\circ}{C}M_f & \text{if } k=2s, \\ C^{4-s}M_f & \text{if } k=2s-1. \end{cases};$$

$$\mathrm{A}_{0,\mathrm{k}} = \mathrm{D}_{0,4}\begin{cases} C^{2-s}M_f & \text{if } k=2s \\ C^{2-s}\overset{\circ}{C}M_f & \text{if } k=2s+1 \end{cases}$$

$$D_{1,14} = \begin{pmatrix} 1 & -3 & 1 & -2 & 1 & -2 & 4 \\ 2 & -5 & 2 & -3 & 2 & -3 & 6 \\ 1 & -2 & 0 & -1 & 1 & -2 & 3 \\ 1 & -4 & 1 & -3 & 2 & -3 & 6 \\ 1 & -2 & 1 & -2 & 0 & -1 & 3 \\ 1 & -4 & 2 & -3 & 1 & -3 & 6 \\ 2 & -7 & 2 & -4 & 2 & -4 & 9 \end{pmatrix}, \qquad D_{2,7} = \begin{pmatrix} 1 & -1 & 0 & -1 & 0 & -1 & 2 \\ 2 & -3 & 1 & -2 & 1 & -2 & 4 \\ 1 & -1 & 0 & 0 & 1 & -1 & 1 \\ 1 & -2 & 1 & -1 & 1 & -2 & 3 \\ 1 & -1 & 1 & -1 & 0 & 0 & 1 \\ 1 & -2 & 1 & -2 & 1 & -1 & 3 \\ 2 & -4 & 1 & -2 & 1 & -2 & 5 \end{pmatrix};$$

$$D_{0,4} = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 1 & -1 & 1 & -1 & 2 \\ 0 & 0 & 0 & -1 & 0 & 0 & 1 \\ 1 & -1 & 0 & -1 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ 1 & -1 & 1 & -1 & 0 & -1 & 2 \\ 1 & -2 & 1 & -2 & 1 & -2 & 4 \end{pmatrix}$$

7. $\delta_{alg}(\widetilde{E}_6) = (1,1;1,1;1,1;3)$.
Hyperplane conditions $H_{\widetilde{E}_6}$:

$$\alpha_1+\beta_1>\gamma, \alpha_1+\beta_2+\delta_2>\gamma, \alpha_1+\delta_1>\gamma, \alpha_2+\beta_1+\delta_2>\gamma,$$
$$\alpha_2+\beta_2+\delta_1>\gamma, \beta_1+\delta_1>\gamma, \alpha_1+\alpha_2+\beta_1+\beta_2+\delta_2>2\gamma,$$
$$\alpha_1+\alpha_2+\beta_1+\delta_1>2\gamma, \alpha_1+\alpha_2+\beta_2+\delta_1+\delta_2>2\gamma, \alpha_1+\beta_1+\beta_2+\delta_1>2\gamma,$$
$$\alpha_1+\beta_1+\delta_1+\delta_2>2\gamma, \alpha_2+\beta_1+\beta_2+\delta_1+\delta_2>2\gamma.$$

10 Table. C-orbits and matrices $\mathrm{A}_{i,j}$ for $\widetilde{E}_7$.

1. Δ consists of 63 δ-orbits.
2. Δ is a union of 7 C-series consisting of singular roots and 11 δ-series of regular roots from the set

$$\{00011111, 00100112, 00101101, 00112212, 01100101,$$
$$01101111, 01111212, 01201223, 01211112, 01212313\}.$$

and one δ-series of imaginary roots.
3. C_1 :

$$10000000, 11000000, 01100000, 00100001, 00000111,$$
$$00001111, 00111101, 01111101, 11101111, 11100112,$$
$$01200112, 01201112, 11111212, 11112212, 01212212,$$
$$01211213, 11201223, 12201223, 12311213, 12312213,$$
$$12212323, 11212324, 01312324, 02312324.$$

$\overset{\circ}{C}v_0 = v_1, \ldots, \overset{\bullet}{C}v_{23} = v_0+\delta$.
4. C_2 :

$$01000000, 11100000, 11100001, 01100111, 00101112,$$
$$00111212, 01112212, 11212212, 12211213, 12301223,$$
$$12301224, 12311324, 12313324, 12323424, 12323425,$$
$$12413435, 13412436, 23512436, 24513436, 24523536,$$
$$23524537, 13524647, 13524648, 23624648.$$

