THESIS FOR THE DEGREE OF LICENTIATE OF PHILOSOPHY

Carleson measures for Hardy Sobolev spaces and Generalized Bergman spaces

EDGAR TCHOUNDJA

CHALMERS | GÖTEBORG UNIVERSITY

Department of Mathematical Sciences
Chalmers University of Technology and Göteborg University
SE - 412 96 Göteborg, Sweden

Göteborg, March 2007

Carleson measures for Hardy Sobolev spaces
and Generalized Bergman spaces
Edgar Tchoundja
ISSN:1652-9715 /NO 2007:7

©Edgar Tchoundja, 2007

Department of Mathematical Sciences
Division of Mathematics
Chalmers University of Technology and Göteborg University
SE-412 96 Göteborg
Sweden
Telephone +46 (0)31 772 10 00

Printed in Göteborg, Sweden 2007

Abstract

Carleson measures for various spaces of holomorphic functions in the unit ball have been studied extensively since Carleson's original result for the disk. In the first paper of this thesis, we give, by means of Green's formula, an alternative proof of the characterization of Carleson measures for some Hardy Sobolev spaces (including Hardy space) in the unit ball. In the second paper of this thesis, we give a new characterization of Carleson measures for the Generalized Bergman spaces on the unit ball using singular integral techniques.

Keywords: Carleson measures, Hardy Sobolev spaces, Generalized Bergman spaces, Besov Sobolev spaces, Arveson's Hardy space, T(1)-Theorem.

AMS 2000 Subject Classification: 26B20, 32A35, 32A37, 32A55, 46E35.

Acknowledgments

First of all I would like to thank my advisor Bo Berndtsson, at Chalmers, for introducing me to the topic of this thesis. I have since then benefited from his valuable explanations and comments. He has taught me the mathematical spirit. He has greatly contributed to making my stay here enjoyable, with his friendly attitude.

I would like to thank my co-advisor Genkai Zhang, at Chalmers, for stimulating discussions on this topic, as well as many others topics. Thanks for always taking time, even when busy, to answer my questions.

I would like to thank my advisor David Békollé, at Yaoundé I, for being the one who introduced me to the complex analysis, and for his constant solicitude. I have benefited from his numerous comments on the singular integral techniques.

I deeply thank the International Science Program (ISP), Uppsala, Sweden for the financial support through a fellowship which gave me the opportunity to carry out part of my posgraduate studies here at Chalmers.

In addition, I thank the Chalmers University of Technology for providing me with an excellent academic environment and good logistics. In particular, I thank the complex analysis group at the department of Mathematical sciences. I express my gratitude to everyone at the Mathematical department of the Faculty of Sciences at the University of Yaoundé I. Especially to my fellows at the classical analysis group.

At last but not least, I would like to thank my family for their love, and numerous friends I have met for their kind support.

This thesis consists of an introduction and the following two papers.

Paper I. E. Tchoundja: Carleson measures for Hardy Sobolev spaces.

Paper II. E. Tchoundja: Carleson measures for the Generalized Bergman spaces via a T(1)- Theorem type.

CARLESON MEASURES FOR HARDY SOBOLEV SPACES AND GENERALIZED BERGMAN SPACES

EDGAR TCHOUNDJA

INTRODUCTION

Let Ω be a region in $\mathbb{C}^n$ and X a Banach space of continuous functions in Ω. We raise the following problem: Characterize positive measures μ in Ω such that there exists a constant $C = C(\mu)$ with the property that

$$\int_{\Omega} |f(z)|^p \leq C\|f\|_X^p$$

for all $f \in X$. Such problems are now known in the literature as Carleson measures problem for X. The purpose of this work is to study Carleson measures for Hardy Sobolev spaces, see Paper I, and also Carleson measures for the Generalized Bergman spaces, see Paper II. In this introduction, we briefly give a literature overview and motivation of this question. We outline our work in the last two sections .

0.1. **Overview and motivation.** Carleson [CA58] was the first to study this question in the case of the unit disk $\mathbf{U}$ of $\mathbb{C}$ and for the Hardy space $\mathbf{H}^p(\mathbf{U})$ which is the space of holomorphic functions f in $\mathbf{U}$ with the property that:

$$\sup_{r<1} \int_{-\pi}^{\pi} |f(re^{i\theta})|^p d\theta < \infty.$$

Carleson measures arise in many questions involving analysis in functions spaces. We could mention here the problem of multipliers. That is, given a space, X, of functions on a set Ω, we want to describe those functions f for which the map of multiplication by f is a continuous map of X to itself. We mention also the interpolating sequences problem. That is, to characterize those sequences $\{y_i\}$ in Ω with the property that, as f ranges over X, the set of sequences of values, $\{f(y_i)\}$ ranges over the space of all sequences which satisfy a natural growth condition. Carleson measures have played an important role in the solution of some famous problems such as the Corona problem and the duality theory for $\mathbf{H}^1$. See Fefferman and Stein [FS72], Carleson [CA62]. This explains why Carleson measures have been studied extensively since the characterization obtained by Carleson [CA62] in 1962.

In 1967, Hörmander [H67] extended Carleson's result to the unit ball of $\mathbb{C}^n$. Since then, for various spaces, this problem as well as its applications have attracted many authors. Among them, we will mention Stegenga [ST80] who characterized Carleson measures for the space, D_α, of analytic functions f in the unit disk for which

$$\sum_{n=0}^{\infty}(1+n^2)^\alpha |a_n|^2 < \infty$$

where $f(z) = \sum_{n\geq 0} a_n z^n$ is the Taylor expansion of f.

Cima and Wogen [CW82] gave the characterization of Carleson measures for the weighted Bergman spaces, $\mathbf{A}^p_\alpha$ ($\alpha > -1$), in the unit ball, $\mathbf{D}_n$, of $\mathbb{C}^n$. This is the space of holomorphic functions f in $\mathbf{D}_n$ such that

$$\int_{\mathbf{D}_n} |f(z)|^p (1-|z|^2)^\alpha d\lambda(z) < \infty.$$

The Hardy Sobolev space $\mathbf{H}^p_\beta(\mathbf{D}_n)$ is the space of holomorphic functions f in $\mathbf{D}_n$ such that

$$\sup_{0\leq r<1} \int_{\mathbf{S}_n} \left|(I+\mathcal{R})^\beta f(r\xi)\right|^p d\sigma(\xi) < \infty,$$

where $(I+\mathcal{R})^\beta f(z) = \sum(1+k)^\beta f_k(z)$ if $f = \sum f_k$ is the homogeneous expansion of f. P.Ahern [A88] characterized Carleson measures for this space when $0 < p \leq 1$. For $p > 1$, Cascante and Ortega [CO95] studied the same question and gave some necessary and some sufficient conditions. They also studied Carleson measures for the holomorphic Besov Sobolev space. The Carleson measures for the analytic Besov spaces in the unit ball have been characterized by Arcozzi, Rochberg and Sawyer [ARS02, ARS06]. We mention finally the generalized Bergman space $\mathbf{A}^p_\alpha$ ($\alpha \in \mathbb{R}$), introduced in [ZZ05], that is the space of holomorphic functions f in the unit ball such that for some integer m with $mp + \alpha > -1$

$$\int_{\mathbf{D}_n} |(I+\mathcal{R})^m f(z)|^p (1-|z|^2)^{pm+\alpha} d\lambda(z) < \infty.$$

Zhu and Zhao [ZZ05] characterized Carleson measures for this space when $0 < p \leq 1$ and Arcozzi, Rochberg and Sawyer [ARS06$_2$] recently obtained results for the case $p = 2$ and $\alpha \in (-n-1, -n]$.

0.2. **Carleson measures on Hardy Sobolev spaces.** We say that a positive Borel measure μ on $\mathbf{D}_n$ is a Carleson measure for $\mathbf{H}^p_\beta = \mathbf{H}^p_\beta(\mathbf{D}_n)$ if there exists a constant $C = C(\mu)$ such that:

$$(1) \qquad \|f\|_{L^p(\mu)} \leq C\|f\|_{\mathbf{H}^p_\beta}$$

for all $f \in \mathbf{H}^p_\beta$.

Observe that when $\beta = 0$ we have the usual Hardy space. For $0 < p \leq 1$ and $0 < \beta < \frac{n}{p}$, Ahern[A88] described Carleson measures using atomic decomposition for functions in $\mathbf{H}^p$ (see [A88, Lemma 1.1]). For $\beta = 0$ and $p > 0$, Hörmander [H67] solved the problem using a covering lemma and boundedness of a maximal function. In paper I, inspired by results in [An97, B87], we give a new proof of these results. We use Green's formula with respect to an appropriate positive closed (1.1) form to reduce the problem to pointwise estimates for functions in these spaces. Moreover we show, by giving a counterexample, that the natural growth condition is no longer sufficient for $p > 1$ and $\beta > 0$.

0.3. **Carleson measures on the Generalized Bergman space.** We restrict ourselves to $p = 2$. The Generalized Bergman spaces $\mathbf{A}^2_\alpha$ [ZZ05] consist of all holomorphic functions f in the unit ball $\mathbf{D}_n$ with the property that:

$$\|f\|^2_\alpha = \int_{\mathbf{D}_n} |(I + \mathcal{R})^m f(z)|^2 (1 - |z|^2)^{2m+\alpha} d\lambda(z)$$

where $2m + \alpha > -1$. Note that this definition is independent of m.
In this case a positive Borel measure μ on $\mathbf{D}_n$ is a Carleson measure for $\mathbf{A}^2_\alpha$ if there exists a constant $C = C(\mu)$ such that:

$$\int_{\mathbf{D}} |f|^2 d\mu \leq C\|f\|^2_\alpha, \quad f \in \mathbf{A}^2_\alpha.$$

One can easily observe that when $\alpha > -1$ we have the usual weighted Bergman spaces (take $m = 0$), and when $\alpha = -1$, we have the usual Hardy space. Thus Carleson type embedding problems have been settled in these cases by Carleson, Hörmander, Stegenga, Cima and Wogen. The range $\alpha \in (-n-1, -1)$ was unsolved until 2006 when Arcozzi, Rochberg and Sawyer [ARS06$_2$] obtained results for the range $\alpha \in (-n-1, -n]$. Their characterization used certain tree condition. They have transformed the question, equivalently to a question of boundedness of an operator T_α in $L^2(\mu)$ where

$$T_\alpha f(z) = \int_{\mathbf{D}} f(w) K_\alpha(z, w) d\mu(w), \ z \in \mathbf{D},$$

with K_α defined by $K_\alpha(z, w) = \Re\left\{\frac{1}{(1-z\overline{w})^{n+1+\alpha}}\right\}$.

In paper II, we use singular integral techniques to characterize measures μ for which this operator is bounded. To be precise, recall that for a topological space X with a pseudo distance d, a kernel $K(x,y)$ is called an n Calderón-Zygmund kernel with respect to the pseudo distance d if

a) $|K(x,y)| \leq \frac{C_1}{d(x,y)^n}$ and
b) There exists $0 < \delta \leq 1$ such that

$$|K(x,y) - K(x',y)| + |K(y,x) - K(y,x')| \leq C_2 \frac{d(x,x')^\delta}{d(x,y)^{n+\delta}}$$

if $d(x,x') \leq C_3 d(x,y)$, $x, x', y \in X$.

Given a Calderón-Zygmund kernel K, we can define (at least formally) a Calderón-Zygmund operator (CZO) associated with this kernel by

$$Tf(x) = \int_X K(x,y) f(y) d\mu(y).$$

One important question in the Calderón-Zygmund theory is to find a criterion for boundedness of the CZO in $L^2(\mu)$. *(In this context, such problem is called a T(1)-Theorem problem)*

Many authors studied this problem. In fact, when $X = \mathbb{R}^m$, $\mu = dx$ (the usual Lebesgue measure) and d is the Euclidean distance, a famous criterion called "T(1)-Theorem" was obtained by Journé and G. David [DJ84]. This criterion states that a CZO is bounded in $L^2(d\mu)$ if and only if it is weakly bounded (in some sense), and the operator and its adjoint send the function 1 in BMO. This result was extended to space of homogeneous type in an unpublished work by R. Coifman. It was then an interesting question to extend this T(1)-Theorem in the case where the space is not of homogeneous type (This essentially means that the measure μ does not satisfy the doubling condition). Several authors such as Tolsa, Nazarov, Treil, Volberg and Verdera [T99, NTV97, V00] treated this situation in the setting of $\mathbb{R}^m$ with the Euclidean distance. One good example of such an operator was the Cauchy integral operator. We say that the Cauchy integral operator is bounded in $L^2(d\mu)$ whenever for some positive constant C, one has for every $\epsilon > 0$

$$\int |\mathcal{C}_\epsilon(f\mu)|^2 d\mu \leq C \int |f|^2 d\mu, \; f \in L^2(d\mu),$$

where

$$\mathcal{C}_\epsilon(f\mu)(z) = \int_{|\zeta - z| > \epsilon} \frac{f(\zeta)}{\zeta - z} d\mu(\zeta), \; z \in \mathbb{C}.$$

Their result is that the Cauchy integral operator is bounded in $L^2(d\mu)$ if and only if

i) $\mu(D) \leq Cr(D)$, for each disc D with radius $r(D)$.
ii) $\int_D |\mathcal{C}_\epsilon(\chi_D \mu)|^2 d\mu \leq C\mu(D)$, for each disc D, $\epsilon > 0$.

Similarly, when we consider the kernel K_α associated with the operator T_α, we first show that this kernel is an $n+1+\alpha$ Calderón Zygmund kernel in the unit ball associated with a pseudo distance d. To characterize the boundedness of this operator, we adapt the idea used by Verdera [V00] to give an alternative proof of the $T(1)-$ Theorem for the Cauchy integral operator. For this, we introduce a sort of Menger curvature [M95] in the unit ball and establish a good λ inequality.

References

[A88] P.Ahern: Exceptional sets for holomorphic sobolev functions. *Michigan Math.J.* **35** (1988), pp 29-41.

[An97] M. Andersson: *Topic in complex analysis*, Springer-Verlag New York, (1997).

[ARS02] N. Arcozzi, R. Rochberg & E. Sawyer: Carleson measures for analytic Besov spaces, *Rev. Mat. Iberoamericana* **18** (2002), pp 443 -510.

[ARS06] N. Arcozzi, R. Rochberg & E. Sawyer: Carleson measures and interpolating sequences for Besov spaces on complex balls. *Mem. Amer. Math. Soc.***182** (2006), no. 859, vi+163.

[ARS06$_2$] N. Arcozzi, R. Rochberg & E. Sawyer: Carleson measures for the Drury-Arveson Hardy space and other Besov-Sobolev spaces on complex balls. Preprint. (2006) submitted.

[B87] B. Berndtsson: $\overline{\partial}_b$ and Carleson type inequalities. *Complex analysis, II (College Park, Md 1985-86)*, 42-54, Lecture Notes in Math, 1276, Springer, Berlin, 1987.

[CA58] L. Carleson: An interpolation problem for bounded analytic functions, *Amer. J. Math.*, **80** (1958), pp 921-930.

[CA62] L. Carleson: Interpolation by bounded analytic functions and the corona problem, *Ann. of Math.*, **76** (1962), pp 547-559.

[CO95] C. Cascante & J. M. Ortega : Carleson measures on spaces of Hardy-sobolev type, *Can. J. Math.*, Vol 47 (**6**) 1995, pp 1177 - 1200.

[CW82] J. A. Cima & W. Wogen: A Carleson measure theorem for the Bergman space on the Ball, *J. Operator theory,* **7** (1982) pp 157-165.

[DJ84] G. David & J.L. Journé: A boundedness criterion for generalized Calderón-Zygmund operators, *Annals of Math.*, **120** (1984), pp 371-397.

[FS72] C. Fefferman & E. M. Stein: H^p spaces of several variables, *Acta Math.* **129** (1972), pp 137-193.

[H67] L. Hörmander: L^p-estimates for (pluri)subharmonic functions, *Math. Scand.* **20** (1967,) p 65 - 78.

[M95] M. S. Melnikov: Analytic capacity: a discrete approach and the curvature of a measure, (Russian), *Mat. Sb.* **186**(6) (1995), 57-76; translation in *Sb. Math.*, **186**: 6 (1995), pp 827-846.

[NTV97] F. Nazarov, S. Treil & A. Volberg: Cauchy Integral and Calderón-Zygmund operators on non-homogeneous spaces, *Internt. Math. Res. Notices* (1997), $n°15$, pp 703-726.

[ST80] D. A. Stegenga: Multipliers of the Dirichlet space, *Illinois J. Math.* **24** (1980), pp 113-139.

[T99] X. Tolsa: L^2-boundedness of the Cauchy integral operator for continuous measures, *Duke Math. J.*, **98** (1999) pp 269-304.

[V00] J. Verdera: On the T(1)-Theorem for the Cauchy integral. *Ark. Mat.*, **38** (2000) pp 183-199.

[ZZ05] R. Zhao & K. Zhu: Theory of Bergman spaces in the unit ball of $\mathbb{C}^n$, Univ. Albany. Preprint. (2005).

CARLESON MEASURES FOR HARDY SOBOLEV SPACES

EDGAR TCHOUNDJA

Abstract. *The main goal of this paper is to present an alternative proof of Carleson measures for Hardy Sobolev spaces in the unit ball (Hardy spaces included)* $\mathbf{H}^p_\beta$. *We simply use Green's formula and pointwise estimates for functions in these spaces. An example shows that for* $p > 1$, *the natural necessary condition is no longer sufficient.*

1. Introduction

In this note, we give a new proof of the characterization of Carleson measures for some Hardy Sobolev spaces in the unit ball. Our method avoids some technicalities used by previous methods such as maximal functions, covering lemma and atomic decomposition; see [7, 1]. It seems therefore easy and can be used for other purposes. We just use Green's formula associated to an appropriate positive closed $(1.1)-$ form. This method is inspired by results in [4, 2]. Let $\mathbf{D} = \{z \in \mathbb{C}^n : \quad |z| < 1\}$ be the unit ball of $\mathbb{C}^n$; and $\mathbf{S} = \partial\mathbf{D}$ its boundary. We will denote by $d\lambda$ the normalized Lebesgue measure on $\mathbf{D}$, and by $d\sigma$ the normalized Lebesgue measure on $\mathbf{S}$. For $\beta \in \mathbb{R}$ and $0 < p < +\infty$, the Hardy Sobolev space $\mathbf{H}^p_\beta$ consists of holomorphic functions f in $\mathbf{D}$ such that $(I + \mathcal{R})^\beta f \in \mathbf{H}^p(\mathbf{D})$ (the usual Hardy Space), where if $f = \sum_k f_k$ is its homogeneous expansion (see [8]), $(I + \mathcal{R})^\beta f = \sum_k (1 + k)^\beta f_k$. Observe that for $\beta = 0$, $\mathbf{H}^p_\beta = \mathbf{H}^p(\mathbf{D})$. We say that a positive Borel measure μ on $\mathbf{D}$ is a Carleson measure for $\mathbf{H}^p_\beta$ if there exists a constant $C = C(\mu)$ such that:

$$\|f\|_{L^p(\mu)} \leq C\|f\|_{\mathbf{H}^p_\beta} \tag{1}$$

for all $f \in \mathbf{H}^p_\beta$. We give a new proof of the following theorem.

