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Abstract

We propose an interior penalty discontinuous finite element method for small strain elasto–plasticity using triangular or tetrahedral meshes. A new penalty formulation suitable for plasticity, in particular allowing for inter-element slip, is introduced. The method is also locking free, which is crucial since the plastic zone may exhibit an incompressible response. Numerical results are presented.

1 Introduction

Near incompressibility, typical in elasto-plastic computations, displays severe locking problems when low order standard nodal-based displacement methods are used. One approach to alleviating this problem is to use non-conforming finite element methods with relaxed continuity requirements, as for example the discontinuous Galerkin method of Hansbo and Larson [8]. In this paper, a closely related penalty method for elasto–plastic problems is introduced. The penalty method has the drawback of not being (weakly) consistent, unlike the discontinuous Galerkin method proposed in [8], which means that the condition number has to be degraded in order to retain optimal convergence for higher order methods. This may not be so restrictive in practice, since it is common to use low order finite element methods for elasto–plasticity which can have very non–smooth solutions. Furthermore, the penalty method yields a formulation that is easily extended to any type of plasticity model and also allows for a clear analogy with the mathematical work on slip lines in classical total deformation theory initiated by Temam and Strang [13].

2 A penalty method for linear elasticity

2.1 Problem formulation

We first consider the equations of linear elasticity: Find the displacement $\boldsymbol{u} = [u_i]_{i=1}^n$ and the symmetric stress tensor $\boldsymbol{\sigma} = [\sigma_{ij}]_{i,j=1}^n$ such that

$$\begin{aligned} \boldsymbol{\sigma} = \lambda \operatorname{tr}\boldsymbol{\varepsilon}(\boldsymbol{u})\,\boldsymbol{I} + 2\mu\boldsymbol{\varepsilon}(\boldsymbol{u}) \quad &\text{in} \quad \Omega, \\ -\nabla\cdot\boldsymbol{\sigma} = \boldsymbol{f} \quad &\text{in} \quad \Omega, \\ \boldsymbol{u} = \boldsymbol{g} \quad &\text{on} \quad \partial\Omega_{\mathrm{D}}, \\ \boldsymbol{\sigma}\cdot\boldsymbol{n} = \boldsymbol{h} \quad &\text{on} \quad \partial\Omega_{\mathrm{N}}. \end{aligned} \tag{1}$$

Here Ω is a closed subset of $\mathbb{R}^n$, $n = 2$ or $n = 3$, λ and μ are the Lamé constants, and $\boldsymbol{\varepsilon}(\boldsymbol{u}) = [\varepsilon_{ij}(\boldsymbol{u})]_{i,j=1}^n$ is the strain tensor with components

$$\varepsilon_{ij}(\boldsymbol{u}) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right),$$

with trace

$$\operatorname{tr} \boldsymbol{\varepsilon}(\boldsymbol{u}) = \sum_i \varepsilon_{ii}(\boldsymbol{u}) = \nabla \cdot \boldsymbol{u}.$$

Furthermore, $\nabla \cdot \boldsymbol{\sigma} = \Big[\sum_{j=1}^n \partial\sigma_{ij}/\partial x_j\Big]_{i=1}^n$, $\boldsymbol{I} = [\delta_{ij}]_{i,j=1}^n$ with $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$, $\boldsymbol{f}$ and $\boldsymbol{h}$ are given loads, $\boldsymbol{g}$ is a given boundary displacement, and $\boldsymbol{n}$ is the outward unit normal to $\partial\Omega$.

2.2 Penalty finite element methods

Consider a subdivision of Ω into a geometrically conforming simplicial finite element partitioning $\mathcal{T}^h = \{T\}$ of Ω. Let

$$P^k(T) = \quad \{\boldsymbol{v}\text{: each component of } \boldsymbol{v} \text{ is a polynomial of degree} \leq k \text{ on } T\},$$

$$\boldsymbol{W}^h = \{\boldsymbol{v} \in [L^2(\Omega)]^n : \ \boldsymbol{v}|_T \in P^k(T) \ \ \forall T \in \mathcal{T}^h\},$$

let ∂T_{int} denote the sides of the element T neighboring to other elements, ∂T_{N} the sides neighboring to $\partial\Omega_{\text{N}}$, and ∂T_{D} the sides neighboring to $\partial\Omega_{\text{D}}$. Further, let $\boldsymbol{n}_T$ denote the outward pointing normal to ∂T, and, for $\boldsymbol{x} \in \partial T$, let

$$[\![\boldsymbol{u}]\!] := \boldsymbol{u}^+ - \boldsymbol{u}^-, \quad \text{where} \quad \boldsymbol{u}^\pm := \lim_{\epsilon \downarrow 0} \boldsymbol{u}(\boldsymbol{x} \mp \epsilon\, \boldsymbol{n}_T).$$

A straightforward penalty method for (1) is to seek a function $\boldsymbol{u}^h \in \boldsymbol{W}^h$ such that

$$a_h(\boldsymbol{u}^h, \boldsymbol{v}^h) = L_h(\boldsymbol{v}^h) \quad \forall \boldsymbol{v}^h \in \boldsymbol{W}^h \text{ and } T \in \mathcal{T}^h, \tag{2}$$

where the bilinear form $a_h(\cdot,\cdot)$ is given by

$$a_h(\boldsymbol{u}, \boldsymbol{v}) \ \ := \ \ \sum_T \Big(\int_T \boldsymbol{\sigma}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{v})\, dx + \frac{1}{2}\int_{\partial T_{\text{int}}} \frac{\gamma_0}{h^s} [\![\boldsymbol{u}]\!] \cdot [\![\boldsymbol{v}]\!]\, ds + \int_{\partial T_{\text{D}}} \frac{\gamma_0}{h^s} \boldsymbol{u}^h \cdot \boldsymbol{v}\, ds\Big), \tag{3}$$