$\overset{\bullet}{C}v_0 = v_1, \ldots, \overset{\circ}{C}v_{23} = v_0+2\delta$.
5. C_3 :

$$00100000, 01100001, 11100111, 11101112, 01211212,$$
$$01212213, 11212323, 12212324.$$

$\overset{\circ}{C}v_0 = v_1, \ldots, \overset{\bullet}{C}v_7 = v_0+\delta$.
6. C_7 :

$$00000010, 00000011, 00100101, 01101101, 11111111,$$
$$11111112, 01201212, 01201213, 11211223, 12212223,$$
$$12312313, 12312314.$$

$\overset{\bullet}{C} v_0 = v_1, \ldots, \overset{\circ}{C} v_{11} = v_0 + \delta.$

7. C_0 :

$$00000001, 00100111, 01101112, 11211212, 12212213, 12312323.$$

$\overset{\bullet}{C} v_0 = v_1, \ldots, \overset{\circ}{C} v_5 = v_0 + \delta.$

8.

$\mathrm{A_{1,k}} = D_{1,27} \begin{cases} C^{s-14}\overset{\bullet}{C} M_f & \text{if } k = 2s \\ C^{s-14} M_f & \text{if } k = 2s-1 \end{cases}$, $\quad \mathrm{A_{2,k}} = D_{2,14} \begin{cases} C^{7-s} M_f & \text{if } k = 2s \\ \overset{\bullet}{C} C^{s-7} M_f & \text{if } k = 2s+1 \end{cases}$;

$\mathrm{A_{3,k}} = D_{3,9} \begin{cases} C^{s-5}\overset{\bullet}{C} M_f & \text{if } k = 2s \\ C^{s-5} M_f & \text{if } k = 2s-1 \end{cases}$, $\quad \mathrm{A_{7,k}} = D_{7,11} \begin{cases} \overset{\circ}{C} C^{s-6} M_f & \text{if } k = 2s \\ C^{6-s} M_f & \text{if } k = 2s-1 \end{cases}$;

$$\mathrm{A_{8,k}} = \mathrm{D_{8,6}} \begin{cases} C^{s-3} M_f & \text{if } k = 2s \\ C^{s-3}\overset{\bullet}{C} M_f & \text{if } k = 2s+1 \end{cases}$$

$D_{1,27} = \begin{pmatrix} -1 & 2 & -4 & -1 & 2 & -3 & -2 & 5 \\ -2 & 4 & -7 & -2 & 4 & -5 & -3 & 8 \\ -2 & 5 & -10 & -2 & 5 & -7 & -5 & 12 \\ -1 & 2 & -3 & -1 & 1 & -2 & -2 & 4 \\ -1 & 3 & -6 & -2 & 3 & -5 & -3 & 8 \\ -2 & 4 & -9 & -3 & 5 & -7 & -5 & 12 \\ -2 & 3 & -6 & -1 & 3 & -5 & -3 & 8 \\ -3 & 6 & -13 & -3 & 6 & -9 & -6 & 16 \end{pmatrix}$, $\quad D_{2,14} = \begin{pmatrix} -1 & 1 & -1 & -1 & 2 & -2 & -1 & 2 \\ -1 & 2 & -3 & -2 & 3 & -4 & -2 & 5 \\ -1 & 3 & -4 & -3 & 4 & -5 & -2 & 6 \\ 0 & 0 & -1 & -1 & 1 & -1 & -1 & 2 \\ 0 & 1 & -2 & -1 & 2 & -3 & -2 & 4 \\ -1 & 2 & -3 & -2 & 3 & -5 & -2 & 6 \\ -1 & 2 & -3 & -1 & 2 & -3 & -1 & 4 \\ -2 & 4 & -5 & -3 & 4 & -6 & -3 & 8 \end{pmatrix}$;