Theorem 1.1. *Let* μ *be a positive Borel measure on* $\mathbf{D}$. *Assume that* $0 < p \leq 1$ *and* $0 < \beta < \frac{n}{p}$ *or* $p > 0$ *and* $\beta = 0$. *Then the following are equivalent:*

2000 *Mathematics Subject Classification.* 26B20, 32A35.

Key words and phrases. Carleson measures, Hardy Sobolev spaces, Arveson's Hardy space, Green's formula.

i) *μ is a Carleson measure for* $\mathbf{H}^p_\beta$.
ii) *There exists a constant C such that*

$$\mu\left(Q_\delta(\xi)\right) \leq C\delta^m$$

for any $\xi \in \mathbf{S}$ and $\delta > 0$,

where $m = n - \beta p$ and

$$Q_\delta(\xi) = \left\{z \in \mathbf{D} : \ |1 - z\overline{\xi}| < \delta\right\}.$$

The sets Q_δ are the high dimensional analogues of Carleson boxes in the unit disk. They are also called non isotropic balls. See [8] for more informations.

2. Preliminary results

Let Ω be a smooth pseudo-convex domain in $\mathbb{C}^n$ and ρ a function such that

$$\Omega = \{z \in \mathbb{C}^n : \rho(z) < 0\}$$

with $|d\rho| > 0$ in $\partial\Omega = \{\rho(z) = 0\}$, the boundary of Ω.

If ω is a positive closed $(1,1)-$form in Ω, the form ω defined a Kähler metric in Ω and the volume form with respect to this metric is given by $dV = \frac{\omega^n}{n!}$.

Definition 2.1. *Suppose that ω is a positive closed $(1,1)-$form in Ω. The Laplacian with respect to the metric induced by ω is defined by:*

$$\Delta_\omega u \frac{\omega^n}{n!} = i\partial\overline{\partial}u \wedge \frac{\omega^{n-1}}{(n-1)!},$$

for any function u of class $\mathcal{C}^2$ in Ω.

An application of Stoke's formula gives this Green's formula.

Theorem 2.2. *Suppose that ω is a positive closed $(1,1)-$form in Ω. Let u,$v \in \mathcal{C}^\infty\left(\overline{\Omega}\right)$, and u,v real. Then*

$$\Re\left(\int_\Omega \left(v\Delta_\omega u - u\Delta_\omega v\right) dV\right) = \Re\left(i\int_{\partial\Omega}\left(v\overline{\partial}u - u\overline{\partial}v\right) \wedge \frac{\omega^{n-1}}{\Gamma(n)}\right).$$

Recall that the Euclidean volume form is determined by the closed $(1,1)-$form $\beta = i\partial\overline{\partial}|z|^2$; precisely, $d\lambda = \frac{\beta^n}{n!}$.

In the sequel, in view of our objective, we suppose that $\overline{\Omega}$ is contained in $\mathbf{D}$.

We consider now the closed $(1,1)-$form on $\mathbf{D}$ given by $\omega = i\partial\overline{\partial}\log\left(\frac{1}{1-|z|^2}\right)$. Observe that:

$$\omega \;=\; \frac{\beta}{1-|z|^2} + \frac{\gamma}{(1-|z|^2)^2}$$

where $\beta = i\partial\overline{\partial}|z|^2$ is the usual Euclidean closed $(1,1)-$form and

$$\gamma = i\sum_{j,k}\overline{z}_j z_k dz_j \wedge d\overline{z}_k = i\partial|z|^2 \wedge \overline{\partial}|z|^2.$$

We can see that $\gamma^2 = 0$ and $\beta\wedge\gamma = \gamma\wedge\beta$. Therefore for any integer p with $1 < p \leq n$, we have by the binomial formula

$$\omega^p = \frac{\beta^p}{(1-|z|^2)^p} + \frac{p\beta^{p-1}\wedge\gamma}{(1-|z|^2)^{p+1}}. \tag{2}$$

In particular,

$$\omega^n = \frac{\beta^n}{(1-|z|^2)^{n+1}} \tag{3}$$

and

$$\omega^{n-1} = \frac{\beta^{n-1}}{(1-|z|^2)^{n-1}} + \frac{(n-1)\beta^{n-2}\wedge\gamma}{(1-|z|^2)^{n}}. \tag{4}$$

Using the definition of the Laplacian together with (3) and (4), we obtain by straightforward computation, the next lemma which gives us a formula to compute the Laplacian with respect to the above closed form ω.

Lemma 2.3. *Suppose that $\omega = i\partial\overline{\partial}\log\left(\frac{1}{1-|z|^2}\right)$. Then for a $\mathcal{C}^2$ function u on $\mathbf{D}$, the Laplacian with respect to ω is given by :*

$$\Delta_\omega u(z) = \left(1-|z|^2\right) Lu(z) \tag{5}$$

where $Lu(z) = \sum_k \frac{\partial^2 u}{\partial z_k \partial\overline{z}_k} - \sum_{j,k}\frac{\partial^2 u}{\partial z_j\partial\overline{z}_k} z_j\overline{z}_k$.

From this we see that the Laplacian with respect to this form is the same as the Bergman Laplacian or the so called invariant Laplacian. So $\Delta_\omega\left(u\circ\varphi_a\right) = \left(\Delta_\omega u\right)\circ\varphi_a$, where φ_a is the involutive automorphism of $\mathbf{D}$ that interchanges the points 0 and $a\in\mathbf{D}$. This invariance also follows from the fact that $\varphi_a^*(\omega) = \omega$; just use the identity (9) and the definition of the Laplacian to see this. The

Green's formula in this case is given by the following result (see [10, p28]and the references therein) .

Proposition 2.4. *Suppose* $0 < r < 1$. *If* u *and* v *are twice differentiable in* $\mathbf{D}$ *then*

$$\int_{|z|<r} (v\Delta_\omega u - u\Delta_\omega v) \frac{d\lambda}{(1-|z|^2)^{n+1}} = 2n \int_{|z|=r} \left(v\frac{\partial u}{\partial \eta} - u\frac{\partial v}{\partial \eta} \right) \frac{r^{2n-1}d\sigma}{(1-r^2)^{n-1}}.$$

where $\frac{\partial u}{\partial \eta}$ is a radial derivative of u. Note that we can obtain Proposition 2.4 directly from Theorem 2.2. A direct computation gives the following lemma.

Lemma 2.5. *Let* $s > 0$. *Consider the function* $\phi(z) = (1-|z|^2)^s$, *then*

$$\Delta_\omega \phi(z) = -\left\{ s(n-s)(1-|z|^2) + s^2(1-|z|^2)^2 \right\} (1-|z|^2)^{s-1}. \tag{6}$$

Proof: Let $u(z) = (1-|z|^2)^s$, we have

$$\frac{\partial^2 u}{\partial z_j \partial \overline{z}_k} = -s\left(1-|z|^2\right)^{s-1} \delta_{jk} + s(s-1)\overline{z}_j z_k \left(1-|z|^2\right)^{s-2}.$$

We use (5) to obtain

$$\begin{aligned} \Delta_\omega \phi(z) &= (1-|z|^2)^{s-1}\left(-sn(1-|z|^2) + s(s-1)|z|^2\right) \\ &\quad +(1-|z|^2)^{s-1}\left(s(1-|z|^2)|z|^2 - s(s-1)|z|^4\right) \\ &= (-s(n-s)(1-|z|^2) - s^2(1-|z|^2)^2)(1-|z|^2)^{s-1}. \end{aligned}$$

□

From Proposition 2.4 we obtain the following result.

Theorem 2.6. *There exists a positive constant* C *such that for all* $u \in \mathcal{C}^\infty\left(\overline{\mathbf{D}}\right)$, *u real and positive, we have*

$$\int_{\mathbf{D}} Lu\, d\lambda \leq C \int_{\mathbf{S}} u\, d\sigma. \tag{7}$$

Proof: Let $u \in \mathcal{C}^\infty\left(\overline{\mathbf{D}}\right)$, fix $\delta \in]0,1[$ small and apply Proposition 2.4 with $r^2 = 1-\delta$. Let $v(z) = (1-|z|^2)^n$, by (6), we have $\Delta_\omega v(z) = -n^2\left(1-|z|^2\right)^{n+1}$.
Proposition 2.4 gives

$$\int_{|z|^2<1-\delta} Lu\, d\lambda + n^2 \int_{|z|^2<r} ud\lambda \;=\; \frac{n}{2} \int_{|z|^2=1-\delta} \left(\delta\frac{\partial u}{\partial \eta} + 2n(1-\delta)^n u \right) d\sigma.$$

Letting now δ tend to 0 we obtain (7).

□

From this, we obtain an important estimate in the following theorem.

Theorem 2.7. *There exists a positive constant* C *such that, if* $g, \psi \in \mathcal{C}^\infty\left(\overline{\mathbf{D}}\right), \psi$ *is real, and* g *is analytic in* $\mathbf{D}$*, then for* $0 < p < +\infty$

$$\int_{\mathbf{D}} L\psi(z)|g|^p e^\psi d\lambda \ \leq \ C\int_{\mathbf{S}} |g|^p e^\psi d\sigma. \tag{8}$$

Proof: Let g, ψ be as in the theorem. Fix a positive ϵ small. A straightforward computation shows that $L\left(\log|g|^2+\epsilon\right) \geq 0$. Since for any smooth function h, we have $L(e^h) \geq e^h Lh$, if we apply (7) to $u = \left(|g|^2+\epsilon\right)^{\frac{p}{2}} e^\psi$, we obtain

$$\int_{\mathbf{D}} L\psi\left(|g|^2+\epsilon\right)^{\frac{p}{2}} e^\psi d\lambda \leq C\int_{\mathbf{S}} \left(|g|^2+\epsilon\right)^{\frac{p}{2}} e^\psi d\sigma$$

for any ϵ.

Letting ϵ tend to 0, this gives (8) and this ends the proof of the theorem.

□

We will also need the following lemma.

Lemma 2.8. *Let* $s > 0$ *and* $a \in \mathbf{D}$*. Consider the function* $\xi(z) = \frac{\left(1-|z|^2\right)^s}{|1-z\overline{a}|^{2s}}$*; then*

$$L\xi(z) = -\frac{s^2\left(1-|z|^2\right)^s\left(1-|a|^2\right)}{|1-z\overline{a}|^{2(s+1)}} - \frac{s\left(n-s\right)\left(1-|z|^2\right)^{s-1}}{|1-z\overline{a}|^{2s}}$$

Proof: Let ξ be as in the lemma. Recall that

$$1-|\varphi_a(z)|^2 = \frac{\left(1-|z|^2\right)\left(1-|a|^2\right)}{|1-z\overline{a}|^2} \tag{9}$$

and observe then that $\xi(z)\left(1-|a|^2\right)^s = \left(1-|\varphi_a(z)|^2\right)^s$. By the invariance of the Laplacian, we have

$$\begin{aligned}\left(1-|a|^2\right)^s \Delta_w\left(\xi(z)\right) &= \Delta_w\left(\xi(z)\left(1-|a|^2\right)^s\right)\\ &= \Delta_w\left(1-|\varphi_a(z)|^2\right)\\ &= \Delta_w\left(\left(1-|z|^2\right)^s\right)\circ\varphi_a(z).\end{aligned}$$

We use now (6) and (9) to obtain the result.

□

3. Carleson measures for Hardy sobolev

In this section, we will give a proof of Theorem 1.1. For $\beta \in \mathbb{R}$ and $0 < p < +\infty$, we recall that the Hardy Sobolev space $\mathbf{H}^p_\beta(\mathbf{D})$ is the space of all holomorphic functions f in $\mathbf{D}$ such that

$$\|f\|^p_{\mathbf{H}^p_\beta(\mathbf{D})} = \sup_{r<1} \int_{\mathbf{S}} \left|(I+\mathcal{R})^\beta f(r\xi)\right|^p d\sigma(\xi) < \infty.$$

By the definition, we can see that the operator $(I+\mathcal{R})^\beta$ is an invertible operator with inverse $(I+\mathcal{R})^{-\beta}$. For $\beta > 0$, by simple change of variables, we see that $(1+k)^{-\beta} = \frac{1}{\Gamma(\beta)} \int_0^1 \left(\log \frac{1}{t}\right)^{\beta-1} t^k dt$, where k is an integer. Thus for a holomorphic function f and $\beta > 0$ we have

$$f(z) = \frac{1}{\Gamma(\beta)} \int_0^1 \left(\log \frac{1}{t}\right)^{\beta-1} (I+\mathcal{R})^\beta f(tz) dt. \tag{10}$$

On the other hand, using integration by parts, we observe that

$$\int_0^1 \left(\log \frac{1}{t}\right)^{\beta-1} t^k dt = \frac{1+k}{\beta} \int_0^1 \left(\log \frac{1}{t}\right)^{\beta} t^k dt.$$

So we can extend the formula (10) for $\beta > -1$ and $\beta \neq 0$. We obtain

$$f(z) = \frac{1}{\Gamma(\beta+1)} \int_0^1 \left(\log \frac{1}{t}\right)^{\beta} (I+\mathcal{R})^{\beta+1} f(tz) dt.$$

We iterate the procedure n times and obtain that for a holomorphic function f and $\beta > -n$, $\beta \neq 0, -1, -2, ...$

$$f(z) = \frac{1}{\Gamma(\beta+n)} \int_0^1 \left(\log \frac{1}{t}\right)^{\beta+n-1} (I+\mathcal{R})^{\beta+n} f(tz) dt. \tag{11}$$

From this discussion we will obtain the following estimates.

Lemma 3.1. *Let m, s be two positive real numbers such that $m-s > 0$. Consider the function $f(z) = \frac{1}{(1-z\overline{w})^m}$. Then the following holds:*

i) *There exists a constant C such that*

$$|(I+\mathcal{R})^s f(z)| \leq C \frac{1}{|1-z\overline{w}|^{m+s}},$$

for all $w \in \mathbf{D}$.

ii) *There exists a constant* C *such that*

$$\left|(I+\mathcal{R})^{-s} f(z)\right| \leq C \frac{1}{|1-z\overline{w}|^{m-s}},$$

for all $w \in \mathbf{D}$.

Proof: Suppose that s is an integer, then a straightforward computation shows that

$$(I+\mathcal{R})^{s} f(z)=\frac{P_s(z\overline{w})}{(1-z\overline{w})^{m+s}},$$

where P_s is a polynomial of degree s. We then obtain i) in this case.

Suppose now that s is not an integer, write $s=n+\gamma$ where n is an integer and γ a real number such that $\gamma \in(-1,0)$. If we take $\beta=-s$, we observe that $\beta \in(-n ;-n+1)$. Using (11) we have

$$(I+\mathcal{R})^{s} f(z)=\frac{1}{\Gamma(-\gamma)} \int_0^1\left(\log \frac{1}{t}\right)^{-\gamma-1}(I+\mathcal{R})^{+n} f(t z) d t.$$

Hence

$$\begin{aligned} \left|(I+\mathcal{R})^{s} f(z)\right| & \leq C \int_0^1\left(\log \frac{1}{t}\right)^{-\gamma-1}\left|(I+\mathcal{R})^{+n} f(t z)\right| d t \\ & \leq C \int_0^1 \frac{(1-t)^{-\gamma-1}}{|1-t z \overline{w}|^{m+n}} d t, \end{aligned}$$

by the first case of i). We then obtain part i) of the lemma because

$$\int_0^1 \frac{(1-t)^r}{|1-t z|^k} d t \simeq|1-z|^{-k+r+1},$$

for $k-r-1>0$ and $-1<r<0$. This finishes part i) of the lemma.

For the second part of the lemma, we argue in the same way. If s is an integer, then by iteration, we can prove that

$$(I+\mathcal{R})^{-s} f(z)=\frac{Q_s(z\overline{w})}{(1-z\overline{w})^{m-s}},$$

where Q_s is a polynomial of degree s. We then obtain ii) in this case. Suppose now that s is not an integer and write $-s=-n+\gamma$, where n is an integer and γ a real number such that $\gamma \in(-1,0)$. Substituting $\beta=-n-\gamma$ in (11), we have

$$(I+\mathcal{R})^{-s} f(z) = \frac{1}{\Gamma(-\gamma)} \int_0^1 \left(\log \frac{1}{t}\right)^{-\gamma-1} (I+\mathcal{R})^{-n} f(tz)dt.$$

The proof of ii) follows as in i).

□

Definition 3.2. *We say that a positive Borel measure μ on $\mathbf{D}$ is a Carleson measure for $\mathbf{H}^p_\beta$ if there exists a constant $C = C(\mu)$ such that:*

$$\|f\|_{L^p(\mu)} \leq C\|f\|_{\mathbf{H}^p_\beta} \tag{12}$$

for all $f \in \mathbf{H}^p_\beta$.

Recall also that, for any $\xi \in \mathbf{S}$ and $\delta > 0$, the Carleson box is the set

$$Q_\delta(\xi) = \left\{z \in \mathbf{D} : \ |1 - z\overline{\xi}| < \delta\right\}.$$

The following theorem, which is proved in [10], gives a characterization of measures which satisfy certain conditions on the Carleson boxes.

