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Matrix Ordered Operator Algebras

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Abstract

We study the question when a sequence of cones $C_n \in M_n(\mathcal{A})$ for a given $*$-algebra $\mathcal{A}$ can be realized as cones of positive operators in a faithful $*$-representation of $\mathcal{A}$ in a Hilbert space. A characterization of operator algebras which are completely boundedly isomorphic to C^*-algebras is presented.

KEYWORDS: $*$-algebra, operator algebra, C^*-algebra, completely bounded homomorphism, Kadison problem.

1 Introduction

Effros and Choi [2] gave an abstract characterization of the self-adjoint subspaces S in C^*-algebras with hierarchy of cones of positive elements in $M_n(S)$. In s.1 of the present paper we are concerned with the same question for $*$-subalgebras of C^*-algebras. More precisely, let $\mathcal{A}$ be an associative $*$-algebra with unit. We present a characterization of the collections of cones $C_n \subseteq M_n(\mathcal{A})$ such that there exist faithful $*$-representation π of $\mathcal{A}$ on Hilbert space H such that C_n coincides with the cone of positive operators contained in $\pi^{(n)}(M_n(\mathcal{A}))$. Here $\pi^{(n)}$ is a n-fold amplification of π. Note that we do not assume that $\mathcal{A}$ has any faithful $*$-representation it follows from the requirements imposed on the cones. In terms close to Effros and Choi we give an abstract characterizations of matrix ordered (not necessary closed) operator $*$-algebras up to complete order $*$-isomorphism.

Based on this characterization we study the question when an operator algebra is similar to a C^*-algebra.

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Let $\mathcal{B}$ be a unital (closed) operator algebra in $B(H)$. In [7] C. Le Merdy presented necessary and sufficient conditions for $\mathcal{B}$ to be self-adjoint. These conditions involve all completely isometric representations of $\mathcal{B}$ on Hilbert space. Our characterization is different in following respect. If S is a bounded invertible operator in $B(H)$ and $\mathcal{A}$ is a C^*-algebra then the operator algebra $S^{-1}\mathcal{A}S$ is not necessarily self-adjoint but only isomorphic to a C^*-algebra via completely bounded isomorphism with completely bounded inverse.

By Haagerup's theorem every completely bounded isomorphism π from a C^*-algebra $\mathcal{A}$ to a operator algebra $\mathcal{B}$ has the form $\pi(a) = S^{-1}\rho(a)S$, $a \in \mathcal{A}$, for some $*$-homomorphism $\rho : \mathcal{A} \to B(H)$ and invertible $S \in B(H)$. Thus the question whether an operator algebra $\mathcal{B}$ is c.b. isomorphic to a C^*-algebra via isomorphism which has c.b. inverse is equivalent to the one if there is bounded invertible operators S s.t. $S\mathcal{B}S^{-1}$ is a C^*-algebra. For instance in case $\mathcal{B}$ is an image under bounded homomorphism of a C^*-algebra it is a famous open problem raised by R. Kadison whether the answer to the above question is affirmative.

We will present a criterion for an operator algebra $\mathcal{B}$ to be completely boundedly isomorphic to a C^*-algebra in terms of the existence of a collection of cones $C_n \in M_n(\mathcal{B})$ satisfying certain axioms (see def. 3). The axioms are derived from the properties of cones of positive elements of a C^*-algebra preserved under completely bounded isomorphisms.

The main results are contained in s.2. We define a $*$-admissible sequence of cones in an operator algebra and present a criterion Theorem 4 for an operator algebra to be c.b. isomorphic to a C^*-algebra.

The last section we consider the operator algebras and a collection of cones associated with Kadison similarity problem.

2 Operator realizations of matrix-ordered $*$-algebras.

The aim of this section is to give necessary and sufficient conditions on a sequences of cones $C_n \subseteq M_n(\mathcal{A})_{sa}$ for unital $*$-algebra $\mathcal{A}$ such that C_n coincides with cone $M_n(\mathcal{A}) \cap M_n(B(H))^+$ for some realization of $\mathcal{A}$ as a $*$-subalgebra of $B(H)$, where $M_n(B(H))^+$ denotes the set of positive operators acting on $H^n = H \oplus \ldots \oplus H$.

In [10] it was proved that a $*$-algebra $\mathcal{A}$ with unit e is a $*$-subalgebra of

$B(H)$ if and only if there is an algebraically admissible cone on $\mathcal{A}$ such that e is an Archimedean order unit. Applying this result to some inductive limit of $M_{2^n}(\mathcal{A})$ we obtain the desired characterization in Theorem 2.

First we give necessary definitions and fix notations. Let $\mathcal{A}_{sa}$ denote the set of self-adjoint elements in $\mathcal{A}$. A subset $C \subset \mathcal{A}_{sa}$ containing unit e of $\mathcal{A}$ is *algebraically admissible* cone, see [11], provided that

(i) C is a cone in $\mathcal{A}_{sa}$, i.e. $\lambda x + \beta y \in C$ for all x, $y \in C$ and $\lambda \geq 0$, $\beta \geq 0$, $\lambda, \beta \in \mathbb{R}$;

(ii) $C \cap (-C) = \{0\}$;

(iii) $xCx^* \subseteq C$ for every $x \in \mathcal{A}$;

We call $e \in \mathcal{A}_{sa}$ an *order unit* if for every $x \in \mathcal{A}_{sa}$ there exists $r > 0$ such that $re + x \in C$. An order unit e is *Archimedean* if $re + x \in C$ for all $r > 0$ implies that $x \in C$

In what follows we will use the following modification of Theorem 1 of [10].

Theorem 1. *Let $\mathcal{A}$ be a $*$-algebra with unit e and $C \subseteq \mathcal{A}_{sa}$ be a cone containing e. If $xCx^* \subseteq C$ for every $x \in \mathcal{A}$ and e is an Archimedean order unit then there is a unital $*$-representation $\pi : \mathcal{A} \to B(H)$ such that $\pi(C) = \pi(\mathcal{A}_{sa}) \cap B(H)^+$. Moreover*

1. $\|\pi(x)\| = \inf\{r > 0 : r^2 \pm x^*x \in C\}$.

2. $\ker \pi = \{x : x^*x \in C \cap (-C)\}$.