$D_{3,9} = \begin{pmatrix} 1 & -1 & 1 & 1 & -1 & 2 & 1 & -2 \\ 1 & -2 & 3 & 2 & -3 & 4 & 2 & -4 \\ 2 & -3 & 5 & 2 & -4 & 6 & 4 & -7 \\ 0 & 0 & 1 & 0 & -1 & 2 & 1 & -2 \\ 1 & -1 & 3 & 1 & -2 & 3 & 2 & -4 \\ 2 & -3 & 5 & 1 & -3 & 5 & 3 & -6 \\ 1 & -2 & 3 & 1 & -2 & 3 & 3 & -4 \\ 3 & -4 & 6 & 2 & -5 & 7 & 5 & -8 \end{pmatrix}$, $\quad D_{7,11} = \begin{pmatrix} 0 & 1 & -1 & -1 & 1 & -1 & 0 & 1 \\ -1 & 2 & -2 & -1 & 1 & -2 & -1 & 3 \\ -1 & 2 & -3 & -1 & 2 & -4 & -1 & 5 \\ -1 & 1 & -1 & 0 & 1 & -1 & 0 & 1 \\ -1 & 1 & -2 & -1 & 2 & -2 & -1 & 3 \\ -1 & 2 & -4 & -1 & 2 & -3 & -1 & 5 \\ -1 & 2 & -3 & -1 & 2 & -3 & -1 & 4 \\ -1 & 3 & -5 & -1 & 3 & -5 & -2 & 7 \end{pmatrix}$;

$$D_{8,6} = \begin{pmatrix} 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 1 & -1 & -1 & 2 \\ 0 & 1 & -2 & -1 & 2 & -2 & -1 & 3 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 1 & -1 & 0 & 0 & -1 & -1 & 2 \\ -1 & 2 & -2 & 0 & 1 & -2 & -1 & 3 \\ -1 & 1 & -1 & -1 & 1 & -1 & -1 & 2 \\ -1 & 2 & -3 & -1 & 2 & -3 & -2 & 5 \end{pmatrix}$$

$\delta_{alg} = (1, 1, 1, 1, 1, 1, 2; 4).$

Hyperplane conditions $H_{\widetilde{E}_7}$:

$$\alpha_1 + \beta_2 > \gamma, \delta + \alpha_1 > \gamma, \alpha_2 + \beta_1 > \gamma, \delta + \alpha_2 + \beta_3 > \gamma,$$
$$\delta + \alpha_3 + \beta_2 > \gamma, \delta + \beta_1 > \gamma, \delta + \alpha_1 + \alpha_2 + \beta_2 > 2\gamma,$$
$$\delta + \alpha_1 + \alpha_3 + \beta_1 > 2\gamma, \delta + \alpha_1 + \alpha_3 + \beta_2 + \beta_3 > 2\gamma, \delta + \alpha_1 + \beta_1 + \beta_3 > 2\gamma,$$
$$\delta + \alpha_2 + \alpha_3 + \beta_1 + \beta_3 > 2\gamma, \delta + \alpha_2 + \beta_1 + \beta_2 > 2\gamma,$$
$$\delta + \alpha_1 + \alpha_2 + \alpha_3 + \beta_1 + \beta_2 > 3\gamma, 2\delta + \alpha_1 + \alpha_2 + \alpha_3 + \beta_2 + \beta_3 > 3\gamma,$$
$$\delta + \alpha_1 + \alpha_2 + \beta_1 + \beta_2 + \beta_3 > 3\gamma, 2\delta + \alpha_1 + \alpha_2 + \beta_1 + \beta_3 > 3\gamma,$$
$$2\delta + \alpha_1 + \alpha_3 + \beta_1 + \beta_2 > 3\gamma, 2\delta + \alpha_2 + \alpha_3 + \beta_1 + \beta_2 + \beta_3 > 3\gamma$$

11 Table. C-orbits and matrices $\mathrm{A}_{i,j}$ for $\widetilde{E}_8$.

1. Δ consists of 241 δ-orbits.
2. Δ is a union of 8 C-series consisting of singular roots and 28 δ-series of regular roots from the set

$$\pm\{000000000, 000010112, 000111111, 000121223, 001111101, 001121213, 001220112, 001232324, 011110101, 011121112, 011221212, 011231324, 012221223, 012341224, 012342425\}.$$

and one δ-series of imaginary roots.
3. C_1 :

$$100000000, 110000000, 011000000, 001100000, 000110000, 000010001, 000000111, 000001111, 000011101, 000110101, 001110011, 011110011, 111110101, 111111101, 011111111, 001110112, 000120112, 000121112, 001111212, 011111212, 111121112, 111220112, 012220112, 012221112, 111221212, 111121213, 011121223, 001221223, 001231213, 011231213, 112221223, 122221223, 122231213, 112331213, 012331223, 012231224, 111231324, 111232324, 012232324, 012331324, 112341224, 122341224, 123331324, 123332324, 122342324, 112341325, 012341335, 012342335, 112342425, 122342425, 123342335, 123441335, 123451325, 123452325, 123442435, 123342436, 122352436, 112452436, 013452436, 023452436.$$