Theorem 3.3. *Let t be a strictly positive real and μ be a positive Borel measure on $\mathbf{D}$. Then the following conditions are equivalent:*

a) *There exists a positive constant C such that*

$$\mu\left(Q_\delta(\xi)\right) \leq C\delta^t$$

for all $\xi \in \mathbf{S}$ and all $\delta > 0$.

b) *For each $s > 0$ there exists a positive constant C such that*

$$\sup_{z\in\mathbf{D}} \int_{\mathbf{D}} \frac{(1-|z|^2)^s}{|1-z\overline{w}|^{t+s}} d\mu(w) \leq C < \infty. \tag{13}$$

c) *For some $s > 0$ there exists a positive constant C such that the inequality in (13) holds.*

From this we obtain the following necessary condition.

Lemma 3.4. *Suppose $m = n - \beta p > 0$. Let μ be a positive Borel measure on $\mathbf{D}$. If μ is a Carleson measure for $\mathbf{H}^p_\beta$, then there exists a positive constant C such that*

$$\mu\left(Q_\delta(\xi)\right) \leq C\delta^m \tag{14}$$

for all $\xi \in \mathbf{S}$ *and all* $\delta > 0$.

Proof:

Suppose that μ is a Carleson measure for $\mathbf{H}^p_\beta$. If for sufficiently large k we take $f(z) = \frac{1}{(1-z\overline{w})^k}$, then by Lemma 3.1 part i), there is an absolute constant C such that $\|f\|^p_{\mathbf{H}^p_\beta} \leq \frac{C}{(1-|z|^2)^{kp+\beta p-n}}$.

This fact together with the Carleson condition gives

$$\sup_{z\in\mathbf{D}} \int_{\mathbf{D}} \frac{\left(1-|z|^2\right)^{kp+\beta p-n}}{|1-z\overline{w}|^{kp}} d\mu(w) \leq C,$$

which by Theorem 3.3 is equivalent to (14).

□

We are ready to give the ***Proof of Theorem 1.1***. It is enough to proof the Carleson inequality for f holomorphic in $\mathbf{D}$, and smooth enough up to the boundary. For such f, we consider the functions

$$g(z) = (I+\mathcal{R})^\beta f(z) \ \text{ and } \ \psi(z) = -\int_{\mathbf{D}} \frac{\left(1-|z|^2\right)^m}{|1-z\overline{w}|^{2m}} d\mu(w).$$

Since μ satisfies (14), by Theorem 3.3, we get that $e^\psi \simeq 1$. By Theorem 2.7 and Lemma 2.8 we have

$$\iint_{\mathbf{D}} \left(\frac{m^2\left(1-|z|^2\right)^m\left(1-|w|^2\right)}{|1-z\overline{w}|^{2m+2}} + \frac{m\,(n-m)\left(1-|z|^2\right)^{m-1}}{|1-z\overline{w}|^{2m}} \right) d\mu(w)|g(z)|^p d\lambda(z) \leq C\int_{\mathbf{S}} |g(\xi)|^p d\sigma(\xi). \tag{15}$$

Therefore in view of Fubini's theorem, (12) follows if we can prove that

$$\begin{aligned} |f(z)|^p \ &\leq \ C\int_{\mathbf{D}} \left(\frac{m^2\left(1-|w|^2\right)^m\left(1-|z|^2\right)}{|1-z\overline{w}|^{2m+2}} \right) |g(w)|^p d\lambda(w) \\ &\quad + C\int_{\mathbf{D}} \left(\frac{m\,(n-m)\left(1-|w|^2\right)^{m-1}}{|1-z\overline{w}|^{2m}} \right) |g(w)|^p d\lambda(w) \end{aligned} \tag{16}$$

for all $z \in \mathbf{D}$. We proceed to prove (16). Observe that for $n-m > 0$, (16) is equivalent to

$$|f(z)|^p \leq C \int_{\mathbf{D}} \frac{(1-|w|^2)^{m-1}}{|1-z\overline{w}|^{2m}} |g(w)|^p d\lambda(w). \tag{17}$$

For $0 < p \leq 1$ *and* $0 < \beta < \frac{n}{p}$, by the reproducing formula for Bergman spaces (see [10, Theorem 2.2]), we have for any strictly positive real number s

$$(I+\mathcal{R})^{\beta} f(z) = g(z) = C_{s.m} \int_{\mathbf{D}} \frac{g(w)\,(1-|w|^2)^s}{(1-z\overline{w})^{n+1+s}} d\lambda(w).$$

Hence by Lemma 3.1 part ii), we have

$$|f(z)| \leq C \int_{\mathbf{D}} \frac{|g(w)|\,(1-|w|^2)^s}{|1-z\overline{w}|^{n+1+s-\beta}} d\lambda(w). \tag{18}$$

Assume s is sufficiently large so that we can set

$$s = \frac{2m}{p} + \beta - (n+1).$$

Thus (18) becomes

$$|f(z)| \leq C \int_{\mathbf{D}} \left| \frac{g(w)}{(1-z\overline{w})^{\frac{2m}{p}}} \right| \left(1-|w|^2\right)^s d\lambda(w).$$

We observe that $s = \frac{n+1+m-1}{p} - (n+1)$; therefore we apply Lemma 2.15 in [10] to obtain

$$|f(z)|^p \leq C \int_{\mathbf{D}} \frac{(1-|w|^2)^{m-1}}{|1-z\overline{w}|^{2m}} |g(w)|^p d\lambda(w),$$

which is (17).

For $p > 0$ *and* $\beta = 0$, by the same reproducing formula applied to $\frac{f(z)}{(1-z\overline{a})^{\alpha}}$ for some $a \in \mathbf{D}$ and $\alpha > 0$ to be chosen, we have

$$\frac{f(z)}{(1-z\overline{a})^{\alpha}} = C_{s,m} \int_{\mathbf{D}} \frac{(1-|w|^2)^s}{(1-z\overline{w})^{n+1+s}} \frac{f(w)}{(1-w\overline{a})^{\alpha}} d\lambda(w).$$

Hence we take $a = z$ and obtain

$$\frac{|f(z)|}{(1-|z|^2)^{\alpha}} \leq C \int_{\mathbf{D}} \frac{(1-|w|^2)^s}{|1-z\overline{w}|^{n+1+s+\alpha}} |f(w)| d\lambda(w).$$

If $0 < p \le 1$, we use Lemma 2.15 as in the previous case to obtain, by choosing $\alpha = \frac{1}{p}$ and $s = \frac{2n+1}{p} - (n+1)$,

$$\frac{|f(z)|^p}{(1-|z|^2)} \;\le\; C\int_{\mathbf{D}} \frac{(1-|w|^2)^n}{|1-z\overline{w}|^{2n+2}}|f(w)|^p d\lambda(w),$$

which is (16).

Finally if $p > 1$ and $\alpha > \frac{1}{p}$, we simply apply Hölder estimates and Theorem 1.12 in [10] to obtain (16). This ends the proof.

□

4. Comments and further results

As we can see in our approach, the space $\mathbf{H}^p_\beta(\mathbf{D})$ is involved just when we want to obtain the key estimate (16). Therefore to obtain some result for $p > 1$ and $\beta > 0$, the difficulty is to prove (16). It turns out that for this case, the estimate (16) is in general not true. We can see this by taking $p = 2$, $\beta = \frac{n-1}{2}$ $(n > 2)$ and

$$f(z,w) = g(z) = \sum_{\alpha \in \mathbb{N}^{n-2}} \frac{|\alpha|!}{\alpha!} \frac{z^\alpha}{(n-2)^{\frac{|\alpha|}{2}}(1+|\alpha|)\ln^s(1+|\alpha|)},$$

where $(z,w) \in \mathbb{C}^{n-2} \times \mathbb{C}^2$ and $\frac{1}{2} < s < 1$. Indeed, in this case, the estimate (16) will imply that

$$|g(z)|^2 \;\le\; C\int_{\mathbf{D}} \frac{(1-|w|^2)^2}{|1-z\overline{w}|^2}|\,(I+\mathcal{R})^{\frac{n-1}{2}}\, g(w)|^2 d\lambda(w),$$

where $\mathbf{D}$ is now a unit ball of $\mathbb{C}^{n-2}$. This also implies that

$$|g(z)|^2 \;\le\; C\int_{\mathbf{D}} |\,(I+\mathcal{R})^{\frac{n-1}{2}}\, g(w)|^2 d\lambda(w).$$

A straightforward computation using Taylor expansion finally shows that the function g must satisfy

$$|g(z)|^2 \;\le\; C\sum_m \frac{1}{(1+m)\ln^{2s}(1+m)}.$$

This means that g must be bounded since $\frac{1}{2} < s < 1$. But, for $z = r\xi$ where ξ is the point of the boundary of $\mathbf{D}$ given by $\xi = \left(\frac{1}{\sqrt{n-2}}, ..., \frac{1}{\sqrt{n-2}}\right)$, we see that $|g(z)| = \sum_m \frac{r^m}{(1+m)\ln^s(1+m)}$ tends to ∞ as r tends to 1. This gives a contradiction.

On the other hand, if we are able to prove that, for some cases (when terms involved make sense), the inequality

$$\sup_{r<1} \int_{\mathbf{S}} \left|(I+\mathcal{R})^{\beta p} f^p(r\xi)\right| d\sigma(\xi) \quad \leq \quad C\|f\|^p_{\mathbf{H}^p_\beta} \tag{19}$$

holds, for all $f \in \mathbf{H}^p_\beta$. In those cases as before, we apply the same reproducing formula to $(I+\mathcal{R})^{\beta p} f^p(z)$ to obtain

$$(I+\mathcal{R})^{\beta p} f^p(z) \quad = \quad C_{n,m} \int_{\mathbf{D}} \frac{(I+\mathcal{R})^{\beta p} f^p(w)\left(1-|w|^2\right)^{m-1}}{\left(1-z\overline{w}\right)^{n+m}} d\lambda(w).$$

Hence by Lemma (3.1) part ii), we will have

$$|f(z)|^p \quad \leq \quad C \int_{\mathbf{D}} \frac{(I+\mathcal{R})^{\beta p} f^p(w)||\left(1-|w|^2\right)^{m-1}}{|1-z\overline{w}|^{n+m-\beta p}} d\lambda(w),$$

which is (17). Thus (14) will still characterize Carleson measures on such cases.

However, we don't know the cases for which (19) is true apart from the trivial case $p = 1$.

Therefore, if we introduce the space $\mathcal{H}^p_{b,\beta}(\mathbf{D})$ as the space of all holomorphic functions f in $\mathbf{D}$ such that (when terms involved make sense)

$$\|f\|^p_{\mathcal{H}^p_{b,\beta}} = \sup_{r<1} \int_{\mathbf{S}} \left|(I+\mathcal{R})^{\beta p} f^p(r\xi)\right| d\sigma(\xi),$$

then for the analogue question of Carleson measures for $\mathbf{H}^p_{b,\beta}(\mathbf{D})$, with our approach, we obtain the following theorem.

Theorem 4.1. *Let μ be a positive Borel measure on $\mathbf{D}$. Suppose that $m = n - \beta p > 0$. Then the following are equivalent:*

i) *μ is a Carleson measure for $\mathbf{H}^p_{b,\beta}(\mathbf{D})$*

ii) *There exists a constant C such that $\mu\left(Q_\delta(\xi)\right) \leq C\delta^m$ for any $\xi \in \mathbf{S}$ and $\delta > 0$.*

We note that our approach easily gives the following closed, computable and sufficient condition.

Proposition 4.2. *Let μ be a positive Borel measure on $\mathbf{D}$ and assume that $m = n - \beta p > 0$. If for some $\epsilon > 0$, the measure μ satisfies*

$$\mu\left(Q_\delta(\xi)\right) \leq C\delta^{m+\epsilon}$$

for any $\xi \in \mathbf{S}$ and $\delta > 0$, then μ is a Carleson measure for $\mathbf{H}^p_\beta$.

Proof: In this case, we simply need to prove that

$$|f(z)|^p \quad \leq \quad C \int_{\mathbf{D}} \frac{\left(1-|w|^2\right)^{m-1+\epsilon}}{|1-z\overline{w}|^{2m+2\epsilon}} |\left(I+\mathcal{R}\right)^\beta f(w)|^p d\lambda(w).$$

It is enough to we just use the same reproducing formula once more and Hölder's inequality.

□

Finally, the following result shows that when $p > 1$, the necessary condition (14) is no longer sufficient. This means that we can not expect to remove ϵ in the previous proposition. This result is proved here for $p = 2$ and $\beta = \frac{n-1}{2}$. However, it can be extended to other cases in the same manner.

Theorem 4.3. *Suppose $p = 2$ and $\beta = \frac{n-1}{2}$ $(n > 1)$. There exists a finite positive measure μ in $\mathbf{D}$ such that μ satisfies the growth condition (14), but μ is not a Carleson measure for $\mathbf{H}^2_{\frac{n-1}{2}}$.*

Before giving the proof of this theorem, let us note that the space $\mathbf{H}^2_{\frac{n-1}{2}}$ is special. It is now known in the literature as the Arveson Hardy space. Arveson [3] has studied extensively this space in connection with applications in multivariable operator theory. We add a change on notations here to emphasize the dimension of the space we are dealing with. So in this proof, we will denote the unit ball of $\mathbb{C}^n$ by

$$\mathbf{D}_n = \{z \in \mathbb{C}^n : \ |z| < 1\},$$

and the Carleson boxes in $\mathbf{D}_n$ by

$$Q^n_\delta(\xi) = \left\{z \in \mathbf{D}_n : \ |1 - z.\overline{\xi}| < \delta\right\}.$$

Proof of Theorem 4.3:

We will adapt the idea used in [1, p 34] to prove an analogue problem for exceptional sets.

First assume $n = 2$ and consider functions of the form $f(z, w) = g(2zw)$, where g is holomorphic in the unit disk $\mathbf{D}_1$. By means of Taylor expansion and Stirling's formula, one can show that $f \in \mathbf{H}^2_{\frac{1}{2}}(\mathbf{D}_2)$ if and only if $g \in \mathbf{H}^2_{\frac{1}{4}}(\mathbf{D}_1)$. Carleson measures for $\mathbf{H}^2_{\frac{1}{4}}(\mathbf{D}_1)$ have been characterized by Stegenga using a capacity condition; see [9, Theorem 2.3]. The proof of this uses the fact that Carleson measures for $\mathbf{H}^2_{\frac{1}{4}}(\mathbf{D}_1)$ coincide with Carleson measures for the space of Poisson transforms of Bessel potentials of L^2 functions. If we now use Proposition 2.1 in [5] by applying it to the function $\varphi(x) = \sqrt{x}$, we conclude that there exists a finite Borel measure ν in $\mathbf{D}_1^*$ $(= \mathbf{D}_1 \backslash \{0\})$ and a function $g \in \mathbf{H}^2_{\frac{1}{4}}(\mathbf{D}_1)$ such that

$$\nu\left(Q^1_\delta(\xi)\right) \leq C\delta^{\frac{1}{2}},$$

and

$$\int_{\mathbf{D}_1} |g(z)|^2 d\nu = \infty.$$

Let $\mathbf{T}$ denote the unit circle ($\mathbf{T} = \partial\mathbf{D}_1$) with its arclength measure $d\theta$. Define $\Psi : \mathbf{D}_1^* \times \mathbf{T} \to \mathbf{D}_2$ by

$$\Psi(r^2e^{is}, e^{i\theta}) = \left(r\frac{\sqrt{2}}{2}e^{i\theta}, r\frac{\sqrt{2}}{2}e^{i(s-\theta)}\right).$$

If we set $\widetilde{\mathbf{D}} = \left\{\left(r\frac{\sqrt{2}}{2}e^{i\theta}, r\frac{\sqrt{2}}{2}e^{i(s-\theta)}\right) \in \mathbf{D}_2 : \ 0 < r < 1; \ \ \theta, s \in [0, 2\pi[\right\}$, then Ψ is clearly an homeomorphism from $\mathbf{D}_1^* \times \mathbf{T}$ to $\widetilde{\mathbf{D}}$. Let μ be the measure on $\widetilde{\mathbf{D}}$ obtained by transporting $d\nu \times d\theta$ on $\mathbf{D}_1^* \times \mathbf{T}$ to $\widetilde{\mathbf{D}}$. We still call μ the extension of this to $\mathbf{D}_2$ ($\mu = 0$ in $\mathbf{D}_2 \backslash \widetilde{\mathbf{D}}$).