3. *If $C \cap (-C) = \{0\}$ then $\ker \pi = \{0\}$ and $\|\pi(a)\| = \inf\{r > 0 : r \pm a \in C\}$ for all $a = a^* \in \mathcal{A}$. Moreover, $\pi(C) = \pi(\mathcal{A}) \cap B(H)^+$*

Proof. Following the same lines as in [10] one obtain that the function $\|\cdot\| : \mathcal{A}_{sa} \to \mathbb{R}_+$ defined as

$$\|a\| = \inf\{r > 0 : re \pm a \in C\}$$

is a seminorm on $\mathbb{R}$-space $\mathcal{A}_{sa}$ and $|x| = \sqrt{\|x^*x\|}$ for $x \in \mathcal{A}$ defines a pre-C^*-norm on $\mathcal{A}$. If N denote a null-space of $|\cdot|$ then the completion $\mathcal{B} = \overline{\mathcal{A}/N}$ with respect to this norm is a C^*-algebra and canonical epimorphism $\pi : \mathcal{A} \to \mathcal{A}/N$ extends to a unital $*$-homomorphism $\pi : \mathcal{A} \to \mathcal{B}$. We can assume without loss of generality that $\mathcal{B}$ is a concrete C^*-algebra in some $B(H)$. Thus $\pi : \mathcal{A} \to B(H)$ can be regarded as a unital $*$-representation. Clearly,

$$\|\pi(x)\| = |x| \text{ for all } x \in \mathcal{A}.$$

From this follows 1.

To show 2 take $x \in \ker \pi$ then $\|\pi(x)\| = 0$ and $re \pm x^*x \in C$ for all $r > 0$. Since e is an Archimedean unit we have $x^*x \in C \cap (-C)$. Conversely $x^*x \in C \cap (-C)$ then $re \pm x^*x \in C$ for all $r > 0$ hence $\|\pi(x)\| = 0$ and 2 holds.

We need to prove that $\pi(C) = \pi(\mathcal{A}_{sa}) \cap B(H)^+$. Let $x \in \mathcal{A}_{sa}$ and $\pi(x) \geq 0$. Then there exists constant $\lambda > 0$ such that $\|\lambda I_H - \pi(x)\| \leq \lambda$, hence $|\lambda e - x| \leq \lambda$. Since $\|a\| \leq |a|$ for all self-adjoint $a \in \mathcal{A}$, see Lemma 3.3 of [10], we have $\|\lambda e - x\| \leq \lambda$. Thus given $\varepsilon > 0$ we have $(\lambda+\varepsilon)e \pm (\lambda e - x) \in C$. Hence $\varepsilon e + x \in C$. Since e is Archimedean we have $x \in C$.

Conversely, let $x \in C$. To show that $\pi(x) \geq 0$ it is sufficient to find $\lambda > 0$ such that $\|\lambda I_H - \pi(x)\| \leq \lambda$. Since $\|\lambda I_H - \pi(x)\| = |\lambda e - x|$ we will prove that $|\lambda e - x| \leq \lambda$ for some $\lambda > 0$. From the definition of norm $|\cdot|$ we have the following equivalences:

$$\begin{aligned} |\lambda e - x| \leq \lambda \quad &\Leftrightarrow \quad (\lambda + \varepsilon)^2 e - (\lambda e - x)^2 \in C \text{ for all } \varepsilon > 0 && (1) \\ &\Leftrightarrow \quad \varepsilon_1 e + x(2\lambda e - x) \geq 0, \text{ for all } \varepsilon_1 > 0. && (2) \end{aligned}$$

By condition (iii) of algebraically admissible cone we have that $xyx \in C$ and $yxy \in C$ for every $x, y \in C$. If $xy = yx$ then $xy(x + y) \in C$. Since e is order unit we can choose $r > 0$ such that $re - x \in C$. Put $y = re - x$ to obtain $rx(r - x) \in C$. Hence (2) is satisfied with $\lambda = \frac{r}{2}$. Thus $\|\lambda e - \pi(x)\| \leq \lambda$ and $\pi(x) \geq 0$, which proves $\pi(C) = \pi(\mathcal{A}_{sa}) \cap B(H)^+$.

In particular, for $a = a^*$ we have

$$\|\pi(a)\| = \inf\{r > 0 : rI_H \pm \pi(a) \in \pi(C)\}. \tag{3}$$

We now in a position to prove 3. Suppose that $C \cap (-C) = 0$. Then $\ker \pi$ is a $*$-ideal and $\ker \pi \neq 0$ implies that there exists a self-adjoint $0 \neq a \in \ker \pi$, i.e. $|a| = 0$. Inequality $\|a\| \leq |a|$ implies $re \pm a \in C$ for all $r > 0$. Since e is Archimedean, $\pm a \in C$, i.e. $a \in C \cap (-C)$ and, consequently, $a = 0$.

Since $\ker \pi = 0$ the inclusion $rI_H \pm \pi(a) \in \pi(C)$ is equivalent to $re \pm a \in C$, and by (3), $\|\pi(a)\| = \inf\{r > 0 : re \pm a \in C\}$. Moreover if $\pi(a) = \pi(a)^*$ then $a = a^*$. Thus we have $\pi(C) = \pi(A) \cap B(H)^+$. $\square$

We say that a $*$-algebra $\mathcal{A}$ with unit e is a *matrix ordered* if the following conditions hold:

(a) for each $n \geq 1$ we are given a cone C_n in $M_n(\mathcal{A})_{sa}$ and $e \in C_1$,

(b) $C_n \cap (-C_n) = \{0\}$ for all n,

(c) for all n and m and all $A \in M_{n\times m}(\mathcal{A})$, we have that $A^* C_n A \subseteq C_m$,

We call $e \in \mathcal{A}_{sa}$ a *matrix order unit* provided that for every $n \in \mathbb{N}$ and every $x \in M_n(\mathcal{A})_{sa}$ there exists $r > 0$ such that $re_n + x \in C_n$, where $e_n = e \otimes I_n$. A matrix order unit is called *Archimedean matrix order unit* provided that for all $n \in \mathbb{N}$ inclusion $re_n + x \in C_n$ for all $r > 0$ implies that $x \in C_n$.

Let $\pi : \mathcal{A} \to B(H)$ be a $*$-representation. Define $\pi^{(n)} : M_n(\mathcal{A}) \to M_n(B(H))$ by $\pi^{(n)}((a_{ij})) = (\pi(a_{ij}))$.