$\overset{\circ}{C}v_0 = v_1, \ldots, \overset{\bullet}{C}v_{59} = v_0 + \delta$.
4. C_2 :

$$010000000, 111000000, 111100000, 011110000, 001110001, 000110111, 000011112, 000011212, 000111212, 001121112, 011220112, 112220112, 122221112, 122221212, 112221213, 011231223, 001231224, 001231324, 011232324, 112232324, 122331324, 123341224, 123441224, 123441324, 123342325, 122342435, 112342436, 012352436, 012452436, 113452436.$$

$\overset{\bullet}{C}v_0 = v_1, \ldots, \overset{\circ}{C}v_{29} = v_0 + \delta$.
5. C_3 :

$$001000000, 011100000, 111110000, 111110001, 011110111, 001111112, 000121212, 000121213, 001121223, 011221223, 112231213, 122331213, 123331223, 123331224, 122341324, 112342325, 012342435, 012342436, 112352436, 122452436.$$

$\overset{\circ}{C}v_0 = v_1, \ldots, \overset{\bullet}{C}v_{19} = v_0 + \delta$.

6. C_4 :

$$000100000, 001110000, 011110001, 111110111, 111111112, \\ 011121212, 001221213, 001231223, 011231224, 112231324, \\ 122332324, 123342324, 123441325, 123451335, 123452336, \\ 123452536, 123453537, 123463547, 123562548, 124572548, \\ 134673548, 235673648, 245673649, 245683659, 23578365(10), \\ 13579375(10), 13579475(11), 23579486(11), 24579486(12), \\ 2467(10)486(12).$$

$\overset{\bullet}{C}v_0 = v_1, \ldots, \overset{\circ}{C}v_{29} = v_0 + 2\delta.$

7. C_5 :

$$000010000, 000110001, 001110111, 011111112, 111121212, \\ 111221213, 012231223, 012331224, 112341324, 122342325, \\ 123342435, 123442436.$$

$\overset{\circ}{C}v_0 = v_1, \ldots, \overset{\bullet}{C}v_{11} = v_0 + \delta.$

8. C_6 :

$$000001000, 000001100, 000000101, 000010011, 000110011, \\ 001110101, 011111101, 111111111, 111110112, 011120112, \\ 001221112, 001221212, 011121213, 111121223, 111221223, \\ 012231213, 012331213, 112331223, 122231224, 122231324, \\ 112332324, 012342324, 012341325, 112341335, 122342335, \\ 123342425, 123442425, 123452335, 123451336, 123451436.$$

$\overset{\bullet}{C}v_0 = v_1, \ldots, \overset{\circ}{C}v_{29} = v_0 + \delta.$

9. C_7 :

$$000000100, 000001101, 000011111, 000110112, 001120112, \\ 011221112, 112221212, 122221213, 122231223, 112331224, \\ 012341324, 012342325, 112342435, 122342436, 123352436, \\ 123552436, 124562436, 134562437, 234562547, 234563548, \\ 134573648, 124673649, 124683659, 13468365(10), \\ 23568375(10), 24578475(10), 24679475(10), 24689375(11), \\ 2468(10)376(11), 2468(10)476(12).$$

$\overset{\circ}{C}v_0 = v_1, \ldots, \overset{\bullet}{C}v_{29} = v_0 + 2\delta.$

10. C_8 :

$$000000010, 000000011, 000010101, 000111101, 001111111, \\ 011110112, 111120112, 111221112, 012221212, 012221213, \\ 111231223, 111231224, 012231324, 012332324112342324, \\ 122341325, 123341335, 123442335, 123452425, 123452426.$$

$\overset{\circ}{C}v_0 = v_1, \ldots, \overset{\bullet}{C}v_{19} = v_0 + \delta.$

9. C_0 :

$$000000001, 000010111, 000111112, 001121212, 011221213,$$
$$112231223, 122331224, 123341324, 123442325, 123452435.$$

$\overset{\bullet}{C}v_0 = v_1, \ldots, \overset{\circ}{C}v_9 = v_0 + \delta.$

10.