Thus we have a function $f(z, w) = g(2zw) \in \mathbf{H}^2_{\frac{1}{2}}(\mathbf{D}_2)$ and a measure $\mu \in \mathbf{D}_2$ so that

$$\begin{aligned}
\int_{\mathbf{D}_2} |f(z,w)|^2 d\mu &= \int_{\widetilde{\mathbf{D}}} |f(z,w)|^2 d\mu \\
&= \int_{\mathbf{D}_1^* \times \mathbf{T}} |g(r^2e^{is}|^2 d\nu \times d\theta \\
&= 2\pi \int_{\mathbf{D}_1} |g(z)|^2 d\nu = \infty.
\end{aligned}$$

The theorem will be proved in this case if we show that μ satisfies (14). We proceed to prove this. It is enough to prove (14) for $\xi \in \partial\widetilde{\mathbf{D}}$. Moreover, by invariance under rotation, we may assume that $\xi = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$. It is then enough to show that

$$\left|1 - r\frac{e^{i\theta}}{2} - r\frac{e^{is}}{2}e^{-i\theta}\right| < \delta \tag{20}$$

implies that $|1 - r^2 e^{is}| < c\delta$ and $|1 - e^{i\theta}| < C\delta^{\frac{1}{2}}$. First observe that (20) implies that $1 - r < \delta$. On the other hand, (20) is the same as

$$\left|e^{-i\frac{s}{2}} - \frac{r}{2}\left(e^{i\left(\theta - \frac{s}{2}\right)} + e^{i\left(\frac{s}{2} - \theta\right)}\right)\right| < \delta,$$

that is

$$\left(\sin\frac{s}{2}\right)^2 + \left(\cos\frac{s}{2} - r\cos\left(\frac{s}{2} - \theta\right)\right)^2 < \delta^2.$$

Thus

$$\sin\frac{s}{2} < \delta \text{ and } \left|\cos\frac{s}{2} - r\cos\left(\frac{s}{2} - \theta\right)\right| < \delta.$$

Since we may also assume δ small enough, the first of these inequalities implies that $s < \sqrt{2}\delta$; it also implies that

$$\begin{aligned} |1 - r^2 e^{is}| &= \left|1 - r^2 + r^2\left(1 - e^{is}\right)\right| \\ &< 2\delta + |1 - e^{is}| \\ &= 2\delta + 2sin\frac{s}{2} < 4\delta. \end{aligned}$$

The second inequality implies that

$$\begin{aligned} r\left|1 - \cos\left(\frac{s}{2} - \theta\right)\right| &< \delta + \left|r - \cos\frac{s}{2}\right| \\ &< \delta + \delta + \left|1 - \cos\frac{s}{2}\right| < 3\delta. \end{aligned}$$

Consequently, since δ small implies r large, we get

$$\left|1 - \cos\left(\frac{s}{2} - \theta\right)\right| = 2\sin^2\left(\frac{s}{4} - \frac{\theta}{2}\right) < 6\delta.$$

So $\left|\frac{s}{2} - \theta\right| < C\delta^{\frac{1}{2}}$ and then $\theta < C\delta^{\frac{1}{2}}$. This leads to

$$|1 - e^{i\theta}| = 2sin\frac{\theta}{2} < C\delta^{\frac{1}{2}}.$$

This finishes the case $n = 2$. The case $n > 2$ follows in the following way. We identify the following subset $\widetilde{\mathbf{D}_2} = \{z \in \mathbf{D}_n : \ z_j = 0,\ j \geq 3\}$ of the unit ball of $\mathbb{C}^n$ with the unit ball $\mathbf{D}_2$ of $\mathbb{C}^2$. We then consider the measure $\widetilde{\mu}$ in $\mathbf{D}_n$ obtained by transporting the measure μ in $\mathbf{D}_2$ we have just constructed. The extension of this measure to $\mathbf{D}_n$ satisfies the growth condition (14). For the functions

$f(z) = F(z_1, z_2)$ we have that, $f \in \mathbf{H}^2_{\frac{n-1}{2}}(\mathbf{D}_n)$ if and only if $F \in \mathbf{H}^2_{\frac{1}{2}}(\mathbf{D}_2)$. We then obtain from the previous construction a function $f \in \mathbf{H}^2_{\frac{n-1}{2}}(\mathbf{D}_n)$ such that $\int_{\mathbf{D}_n} |f|^2 d\widetilde{\mu} = \infty$. This finishes the proof of the theorem.

References

[1] P.Ahern: Exceptional sets for holomorphic sobolev functions. *Michigan Math.J.* **35** (1988), pp 29-41.

[2] M. Andersson: *Topic in complex analysis*, Springer-Verlag New York, (1997).

[3] W. Arveson: Subalgebras of $\mathcal{C}^*-$ algebras III; multivariable operator theory. *Acta Math.* **181** (1998), p 159-228

[4] B. Berndtsson: $\overline{\partial}_b$ and Carleson type inequalities. *Complex analysis, II (College Park, Md 1985-86)*, 42-54, Lecture Notes in Math, 1276, Springer, Berlin, 1987.

[5] C. Cascante, J. M. Ortega : Carleson measures on spaces of Hardy-sobolev type, *Can. J. Math.*, Vol 47 (**6**) 1995, pp 1177 - 1200.

[6] W.S. Cohn, I.E. Verbitsky: On the trace inequalities for Hardy- Sobolev functions in the Unit Ball of $\mathcal{C}^n$. *Indiana Univ. Math. J.* Vol **43**. $n°4$ (1994), pp 1079-1097.

[7] L. Hörmander: L^p-estimates for (pluri)subharmonic functions, *Math. Scand.* **20** (1967,) p 65 - 78.

[8] W.Rudin: *Function theory on the Unit Ball of* $\mathcal{C}^n$, Springer-verlag, New-York, 1980.

[9] D. A. Stegenga: Multipliers of the Dirichlet space, *Illinois J. Math.* **24** (1980), pp 113-139.

[10] K.Zhu: *Spaces of holomorphic functions in the Unit Ball*, Springer-Verlag, new-York, 2004.

Department of Mathematical Sciences, Division of Mathematics, Chalmers University of Technology and Göteborg University, SE-412 96 Göteborg, Sweden

E-mail address: tchoundj@math.chalmers.se

CARLESON MEASURES FOR THE GENERALIZED BERGMAN SPACES VIA A T(1)-THEOREM TYPE

EDGAR TCHOUNDJA

Abstract. *In this paper, we give a new characterization of Carleson measures on the Generalized Bergman spaces. We show first that this problem is equivalent to a T(1)-Theorem problem type. Using Verdera idea (see* [V]*), we introduce a sort of curvature in the unit ball adapted to our kernel and we establish a good λ inequality which then yields to the solution of this T(1)-Theorem problem.*

Contents

2000 *Mathematics Subject Classification.* 32A37, 32A55, 46E35.
Key words and phrases. Carleson measures, Generalized Bergman spaces, Arveson's Hardy space, T(1)-Theorem.

1. Introduction

Let n be a positive integer and let

$$\mathbb{C}^n = \mathbb{C} \times \cdots \times \mathbb{C}$$

denote the n dimensional complex Euclidean space.

For $z = (z_1, \cdots, z_n)$ and $w = (w_1, \cdots, w_n)$ in $\mathbb{C}^n$, we write

$$z\overline{w} = \langle z, w\rangle = z_1\overline{w_1} + \cdots + z_n\overline{w_n}$$

and

$$|z| = \sqrt{|z_1|^2 + \cdots + |z_n|^2}.$$

The open unit ball in $\mathbb{C}^n$ is the set

$$\mathbf{D} = \{z \in \mathbb{C}^n : \quad |z| < 1\}.$$

We use $\mathbf{H}(\mathbf{D})$ to denote the space of all holomorphic functions in $\mathbf{D}$. Let $\mathbf{S} = \partial\mathbf{D}$ be the boundary of $\mathbf{D}$. For $\alpha \in \mathbb{R}, \alpha > -n-1$, we define the Generalized Bergman spaces $\mathbf{A}^2_\alpha$ [ZZ] to consist of all holomorphic functions f in the unit ball $\mathbf{D}$ with the property that

$$\|f\|^2_\alpha = \sum_{m\in\mathbb{N}^n} |c(m)|^2 \frac{\Gamma(n+1+\alpha)m!}{\Gamma(n+1+|m|+\alpha)} < \infty,$$

where $f(z) = \sum_{m\in\mathbb{N}^n} c(m)z^m$ is the Taylor expansion of f.

For $\beta \in \mathbb{R}$, we define the fractional radial derivative of order β by

$$(I + \mathcal{R})^\beta f(z) := \sum_m (1 + |m|)^\beta c(m)z^m.$$

One then easily observes, by means of Taylor expansion and Stirling's formula, that

$$\|f\|^2_\alpha \cong \int_{\mathbf{D}} |(I + \mathcal{R})^m f(z)|^2 (1 - |z|^2)^{2m+\alpha} d\lambda(z) \tag{1}$$

where $2m+\alpha > -1$. One also observes that the right hand side of (1) is independent of the choice of m. If we let $2\sigma = \alpha+n+1$ then we see by (1) that $\mathbf{A}^2_\alpha = B^\sigma_2$, where B^σ_2 is the analytic Besov-Sobolev spaces defined in [ARS]. Thus this scale of spaces includes the Drury-Arveson Hardy space $\mathbf{A}^2_{-n}$, the usual Hardy space $\mathbf{H}^2(\mathbf{D}) = \mathbf{A}^2_{-1}$ and the weighted Bergman spaces when $\alpha > -1$.

An interesting question about these spaces is to find their Carleson measures, that is characterize positive measures μ on $\mathbf{D}$ such that

$$\int_{\mathbf{D}} |f|^2 d\mu \leq C(\mu)\|f\|_\alpha^2, \quad f \in \mathbf{A}_\alpha^2. \tag{2}$$

(A measure μ which satisfies (2) is called a Carleson measure for $\mathbf{A}_\alpha^2$ or simply an $\mathbf{A}_\alpha^2$ Carleson measure.)

Viewing the space $\mathbf{A}_\alpha^2$ as defined by the relation (1), we see that the literature is now rich with solutions of this question for various values of α. Indeed the first case of interest was the case $\alpha = -1$ (the usual Hardy space). In [CA], Carleson gave the result when $n = 1$ and later in 1967, Hörmander [H] gave a solution for $n > 1$. Stegenga [ST] (when $n = 1$), Cima and Wogen [CW] (when $n > 1$) characterized Carleson measures for $\alpha > -1$.

The range $\alpha \in (-n-1, -1)$ was unsolved until 2006, when Arcozzi, Rochberg and Sawyer [ARS], obtained results for the range $\alpha \in (-n-1, -n]$. Their results use certain tree conditions and seem difficult to handle.

The purpose of this note is to present an alternative characterization of Carleson measures in the same range $\alpha \in (-n-1, -n]$ as in [ARS]. We obtain a result which seems simple in the sense of applications.

To obtain our characterization, we show first that this problem is equivalent to a kind of T(1)-Theorem problem associated with a Calderón-Zygmund type kernel and then we solve the T(1)-Theorem problem type which occurs. To be precise, recall that for a topological space X with a pseudo distance d, a kernel $K(x, y)$ is called an n Calderón-Zygmund kernel (or simply a Calderón-Zygmund kernel) with respect to the pseudo distance d if

a) $|K(x, y)| \leq \frac{C_1}{d(x,y)^n}$, and

b) There exists $0 < \delta \leq 1$ such that

$$|K(x, y) - K(x', y)| + |K(y, x) - K(y, x')| \leq C_2 \frac{d(x, x')^\delta}{d(x, y)^{n+\delta}}$$

if $d(x, x') \leq C_3 d(x, y)$, $x, x', y \in X$.

Given a Calderón-Zygmund kernel K, we can define (at least formally) a Calderón-Zygmund operator (CZO) associated with this kernel by

$$Tf(x) = \int_X K(x, y) f(y) d\mu(y).$$

One important question in the Calderón-Zygmund theory is to find a criterion for boundedness of a CZO in $L^2(\mu)$. *(We will call such a problem a T(1)-Theorem problem)*

Many authors studied this problem. When $X = \mathbb{R}^m$, $\mu = dx$ (the usual Lebesgue measure) and d is the Euclidean distance, a famous criterion called "T(1)-Theorem" was obtained by Journé and G. David [DJ]. This criterion states that a CZO is bounded in $L^2(d\mu)$ if and only if it is weakly bounded (in some sense), and the operator and its adjoint send the function 1 in BMO. This result was extended to space of homogeneous type in an unpublished work by R. Coifman. Later it was an interesting question to extend this T(1)-Theorem in the case where the space is not of homogeneous type (This essentially means that the measure μ does not satisfy the doubling condition). Several authors such as Tolsa, Nazarov, Treil, Volberg and Verdera [T1, NTV, V] treated this situation in the setting of $\mathbb{R}^m$ with the Euclidean distance. One good example of such an operator is the Cauchy integral operator. We say that the Cauchy integral operator is bounded in $L^2(d\mu)$ whenever for some positive constant C, one has for every $\epsilon > 0$,

$$\int |\mathcal{C}_\epsilon(f\mu)|^2 d\mu \leq C \int |f|^2 d\mu,\ f \in L^2(d\mu),$$

where

$$\mathcal{C}_\epsilon(f\mu)(z) = \int_{|\zeta - z| > \epsilon} \frac{f(\zeta)}{\zeta - z} d\mu(\zeta),\ z \in \mathbb{C}.$$

Their result is that the Cauchy integral operator is bounded in $L^2(d\mu)$ if and only if

i) $\mu(D) \leq Cr(D)$, for each disc D with radius $r(D)$;
ii) $\int_D |\mathcal{C}_\epsilon(\chi_D\mu)|^2 d\mu \leq C\mu(D)$, for each disc D, $\epsilon > 0$.

We now return back to the problem (2). We consider the kernel K_α defined by

$$K_\alpha(z, w) = \Re\left\{\frac{1}{(1 - z\overline{w})^{n+1+\alpha}}\right\}.$$

For a positive Borel measure μ in $\mathbf{D}$, we consider the operator T_α associated with this kernel defined by

$$T_\alpha f(z) = \int_{\mathbf{D}} f(w) K_\alpha(z, w) d\mu(w),\ z \in \mathbf{D}.$$

We will prove that if on $\mathbf{D}$ we consider as in [B] the pseudo distance d defined by

$$d(z, w) = ||z| - |w|| + \left|1 - \frac{z}{|z|}\frac{\overline{w}}{|w|}\right|,$$

the kernel K_α is an $(n + 1 + \alpha)$ Calderón-Zygmund kernel in the unit ball $\mathbf{D}$ with respect to the pseudo distance d. Let $B = B(z, r) = \{w \in \mathbf{D};\ d(z, w) < r\}$ be a "pseudo ball" or simply a ball of center z and radius r. We are now ready to state our result.

Theorem 1.1 (**Main theorem**). *Suppose $\alpha \in]-n-1,-n]$ and μ be a positive Borel measure in $\mathbf{D}$. Then the following conditions are equivalent.*

a) *μ is a Carleson measure for $\mathbf{A}^2_\alpha$;*
b) *T_α is bounded in $L^2(\mu)$;*
c) *There exists a constant C such that*
  i) $\mu(B(z,r)) \leq Cr^{n+1+\alpha}$,
  ii) $\int_B |T_\alpha(\chi_B)|^2 d\mu \leq C\mu(B)$,
  for each ball $B = B(z,r)$ which touches the boundary of $\mathbf{D}$.

This theorem is a T(1)-Theorem type result with respect to the CZO T_α ($b \Leftrightarrow c$) and it shows the equivalency of this T(1)-Theorem problem with Carleson measures for $\mathbf{A}^2_\alpha$ ($a \Leftrightarrow b$). Observe that the equivalency $a \Leftrightarrow b$ is proved in [ARS, Lemma 24, p 42] in a more general situation. Nevertheless we have included the proof here for the sake of completeness. Thus to prove Theorem 1.1 , we will essentially prove the hard part $b \Leftrightarrow c$. To prove the hard part, we will adapt to the unit ball the idea used by J. Verdera [V] to give an alternative proof of the T(1)-Theorem for the Cauchy integral operator.

The paper is organized as follows. In section 2 we gather some preliminaries including a key covering lemma, terminology and background. Section 3 is devoted to the study of the generalized Bergman spaces $\mathbf{A}^2_\alpha$ and the proof of $a \Leftrightarrow b$. Section 4 contains the proof of the hard part of the main theorem. Section 5 deals with some extensions, comments and opens questions.

2. Preliminary results

We collect in this section few results which will be useful to our purpose. These concern results on general homogeneous spaces and results for the special case of the unit ball $\mathbf{D}$.

2.1. **Definition and Properties of a space of homogeneous type.**

Definition 2.1. *A pseudo distance on a set X is a map ρ from $X \times X$ to $\mathbb{R}^+$ such that*

1) $\rho(x,y) = 0 \Leftrightarrow x = y$
2) $\rho(x,y) = \rho(y,x)$
3) *there exists a positive constant K ($K \geq 1$) such that, for all $x, y, z \in X$*

$$\rho(x,y) \leq K(\rho(x,z) + \rho(z,y)). \text{ (Quasi triangular inequality)}$$

For $x \in X$ and $r > 0$, the set $B(z,r) = \{y \in X : \ \rho(x,y) < r\}$ is called a pseudo ball or simply a ball of center x and radius r.

Definition 2.2. *A space of homogeneous type is a topological space X with a pseudo distance ρ and a positive Borel measure μ on X such that:*

1) *The balls $B(x,r)$ form a basis of open neighborhoods of x.*
2) ***(Doubling property)*** *There exists a constant $A > 0$ such that, for all $x \in X$ and $r > 0$, we have*

$$0 < \mu(B(x,2r)) \leq A\mu(B(x,r)) < \infty.$$

The triplet (X,ρ,μ) is called a space of homogeneous type or simply a homogeneous space. We will often abuse by calling X a homogeneous space instead of (X,ρ,μ).

Homogeneous spaces have been treated by several authors such as Coifman and Weiss [CWe], Stein [S]. We refer to them for further details.

We will use the following lemma to prove a key type of covering lemma, Lemma 2.4 below. It will be crucial in our argument later.

Lemma 2.3. *There exists a constant C_1 such that if $B(x_1,r_1)$ and $B(x_2,r_2)$ are two non disjoint balls and if $r_1 \leq r_2$ then*

$$B(x_1,r_1) \subset B(x_2,C_1 r_2).$$

Proof: Let $y \in B(x_1,r_1) \cap B(x_2,r_2)$. We have for $x \in B(x_1,r_1)$

$$\begin{aligned} \rho(x,x_2) &\leq K(\rho(x,y) + \rho(y,x_2)) \\ &\leq K(K(\rho(x,x_1) + \rho(x_1,y)) + \rho(y,x_2)) \\ &< K(2Kr_1 + r_2) \\ &< K(2K+1)r_2. \end{aligned}$$

We obtain the desired result if we set $C_1 = K(2K+1)$.

□

Lemma 2.4. *Let (X,d,μ) be an homogeneous space. There exists positive constants K_1, K_2, K_3 with $K_3 > K_2 > K(C_1+1)K_1$ such that:*
for an open set O of X ($O \subsetneq X$), there exists a collection of balls $B_k := B(x_k,\rho_k)$ so that, if $B_k^* = B(x_k,K_1\rho_k)$, $B_k^{**} = B(x_k,K_2\rho_k)$ and $B_k^{***} = B(x_k,K_3\rho_k)$,

a) *the balls B_k are pairwise disjoint*
b) $O = \underset{k}{\cup} B_k^*$
c) $O = \underset{k}{\cup} B_k^{**}$
d) *for each k, $B_k^{***} \cap O^c \neq \emptyset$*

e) *a point $x \in O$ belongs to at most M balls B_k^{**} (bounded overlap property). Moreover, the constant M depends only on constants K_1, K_2, A and K.*

Proof: Let O be an open set of X ($O \subsetneq X$). Let $\epsilon = \frac{1}{16K^2C_1^2(1+C_1)}$ where C_1 is the constant defined in Lemma 2.3. Consider the covering of O by the balls $B(x, \epsilon\delta(x))$ where $\delta(x) = d(x, O^c)$, $x \in O$.