Theorem 2. *If $\mathcal{A}$ is a matrix-ordered $*$-algebra with a unit e which is Archimedean matrix order unit then there exists a Hilbert space H and a faithful unital $*$-representation $\tau : \mathcal{A} \to B(H)$, such that $\tau^{(n)}(C_n) = M_n(\tau(\mathcal{A}))^+$ for all n. Conversely, every unital $*$-subalgebra $\mathcal{D}$ of $B(H)$ is matrix-ordered by cones $M_n(\mathcal{D})^+$ and unit of this algebra is an Archimedean order unit.*

Proof. Consider an inductive system of $*$-algebras and a unital injective $*$-homomorphisms:

$$\phi_n : M_{2^n}(\mathcal{A}) \to M_{2^{n+1}}(\mathcal{A}), \quad \phi_n(a) = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \text{ for all } a \in \mathcal{A}, n \geq 0.$$

Let $\mathcal{B} = \varinjlim M_{2^n}(\mathcal{A})$ be an inductive limit of this system. By (c) in the definition of the matrix ordered algebra we have $\phi_n(C_{2^n}) \subseteq C_{2^{n+1}}$. We will identify $M_{2^n}(\mathcal{A})$ with a subalgebra of $\mathcal{B}$ via canonical inclusions. Let $C = \bigcup_{n\geq 1} C_{2^n} \subseteq \mathcal{B}_{sa}$ and e_∞ be the unit of $\mathcal{B}$.

Let us prove that C is an algebraically admissible cone. Clearly, C satisfies conditions (i) and (ii) of definition of algebraically admissible cone. To prove (iii) suppose that $x \in \mathcal{B}$ and $a \in C$, then for sufficiently large n we have $a \in C_{2^n}$ and $x \in M_{2^n}(\mathcal{A})$. Therefore, by (c), $x^* a x \in C$. Since e is an Archimedean order unit of $\mathcal{A}$ we obviously have that e_∞ is also an Archimedean order unit. Thus $*$-algebra $\mathcal{B}$ satisfies assumptions of Theorem 1 and there is a faithful $*$-representation $\pi : \mathcal{B} \to B(H)$ such that $\pi(C) = \pi(\mathcal{B}) \cap B(H)^+$.

Let $\xi_n : M_{2^n}(\mathcal{A}) \to \mathcal{B}$ be canonical injections, $n \geq 0$. Then $\tau = \pi \circ \xi_0 : \mathcal{A} \to B(H)$ is a injective $*$-homomorphism.

We claim that $\tau^{(2^n)}$ is unitary equivalent to $\pi \circ \xi_n$. By replacing π with π^α, where α is an infinite cardinal, we can assume that π^α is unitary equivalent to π. Then $\pi \circ \xi_n : M_{2^n}(\mathcal{A}) \to B(H)$ is a $*$-homomorphism. Thus there exist unique Hilbert space K_n, $*$-homomorphism $\rho_n : \mathcal{A} \to B(K_n)$ and unitary $U_n : K_n \otimes \mathbb{C}^{2^n} \to H$ such that

$$\pi \circ \xi_n = U_n(\rho_n \otimes id_{M_{2^n}})U_n^*.$$

For $a \in \mathcal{A}$ we have

$$\begin{aligned} \pi \circ \xi_0(a) &= \pi \circ \xi_n(a \otimes E_{2^n}) \\ &= U_n(\rho_n(a) \otimes E_{2^n})U_n^*, \end{aligned}$$

where E_{2^n} is the identity matrix in $M_{2^n}(\mathbb{C})$. Thus $\tau(a) = U_0\rho_0(a)U_0^* = U_n(\rho_n(a) \otimes E_{2^n})U_n^*$. Let $\sim$ stands for the unitary equivalence of representations. Since $\pi \circ \xi_n \sim \rho_n \otimes id_{M_{2^n}}$ and $\pi^\alpha \sim \pi$ we have that $\rho_n^\alpha \otimes id \sim \pi^\alpha \circ \xi_n \sim \rho_n \otimes id_{M_{2^n}}$. Hence $\rho_n^\alpha \sim \rho_n$. Thus $\rho_n \otimes E_{2^n} \sim \rho^{2^n\alpha} \sim \rho_n$. Consequently $\rho_0 \sim \rho_n$ and $\pi \circ \xi_n \sim \rho_0 \otimes id_{M_{2^n}} \sim \tau \otimes id_{M_{2^n}}$. Therefore $\tau^{(2^n)} = \tau \otimes id_{M_{2^n}}$ is unitary equivalent to $\pi \circ \xi_n$.

What is left to show is that $\tau^{(n)}(C_n) = M_n(\tau(\mathcal{A}))^+$. Note that $\pi \circ \xi_n(M_{2^n}(\mathcal{A})) \cap B(H)^+ = \pi(C_{2^n})$. Indeed, the inclusion $\pi \circ \xi(C_{2^n}) \subseteq M_{2^n}(\mathcal{A}) \cap B(H)^+$ is obvious. To show the converse take $x \in M_{2^n}(\mathcal{A})$ such that $\pi(x) \geq 0$. Then $x \in C \cap M_{2^n}(\mathcal{A})$. Using (c) one can easily show that $C \cap M_{2^n}(\mathcal{A}) = C_{2^n}$. Hence $\pi \circ \xi_n(M_{2^n}(\mathcal{A})) \cap B(H)^+ = \pi(C_{2^n})$. Since $\tau^{(2^n)}$ is unitary equivalent to $\pi \circ \xi_n$ we have that $\tau^{(2^n)}(C_{2^n}) = M_{2^n}(\tau(\mathcal{A})) \cap B(H^{2^n})^+$.

Let now show that $\tau^{(n)}(C_n) = M_n(\tau(\mathcal{A}))^+$. For $X \in M_n(\mathcal{A})$ denote

$$\widetilde{X} = \begin{pmatrix} X & 0_{n\times(2^n-n)} \\ 0_{(2^n-n)\times n} & 0_{(2^n-n)\times(2^n-n)} \end{pmatrix} \in M_{2^n}(\mathcal{A}).$$

Then, clearly, $\tau^{(n)}(X) \geq 0$ if and only if $\tau^{(2^n)}(\widetilde{X}) \geq 0$. Thus $\tau^{(n)}(X) \geq 0$ is equivalent to $\widetilde{X} \in C_{2^n}$ which in turn is equivalent to $X \in C_n$ by (c). $\square$

3 Operator Algebras c.b. isomorphic to C^*-algebras.

The algebra $M_n(B(H))$ of $n \times n$ matrices with entries in $B(H)$ has a norm $\|\cdot\|_n$ via the identification of $M_n(B(H))$ with $B(H^n)$, where H^n is the direct sum of n copies of a Hilbert space H. If $\mathcal{A}$ is a subalgebra of $B(H)$ then $M_n(\mathcal{A})$ inherits a norm $\|\cdot\|_n$ via natural inclusion into $M_n(B(H))$. The sequence of norms $\{\|\cdot\|_n\}_{n\geq 1}$ is called matrix norms on the operator algebra $\mathcal{A}$. In the sequel all operator algebras will be assumed to be norm closed.