$$\mathrm{A_{1,k}} = D_{1,65} \begin{cases} C^{s-33}\overset{\bullet}{C}M_f & \text{if } k = 2s \\ C^{s-33}M_f & \text{if } k = 2s-1 \end{cases}, \qquad \mathrm{A_{2,k}} = D_{2,34} \begin{cases} C^{s-17}M_f & \text{if } k = 2s \\ C^{s-17}\overset{\bullet}{C}M_f & \text{if } k = 2s+1 \end{cases};$$

$$\mathrm{A_{3,k}} = D_{3,23} \begin{cases} C^{s-12}\overset{\bullet}{C}M_f & \text{if } k = 2s \\ C^{s-12}M_f & \text{if } k = 2s-1 \end{cases}, \qquad \mathrm{A_{4,k}} = D_{4,14} \begin{cases} C^{s-7}M_f & \text{if } k = 2s \\ C^{s-7}\overset{\bullet}{C}M_f & \text{if } k = 2s+1 \end{cases};$$

$$\mathrm{A_{5,k}} = D_{5,13} \begin{cases} C^{s-7}\overset{\bullet}{C}M_f & \text{if } k = 2s, \\ C^{s-7}M_f & \text{if } k = 2s-1. \end{cases}, \qquad \mathrm{A_{6,k}} = D_{6,28} \begin{cases} C^{s-14}M_f & \text{if } k = 2s \\ C^{s-14}\overset{\bullet}{C}M_f & \text{if } k = 2s+1 \end{cases};$$

$$\mathrm{A_{7,k}} = D_{7,17} \begin{cases} C^{s-9}\overset{\bullet}{C}M_f & \text{if } k = 2s, \\ C^{s-9}M_f & \text{if } k = 2s-1. \end{cases}, \qquad \mathrm{A_{8,k}} = D_{8,19} \begin{cases} C^{s-10}\overset{\bullet}{C}M_f & \text{if } k = 2s, \\ C^{s-10}M_f & \text{if } k = 2s-1. \end{cases};$$

$$\mathrm{A_{0,k}} = \mathrm{D_{0,10}} \begin{cases} C^{s-5}M_f & \text{if } k = 2s \\ C^{s-5}\overset{\bullet}{C}M_f & \text{if } k = 2s+1 \end{cases}$$

$$D_{1,65} = \begin{pmatrix} -1 & 2 & -3 & 4 & -6 & 2 & -4 & -3 & 7 \\ -2 & 4 & -6 & 8 & -11 & 4 & -7 & -5 & 12 \\ -3 & 6 & -8 & 11 & -16 & 5 & -10 & -8 & 18 \\ -3 & 7 & -10 & 14 & -21 & 7 & -14 & -10 & 24 \\ -4 & 8 & -12 & 17 & -26 & 9 & -17 & -13 & 30 \\ -2 & 3 & -5 & 7 & -10 & 3 & -7 & -5 & 12 \\ -4 & 7 & -10 & 13 & -20 & 7 & -14 & -10 & 24 \\ -2 & 5 & -8 & 10 & -15 & 5 & -10 & -8 & 18 \\ -5 & 10 & -15 & 20 & -31 & 10 & -20 & -15 & 36 \end{pmatrix}, \qquad D_{2,34} = \begin{pmatrix} -1 & 1 & -2 & 3 & -3 & 1 & -2 & -1 & 3 \\ -1 & 2 & -4 & 5 & -6 & 2 & -4 & -3 & 7 \\ -1 & 3 & -6 & 7 & -8 & 3 & -5 & -4 & 9 \\ -2 & 4 & -7 & 8 & -10 & 4 & -7 & -5 & 12 \\ -2 & 4 & -8 & 10 & -13 & 4 & -8 & -6 & 15 \\ -1 & 2 & -3 & 4 & -5 & 1 & -3 & -3 & 6 \\ -1 & 3 & -6 & 8 & -10 & 3 & -7 & -5 & 12 \\ -1 & 2 & -5 & 6 & -7 & 2 & -5 & -4 & 9 \\ -2 & 4 & -9 & 12 & -15 & 5 & -10 & -8 & 18 \end{pmatrix};$$