We have $d(x, O^c) > 0$ since O^c is a closed set. $d(x, O^c) > 0$ is finite. We now select a maximal disjoint subcollection of $\{B(x, \epsilon\delta(x))\}_{x\in O}$; for this subcollection $B_1, B_2, \ldots, B_k, \ldots$ with $B_k \doteq B(x_k, \epsilon\delta(x_k)) = B(x_k, \rho_k)$, we shall prove assertions a), b), c), d), and e) above. We set

$$K_1 = \frac{1}{4K^2(C_1+1)\epsilon}, \quad K_2 = \frac{1}{2\epsilon K} \textit{ and } K_3 = \frac{2}{\epsilon}.$$

Observe that our choice makes these constants satisfy our hypothesis. Observe that a) and d) hold automatically by our choice of B_k. It is also clear that $B_k^* = B\left(x_k, \frac{\delta(x_k)}{4K^2(C_1+1)}\right) \subset B\left(x_k, \frac{\delta(x_k)}{2K}\right) = B_k^{**} \subset O$;
what remains to be shown is that $O \subset \underset{k}{\cup} B_k^*$ (in this case b) and c) will be valid) and that e) is true.

Let us prove that $O \subset \underset{k}{\cup} B_k^*$.
Let $x \in O$; by the maximality of the collection B_k,

$$B(x_k, \epsilon\delta(x_k)) \cap B(x, \epsilon\delta(x)) \neq \emptyset \text{ for some k.}$$

We claim that $\delta(x_k) \geq \frac{\delta(x)}{4C_1}$. If not, since $\epsilon < \frac{1}{2C_1} < 1$, we have

$$B(x_k, 2\delta(x_k)) \cap B\left(x, \frac{\delta(x)}{2C_1}\right) \neq \emptyset.$$

Since $2\delta(x_k) < \frac{\delta(x)}{2C_1}$, by the Lemma 2.3, $B(x_k, 2\delta(x_k)) \subset B\left(x, \frac{\delta(x)}{2}\right)$, which gives a contradiction since $B(x_k, 2\delta(x_k))$ meets O^c, while $B\left(x, \frac{\delta(x)}{2}\right) \subset O$. Using the fact that $4C_1\epsilon\delta(x_k) \geq \epsilon\delta(x)$, Lemma 2.3 gives

$$x \in B(x, \epsilon\delta(x)) \subset B(x_k, 4\epsilon C_1^2\delta(x_k)) = B_k^*.$$

This proves b) and c).

We proceed to prove e).

Assume that $x \in \bigcap_{k=1}^{M} B_k^{**} = \bigcap_{k=1}^{M} B(x_k, K_2\rho_k)$. We have

$$d(x_k, O^c) \leq K(d(x, O^c) + d(x, x_k)) \leq K(d(x, O^c) + K_2\rho_k);$$

this implies that $d(x,O^c) \geq \frac{1}{K}(d(x_k,O^c) - KK_2\rho_k)$. But

$$KK_2\rho_k = KK_2\epsilon d(x_k,O^c) = \frac{d(x_k,O^c)}{2}$$

and thus $d(x,O^c) \geq \frac{d(x_k,O^c)}{2K} = \frac{\rho_k}{2K\epsilon}$. Hence $\rho_k \leq 2K\epsilon d(x,O^c)$.

On the other hand,

$$\begin{aligned} d(x,O^c) &\leq K(d(x_k,O^c) + d(x,x_k)) \\ &\leq K\left(K_2\rho_k + \frac{\rho_k}{\epsilon}\right) = K(K_2+\epsilon^{-1})\rho_k. \end{aligned}$$

So if $x \in B_k^{**}$, the radius ρ_k satisfies

$$\frac{d(x,O^c)}{K(K_2+\epsilon^{-1})} \leq \rho_k \leq 2K\epsilon d(x,O^c).$$

From this, we have

$$B(x_k,\rho_k) \subset B(x, C_2 d(x,O^c))$$

where $C_2 = 2K^2(K_2+1)\epsilon$. We also have, for each k

$$B(x, C_2 d(x,O^c)) \subset B(x_k, C_3\rho_k)$$

where $C_3 = K(C_2K(K_2+\epsilon^{-1}) + K_2)$. Thus

$$\bigcup_k B(x_k,\rho_k) \subset B(x, C_2 d(x,O^c))$$

and

$$B(x, C_2 d(x,O^c)) \subset B(x_k, C_3\rho_k) \; for \; each \; k.$$

Therefore, by the doubling property and the disjointness of B_k, we have

$$\begin{aligned} \sum_{k=1}^{M} \mu(B(x_k,\rho_k)) &\leq \mu(B(x, C_2 d(x,O^c))) \\ &\leq \mu(B(x_k, C_3\rho_k)) \\ &\leq C\mu(B(x_k,\rho_k)). \end{aligned}$$

Thus $M \leq C$ and we are done.

□

One key result in the real variable theory is that, by means of Besicovitch covering Lemma, the usual central Hardy-Littlewood maximal function is bounded in $L^p(\mathbb{R}^m, d\mu)$ $(1 < p < \infty)$ where the measure μ is not assumed to be doubling. Since Besicovitch covering Lemma is no longer true in general homogeneous spaces

[KR], we will obtain the same L^p estimates for a certain "contractive" central Hardy-Littlewood maximal function to be defined later, via the following lemma

Lemma 2.5 (**ϵ-Besicovitch**). *Let (X, d, μ) be an homogeneous space. Let E be a bounded set, fix a positive number M and denote by $\mathcal{F}$ the family of balls $B(a, r)$ with center $a \in E$ and radius $r \leq M$. Then there exists a countable subfamily $\{B(a_k, r_k)\}_{k=1}^{\infty}$ of $\mathcal{F}$ with the following properties.*

i) $E \subset \bigcup_{k=1}^{\infty} B(a_k, r_k)$

ii) *the family $\{B(a_k, \frac{r_k}{\alpha})\}_{k=1}^{\infty}$ is disjoint, where $\alpha = \frac{7K}{3}$, and K is the constant in the quasi triangle inequality for d;*

iii) *for all $0 < \epsilon < 1$, the family $\mathcal{F}_\epsilon = \{B(a_k, (1-\epsilon)r_k)\}_{k=1}^{\infty}$ has bounded overlaps, namely*

$$\sum_{k=1}^{\infty} \chi_{B(a_k,(1-\epsilon)r_k)}(x) \leq C \log \frac{1}{\epsilon},$$

where C depends only on constants of X and χ_A denotes the characteristic function of the set A.

Proof: See [FGL, Lemma 3.1].

□

We now turn our attention to the special domain of interest, the unit ball

$$\mathbf{D} = \{z \in \mathbb{C}^n : |z| < 1\}.$$

In [B] it is defined a map d on $\mathbf{D} \times \mathbf{D}$ by

$$d(z, w) = \begin{cases} ||z| - |w|| + \left|1 - \frac{z}{|z|}\frac{\overline{w}}{|w|}\right| & if\ z, w \in \mathbf{D}^*, \\ |z| + |w| & \text{otherwise}, \end{cases}$$

where $\mathbf{D}^* = \mathbf{D} \setminus \{0\}$.

2.2. **Properties of the pseudo distance d.**

Lemma 2.6. *The following assertions hold:*

i) *d is a pseudo distance on $\mathbf{D}$*

ii) *d is invariant under rotation.*

Proof:

Assertion *i)* will follow essentially from the fact that the map $(\xi, \zeta) \longmapsto |1-\xi\overline{\zeta}|^{\frac{1}{2}}$ is a distance in $\partial\mathbf{D}$ (see [R, Proposition 5.1.2]).

Indeed, the first property of a pseudo distance follows easily from this assumption; the second property is obvious. For the third property, let z, w and $\zeta \in \mathbf{D}$. We have

$$\begin{aligned} d(z,w) &= ||z|-|w|| + \left|1 - \frac{w}{|w|}\frac{\overline{z}}{|z|}\right| \\ &\leq ||z|-|\zeta|| + ||\zeta|-|w|| + 2\left(\left|1 - \frac{w}{|w|}\frac{\overline{\zeta}}{|\zeta|}\right| + \left|1 - \frac{\zeta}{|\zeta|}\frac{\overline{z}}{|z|}\right|\right) \end{aligned}$$

(by the triangular inequality for the distance in the boundary).

So

$$d(z,w) \leq 2(d(z,\zeta) + d(\zeta,w)).$$

Assertion $ii)$ follows from the fact that the inner product $z\overline{w}$ is invariant under rotation.

□

The pseudo balls associated with this pseudo distance satisfy this important observation.

Lemma 2.7. *The pseudo ball $B(z,r) = \{w \in \mathbf{D} : \ d(z,w) < r\}$ touches the boundary of $\mathbf{D}$ if and only if $r > 1 - |z|$.*

Proof: Fix a pseudo ball $B(z,r)$. Let $\epsilon = r-(1-|z|)$. Since we are interested in points which touch the boundary, we have to find conditions on points $w \in \overline{B(z,r)}$ such that $|w| > |z|$. For such w, we have $d(z,w) = |w| - |z| + \left|1 - \frac{z}{|z|}\frac{\overline{w}}{|w|}\right|$. So

$$d(z,w) < r \Leftrightarrow \left|1 - \frac{z}{|z|}\frac{\overline{w}}{|w|}\right| < \epsilon + 1 - |z| - |w| + |z| = \epsilon + 1 - |w|. \tag{3}$$

From this we have our result. In fact, (3) shows that $B(z,r)$ touches the boundary of $\mathbf{D}$ if and only if $\epsilon > 0$.

□

These pseudo balls have close relations with the so called Korányi balls. Precisely, for $\xi \in \partial\mathbf{D} = \mathbf{S}$ and $\delta > 0$, if we set

$$Q_\delta(\xi) := Q(\xi,\delta) = \{z \in \mathbf{D} : \ |1 - z\overline{\xi}| < \delta\},$$

then we have the following proposition.

Proposition 2.8. *There exists positive constants a_1 and a_2 such that, for every pseudo ball $B(z,r)$ which touches the boundary of* $\mathbf{D}$,

$$Q\left(\frac{z}{|z|}, r\right) \subset B(z, a_1 r),$$

and

$$B(z,r) \subset Q\left(\frac{z}{|z|}, a_2 r\right).$$

Proof: Let $B(z,r)$ be a pseudo ball which touches the boundary of $\mathbf{D}$. Let $w \in B(z,r)$. Since $||z|-|w|| < r$ we have for $|w| \leq |z|$,

$$|z|-|w| < r \Rightarrow 1-|w| < 2r$$

and for $|w| > |z|$

$$1-|w| < 1-|z| < r.$$

So if $w \in B(z,r)$, $1-|w| < 2r$. Therefore for $w \in B(z,r)$,

$$\begin{aligned} \left|1 - w\frac{\overline{z}}{|z|}\right| &= \left|1-|w|+|w|-w\frac{\overline{z}}{|z|}\right| \\ &\leq 1-|w|+|w|\left|1-\frac{w}{|w|}\frac{\overline{z}}{|z|}\right| \\ &< 2r + \left|1-\frac{w}{|w|}\frac{\overline{z}}{|z|}\right| < 3r. \end{aligned}$$

So $B(z,r) \subset Q\left(\frac{z}{|z|}, 3r\right)$. Take $a_2 = 3$.

For the first inclusion, we can suppose that $0 < r < \frac{1}{2}$. In fact, for $r \geq \frac{1}{2}$, $Q\left(\frac{z}{|z|}, r\right) \subset B(z, 6r)$. Let $w \in Q\left(\frac{z}{|z|}, r\right)$. We have

$$\begin{aligned} \left|1-\frac{w}{|w|}\frac{\overline{z}}{|z|}\right| &\leq \left|1-\frac{1}{|w|}\right| + \frac{1}{|w|}\left|1-w\frac{\overline{z}}{|z|}\right| \\ &= \frac{1-|w|+\left|1-w\frac{\overline{z}}{|z|}\right|}{|w|} \\ &< \frac{2r}{|w|} < \frac{2r}{1-r} < 4r. \end{aligned}$$

On the other hand, if $|z| > |w|$, then

$$||z|-|w|| = |z|-|w| < ||z|-w\overline{z}| < r|z| < r.$$

If $|z| \leq |w|$, then $|w| - |z| < 1 - |z| < r$. So $Q\left(\frac{z}{|z|}, r\right) \subset B(z, 5r)$. Take $a_1 = 6$.

□

For $\alpha > -1$, let $d\lambda_\alpha(z) = (1 - |z|^2)^\alpha d\lambda(z)$ where $d\lambda(z)$ is the usual Lebesgue measure of $\mathbb{C}^n \sim \mathbb{R}^{2n}$. We then have the following result.

Lemma 2.9. *For each fixed $\alpha > -1$, the triplet $(\mathbf{D}, d, d\lambda_\alpha)$ is an homogeneous space.*

Proof: Since d is already a pseudo distance on $\mathbf{D}$, we need only to prove that $d\lambda_\alpha$ is a doubling measure. One can prove that for $0 < R < 3$, $\zeta = (r, 0, \cdots, 0)$, $0 < r < 1$

$$\lambda_\alpha(B(\zeta, R)) \cong R^{n+1}\{\max(R, 1 - r)\}^\alpha. \tag{4}$$

This ends the proof of the lemma.

□

Remark 2.10. *This lemma shows that we can apply Lemmas 2.4 and 2.5 in the unit ball $\mathbf{D}$.*

We will make use of these others properties of d.

Lemma 2.11. *For every $z \in \mathbf{D}$ and r_0, $0 < r_0 < 1$, if we denote by $z_0 = (r_0, 0, \cdots, 0)$ we have*

1) $|1 - z_1 r_0| \geq \frac{1}{3} d(z, z_0)$
2) $|z_1 - r_0| \leq d(z, z_0)$
3) $\sum_{k=2}^{n} |z_k|^2 \leq 2d(z, z_0)$
4) $|1 - z z_0| \leq 1 - r_0^2 + d(z, z_0)$.

Proof:

1) We have

$$\begin{aligned} \left|1 - \frac{z_1}{|z|}\right| &\leq |1 - z_1 r_0| + \left|z_1 r_0 - \frac{z_1}{|z|}\right| \\ &\leq |1 - z_1 r_0| + 1 - r_0|z| \\ &\leq 2\,|1 - z_1 r_0|\,. \end{aligned}$$

If $|z| \leq r_0$ then

$$||z| - r_0| = r_0 - |z| \leq 1 - |z_1| \leq |1 - z_1 r_0|.$$

If $|z| > r_0$ then

$$||z| - r_0| = |z| - r_0 \leq 1 - r_0 \leq |1 - z_1 r_0|.$$

Hence $d(z,z_0) = \left|1-\frac{z_1}{|z|}\right| + ||z|-r_0| \leq 3|1-z_1r_0|$.

2) We have

$$\begin{aligned} |z_1 - r_0| &\leq |z_1 - |z|| + ||z| - r_0| \\ &\leq |z|\left|1-\frac{z_1}{|z|}\right| + ||z|-r_0| \leq d(z,z_0). \end{aligned}$$

3) We have

$$\begin{aligned} \sum_{k=2}^{n} |z_k|^2 = |z|^2 - |z_1|^2 &\leq 2||z| - |z_1|| \\ &\leq 2|z_1 - |z|| \leq 2\left|1-\frac{z_1}{|z|}\right| \leq 2d(z,z_0). \end{aligned}$$

4) We have

$$\begin{aligned} |1 - zz_0| = |1 - z_1r_0| &\leq 1 - r_0^2 + |r_0^2 - z_1r_0| \\ &\leq 1 - r_0^2 + |r_0 - z_1| \underset{2)}{\leq} 1 - r_0^2 + d(z,z_0). \end{aligned}$$

□

For $\alpha > -n-1$ fixed, set $k = n+1+\alpha$. We consider the following kernel

$$K_\alpha(z,w) = \Re\left\{\frac{1}{(1-z\overline{w})^k}\right\}$$

and obtain the following this important result about this kernel.

Proposition 2.12. 1) *There exists a constant C_3 such that for all $z, w \in \mathbf{D}$,*

$$|K_\alpha(z,w)| \leq \frac{C_3}{d(z,w)^k}.$$

2) *There exists two constants C_1, C_2 such that for all $z, w, \zeta \in \mathbf{D}$ satisfying*

$$d(z,\zeta) > C_1 d(w,\zeta),$$

we have

$$|K_\alpha(z,w) - K_\alpha(z,\zeta)| \leq C_2 \frac{d(w,\zeta)^{\frac{1}{2}}}{d(z,\zeta)^{k+\frac{1}{2}}}.$$

Proof:

Assertion 1) follows from Lemma 2.11 and the invariance under rotation. Indeed for $z, w \in \mathbf{D}$, if Δ is the rotation such that $\Delta(w) = (|w|, 0, \cdots, 0)$ then we have

$$d(z,w) = d(\Delta(z), \Delta(w)) \leq 3|1 - \Delta(z)\overline{\Delta(w)}| \leq 3|1 - z\overline{w}|. \tag{5}$$

So $d(z,w)^k \leq 3^k|1 - z\overline{w}|^k$. Take $C_3 = 3^k$.