Operator algebras $\mathcal{A}$ and $\mathcal{B}$ are called completely boundedly isomorphic if there is a completely bounded isomorphism $\tau : \mathcal{A} \to \mathcal{B}$ with completely bounded inverse. The aim of this section is to give necessary and sufficient conditions for an operator algebra to be completely boundedely isomorphic to a C^*-algebra. To do this we introduce a concept of $*$-admissible cones which reflect the properties of the cones of positive elements of a C^*-algebra preserved under completely bounded isomorphism.

Definition 3. *Let $\mathcal{B}$ be an operator algebra with unit e. A sequence $C_n \subseteq M_n(\mathcal{B})$ of closed (in the norm $\|\cdot\|_n$) cones will be called $*$-admissible if it satisfies the following conditions:*

1. $e \in C_1$;
2. *(i)* $M_n(\mathcal{B}) = (C_n - C_n) + i(C_n - C_n)$, *for all* $n \in \mathbb{N}$,

   (ii) $C_n \cap (-C_n) = \{0\}$, *for all* $n \in \mathbb{N}$,

   (iii) $(C_n - C_n) \cap i(C_n - C_n) = \{0\}$, *for all* $n \in \mathbb{N}$;
3. *(i) for all* c_1, $c_2 \in C_n$ *and* $c \in C_n$, *we have that* $(c_1-c_2)c(c_1-c_2) \in C_n$,

   (ii) for all n, m *and* $B \in M_{n\times m}$ *we have that* $B^*C_nB \subseteq C_m$;
4. *for every net* $c_j \in C_{n_j} - C_{n_j}$ *the condition* $\sup\limits_j \|c_j\|_{n_j} < \infty$ *implies that there exists* $r > 0$ *such that* $re_{n_j} + c_j \in C_{n_j}$ *for all* j,
5. *there exists a constant* $K > 0$ *such that for all* $n \in \mathbb{N}$ *and* $a, b \in C_n - C_n$ *we have* $\|a\|_n \leq K \cdot \|a + ib\|_n$.

Theorem 4. *If an operator algebra $\mathcal{B}$ has a $*$-admissible sequence of cones then there is a completely bounded homomorphism τ from $\mathcal{B}$ onto a C^*-algebra $\mathcal{A}$. In addition if one of the following condition holds*

(1) for every two nets c_α, $d_\alpha \in C_{n_\alpha}$ *such that* $\lim\limits_\alpha \|c_\alpha + d_\alpha\| = 0$ *we have* $\lim\limits_\alpha \|c_\alpha\| = 0$

(2) $\|(x - iy)(x + iy)\| \geq \alpha\|x - iy\|\|x + iy\|$ *for all* $x, y \in C_n - C_n$

then the inverse $\tau^{-1} : \mathcal{A} \to \mathcal{B}$ is also completely bounded.

Conversely, if such homomorphism τ exists then $\mathcal{B}$ possesses a $*$-admissible sequence of cones and conditions (1) *and* (2) *are satisfied.*

The proof will be divided into 3 lemmas.

Let $\{C_n\}_{n\geq 1}$ be a $*$-admissible sequence of cones of $\mathcal{B}$. Let $\mathcal{B}_{2^n} = M_{2^n}(\mathcal{B})$, $\phi_n : \mathcal{B}_{2^n} \to \mathcal{B}_{2^{n+1}}$ be unital homomorphisms given by $\phi_n(x) = \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$, $x \in \mathcal{B}_{2^n}$. Denote by $\mathcal{B}_\infty = \varinjlim \mathcal{B}_{2^n}$ the inductive limit of the system $(\mathcal{B}_{2^n}, \phi_n)$. As all inclusions ϕ_n are unital $\mathcal{B}_\infty$ has a unit, denoted by e_∞. Since $\mathcal{B}_\infty$ can be considered as a subalgebra of a C^*-algebra of the corresponding inductive limit of $M_{2^n}(B(H))$ we can define the closure of $\mathcal{B}_\infty$, $\overline{\mathcal{B}}_\infty$, in this C^*-algebra.

Now we will define an involution on $\mathcal{B}_\infty$. Let $\xi_n : M_{2^n}(\mathcal{B}) \to \mathcal{B}_\infty$ be canonical morphisms. By $(3ii)$, $\phi_n(C_{2^n}) \subseteq C_{2^{n+1}}$. Hence $C = \bigcup\limits_n \xi_n(C_{2^n})$ is a well defined cone in $\mathcal{B}_\infty$. Denote by $\overline{C}$ its completion. By $(2i)$ and $(2iii)$ for every $x \in \mathcal{B}_{2^n}$ we have $x = x_1 + ix_2$ for unique x_1, $x_2 \in C_{2^n} - C_{2^n}$. By $(3ii)$ we have $\begin{pmatrix} x_i & 0 \\ 0 & x_i \end{pmatrix} \in C_{2^{n+1}} - C_{2^{n+1}}$, $i = 1, 2$. Thus for every $x \in B_\infty$ we have unique decomposition $x = x_1 + ix_2$, $x_1 \in C - C$, $x_2 \in C - C$. Hence the mapping $x \mapsto x^\sharp = x_1 - ix_2$ is a well defined involution on $\mathcal{B}_\infty$.

Lemma 5. *Involution on $\mathcal{B}_\infty$ is compatible with the one on $\mathcal{B}$, i.e. for all $A = (a_{ij})_{i,j} \in M_{2^n}(\mathcal{B})$*

$$A^\sharp = (a_{ji}^\sharp)_{i,j}.$$

Proof. Assignment $A^\circ = (a_{ji}^\sharp)_{i,j}$, clearly, defines an involution on $M_{2^n}(\mathcal{B})$. We need to prove that $A^\sharp = A^\circ$.