$$D_{3,23} = \begin{pmatrix} -1 & 1 & -1 & 2 & -2 & 0 & -1 & -1 & 2 \\ -1 & 2 & -3 & 4 & -4 & 1 & -2 & -2 & 4 \\ -2 & 3 & -4 & 5 & -6 & 2 & -4 & -3 & 7 \\ -3 & 4 & -5 & 6 & -7 & 3 & -5 & -3 & 8 \\ -3 & 4 & -6 & 7 & -8 & 3 & -6 & -4 & 10 \\ -1 & 2 & -2 & 2 & -3 & 1 & -2 & -2 & 4 \\ -2 & 3 & -4 & 5 & -7 & 2 & -4 & -3 & 8 \\ -1 & 2 & -3 & 4 & -5 & 2 & -4 & -2 & 6 \\ -3 & 4 & -6 & 8 & -10 & 4 & -7 & -5 & 12 \end{pmatrix}, \qquad D_{4,14} = \begin{pmatrix} 0 & 1 & -1 & 1 & -1 & 0 & 0 & -1 & 1 \\ -1 & 2 & -2 & 2 & -2 & 1 & -1 & -1 & 2 \\ -1 & 2 & -3 & 3 & -3 & 1 & -2 & -2 & 4 \\ -1 & 2 & -3 & 4 & -5 & 2 & -3 & -3 & 6 \\ -1 & 2 & -3 & 4 & -6 & 2 & -3 & -3 & 7 \\ 0 & 0 & -1 & 2 & -2 & 1 & -1 & -1 & 2 \\ 0 & 1 & -2 & 3 & -4 & 2 & -3 & -2 & 5 \\ -1 & 1 & -1 & 2 & -3 & 1 & -2 & -2 & 4 \\ -1 & 2 & -3 & 4 & -6 & 3 & -4 & -4 & 8 \end{pmatrix};$$

$$D_{5,13} = \begin{pmatrix} 0 & 0 & 0 & 1 & -1 & 0 & -1 & 0 & 1 \\ 0 & 1 & -1 & 2 & -2 & 0 & -1 & -1 & 2 \\ -1 & 2 & -2 & 3 & -3 & 1 & -2 & -1 & 3 \\ -1 & 2 & -3 & 4 & -4 & 1 & -3 & -2 & 5 \\ -1 & 2 & -3 & 5 & -6 & 2 & -4 & -3 & 7 \\ 0 & 0 & -1 & 2 & -2 & 1 & -1 & -1 & 2 \\ 0 & 1 & -2 & 3 & -4 & 2 & -3 & -2 & 5 \\ -1 & 1 & -1 & 2 & -3 & 1 & -2 & -2 & 4 \\ -1 & 2 & -3 & 5 & -7 & 2 & -4 & -3 & 8 \end{pmatrix}, \quad D_{6,28} = \begin{pmatrix} -1 & 1 & -1 & 2 & -2 & 0 & -1 & -1 & 2 \\ -1 & 2 & -3 & 4 & -4 & 1 & -2 & -2 & 4 \\ -1 & 3 & -4 & 5 & -6 & 1 & -3 & -4 & 7 \\ -2 & 4 & -5 & 6 & -8 & 2 & -5 & -5 & 10 \\ -2 & 4 & -6 & 8 & -11 & 2 & -6 & -6 & 13 \\ -1 & 2 & -3 & 4 & -5 & 1 & -3 & -3 & 6 \\ -1 & 3 & -5 & 7 & -9 & 2 & -6 & -5 & 11 \\ -1 & 2 & -4 & 5 & -6 & 1 & -4 & -4 & 8 \\ -2 & 4 & -7 & 10 & -13 & 3 & -8 & -8 & 16 \end{pmatrix};$$

$$D_{7,17} = \begin{pmatrix} 0 & 1 & -1 & 1 & -1 & 0 & 0 & -1 & 1 \\ -1 & 2 & -2 & 2 & -2 & 1 & -1 & -1 & 2 \\ -1 & 2 & -3 & 3 & -3 & 1 & -2 & -2 & 4 \\ -1 & 2 & -3 & 4 & -5 & 2 & -3 & -3 & 6 \\ -1 & 3 & -4 & 5 & -7 & 2 & -4 & -3 & 8 \\ 0 & 1 & -2 & 3 & -3 & 1 & -2 & -1 & 3 \\ -1 & 3 & -4 & 5 & -6 & 2 & -4 & -3 & 7 \\ -1 & 2 & -2 & 3 & -4 & 1 & -3 & -2 & 5 \\ -1 & 4 & -5 & 6 & -8 & 3 & -6 & -4 & 10 \end{pmatrix}, \quad D_{8,19} = \begin{pmatrix} -1 & 1 & -1 & 1 & -1 & 1 & -1 & 0 & 1 \\ -1 & 1 & -2 & 2 & -2 & 1 & -2 & -1 & 3 \\ -1 & 1 & -2 & 3 & -4 & 2 & -3 & -2 & 5 \\ -1 & 2 & -3 & 4 & -6 & 2 & -4 & -2 & 7 \\ -1 & 3 & -4 & 5 & -7 & 3 & -6 & -3 & 9 \\ 0 & 1 & -2 & 3 & -3 & 1 & -2 & -1 & 3 \\ -1 & 2 & -3 & 5 & -6 & 2 & -5 & -2 & 7 \\ -1 & 2 & -3 & 4 & -5 & 2 & -4 & -2 & 6 \\ -2 & 4 & -5 & 7 & -9 & 3 & -7 & -4 & 11 \end{pmatrix};$$