Let us prove 2). By the invariance under rotation, we can suppose $\zeta = (r_0, 0, \cdots, 0)$. We use the identity

$$K_\alpha(z,w) - K_\alpha(z,\zeta) = \int_0^1 \Re\left(\frac{kz(\overline{w} - \overline{\zeta})}{(1 - z\overline{w} - tz(\overline{\zeta} - \overline{w}))^{k+1}}\right) dt$$

to obtain

$$|K_\alpha(z,w) - K_\alpha(z,\zeta)| \leq \int_0^1 \frac{k|z(\overline{w} - \overline{\zeta})|}{|1 - z\overline{w} - tz(\overline{\zeta} - \overline{w})|^{k+1}} dt. \tag{6}$$

We have

$$\begin{aligned} |z(\overline{w} - \overline{\zeta})| &\leq |z_1(\overline{w_1} - r_0)| + \left(\sum_{k=2}^n |z_k|^2\right)^{\frac{1}{2}} \left(\sum_{k=2}^n |w_k|^2\right)^{\frac{1}{2}} \\ &\leq |w_1 - r_0| + \left(\sum_{k=2}^n |z_k|^2\right)^{\frac{1}{2}} \left(\sum_{k=2}^n |w_k|^2\right)^{\frac{1}{2}}. \end{aligned}$$

So by Lemma 2.11 and (5), we have

$$\begin{aligned} |z(\overline{w} - \overline{\zeta})| &\leq 2d(w,\zeta)^{\frac{1}{2}} \left(d(w,\zeta)^{\frac{1}{2}} + d(z,\zeta)^{\frac{1}{2}}\right) \\ &\leq \frac{4}{\sqrt{C_1}} d(w,\zeta)^{\frac{1}{2}} d(z,\zeta)^{\frac{1}{2}} \\ &\leq \frac{C}{\sqrt{C_1}} d(w,\zeta)^{\frac{1}{2}} |1 - z\overline{\zeta}|^{\frac{1}{2}}. \qquad (7) \end{aligned}$$

This shows that for C_1 large enough, we have $|z(\overline{w} - \overline{\zeta})| \leq \frac{1}{2}|1 - z\overline{\zeta}|$. On the other hand, observe that

$$|1 - z\overline{w} - tz(\overline{\zeta} - \overline{w})| = |1 - z\overline{\eta}|$$

where $\eta = (1-t)w + t\zeta$. Since

$$|(1 - z\overline{\zeta}) - (1 - z\overline{\eta})| = |z(\overline{\eta} - \overline{\zeta})|$$

and

$$|(1-z\overline{\zeta})-(1-z\overline{\eta})| = (1-t)|z(\overline{w}-\overline{\zeta})| \leq |z(\overline{w}-\overline{\zeta})|,$$

we conclude that for large C_1, $|1-z\overline{\eta}| > \frac{1}{2}|1-z\overline{\zeta}|$.

Therefore, from (6), (7) and (5), we have

$$\begin{aligned} |K_\alpha(z,w)-K_\alpha(z,\zeta)| &\leq 2^{k+1}k\frac{C}{\sqrt{C_1}}\frac{|1-z\overline{\zeta}|^{\frac{1}{2}}d(w,\zeta)^{\frac{1}{2}}}{|1-z\overline{\zeta}|^{k+1}} \\ &\leq C_2\frac{d(w,\zeta)^{\frac{1}{2}}}{d(z,\zeta)^{k+\frac{1}{2}}}. \end{aligned}$$

□

Remarks 1. *This Proposition shows that the kernel K_α is a k Calderón-Zygmund kernel with respect to the pseudo distance d.*

We can observe from the proof of Proposition 2.12 that the assertions are still true if we replace the kernel K_α by the kernel $\frac{1}{(1-z\overline{w})^k}$.

3. Bergman-Sobolev type space $\mathbf{A}^2_\alpha$.

In this section, we define the space $\mathbf{A}^2_\alpha$. We give some properties of this space. Finally we show that the Carleson measures problem for these space is equivalent to the T(1)-Theorem problem associated with the Calderón-Zygmund kernel K_α.

Definition 3.1. *Let $\alpha \in \mathbb{R}, \alpha > -n-1$. We denote by $\mathbf{A}^2_\alpha$ the space of all holomorphic functions f in the unit ball $\mathbf{D}$ with the property that*

$$\|f\|^2_\alpha = \sum_{m\in\mathcal{N}^n}|c(m)|^2\frac{\Gamma(n+1+\alpha)m!}{\Gamma(n+1+|m|+\alpha)} < \infty,$$

where $f(z) = \sum\limits_{m\in\mathbb{N}^n} c(m)z^m$ is the Taylor expansion of f.

Theorem 3.2. *The space $\mathbf{A}^2_\alpha$ is equipped with an inner product such that the associated reproducing kernel is given by*

$$B_\alpha(z,w) = \frac{1}{(1-z\overline{w})^{n+1+\alpha}}.$$

Proof:

For $f(z) = \sum\limits_m c(m)z^m$ and $g(z) = \sum\limits_m d(m)z^m$, define the product by

$$\langle f,g\rangle_\alpha = \sum_m c(m)\overline{d(m)}\frac{\Gamma(n+1+\alpha)m!}{\Gamma(n+1+|m|+\alpha)}.$$

This clearly defines an inner product in $\mathbf{A}^2_\alpha$. Let $f \in \mathbf{A}^2_\alpha$ with $f(z) = \sum_m c(m) z^m$. Since

$$B_\alpha(z,w) = \sum_m \frac{\Gamma(n+1+|m|+\alpha)}{\Gamma(n+1+\alpha)m!} z^m \overline{w}^m,$$

we have for $w \in \mathbf{D}$,

$$\begin{aligned}\langle f, B_\alpha(.,w)\rangle_\alpha &= \sum_m c(m) \frac{\Gamma(n+1+|m|+\alpha)}{\Gamma(n+1+\alpha)m!} w^m \frac{\Gamma(n+1+\alpha)m!}{\Gamma(n+1+|m|+\alpha)} \\ &= \sum_m c(m) w^m = f(w).\end{aligned}$$

We are done.

□

Remark 3.3. *The space $\mathbf{A}^2_\alpha$ is a Hilbert space with the Hilbert norm $\|.\|_\alpha$.*

Proposition 3.4. *Let $w \in \mathbb{B}_n$ and set $f(z) = \frac{1}{(1-z\overline{w})^s}$. If $2s > n+1+\alpha$ then $f \in \mathbf{A}^2_\alpha$. Moreover,*

$$\|f\|^2_\alpha \cong \frac{1}{(1-|w|^2)^{2s-n-1-\alpha}}.$$

Proof: It is enough to verify the last assertion of the proposition. Let

$$f(z) = \frac{1}{(1-z\overline{w})^s} = \sum_{m \in \mathcal{N}^n} \frac{\Gamma(|m|+s)}{\Gamma(s)m!} z^m \overline{w}^m.$$

we have by Stirling's formula,

$$\begin{aligned}
\|f\|_\alpha^2 &\cong \sum_m \frac{\Gamma^2(|m|+s)}{\Gamma^2(s)(m!)^2}|\overline{w}^m|^2 \frac{\Gamma(n+1+\alpha)m!}{\Gamma(n+1+|m|+\alpha)} \\
&= \sum_k \frac{\Gamma^2(k+s)\Gamma(n+1+\alpha)}{\Gamma^2(s)\Gamma(n+1+k+\alpha)\Gamma(k+1)} \sum_{|m|=k} \frac{\Gamma(k+1)}{m!}|\overline{w}^m|^2 \\
&= \sum_k \frac{\Gamma^2(k+s)\Gamma(n+1+\alpha)}{\Gamma^2(s)\Gamma(n+1+k+\alpha)\Gamma(k+1)}|w|^{2k} \\
&\cong \sum_k (1+k)^{2s-2-\alpha-n}|w|^{2k} \\
&\cong \sum_k \frac{\Gamma(2s-n-1-\alpha+k)}{\Gamma(2s-n-\alpha-1)\Gamma(k+1)}|w|^{2k} \\
&= \frac{1}{(1-|w|^2)^{2s-n-1-\alpha}}.
\end{aligned}$$

□

For $\beta \in \mathbb{R}$, we define the fractional radial derivative of order β by

$$(I+\mathcal{R})^\beta f(z) := \sum_m (1+|m|)^\beta c(m) z^m,$$

where $f(z) = \sum_{m\in\mathbb{N}^n} c(m)z^m$ is the Taylor expansion of f.

The following lemma follows by the use of Taylor's expansion and Stirling's formula.

Lemma 3.5.

$$\|f\|_\alpha^2 \cong \int_{\mathbf{D}} |(I+\mathcal{R})^m f(z)|^2 (1-|z|^2)^{2m+\alpha} d\lambda(z) \tag{8}$$

where $2m+\alpha > -1$*.*

We also observe that the right hand side of (8) is independent of the choice of m.

Remarks 2.

- *The space* $\mathbf{A}_\alpha^p$ *are introduced in* [ZZ] *for general values of* p*, we refer there for further details about this space.*

- *For* $\alpha = -1$*, the space* $\mathbf{A}^2_\alpha$ *is the usual Hardy space*

$$\mathbf{H}^2(\mathbf{D}) = \left\{ f \in \mathbf{H}(\mathbf{D}) : \|f\|^2_{H^2(\mathbf{D})} = \sup_{r<1} \int_{\mathbb{S}} |f(r\xi)|^2 d\sigma(\xi) < \infty \right\}.$$

- *For* $\alpha > -1$*, the space* $\mathbf{A}^2_\alpha$ *is the usual weighted Bergman space. When* $\alpha = -n$*,* $\mathbf{A}^2_\alpha$ *is the so called Arveson's Hardy space; for other values of* α*, we obtain analytic Besov-Sobolev spaces.*

We recall that we want to characterize positive Borel measures μ on $\mathbf{D}$ such that

$$\int_{\mathbf{D}} |f|^2 d\mu \le C(\mu)\|f\|^2_\alpha, \quad f \in \mathbf{A}^2_\alpha. \tag{9}$$

(A measure μ which satisfies (9) is called a Carleson measure for $\mathbf{A}^2_\alpha$ or simply an $\mathbf{A}^2_\alpha$ Carleson measure.)

As we have mentioned in the introduction the solution of this question is well known for $\alpha \ge -1$. The result from these cases is the following theorem.

Theorem 3.6 (**Carleson, Hörmander, Stegenga, Cima and Wogen**)**.** *Let* $\alpha \ge -1$ *and* μ *be a positive Borel measure on* $\mathbf{D}$*. The following conditions are equivalent.*

a) *There exists a positive constant* C *such that*

$$\mu(Q_\delta(\xi)) \le C\delta^{n+1+\alpha} \tag{10}$$

for all $\xi \in \mathbf{S}$ *and all* $\delta > 0$*.*

b) *The measure* μ *is an* $\mathbf{A}^2_\alpha$ *Carleson measure.*

The range $\alpha \in (-n-1, -1)$ is difficult. In this note our approach yields a new characterization for Carleson measures for this space in the range $\alpha \in (-n-1, -n]$. A characterization of Carleson measures in this range has been previously obtained by Arcozzi, Rochberg and Sawyer [ARS]. It seems likely that our characterization could be extended to the remaining range $\alpha \in (-n, -1)$. However, we have not yet succeeded to do this.

Nevertheless, observe that for $\alpha > -n-1$, condition (10) remains a necessary condition for Carleson measures for $\mathbf{A}^2_\alpha$. This can be seen by using Proposition 3.4, (9) and the following result [ZZ, Theorem 45].

Theorem 3.7. *Let* α *be a real such that* $n + 1 + \alpha$ *and* μ *be a positive Borel measure on* $\mathbf{D}$ *. Then the following conditions are equivalent.*

a) *There exists a positive constant* C *such that*

$$\mu(Q_\delta(\xi)) \le C\delta^{n+1+\alpha}$$

for all $\xi \in \mathbf{S}$ and all $\delta > 0$.

b) *For each $s > 0$ there exists a positive constant C such that*

$$\sup_{z\in\mathbf{D}} \int_{\mathbf{D}} \frac{(1-|z|^2)^s}{|1-z\overline{w}|^{n+1+\alpha+s}} \leq C < \infty. \tag{11}$$

c) *For some $s > 0$ there exists a positive constant C such that the inequality in (11) holds.*

We end this section by proving, in a general setting, the equivalence $a \Leftrightarrow b$ of Theorem 1.1. Recall that the CZO T_α is defined by

$$T_\alpha f(z) = \int_{\mathbf{D}} f(w) K_\alpha(z,w) d\mu(w),\ z \in \mathbf{D},$$

where the kernel K_α is defined by

$$K_\alpha(z,w) = \Re\left\{\frac{1}{(1-z\overline{w})^{n+1+\alpha}}\right\}.$$

Proposition 3.8 (cf [ARS]). *Suppose $n+1+\alpha > 0$ and let μ be a positive Borel measure on $\mathbf{D}$. Then the following conditions are equivalent.*

a) *The measure μ is an $\mathbf{A}^2_\alpha$ Carleson measure.*

b) *The operator T_α is bounded in $L^2(\mu)$.*

Proof: Let $\mathcal{I}$ be the linear map defined by:

$$\begin{array}{cccc} \mathcal{I}: & \mathbf{A}^2_\alpha & \rightarrow & L^2(\mu) \\ & f & \mapsto & f \end{array}.$$

The adjoint of $\mathcal{I}$ is given by

$$\mathcal{I}^* f(z) = \int_{\mathbf{D}} \frac{f(w)d\mu(w)}{(1-z\overline{w})^{n+1+\alpha}}.$$

Indeed for $f \in L^2(\mu)$ with $\mathcal{I}^* f \in \mathbf{A}^2_\alpha$ we have, by the reproducing property,

$$\begin{aligned} \mathcal{I}^* f(z) &= \langle \mathcal{I}^* f, B_\alpha(.,z)\rangle_\alpha \\ &= \langle f, \mathcal{I}(B_\alpha(.,z))\rangle_\mu \\ &= \langle f, (B_\alpha(.,z))\rangle_\mu = \int_{\mathbf{D}} \frac{f(w)d\mu(w)}{(1-z\overline{w})^{n+1+\alpha}}. \end{aligned}$$

On the other hand, observe that $\mathcal{I}$ is bounded if and only if the measure μ is an $\mathbf{A}^2_\alpha$ Carleson measure. It is well known that

$$\mathcal{I} \text{ is bounded} \Leftrightarrow \mathcal{I}^* \text{ is bounded}.$$

Note that $\mathcal{I}^*$ is bounded means that

$$\|\mathcal{I}^* f\|_\alpha^2 = \langle \mathcal{I}^* f, \mathcal{I}^* f \rangle_\alpha \leq C\|f\|_\mu^2. \tag{12}$$

Then

$$\begin{aligned}
\|\mathcal{I}^* f\|_\alpha^2 &= \langle \mathcal{I}^* f, \mathcal{I}^* f \rangle_\alpha \\
&= \left\langle \int B_\alpha(., w) f(w) d\mu(w), \int B_\alpha(., z) f(z) d\mu(z) \right\rangle_\alpha \\
&= \int\int \langle B_\alpha(., w), B_\alpha(., z) \rangle_\alpha f(w) d\mu(w) \overline{f(z)} d\mu(z) \\
&= \int\int B_\alpha(z, w) f(w) d\mu(w) \overline{f(z)} d\mu(z).
\end{aligned}$$

Having (12) for general f is equivalent to having it for real f. We now suppose f is real. In this case we continue with

$$\begin{aligned}
\|\mathcal{I}^* f\|_\alpha^2 &= \int\int \Re\{B_\alpha(z, w)\} f(w) d\mu(w) f(z) d\mu(z) \\
&= \langle T_\alpha f, f \rangle_{L^2(\mu)}.
\end{aligned}$$

The last quantity satisfies the required estimates exactly if T_α is bounded. The proof is complete.

□

4. Proof of the equivalence $(b) \Leftrightarrow (c)$ of Theorem 1.1

This section is devoted to the proof of the T(1)-Theorem. That is the characterization of positive Borel measures μ on $\mathbf{D}$ such that the operator T_α is bounded in $L^2(\mu)$. To get the equivalence $(b) \Leftrightarrow (c)$ in Theorem 1.1, it suffices to prove the following theorem.

Theorem 4.1. *Let $k = n + 1 + \alpha$ and μ be a positive Borel measure on $\mathbf{D}$. Then the following conditions are equivalent.*

1) *The operator T_α is bounded in $L^2(\mu)$.*
2) *The operator T_α is bounded in $L^p(\mu)$ for some $p > 2$.*
3) i) *There exists a constant $C > 0$ such that*

$$\mu(B(z, r)) \leq Cr^k \tag{13}$$

for all pseudo balls $B(z, r)$ which touch the boundary, and

ii) *There exists a constant $C > 0$ such that*

$$\int_B \left(\int_B \Re \left\{ \frac{1}{(1 - z\overline{w})^k} \right\} d\mu(w) \right)^2 d\mu(z) \leq C\mu(B)$$

for all pseudo balls B which touch the boundary.

4.1. **Proof of** 1) $\Rightarrow$ 3)**.** Assertion *i*) follows from the discussion after Theorem 3.6 and the fact that the sets $B(z, r)$ and $Q_r\left(\frac{z}{|z|}\right)$ are comparable when $B(z, r)$ touches the boundary (in the sense of Proposition 2.8), and also by Proposition 3.8. Assertion *ii*) is easily obtained by testing the boundedness on the characteristic function $f = \chi_B$.

□

4.2. **Related maximal functions.**

Definition 4.2. *We say that a measure μ* ***satisfies the growth condition*** *when μ satisfies inequality (13).*

We proceed now to prove that *i*) and *ii*) are sufficient for the boundedness of T_α for some $p > 2$ that is the proof of the implication 3) $\Rightarrow$ 2). We focus our attention first to the special case $\alpha = -n$. We then set $T = T_{-n}$ and $K = K_{-n}$.

As we have mentioned in the introduction, we follow the same idea as in [V]. Indeed, we first introduce a sort of curvature which plays the role of the Menger curvature. This curvature is adapted to our domain and has a close relation with our operator. Next, we proceed to construct for every ball which touches the boundary, a "big piece" associated with this ball. This is the first crucial step of our proof. Finally, the next crucial step is to prove an appropriate good λ inequality without resorting to a doubling property on μ. Lemma 2.4 is used in those steps.

We suppose that a measure μ satisfies the growth condition. In our estimates we use two variants of the central Hardy-Littlewood maximal operator acting on a complex Radon measure ν, namely,

$$M\nu(z) = \sup_{r > 1-|z|} \frac{|\nu|(B(z, r))}{r},$$

and for a positive constant $\rho \geq 1$

$$M_\mu^\rho \nu(z) = \sup_{r > 1-|z|} \frac{|\nu|(B(z, r))}{\mu(B(z, \rho r))} \quad z \in supp\, \mu\,,$$

where $B(z,r)$ is the pseudo ball centered at z of radius r which touches the boundary and $supp\,\mu$ is the closed support of μ.