Let $A = (a_{ij})_{i,j} \in M_{2^n}(\mathcal{B})$ be self-adjoint $A^\circ = A$. Then $A = \sum\limits_i a_{ii} \otimes E_{ii} + \sum\limits_{i<j}(a_{ij} \otimes E_{ij} + a_{ij}^\sharp \otimes E_{ji})$ and $a_{ii}^\sharp = a_{ii}$, for all i. By $(3ii)$ we have $\sum\limits_i a_{ii} \otimes E_{ii} \in C_{2^n} - C_{2^n}$. Since $a_{ij} = a'_{ij} + ia''_{ij}$ for some a'_{ij}, $a''_{ij} \in C_{2^n} - C_{2^n}$

we have

$$\begin{aligned}
a_{ij}\otimes E_{ij}+a^\sharp_{ij}\otimes E_{ji} &= (a'_{ij}+ia''_{ij})\otimes E_{ij}+(a'_{ij}-ia''_{ij})\otimes E_{ji}\\
&= (a'_{ij}\otimes E_{ij}+a'_{ij}\otimes E_{ji})+(ia''_{ij}\otimes E_{ij}-ia''_{ij}\otimes E_{ji})\\
&= (E_{ii}+E_{ji})(a'_{ij}\otimes E_{ii}+a'_{ij}\otimes E_{jj})(E_{ii}+E_{ij})\\
&- (a'_{ij}\otimes E_{ii}+a'_{ij}\otimes E_{jj})\\
&+ (E_{ii}-iE_{ji})(a''_{ij}\otimes E_{ii}+a''_{ij}\otimes E_{jj})(E_{ii}+iE_{ij})\\
&- (a''_{ij}\otimes E_{ii}+a''_{ij}\otimes E_{jj})\in C_{2^n}-C_{2^n}.
\end{aligned}$$

Thus $A\in C_{2^n}-C_{2^n}$ and $A^\sharp=A$. Since for every $x\in M_{2^n}(\mathcal{B})$ there exist unique $x_1=x_1^\circ$ and $x_2=x_2^\circ$ in $M_{2^n}(\mathcal{B})$, such that $x=x_1+ix_2$, and unique $x'_1=x_1'^\sharp$ and $x'_2=x_2'^\sharp$, such that $x=x'_1+ix'_2$, we have that $x_1=x_1^\sharp=x'_1$, $x_2=x_2^\sharp=x'_2$ and involutions $\sharp$ and $\circ$ coincide. $\square$

Lemma 6. *Involution $x\to x^\sharp$ is continuous on $\mathcal{B}_\infty$ and extends to the involution on $\overline{\mathcal{B}}_\infty$. With respect to this involution $\overline{C}\subseteq(\overline{\mathcal{B}}_\infty)_{sa}$ and $x^\sharp\overline{C}x\subseteq\overline{C}$ for every $x\in\overline{\mathcal{B}}_\infty$.*

Proof. Consider a convergent net $\{x_i\}\subseteq\mathcal{B}_\infty$ with the limit $x\in\overline{\mathcal{B}}_\infty$. Decompose $x_i=x'_i+ix''_i$. By (5) the nets $\{x'_i\}$ and $\{x''_i\}$ are also convergent. Thus $x=a+ib$, where $a=\lim x'_i\in\overline{C}$, $b=\lim x''_i\in\overline{C}$. Therefore the involution defined on $\mathcal{B}_\infty$ can be extended by continuity to $\overline{\mathcal{B}}_\infty$.

Under this involution $\overline{C}\subseteq(\overline{\mathcal{B}}_\infty)_{sa}=\{x\in\overline{\mathcal{B}}_\infty : x=x^\sharp\}$.

Let us show that for every $x\in\overline{\mathcal{B}}_\infty$ and $c\in\overline{C}$ we have that $x^\sharp cx\in\overline{C}$. Take firstly $c\in C_{2^n}$ and $x\in\mathcal{B}_{2^n}$. Then $x=x_1+ix_2$ for some x_1, $x_2\in C_{2^n}-C_{2^n}$ and

$$(x_1+ix_2)^\sharp c(x_1+ix_2)=(x_1-ix_2)c(x_1+ix_2)$$
$$=\frac{1}{2}\begin{pmatrix}1 & 1\end{pmatrix}\begin{pmatrix}-x_1 & -ix_2\\ ix_2 & x_1\end{pmatrix}\begin{pmatrix}c & 0\\ 0 & c\end{pmatrix}\begin{pmatrix}-x_1 & -ix_2\\ ix_2 & x_1\end{pmatrix}\begin{pmatrix}1\\ 1\end{pmatrix}$$

By (3i), Lemma 5 and (3ii) $x^\sharp cx\in C_{2^n}$.

We let now $c\in\overline{C}$ and $x\in\overline{\mathcal{B}}_\infty$. Suppose that $c_i\to c$ and $x_i\to x$, where $c_i\in C$, $x_i\in\mathcal{B}_\infty$. We can assume that c_i, $x_i\in B_{2^{n_i}}$. Then $x_i^\sharp c_ix_i\in C_{2^{n_i}}$ for all i and since it is convergent we have $x^\sharp cx\in\overline{C}$. $\square$

Lemma 7. *The unit of $\overline{\mathcal{B}}_\infty$ is an Archimedean order unit and $(\overline{\mathcal{B}}_\infty)_{sa}=\overline{C}-\overline{C}$.*

Proof. Firstly let show that e_∞ is an order unit. Clearly, $(\overline{\mathcal{B}}_\infty)_{sa} = \overline{C - C}$. For every $a \in \overline{C - C}$ there is a net $a_i \in C_{2^{n_i}} - C_{2^{n_i}}$ convergent to a. Since $\sup\limits_i \|a_i\| < \infty$ there exists $r_1 > 0$ such that $r_1 e_{n_i} - a_i \in C_{2^{n_i}}$, i.e. $r_1 e_\infty - a_i \in C$. Passing to the limit we get $r_1 e_\infty - a \in \overline{C}$. Replacing a by $-a$ we can find $r_2 > 0$ such that $r_2 e_\infty + a \in \overline{C}$. If $r = \max(r_1, r_2)$ then $re_\infty \pm a \in \overline{C}$. This proves that e_∞ is an order unit and that for all $a \in \overline{C - C}$ we have $a = re_\infty - c$ for some $c \in \overline{C}$. Thus $\overline{C - C} \in \overline{C} - \overline{C}$. The converse inclusion, clearly, holds. Thus $\overline{C - C} = \overline{C} - \overline{C}$.