$$D_{0,10} = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 1 & -1 & -1 & 2 \\ 0 & 0 & 0 & 1 & -2 & 1 & -2 & -1 & 3 \\ 0 & 1 & -1 & 2 & -3 & 1 & -2 & -2 & 4 \\ -1 & 2 & -2 & 3 & -4 & 2 & -3 & -2 & 5 \\ -1 & 1 & -1 & 1 & -1 & 0 & -1 & -1 & 2 \\ -1 & 1 & -2 & 3 & -3 & 1 & -2 & -2 & 4 \\ 0 & 1 & -2 & 2 & -2 & 1 & -2 & -1 & 3 \\ -1 & 2 & -3 & 4 & -5 & 2 & -4 & -3 & 7 \end{pmatrix}$$

11. $\delta_{alg} = (1,1,1,1,1,2,2,3,6)$.
Hyperplane conditions $H_{\widetilde{E}_8}$:
$\delta+\alpha_2 > \gamma$, $\delta+\beta_1 > \gamma$, $\alpha_3+\beta_1 > \gamma$, $3\delta+\alpha_2+\alpha_3+\alpha_4+\alpha_5+2\beta_1+2\beta_2 > 5\gamma$, $\delta+\alpha_1+\alpha_5+\beta_1 > 2\gamma$, $\alpha_1+\beta_2 > \gamma$, $\delta+\alpha_4+\beta_2 > \gamma$, $\delta+\alpha_2+\alpha_4+\beta_1 > 2\gamma$, $\delta+\alpha_2+\beta_1+\beta_2 > 2\gamma$, $2\delta+\alpha_2+\alpha_3+\beta_1+\beta_2 > 3\gamma$, $2\delta+\alpha_1+\alpha_4+\beta_1+\beta_2 > 3\gamma$, $\delta+\alpha_1+\alpha_3+\alpha_4+\beta_1+\beta_2 > 3\gamma$, $\delta+\alpha_1+\alpha_2+\alpha_5+\beta_1+\beta_2 > 3\gamma$, $\delta+\alpha_3+\alpha_5+\beta_1+\beta_2 > 2\gamma$, $2\delta+\alpha_1+\alpha_2+\alpha_3+\alpha_5+\beta_1+\beta_2 > 4\gamma$, $2\delta+\alpha_2+\alpha_4+\alpha_5+\beta_1+\beta_2 > 3\gamma$, $2\delta+\alpha_2+\alpha_3+\alpha_4+2\beta_1+\beta_2 > 4\gamma$, $2\delta+\alpha_1+\alpha_3+\alpha_5+2\beta_1+\beta_2 > 4\gamma$, $3\delta+\alpha_1+\alpha_2+\alpha_4+\alpha_5+2\beta_1+\beta_2 > 5\gamma$, $2\delta+\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5+2\beta_1+\beta_2 > 5\gamma$, $2\delta+\alpha_1+\alpha_2+\alpha_4+\beta_1+2\beta_2 > 4\gamma$, $3\delta+\alpha_1+\alpha_2+\alpha_3+\alpha_4+\beta_1+2\beta_2 > 5\gamma$, $2\delta+\alpha_1+\alpha_3+\alpha_4+\alpha_5+\beta_1+2\beta_2 > 4\gamma$, $2\delta+\alpha_1+\alpha_2+\alpha_3+\alpha_5+2\beta_1+2\beta_2 > 5\gamma$, $\delta+\alpha_1+\alpha_3+\beta_2 > 2\gamma$

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