Proposition 4.3. *Let μ be a positive Borel measure which satisfies the growth condition. For every $\rho > 1$, there exists a positive constant $C(\rho)$ such that for any $f \in L^p(\mu)$,*

$$\int_{\mathbf{D}} M_\mu^\rho (f\mu)^p d\mu \leq C \int |f|^p d\mu, \ 1 < p < \infty. \tag{14}$$

Proof: Fix $\rho > 1$. Let $E_\lambda^\rho = \{z \in \mathbf{D} : \ M_\mu^\rho(f\mu)(z) > \lambda\}$. Observe first that, for each $z \in E_\lambda^\rho$, there exists a pseudo ball $B(z, r_z)$ such that

$$\mu(B(z, \rho r_z)) \leq \frac{1}{\lambda} \int_{B(z,r_z)} |f| d\mu. \tag{15}$$

Consider the family $\mathcal{F} = \{B(z, \rho r_z)\}_{z \in E_\lambda^\rho}$. Applying Lemma 2.5 to this family with $\epsilon = 1 - \frac{1}{\rho}$, we obtain a subfamily $\{B(z_k, \rho r_k)\}$ of $\mathcal{F}$ such that $E_\lambda^\rho \subset \underset{k}{\cup} B(z_k, \rho r_k)$, and the family $\{B(z_k, r_k)\}$ has bounded overlaps. Therefore, from (15) and this bounded overlap property, we have

$$\begin{aligned} \mu(E_\lambda^\rho) &\leq \sum_k \mu(B(z_k, \rho r_k)) \\ &\leq \frac{1}{\lambda} \sum_k \int_{B(z_k, r_k)} |f| d\mu \\ &\leq \frac{C(\rho)}{\lambda} \int_{\mathbf{D}} |f| d\mu. \end{aligned}$$

Hence, M_μ^ρ is of weak type $(1,1)$. We obtain the desired result from the obvious L^∞ estimate and the Marcinkiewicz interpolation.

□

Remarks 3.

- *Observe that for some constant $C(\rho) > 0$, we have*

$$M\nu(z) \leq C(\rho) M_\mu^\rho \nu(z), \ z \in supp\ \mu. \tag{16}$$

  So (14) remains true if we replace M_μ^ρ by M.
- *The weak estimate is valid if one replaces $f\mu$ by any finite measure ν.*

Lemma 4.4. *Let μ be a positive Borel measure which satisfies the growth condition. There exists a constant C such that, for all $\beta > 0$, z^0, $R > 1 - |z^0|$ and a positive function f:*

$$R^\beta \int_{d(z^0,w)>R} \frac{f(w)d\mu(w)}{d(z^0,w)^{1+\beta}} \le CM(f\mu)(z)$$

for all $z \in B(z^0, R)$*. In particular, we have*

$$\mu^\beta(B(z^0,R)) \int_{d(z^0,w)>R} \frac{d\mu(w)}{|1-z^0\overline{w}|^{1+\beta}} \le C.$$

Proof: Fix $\beta > 0$, z^0, $R > 1 - |z^0|$ and a positive function f. Let $z \in B(z^0, R)$. We have

$$\begin{aligned}
\int_{d(z^0,w)>R} \frac{f(w)d\mu(w)}{d(z^0,w)^{1+\beta}} &\le \sum_{k=0} \int_{2^kR<d(z^0,w)\le 2^{k+1}R} \frac{f(w)d\mu(w)}{d(z^0,w)^{1+\beta}} \\
&\le \sum_{k=0} \frac{1}{(2^kR)^{1+\beta}} \int_{d(z^0,w)<2^{k+1}R} f(w)d\mu(w) \\
&\le \sum_{k=0} \frac{1}{(2^kR)^{1+\beta}} \int_{d(z,w)<2K2^{k+1}R} f(w)d\mu(w) \\
&\le CM(f\mu)(z) \sum_{k=0} \frac{2^kR}{(2^kR)^{1+\beta}} \\
&\le CR^{-\beta}M(f\mu)(z).
\end{aligned}$$

The particular case follows from the fact that

$$\mu^\beta(B(z^0,R)) \le CR^\beta, \ \frac{1}{|1-z^0\overline{w}|^{1+\beta}} \le \frac{C}{d(z^0,w)^{1+\beta}}, \text{ and } M(f\mu)(z^0) \le 1 \text{ for } f \equiv 1.$$

□

For a Radon measure ν, set, for $z \in \mathbf{D}$,

$$T^*\nu(z) = \int_{\mathbf{D}} \frac{d|\nu|(w)}{|1-z\overline{w}|}.$$

Lemma 4.5. *Let* Ω *be an open pseudo ball which touches the boundary and let* μ *be a positive Borel measure on* $\mathbf{D}$ *satisfying the growth condition.*

If we set $\nu = \chi_{\Omega^c}\mu$*, then*

$$\int_\Omega T^*(f\nu)^2 d\mu \le C \int |f|^2 d\nu, \ f \in L^2(\nu).$$

Proof: It is enough to prove that for some $\eta > 0$, there exists $\rho > 1$, $\gamma > 0$ and $C > 0$ such that

$$\mu(\{z \in \Omega : \ T^*(f\nu)(z) > (1+\eta)t\}) \leq \mu(\{z \in \Omega : \ M_\mu^\rho(f\nu)(z) > \gamma t\}). \tag{17}$$

Indeed if (17) is true, then

$$\begin{aligned}
\int_\Omega T^*(f\nu)^2 d\mu &= \int_0^{+\infty} \mu(\{z \in \Omega : \ T^*(f\nu)(z) > t\}) 2t dt \\
&= (1+\eta)^2 \int_0^{+\infty} \mu(\{z \in \Omega : \ T^*(f\nu)(z) > (1+\eta)t\}) 2t dt \\
&\leq C \int_0^{+\infty} \mu(\{z \in \Omega : \ M_\mu^\rho(f\nu)(z) > \gamma t\}) 2t dt \\
&\leq C \int_\Omega M_\mu^\rho(f\nu)^2(z) d\mu(z) \\
&\leq C \int |f|^2 d\nu, \qquad \textit{by } (14).
\end{aligned}$$

We prove (17) using Lemma 2.4 applied to the open set

$$E_t = \{z \in \Omega : \ T^*(f\nu)(z) > t\}.$$

We obtain (17) once we prove that for each j

$$\mu(\{z \in B_j^* : \ T^*(f\nu)(z) > (1+\eta)t, \ M_\mu^\rho(f\nu)(z) \leq \gamma t\}) = 0, \tag{18}$$

where B_j^* is a term of the first decomposition of the open set E_t with respect to Lemma 2.4.

In fact we will have

$$\begin{aligned}
&\mu(\{z\in\Omega:\ T^*(f\nu)(z)>(1+\eta)t\})\\
&\quad\le\sum_j\mu(\{z\in B_j^*:\ T^*(f\nu)(z)>(1+\eta)t\})\\
&\quad\le\sum_j\big(\mu(\{z\in B_j^*:\ T^*(f\nu)(z)>(1+\eta)t,\ M_\mu^\rho(f\nu)(z)\le\gamma t\})\big)\\
&\qquad\quad+\sum_j\big(\mu(\{z\in B_j^*:\ M_\mu^\rho(f\nu)(z)>\gamma t\})\big)\\
&\quad\le\sum_j\mu(\{z\in B_j^*:\ M_\mu^\rho(f\nu)(z)>\gamma t\})\\
&\quad\le C\mu(\{z\in\Omega:\ M_\mu^\rho(f\nu)(z)>\gamma t\}).
\end{aligned}$$

by the bounded overlap property.

So it remains to prove (18).

Set $B=B\left(z^B,K_1\varrho\right)=B_j^*$ and $B'=B\left(z^B,K_2\varrho\right)=B_j^{**}$.

Suppose without loss of generality that there exists $\xi^0\in B$ such that

$$M_\mu^\rho(f\nu)(\xi^0)\le\gamma t.$$

Let z^0 be a point in $E_t^c\cap B\left(z^B,K_3\varrho\right)$. Set $\overline{B}$ to be a ball centered at z^0 whose radius is equal to $\max\left(2(1-|z^0|),C\varrho\right)$, where C is a constant greater than or equal to K_3 to be precised later. Then $\overline{B}$ touches the boundary of $\mathbf{D}$.

Let $f_1=f\chi_{\overline{B}}$ and $f_2=f-f_1=f\chi_{\overline{B}^c}$. There exists a constant A_1 such that

$$T^*(f\nu)(z)\le T^*\left(f\nu\chi_{\overline{B}}\right)(z)+(1+A_1\gamma)t,\ z\in B. \tag{19}$$

To prove (19), let $z\in B$. Then

$$\begin{aligned}
T^*\left(f\nu\chi_{\overline{B}^c}\right)(z) &= \int_{\overline{B}^c}\frac{|f(w)|d\nu(w)}{|1-z\overline{w}|}\\
&\le \int_{\overline{B}^c}\frac{|f(w)|d\nu(w)}{|1-z^0\overline{w}|}+\int_{\overline{B}^c}|f(w)|d\nu(w)\left|\frac{1}{1-z\overline{w}}-\frac{1}{1-z^0\overline{w}}\right|\\
&\le t+\int_{\overline{B}^c}|f(w)|d\nu(w)\frac{d(z,z^0)^{\frac12}}{d(w,z^0)^{\frac32}}
\end{aligned}$$

provided that C is chosen large enough so that we can use Proposition 2.12. Hence by Lemma 4.4 we have

$$T^*\left(f\nu\chi_{\overline{B}^c}\right)(z)\le(1+A_1\gamma)t.$$

This proves (19).

Set $\widetilde{B} = B(\xi^0, C_1K_1\varrho)$ and observe that $B \subset \widetilde{B} \subset B' \subset E_t$.

Now, if $C\varrho \geq 2(1 - |z^0|)$, there exists a constant $A_2 > 0$ such that for $z \in B$

$$T^*\left(f\nu\chi_{\overline{B}}\right)(z) \leq T^*\left(f\nu\chi_{\widetilde{B}}\right)(z) + A_2\gamma t. \tag{20}$$

To prove (20), we have

$$\begin{aligned} T^*\left(f\nu\chi_{\overline{B}}\right)(z) &\leq T^*\left(f\nu\chi_{\widetilde{B}}\right)(z) + \int_{\overline{B}\setminus\widetilde{B}} \frac{|f(w)|d\nu(w)}{|1 - z\overline{w}} \\ &= T^*\left(f\nu\chi_{\widetilde{B}}\right)(z) + I, \end{aligned}$$

where $I = \int_{\overline{B}\setminus\widetilde{B}} \frac{|f(w)|d\nu(w)}{|1-z\overline{w}}$.

By (5), $\frac{1}{|1-z\overline{w}|} \leq \frac{C}{d(z,w)}$, and on the other hand, for $w \in \overline{B}\setminus\widetilde{B}$ we have

$$C_1K_1\varrho \leq d(\xi^0, w) \leq K(d(z^B, \xi^0) + d(z^B, w)) < K(K_1\varrho + K(d(z^B, z) + d(z, w))).$$

Thus

$$K^2d(z, w) > C_1K_1\varrho - KK_1\varrho(K + 1) = K_1K^2\varrho.$$

Therefore

$$I \leq CM(f\nu)(\xi^0) \leq A_2\gamma t.$$

This proves (20). Since $\widetilde{B} \subset \Omega$, we have $T^*\left(f\nu\chi_{\widetilde{B}}\right)(z) = 0$.

For the case $C\varrho < 2(1 - |z^0|)$, we have for $z \in B$ and $w \in \overline{B}$,

$$|1 - z\overline{w}| > 1 - |z| > C'(1 - |z^0|).$$

Hence

$$T^*\left(f\nu\chi_{\overline{B}}\right)(z) \leq \frac{C}{1 - |z^0|} \int_{\overline{B}} |f|d\nu \leq CM^{\rho}_{\mu}(f\nu)(\xi^0) \leq C"\gamma t.$$

So we finally conclude that there exists a constant $A > 0$ such that

$$T^*(f\nu)(z) \quad \leq \quad (1 + A\gamma)t, \quad z \in B.$$

From this we have

$$\begin{aligned} &\mu(\{z \in B : T^*(f\nu)(z) > (1 + \eta)t, \ M^{\rho}_{\mu}(f\nu)(z) \leq \gamma t\}) \\ &\qquad \leq \mu(\{z \in B : (1 + A\gamma)t > (1 + \eta)t, \ M^{\rho}_{\mu}(f\nu)(z) \leq \gamma t\}); \end{aligned}$$

so if we choose $0 < \gamma \leq \frac{\eta}{2A}$, we obtain (18). This ends the proof of the Lemma.

□

4.3. **Curvature in the unit ball.**

Definition 4.6. *Given three points* $z_1, z_2, z_3 \in \mathbf{D}$, *we define their curvature* $c(z_1, z_2, z_3)$ *by*

$$c^2(z_1, z_2, z_3) = \sum_{\sigma} K\left(z_{\sigma(2)}, z_{\sigma(1)}\right) K\left(z_{\sigma(3)}, z_{\sigma(1)}\right)$$

where the sum is taken over the six permutations of $1, 2, 3$.

For a positive Borel measure ν the quantity

$$c^2(\nu) = \iiint c^2(z_1, z_2, z_3) d\nu(z_1) d\nu(z_2) d\nu(z_3)$$

is called the Total curvature of ν. One important fact about this curvature is that $c^2(z_1, z_2, z_3) > 0$. Indeed for z and w in $\mathbf{D}$

$$K(z, w) = \frac{\Re(1 - z\overline{w})}{|1 - z\overline{w}|^2} = \frac{1 - \Re(z\overline{w})}{|1 - z\overline{w}|^2} > 0$$

since $\Re(z\overline{w}) < 1$.

The next Lemma gives a relation between this curvature and our operator T.

Lemma 4.7. *Let* ν_j, $j = 1, 2, 3$ *be three Borel measures. Then*

$$\sum_{\sigma} \int T\left(\nu_{\sigma(1)}\right) T\left(\nu_{\sigma(2)}\right) d\nu_{\sigma(3)} = \iiint c^2(z_1, z_2, z_3) d\nu_1(z_1) d\nu_2(z_2) d\nu_3(z_3).$$

Proof: We have

$$\int T\left(\nu_{\sigma(1)}\right) T\left(\nu_{\sigma(2)}\right) d\nu_{\sigma(3)}$$
$$= \iiint K\left(z_{\sigma(3)}, z_{\sigma(1)}\right) K\left(z_{\sigma(3)}, z_{\sigma(2)}\right) d\nu_{\sigma(1)}\left(z_{\sigma(1)}\right) d\nu_{\sigma(2)}\left(z_{\sigma(2)}\right) d\nu_{\sigma(3)}\left(z_{\sigma(3)}\right).$$

Since for each σ

$$d\nu_{\sigma(1)}\left(z_{\sigma(1)}\right) d\nu_{\sigma(2)}\left(z_{\sigma(2)}\right) d\nu_{\sigma(3)}\left(z_{\sigma(3)}\right) = d\nu_1(z_1) d\nu_2(z_2) d\nu_3(z_3),$$

summing over the six permutations we obtain

$$\sum_{\sigma} \int T\left(\nu_{\sigma(1)}\right) T\left(\nu_{\sigma(2)}\right) d\nu_{\sigma(3)} = \iiint c^2(z_1, z_2, z_3) d\nu_1(z_1) d\nu_2(z_2) d\nu_3(z_3).$$

□

We apply Lemma 4.7 to $\nu_1 = \nu_2 = f\mu$ with f (a real function) in $L^2(\mu)$ and $\nu_3 = \chi_B\mu$ with B a fixed pseudo ball which touches the boundary. We then have

$$2\int_B |T(f\mu)|^2 d\mu + 4\int T(f\mu)T(\chi_B\mu)\, f d\mu \\ = \iiint c^2(z,w,\zeta)f(z)f(w)\chi_B(\zeta)d\mu(z)d\mu(w)d\mu(\zeta). \tag{21}$$

In particular taking $f = \chi_B$, one gets

$$6\int_B |T(f\mu)|^2 d\mu = \iiint_{B^3} c^2(z,w,\zeta)d\mu(z)d\mu(w)d\mu(\zeta),$$

and thus

$$\iiint_{B^3} c^2(z,w,\zeta)d\mu(z)d\mu(w)d\mu(\zeta) \le C\mu(B), \tag{22}$$

provided μ satisfies condition $ii)$ of 3) in Theorem 4.1 (case $k = 1$).

We are now ready to produce a "big piece" inside a given pseudo ball B which touches the boundary. As in [V], set

$$c_B^2(z) = \iint_{B^2} c^2(z,w,\zeta)d\mu(w)d\mu(\zeta), \quad z \in B.$$

By Chebyschev's inequality, condition $ii)$ of 3) in Theorem 4.1 and (22)

$$\mu(\{z \in B:\ c_B(z) > t \text{ or } |T(\chi_B\mu)(z)| > t\}) \\ \le \frac{1}{t^2}\left(\int_B c_B^2(z)d\mu(z) + \int_B |T(\chi_B\mu)(z)|^2 d\mu(z)\right) \\ \le C\frac{\mu(B)}{t^2}. \tag{23}$$

From this we have the following lemma.

Lemma 4.8. *Given $0 < \theta < 1$, there exists a set $E \subset B$ such that*

$$c_B^2(z) \le \frac{C}{\theta} \quad \text{and} \quad |T(\chi_B\mu)(z)|^2 \le \frac{C}{\theta},\ z \in E$$

and

$$\mu(B\backslash E) \le \theta(\mu(B)).$$

Proof: Fix $0 < \theta < 1$. If $\mu(B) = 0$, there is nothing to do. If $\mu(B) \neq 0$, set

$$E = \left\{ z \in B : c_B^2(z) \leq \frac{C}{\theta} \quad and \quad |T(\chi_B \mu)(z)|^2 \leq \frac{C}{\theta} \right\}.$$

It is then easy to verify that this set satisfies our requirements.

□

We set $k(z, w) = \int_B c^2(z, w, \zeta) d\mu(\zeta)$ so that

$$\int_E k(z, w) d\mu(w) = c_B^2(z) \leq \frac{C}{\theta} \quad (z \in E). \tag{24}$$

Since $k(z, w) = k(w, z)$ we obtain the following lemma.