Let show now that e_∞ is Archimedean unit. Take $x \in (\overline{\mathcal{B}}_\infty)_{sa}$ such that for every $r > 0$ we have $r + x \in \overline{C}$. Consider a net $\{x_i\}_i \subseteq (\mathcal{B}_\infty)_{sa}$ converging to x. Then $\lim\limits_i (r + x_i) = \lim\limits_i c_{i,r}$ for some $c_{i,r} \in C$ and $\lim\limits_i (c_{i,r} - x_i) = r$. Taking $r := \frac{1}{n}$ we have for every s there exists N_s such that for all $i \geq N_s$ we have that $\|c_{i,\frac{1}{s}} - x_i\| < \frac{1}{s}$, therefore $\lim\limits_s c_{N_s,\frac{1}{s}} = x$ and $x \in \overline{C}$. □

Lemma 8. $\mathcal{B}_\infty \cap \overline{C} = C$.

Proof. Denote by $\mathcal{D} = \varinjlim M_{2^n}(B(H))$ the C^*-algebra inductive limit corresponding to inductive system ϕ_n and denote $\phi_{n,m} = \phi_{m-1} \circ \ldots \circ \phi_n : M_{2^n}(B(H)) \to M_{2^m}(B(H))$. For $n < m$ we identify $M_{2^{m-n}}(M_{2^n}(B(H)))$ with $M_{2^m}(B(H))$ by omitting superfluous parentheses in a block matrix $B = [B_{ij}]_{ij}$ with $B_{ij} \in M_{2^n}(B(H))$.

Denote by $P_{n,m}$ the operator $diag(I, 0, \ldots, 0) \in M_{2^{m-n}}(M_{2^n}(B(H)))$ and by $V_{n,m} = \sum_{k=1}^{2^{m-n}} E_{k,k-1}$. Here I is the identity matrix in $M_{2^n}(B(H))$ and $E_{k,k-1}$ is $2^n \times 2^n$ block matrix with identity operator at $(k, k-1)$-entry and all other entries being zero. Define an operator $\psi_{n,m}([B_{ij}]) = diag(B_{11}, \ldots, B_{11})$. It is easy to see that

$$\psi_{n,m}([B_{ij}]) = \sum_{k=0}^{2^{m-n}-1} (V_{n,m}^k P_{n,m}) B (V_{n,m}^k P_{n,m})^*.$$

Hence by $(3ii)$

$$\psi_{n,m}(C_{2^m}) \subseteq \phi(C_{2^n}) \subseteq C_{2^m}. \tag{4}$$

Clearly, $\psi_{n,m}$ is linear contraction and

$$\psi_{n,m+k} \circ \phi_{m,m+k} = \phi_{m,m+k} \circ \psi_{n,m}$$

Hence there is a well defined contraction $\psi_n = \lim_m \psi_{n,m} : \mathcal{D} \to \mathcal{D}$ such that

$$\psi_n|_{M_{2^n}(B(H))} = id_{M_{2^n}(B(H))},$$

where $M_{2^n}(B(H))$ is considered as a subalgebra in $\mathcal{D}$. Clearly, $\psi_n(\overline{\mathcal{B}}_\infty) \subseteq \overline{\mathcal{B}}_\infty$ and $\psi_n|_{\mathcal{B}_{2^n}} = id$. Consider C and C_{2^n} as subalgebras in $\mathcal{B}_\infty$, by (4) we have $\psi_n : C \to C_{2^n}$.

To prove that $\mathcal{B}_\infty \cap \overline{C} = C$ take $c \in \mathcal{B}_\infty \cap \overline{C}$. Then there is a net c_j in C such that $\|c_j - c\| \to 0$. Since $c \in \mathcal{B}_\infty$, $c \in \mathcal{B}_{2^n}$ for some n and, consequently, $\psi_n(c) = c$. Thus

$$\|\psi_n(c_j) - c\| = \|\psi_n(c_j - c)\| \leq \|c_j - c\|.$$

Hence $\psi_n(c_j) \to c$. But $\psi_n(c_j) \in C_{2^n}$ and the letter is closed. Thus $c \in C$. The converse inclusion is obvious. □

Remark 9. *Note that for every $x \in \mathcal{D}$*

$$\lim_n \psi_n(x) = x. \tag{5}$$

Indeed, for every $\varepsilon > 0$ there is $x \in \mathcal{B}_{2^n}$ such that $\|x - x_n\| < \varepsilon$. Since ψ_n is a contraction and $\psi_n(x_n) = x_n$ we have

$$\begin{aligned} \|\psi_n(x) - x\| &\leq \|\psi_n(x) - x_n\| + \|x_n - x\| \\ &= \|\psi_n(x - x_n)\| + \|x_n - x\| \leq 2\varepsilon. \end{aligned}$$

Since $x_n \in \mathcal{B}_{2^n}$ also belong to $\mathcal{B}_{2^m}$ for all $m \geq n$ we have that $\|\psi_m(x) - x\| \leq 2\varepsilon$. Thus $\lim_n \psi_n(x) = x$.

Proof of the Theorem 4. By Lemma 6 and 7 the cone $\overline{C}$ and the unit e_∞ satisfies all assumptions of Theorem 1. Thus there is a homomorphism $\tau : \overline{\mathcal{B}}_\infty \to B(\widetilde{H})$ such that $\tau(a^\sharp) = \tau(a)^*$ for all $a \in \overline{\mathcal{B}}_\infty$. Since the image of τ is a $*$-subalgebra of $B(\widetilde{H})$ we have that τ is bounded by [3, (23.11), p. 81]. The arguments at the end of the proof of Theorem 2 show that the restriction of τ to $\mathcal{B}_{2^n}$ is unitary equivalent to the 2^n-amplification of $\tau|_{\mathcal{B}}$. Thus $\tau|_{\mathcal{B}}$ is completely bounded.

Let prove that $\ker(\tau) = \{0\}$. By Theorem 2.3 it is sufficient to show that $\overline{C} \cap (-\overline{C}) = 0$. If $c, d \in \overline{C}$ such that $c + d = 0$ then $c = d = 0$. Indeed, for every $n \geq 1$, $\psi_n(c) + \psi_n(d) = 0$. By Lemma 8 we have

$$\psi_n(\overline{C}) \subseteq \overline{C} \cap \mathcal{B}_{2^n} = C_{2^n}.$$

Therefore $\psi_n(c)$, $\psi_n(d) \in C_{2^n}$. Hence $\psi_n(c) = -\psi_n(d) \in C_{2^n} \cap (-C_{2^n})$ and, consequently, $\psi_n(c) = \psi_n(d) = 0$. Since $\|\psi_n(c)-c\| \to 0$ and $\|\psi_n(d)-d\| \to 0$ by Remark 9 we have that $c = d = 0$. If $x \in \overline{C} \cap (-\overline{C})$ then $x + (-x) = 0$, $x, -x \in \overline{C}$ and $x = 0$. Thus τ is injective.

We will show that image of τ is closed if one of the conditions (1) of the statement holds.

Assume firstly that operator algebra $\mathcal{B}$ satisfies first condition. Since $\tau(\overline{\mathcal{B}}_\infty) = \tau(\overline{C}) - \tau(\overline{C}) + i(\tau(\overline{C}) - \tau(\overline{C}))$ and $\tau(\overline{C})$ is exactly the set of positive operators in the image of τ it is suffices to prove that $\tau(\overline{C})$ is closed. By Theorem 1.3 for self-adjoint (under involution $\sharp$) elements $x \in \overline{\mathcal{B}}_\infty$ we have

$$\|\tau(x)\|_{B(\widetilde{H})} = \inf\{r > 0 : re_\infty \pm x \in \overline{C}\}.$$

If $\tau(c_\alpha) \in \tau(C)$ is a Cauchy net in $B(\widetilde{H})$ then for every $\varepsilon > 0$, $\varepsilon \pm (c_\alpha - c_\beta) \in \overline{C}$ for all $\alpha \geq \gamma$ and $\beta \geq \gamma$ and some γ. Since $\overline{C} \cap \mathcal{B}_\infty = C$, $\varepsilon \pm (c_\alpha - c_\beta) \in C$. Let us denote $c_{\alpha\beta} = \varepsilon + (c_\alpha - c_\beta)$ and $d_{\alpha\beta} = \varepsilon - (c_\alpha - c_\beta)$. The set of pairs (α, β) is directed if $(\alpha, \beta) \geq (\alpha_1, \beta_1)$ iff $\alpha \geq \alpha_1$ and $\beta \geq \beta_1$. Since $c_{\alpha\beta} + d_{\alpha\beta} = 2\varepsilon$ this net converges to zero in the topology of norm of $\overline{\mathcal{B}}_\infty$. Thus by $(*)$, $\|c_{\alpha\beta}\|_{\overline{\mathcal{B}}_\infty} \to 0$. This implies that c_α is a Cauchy net in $\overline{\mathcal{B}}_\infty$. Let $c = \lim c_\alpha$. Clearly, $c \in \overline{C}$. Since τ is continuous $\|\tau(c_\alpha) - \tau(c)\|_{\overline{\mathcal{B}}_\infty} \to 0$. Hence the closure $\overline{\tau(C)}$ is contained in $\tau(\overline{C})$. Since τ is continuous we have $\tau(\overline{C}) \subseteq \overline{\tau(C)}$. Hence $\overline{\tau(\overline{C})} \subseteq \tau(\overline{C})$ and $\tau(\overline{C})$ is closed.

Let now $\mathcal{B}$ satisfies condition (2) of the Theorem. Then for every $x \in \overline{\mathcal{B}}_\infty$ we have $\|x^\sharp x\| \geq \alpha \|x\| \|x^\sharp\|$. By [3, theorem 34.3] $\overline{\mathcal{B}}_\infty$ admits an equivalent C^*-norm $|\cdot|$. Since τ is a faithful $*$-representation of the C^*-algebra $(\overline{\mathcal{B}}_\infty, |\cdot|)$ it is isometric. Therefore $\tau(\overline{\mathcal{B}}_\infty)$ is closed.

Let us show that $(\tau|_{\mathcal{B}})^{-1} : \tau(\mathcal{B}) \to \mathcal{B}$ is completely bounded. The image $\mathcal{A} = \tau(\overline{\mathcal{B}}_\infty)$ is a C^*-algebra in $B(\widetilde{H})$ isomorphic to $\overline{\mathcal{B}}_\infty$. By Johnson's theorem two Banach algebra norms on a semi-simple algebra are equivalent, hence, $\tau^{-1} : \mathcal{A} \to \overline{\mathcal{B}}_\infty$ is bounded homomorphism, say $\|\tau^{-1}\| = R$. Let us show that $\|(\tau|_{\mathcal{B}})^{-1}\|_{cb} = R$. Since

$$\tau|_{\mathcal{B}_{2^n}} = U_n(\rho \otimes id_{M_{2^n}})U_n^*,$$

where representation $\rho : \mathcal{B} \to B(K)$ is unitary equivalent to $\tau|_{\mathcal{B}}$ and $U_n : K \otimes \mathbb{C}^{2^n} \to \widetilde{H}$ is unitary operator.

We have for any $B = [b_{ij}] \in M_{2^n}(\mathcal{B})$

$$\begin{aligned} \|\sum b_{ij} \otimes E_{ij}\| &\leq R\|U_n(\sum \rho(b_{ij}) \otimes E_{ij})U_n^*\| \\ &= R\|\sum \rho(b_{ij}) \otimes E_{ij}\|. \end{aligned}$$

This is equivalent to

$$\|\sum \rho^{-1}(b_{ij}) \otimes e_{ij}\| \leq R\|\sum b_{ij} \otimes E_{ij}\|,$$

hence $\|(\rho^{-1})_{2^n}(B)\| \leq R\|B\|$. This proves that $\|(\tau|_{\mathcal{B}})^{-1}\|_{cb} = R$. $\square$

4 Operator Algebra associated with Kadison's similarity problem.

In 1955 R. Kadison raised the following problem. Is any bounded homomorphism π of a C^*-algebra $\mathcal{A}$ into $B(H)$ similar to a $*$-representation? The similarity above means that there exists invertible operator $T \in B(H)$ such that $x \to T^{-1}\pi(x)T$ is a $*$-representation of $\mathcal{A}$.

The following criterium due to Haagerup (see [4]) is widely used in reformulations of Kadison's problem: non-degenerate homomorphism π is similar to a $*$-representation iff π is completely bounded. Moreover the similarity S can be chosen such that $\|S^{-1}\|\|S\| = \|\pi\|_{cb}$.

The affirmative answer to the Kadison's problem is obtain in many important cases. In particular, for nuclear $\mathcal{A}$, π is automatically completely bounded with $\|\pi\|_{cb} \leq \|\pi\|^2$ (see [1]).