Lemma 4.9. *There exists a constant $C = C(\theta)$ which does not depend on B such that*

$$\iiint c^2(z, w, \zeta) f(z) f(w) \chi_B(\zeta) d\mu(z) d\mu(w) d\mu(\zeta) \leq C \int f^2 d\mu,$$

where $f \in L^2(E) = L^2(E, \mu)$, with f real.

Proof: The result follows from Schur's test since

$$\int_E k(z, w) d\mu(w) \leq \frac{C}{\theta} \quad (z \in E).$$

□

Therefore from (21), Lemma 4.8 and Lemma 4.9, for any $f \in L^2(E)$, we get

$$\int_B |T(f\mu)|^2 d\mu \leq C \left(\int_B |T(f\mu)|^2 d\mu \right)^{\frac{1}{2}} \left(\int f^2 d\mu \right)^{\frac{1}{2}} + C \int f^2 d\mu$$

and consequently

$$\int_B |T(f\mu)|^2 d\mu \leq C \int_E f^2 d\mu, \quad f \in L^2(E).$$

By duality this implies

$$\int_E |T(g\mu)|^2 d\mu \leq C \int_B g^2 d\mu, \quad g \in L^2(B).$$

So by Chebyshev's inequality

$$\mu(\{z \in E : \quad |T(g\mu)(z)| > t\}) \ \leq \ \frac{C}{t^2} \int_B g^2 d\mu, \quad g \in L^2(B). \tag{25}$$

Now, for every $h \in L^2(\mathbf{D}, \mu)$, Lemma 4.5 and (25) give

$$\begin{aligned}
\mu(\{z \in E : \quad |T(h\mu)(z)| > t\}) \\
&\leq \mu\left(\left\{z \in E : \quad |T(h\chi_B\mu)(z)| > \frac{t}{2}\right\}\right) \\
&\qquad + \mu\left(\left\{z \in E : \quad |T(h\chi_{B^c}\mu)(z)| > \frac{t}{2}\right\}\right) \\
&\leq \frac{C}{t^2}\int_B h^2 d\mu + \frac{C}{t^2}\int_B |T(h\chi_{B^c}\mu)(z)|^2 d\mu \\
&\leq \frac{C}{t^2}\int_B h^2 d\mu + \frac{C}{t^2}\int_{B^c} h^2 d\mu \\
&\leq \frac{C}{t^2}\int_{\mathbf{D}} h^2 d\mu. \qquad (26)
\end{aligned}$$

4.4. **A good λ inequality.** We will establish in this section the next crucial argument in the proof of the implication 3) $\Rightarrow$ 2) of Theorem 4.1. The result is the following theorem.

Theorem 4.10. *Let μ be a positive Borel measure on $\mathbf{D}$ with satisfies i) and ii). Then for each $\eta > 0$, there exists $\gamma = \gamma(\eta) > 0$ small enough so that*

$$\mu\left(\left\{z \in \mathbf{D} : |T(f\mu)(z)| > (1+\eta)t \text{ and } M_\mu^\rho(f^2\mu)^{\frac{1}{2}}(z) \leq \gamma t\right\}\right) \leq \frac{1}{2}\mu\left(\{z \in \mathbf{D} : |T(f\mu)(z)| > t\}\right).$$

Proof: Let $\Omega = \{z \in \mathbf{D} : |T(f\mu)(z)| > t\}$. The set Ω is open. By Lemma 2.4 applied to this set, the theorem will follow if we can prove the following lemma.

Lemma 4.11. *Let $\eta > 0$ and $0 < \alpha < 1$. There exists $\gamma = \gamma(\eta, \alpha) > 0$ such that*

$$\mu\left(\left\{z \in B_j^* : |T(f\mu)(z)| > (1+\eta)t \text{ and } M_\mu^\rho(f^2\mu)^{\frac{1}{2}}(z) \leq \gamma t\right\}\right) \leq \alpha\mu\left(B_j^{**}\right), \qquad (27)$$

where B_j^* and B_j^{**} are respectively the first and the second decompositions of the open set Ω with respect to Lemma 2.4.

Indeed, if the lemma is true, then

$$\begin{aligned}
& \mu\left(\left\{z \in \mathbf{D}:\ |T(f\mu)(z)| > (1+\eta)t \text{ and } M_\mu^\rho(f^2\mu)^{\frac{1}{2}}(z) \le \gamma t\right\}\right) \\
= \ & \mu\left(\left\{z \in \Omega:\ |T(f\mu)(z)| > (1+\eta)t \text{ and } M_\mu^\rho(f^2\mu)^{\frac{1}{2}}(z) \le \gamma t\right\}\right) \\
\le \ & \sum_j \mu\left(\left\{z \in B_j^*:\ |T(f\mu)(z)| > (1+\eta)t \text{ and } M_\mu^\rho(f^2\mu)^{\frac{1}{2}}(z) \le \gamma t\right\}\right) \\
\le \ & \alpha \sum_j \mu\left(B_j^{**}\right) \\
\le \ & \alpha M \mu(\Omega) \quad \text{(by the bounded overlap property).}
\end{aligned}$$

We then have to choose α so that $\alpha M = \frac{1}{2}$. We obtain the result.

Let us turn our attention to the proof of the Lemma.

Set $B = B\left(z^B, K_1\varrho\right) = B_j^*$ and $B' = B\left(z^B, K_2\varrho\right) = B_j^{**}$.
We follow with a little change, the proof of the Lemma 4.5.
Suppose without loss of generality that there exists $\xi^0 \in B$ such that

$$M_\mu^\rho(f^2\mu)^{\frac{1}{2}}(\xi^0) \le \gamma t.$$

Let z^0 be a point in $\Omega^c \cap B\left(z^B, K_3\varrho\right)$. Let $\overline{B}$ be a ball centered at z^0 whose radius is equal to $\max\left(2(1-|z^0|), C\varrho\right)$, where C is a constant greater than or equal to K_3 to be precised later. Then $\overline{B}$ touches the boundary of $\mathbf{D}$.

Let $f_1 = f\chi_{\overline{B}}$ and $f_2 = f - f_1 = f\chi_{\overline{B}^c}$. As in the proof of (19) there exists a constant A_1 such that

$$|T(f\mu)(z)| \le \left|T\left(f\mu\chi_{\overline{B}}\right)(z)\right| + (1 + A_1\gamma)t,\ z \in B. \tag{28}$$

On the other hand, if we set $\widetilde{B} = B(\xi^0, C_1K_1\varrho)$, we observe that

$$B \subset \rho\widetilde{B} \subset B' \subset \Omega$$

for some $\rho > 1$.

Now, if $C\varrho \ge 2(1-|z^0|)$, there exists a constant $A_2 > 0$ such that for $z \in B$,

$$\left|T\left(f\mu\chi_{\overline{B}}\right)(z)\right| \le \left|T\left(f\mu\chi_{\widetilde{B}}\right)(z)\right| + A_2\gamma t. \tag{29}$$

We obtain (29) as in the proof of (20).

From (28), we have

$$\mu\left(\left\{z \in B:\ |T(f\mu)(z)| > (1+\eta)t \text{ and } M_{\mu}^{\rho}(f^2\mu)^{\frac{1}{2}}(z) \leq \gamma t\right\}\right)$$
$$\leq \mu\left(\left\{z \in B:\ |T(f\mu\chi_{\overline{B}})(z)| > (\eta - A_1\gamma)t \text{ and } M_{\mu}^{\rho}(f^2\mu)^{\frac{1}{2}}(z) \leq \gamma t\right\}\right).$$

If $C\varrho < 2(1-|z^0|)$, then as in the proof of the Lemma 4.5 we have

$$|T\left(f\mu\chi_{\overline{B}}\right)(z)| \leq C\text{"}\gamma t$$

so that for γ small enough

$$\left\{z \in B:\ |T(f\mu\chi_{\overline{B}})(z)| > (\eta - A_1\gamma)t \text{ and } M_{\mu}^{\rho}(f^2\mu)^{\frac{1}{2}}(z) \leq \gamma t\right\} = \emptyset.$$

Thus (27) is satisfied in this case. If $C\varrho \geq 2(1-|z^0|)$, then from (29), we have

$$\mu\left(\left\{z \in B:\ |T(f\mu)(z)| > (1+\eta)t \text{ and } M_{\mu}^{\rho}(f^2\mu)^{\frac{1}{2}}(z) \leq \gamma t\right\}\right)$$
$$\leq \mu\left(\left\{z \in B:\ |T(f\mu\chi_{\overline{B}})(z)| > (\eta - (A_1+A_2)\gamma)t \text{ and } M_{\mu}^{\rho}(f^2\mu)^{\frac{1}{2}}(z) \leq \gamma t\right\}\right).$$

If we choose γ small enough ($0 < \gamma \leq \frac{\eta}{2(A_1+A_2)}$ will do), we finally have

$$\mu\left(\left\{z \in B:\ |T(f\mu)(z)| > (1+\eta)t \text{ and } M_{\mu}^{\rho}(f^2\mu)^{\frac{1}{2}}(z) \leq \gamma t\right\}\right) \tag{30}$$
$$\leq \mu\left(\left\{z \in B:\ |T(f\mu\chi_{\widetilde{B}})(z)| > \frac{\eta}{2}t \text{ and } M_{\mu}^{\rho}(f^2\mu)^{\frac{1}{2}}(z) \leq \gamma t\right\}\right).$$

We distinguish two cases.

If B does not touch the boundary then we easily obtain,

$$|T(f\mu\chi_{\widetilde{B}})(z)| \leq C^{*}Mf(\xi^0) \leq C\gamma t,$$

such that for γ small enough ($0 < \gamma \leq \frac{\eta}{4C}$), (27) is satisfied.

Finally, suppose that B touches the boundary. Let E be a "big piece" associated with the ball B and the number θ. From (30) and (26) we have

$$\begin{aligned}
&\mu\left(\left\{z\in B:\ |T(f\mu)(z)|>(1+\eta)t \text{ and } M_\mu^\rho(f^2\mu)^{\frac{1}{2}}(z)\leq\gamma t\right\}\right)\\
\leq\ &\mu\left(\left\{z\in B:\ |T(f\mu\chi_{\widetilde{B}})(z)|>\frac{\eta}{2}t \text{ and } M_\mu^\rho(f^2\mu)^{\frac{1}{2}}(z)\leq\gamma t\right\}\right)\\
\leq\ &\mu(B\backslash E)+\mu\left(\left\{z\in E:\ |T(f\mu\chi_{\widetilde{B}})(z)|>\frac{\eta}{2}t\right\}\right)\\
\leq\ &\theta\mu(B)+\frac{C}{\eta^2t^2}\int_{\widetilde{B}}|f|^2d\mu\\
\leq\ &\theta\mu(B)+\frac{C}{\eta^2t^2}\mu\left(\rho\widetilde{B}\right)M_\mu^\rho(f^2\mu)(\xi^0)\\
\leq\ &\theta\mu(B)+\frac{C}{\eta^2t^2}\mu\left(\rho\widetilde{B}\right)\gamma^2t^2\\
\leq\ &\left(\theta+C\eta^{-2}\gamma^2\right)\mu\left(\rho\widetilde{B}\right)\\
\leq\ &\alpha\mu(B'),
\end{aligned}$$

provided θ and γ are chosen small enough so that $(\theta+C\eta^{-2}\gamma^2)\leq\alpha$. This completes the proof of the lemma and consequently the proof of the theorem.

□

4.5. **Proof of the implication** 3) ⇒ 2) **of Theorem 4.1.** Let $f\in L^p(\mu)$, $p>2$. We have

$$\begin{aligned}
&\int|T(f\mu)|^pd\mu\\
&=\int_0^{+\infty}\mu\left(\{z\in\mathbf{D}:\ |T(f\mu)(z)|>t\}\right)dt^p\\
&=(1+\eta)^p\int_0^{+\infty}\mu\left(\{z\in\mathbf{D}:\ |T(f\mu)(z)|>(1+\eta)t\}\right)dt^p\\
&\leq(1+\eta)^p\int_0^{+\infty}\mu\left(\left\{z\in\mathbf{D}:\ |T(f\mu)(z)|>(1+\eta)t;\ M_\mu^\rho(f^2\mu)^{\frac{1}{2}}(z)\leq\gamma t\right\}\right)dt^p\\
&\qquad+(1+\eta)^p\int_0^{+\infty}\mu\left(\left\{z\in\mathbf{D}:M_\mu^\rho(f^2\mu)^{\frac{1}{2}}(z)\geq\gamma t\right\}\right)dt^p\\
&\leq\frac{(1+\eta)^p}{2}\int_0^{+\infty}\mu\left(\{z\in\mathbf{D}:\ |T(f\mu)(z)|>t\}\right)dt^p\\
&\qquad+\frac{(1+\eta)^p}{\gamma^p}\int\left(M_\mu^\rho(f^2\mu)^{\frac{1}{2}}(z)\right)^pd\mu
\end{aligned}$$

by Theorem 4.10. We then choose η small and use Proposition 4.3 to obtain

$$\int |T(f\mu)|^p d\mu \leq C \int |f|^p d\mu. \tag{31}$$

□

4.6. **Proof of the implication** 2) $\Rightarrow$ 1) **in Theorem 4.1.** Since T_α is self adjoint, then by duality, for $1 < p < 2$ the inequality (31) holds, and so for $p = 2$ by interpolation. This finishes the proof.

□

We have proved the result for the case $\alpha = -n$. The same argument holds with minor changes for $-n-1 < \alpha < -n$. In fact one just has to use the following maximal operator

$$M\nu(z) = \sup_{r > 1-|z|} \frac{|\nu|(B(z,r))}{r^k}$$

and the following curvature

$$c^2(z_1, z_2, z_3) = \sum_\sigma K_\alpha\left(z_{\sigma(2)}, z_{\sigma(1)}\right) K_\alpha\left(z_{\sigma(3)}, z_{\sigma(1)}\right).$$

5. Comments

- One reason why we could not carry out our argument in the remaining range $-n < \alpha < -1$ is that the curvature we have defined is no longer strictly positive in this range. Nevertheless, we conjecture that conditions 3) in Theorem 4.1 are sufficient for boundedness in the remaining range.
- Since T_α is a self adjoint CZO, one classical result on the Calderón-Zygmund theory is that a CZO which is bounded in $L^2(\mu)$ is weakly bounded. So a natural question comes:
  is it true that conditions 3) in Theorem 4.1 imply that T_α is weakly bounded?, that is

$$\mu(\{z \in \mathbf{D} : \ |T_\alpha f(z)| > \lambda\}) \leq C \frac{\|f\|_{L^1(\mu)}}{\lambda}, \quad \text{for all } f \in L^1(\mu).$$

References

[ARS] N. Arcozzi, R. Rochberg & E. Sawyer: Carleson measures for the Drury-Arveson Hardy space and other Besov-Sobolev spaces on complex balls. Preprint. (2006) submitted.

[B] D. Békollé: Inégalités à poids pour le projecteur de Bergman dans la boule unité de $\mathbb{C}^n$, *Studia Mathematica*, T.LXXI, (1982) pp 305 - 323.

[CA] L. Carleson: Interpolation by bounded analytic functions and the corona problem, *Ann. of Math.*, **76** (1962), pp 547-559.

[CJ] C. Cascante, J. M. Ortega: Carleson measures on spaces of Hardy-sobolev type, Can. J. Math., Vol 47 (**6**) 1995 pp 1177 - 1200.

[CW] J. A. Cima & W. Wogen: A Carleson measure theorem for the Bergman space on the Ball, *J. Operator theory,* **7** (1982) pp 157-165.

[CWe] R.R. Coifman; G.Weiss: *Analyse harmonique non-commutative sur certains espaces homogénes*, Lectures notes, Springer-Verlag, 1971.

[DJ] G. David & J.L. Journé: A boundedness criterion for generalized Calderón-Zygmund operators, *Annals of Math.*, **120** (1984), pp 371-397.

[DJS] G. David, J.L. Journé & S. Semmes: Opérateurs de Calderón-Zygmund, functions para-accrétives et interpolation. *Revista Mathematica IberoAmericana,* vol, **1** $n°4$ (1985), pp 1-56.

[FGL] G. D. Fazio, C. Gutiérrez & E. Lanconelli: Covering theorems, inequalities on metric spaces and applications to PDE's.Preprint (2005) submitted.

[H] L. Hörmander: L^p-estimates for (pluri)subharmonic functions, *Math. Scand.* **20** (1967) pp 65 - 78.

[KR] A. Koranyi & H. M. Reimann: Foundations for the theory of quasiconformal mappings on the Heisenberg group. *Adv. In Math.*, 111 **1** (1995) pp 1-87.

[NTV] F. Nazarov, S. Treil & A. Volberg: Cauchy Integral and Calderón-Zygmund operators on non-homogeneous spaces, *Internt. Math. Res. Notices* (1997), $n°15$, pp 703-726.

[R] W.Rudin: *Function theory on the Unit Ball of* $\mathcal{C}^n$, Springer-verlag, New-York, 1980.

[S] E.L. Stein: *Harmonic analysis, real-variable methods, orthogonality, and oscillatory integrals.* Princeton University Press, 1993.

[ST] D. A. Stegenga: Multipliers of the Dirichlet space, *Illinois J. Math.* **24** (1980), pp 113-139.

[T1] X. Tolsa: L^2-boundedness of the Cauchy integral operator for continuous measures, *Duke Math. J.*, **98** (1999) pp 269-304.

[T2] X. Tolsa: A T(1)-Theorem for non-doubling measures with atoms. *Proc. London. Math. Soc.*, (3) **82** (2001) pp 195-228.

[V] J. Verdera: On the T(1)-Theorem for the Cauchy integral. *Ark. Mat.*, **38** (2000) pp 183-199.

[ZZ] R. Zhao & K. Zhu: Theory of Bergman spaces in the unit ball of $\mathbb{C}^n$, Univ. Albany. Preprint. (2005).

Department of Mathematical Sciences, Division of Mathematics, Chalmers University of Technology and Göteborg University, SE-412 96 Göteborg, Sweden

E-mail address: tchoundj@math.chalmers.se