About recent state of the problem we refer the reader to [8, 5].

We can associate an operator algebra $\pi(B)$ for every bounded homomorphism π of a C^*-algebra $\mathcal{A}$. That fact that $\pi(B)$ is closed can be seen by restricting π to a nuclear C^*-algebra $C^*(x^*x)$. This restriction is similar to $*$-homomorphism for every $x \in \mathcal{A}$ which gives the estimate $\|x\| \leq \|\pi\|^3\|\pi(x)\|$ (for details see [9, p. 4]). Denote $C_n = \pi^{(n)}(M_n(\mathcal{A})^+)$.

Let J be involution in $B(H)$, i.e. self-adjoint operator such that $J^2 = I$. Clearly, J is also a unitary operator. A representation $\pi : \mathcal{A} \to B(H)$ of a $*$-algebra $\mathcal{A}$ is called J-symmetric if $\pi(a^*) = J\pi(a)^*J$. Such representations are natural analogs of $*$-representations for Krein space with indefinite metric $[x, y] = \langle Jx, y\rangle$.

We will need the following observation due to V. Shulman [12] (see also [6, lemma 9.3, p.131]). If π is arbitrary representations of $\mathcal{A}$ in $B(H)$ then the representation $\rho : \mathcal{A} \to B(H \oplus H)$, $a \mapsto \pi(a) \oplus \pi(a^*)^*$ is J-symmetric with $J(x \oplus y) = y \oplus x$ and representation π is a restriction $\rho|_{K\oplus\{0\}}$. Moreover, if ρ is similar to $*$-representation then so is π. Clearly the converse is also true, thus π and ρ are simultaneously similar to $*$-representations or not. In sequel for an operator algebra $\mathcal{D} \in B(H)$ the algebra $\overline{\varinjlim M_{2^n}(\mathcal{D})}$ will denote the closure of the algebraic direct limit of of $M_{2^n}(\mathcal{D})$ in the C^*-algebra direct limit of inductive system $M_{2^n}(B(H))$ with standard inclusions $x \to \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$.

Theorem 10. *If $\pi : \mathcal{A} \to B(H)$ is bounded unital J-symmmetric isomorphism of a C^*-algebra $\mathcal{A}$. Denote $\mathcal{B} = \pi(\mathcal{A})$. Then π^{-1} is completely bounded homomorphism. Its extension $\widetilde{\pi^{-1}}$ to the homomorphism between the inductive limits $\overline{\mathcal{B}_\infty} = \overline{\varinjlim M_{2^n}(\mathcal{B})}$ and $\overline{\mathcal{A}_\infty} = \overline{\varinjlim M_{2^n}(\mathcal{A})}$ is injective.*

Proof. Let us show that $\{C_n\}_{n\geq 1}$ is a $*$-admissible sequence of cones. It is routine to verify that conditions (1)-(3) in the definition of $*$-admissible cones is satisfied for $\{C_n\}$. To see that condition (4) is also satisfied take $B_j \in C_{n_j} - C_{n_j}$ such that $\|B_j\| \leq r$ for some constant $r > 0$. Take $D_j \in M_{n_j}(A)_{sa}$ such that $B_j = \pi^{(n_j)}(D_j)$. Since $\pi^{(n)} : M_n(\mathcal{A}) \to M_n(\mathcal{B})$ is algebraic isomorphism it preserve spectra $\sigma_{M_n(\mathcal{A})}(x) = \sigma_{M_n(\mathcal{B})}(\pi^{(n)}(x))$. Since $\|B_j\| \leq r$ we have the following estimate for the spectral radius $\mathrm{spr}(B_j) \leq r$ and, consequently, $\mathrm{spr}(D_j) \leq r$. Since D_j is self-adjoint $re_{n_j} + D_j \in M_{n_j}(A)^+$. Applying $\pi^{(n_j)}$ we get $re_{n_j} + B_j \in C_{n_j}$.

Since π is J-symmetric $\|\pi^{(n)}(a)\| = \|(J\otimes E_n)\pi^{(n)}(a)^*(J\otimes E_n)\| = \|\pi^{(n)}(a^*)\|$ for every $a \in M_n(\mathcal{A})$ and

$$\begin{aligned}\|\pi^{(n)}(h_1)\| &\leq 1/2(\|\pi^{(n)}(h_1) + i\pi^{(n)}(h_2)\| + \|\pi^{(n)}(h_1) - i\pi^{(n)}(h_2)\|) \\ &= 1/2\|\pi^{(n)}(h_1) - i\pi^{(n)}(h_2)\|\end{aligned}$$

for all $h_1, h_2 \in C_n - C_n$. Thus condition (5) is satisfied and $\{C_n\}$ is $*$-admissible. By Theorem 4 there is an injective bounded homomorphism $\tau : \overline{\mathcal{B}_\infty} \to B(\widetilde{H})$ such that its restriction to $\mathcal{B}$ is completely bounded, $\tau(b^\sharp) = \tau(b)^*$ and $\tau_n(C_n) = \tau_n(M_n(\mathcal{B}))^+$.

Denote $\rho = \tau \circ \pi : \mathcal{A} \to B(\widetilde{H})$. Since ρ is positive homomorphism then it is a $*$-representation. Moreover, $\ker \rho = \{0\}$ because both π and τ are injective. Therefore ρ^{-1} is $*$-isomorphism. Since $\tau : \mathcal{B} \to B(\widetilde{H})$ extends to an injective homomorphism of inductive limit $\overline{\mathcal{B}_\infty}$ and ρ^{-1} is completely isometric we have that $\pi^{-1} = \rho^{-1} \circ \tau$ extends to injective homomorphism of $\overline{\mathcal{B}_\infty}$. It is also clear that π^{-1} is completely bounded as a superposition of two c.b. maps. $\square$

Remark 11. *The fact that π^{-1} is c.b. also follows from [9, Theorem 2.6]*

Remark 12. *Note that condition 1 in Theorem 4 for cones C_n from the proof of theorem 10 is obviously equivalent to π being completely bounded.*

Remark 13. *It can be easily proved that for every free ultrafilter ω on $\mathbb{N}$ the natural extension of $\widetilde{\pi^{-1}}$ to the ultrapowers: $\widetilde{\pi^{-1}}_\omega : (\overline{\mathcal{B}}_\infty)_\omega \to (\overline{\mathcal{A}}_\infty)_\omega$ will necessarily be non-injective for non completely bounded homomorphism π.*

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