where we have used the notation $\boldsymbol{\sigma} : \boldsymbol{\varepsilon} = \sum_i \sum_j \sigma_{ij}\varepsilon_{ij}$, and the linear functional as

$$L_h(\boldsymbol{v}) := \sum_T \Big(\int_T \boldsymbol{f} \cdot \boldsymbol{v}\, dx + \int_{\partial T_{\text{N}}} \boldsymbol{h} \cdot \boldsymbol{v}\, ds + \int_{\partial T_{\text{D}}} \frac{\gamma_0}{h^s} \boldsymbol{g} \cdot \boldsymbol{v}\, ds\Big). \tag{4}$$

For definiteness, we define the mesh parameter h on each face E by

$$h = \begin{cases} \dfrac{\text{meas}(T^+) + \text{meas}(T^-)}{2\,\text{meas}(E)} & \text{for } E \subset \partial T^+ \cap \partial T^-, \\ \text{meas}(T)/\text{meas}(E) & \text{for } E \subset \partial T \cap \partial\Omega. \end{cases} \tag{5}$$

Further, γ_0 is a penalty parameter and $s \geq 1$ is a number that must be chosen relative to the polynomial order of the finite element method (cf. below). Note that in order for the method to scale correctly, the parameter γ_0 must be dependent on the elasticity parameters. Mesh-dependent penalty methods of this type were first proposed by Babuška and Zlámal [2].

The bilinear form given by (3) is not suitable for the extension to elasto–plasticity, however. The reason is that in a plastic zone terms of the type

$$\int_{\partial T_{\text{int}}} \frac{\gamma_0}{h^s} [\![\boldsymbol{u}^h]\!] \cdot [\![\boldsymbol{v}^h]\!]\, ds \tag{6}$$

(used also for plasticity in [6]) are too strong: only normal continuity can be expected in general, but (6) is penalizing also the tangential interelement displacement. One way to avoid this problem is to split the displacement into a normal and a tangential part, and penalizing the different parts differently. This approach was used in the discontinuous Galerkin method of Hansbo and Larson [8], but it is still not the natural penalty in the setting of elasto–plasticity. In particular, it is difficult to define how to project a tangential stress onto a yield surface in a way consistent with the stress projection in the interior of an element. This problem is also present in standard Lagrange multiplier methods, where the multiplier is typically interpreted as the traction vector.

For the purpose of modeling plasticity also on the interfaces between elements, we propose the following modification of the penalty term. Note first that for an arbitrary symmetric tensor $\boldsymbol{\tau}$ and an arbitrary displacement $\boldsymbol{v}$ there holds

$$\boldsymbol{\tau}\cdot\boldsymbol{n}_T\cdot\boldsymbol{v}\equiv\boldsymbol{\tau}:\mathbf{E}(\boldsymbol{v};\boldsymbol{n}_T), \tag{7}$$

where

$$\mathbf{E}(\boldsymbol{v};\boldsymbol{n}):=\frac{1}{2}\left(\boldsymbol{n}\otimes\boldsymbol{v}+\boldsymbol{v}\otimes\boldsymbol{n}\right)$$

is a strain-like tensor with components

$$\mathrm{E}_{ij}(\boldsymbol{v};\boldsymbol{n})=\frac{1}{2}\left(v_in_j+v_jn_i\right).$$

In the following, we shall suppress the dependence on $\boldsymbol{n}_T$ and write $\mathbf{E}(\boldsymbol{v})$ instead of $\mathbf{E}(\boldsymbol{v};\boldsymbol{n}_T)$, since the dependence on $\boldsymbol{n}_T$ will be clear from the context.

Now, multiplying (1) with a function $\boldsymbol{v}^h\in\boldsymbol{W}^h$ with support only on (for simplicity, an internal) T, applying integration by parts and using (7) gives us the relation

$$\int_T\boldsymbol{\sigma}(\boldsymbol{u}):\boldsymbol{\varepsilon}(\boldsymbol{v}^h)\,dx-\int_{\partial T}\boldsymbol{\sigma}(\boldsymbol{u}):\mathbf{E}(\boldsymbol{v}^h)\,ds=\int_T\boldsymbol{f}\cdot\boldsymbol{v}^h\,dx. \tag{8}$$

For numerical modeling purposes we are thus looking for a "penalty stress" $\boldsymbol{\Sigma}$ such that for $\boldsymbol{x}\in\partial T$

$$\boldsymbol{\Sigma}([\![\boldsymbol{u}^h(\boldsymbol{x})]\!])\approx-\boldsymbol{\sigma}(\boldsymbol{u}(\boldsymbol{x})),$$

and a natural choice is to relate $\boldsymbol{\Sigma}$ to $\mathbf{E}$ in the same way that $\boldsymbol{\sigma}$ is related to $\boldsymbol{\varepsilon}$, i.e.,

$$\boldsymbol{\Sigma}(\boldsymbol{u}):=\frac{\gamma}{h^s}\left(\lambda\ \left(\operatorname{tr}\mathbf{E}(\boldsymbol{u})\right)\boldsymbol{I}+2\mu\,\mathbf{E}(\boldsymbol{u})\right), \tag{9}$$

where we note in particular that

$$\operatorname{tr}\mathbf{E}(\boldsymbol{u})=\boldsymbol{n}_T\cdot\boldsymbol{u}, \tag{10}$$

and that the dimensionless number γ (which controls the size of the discontinuity) can be chosen independently of the elasticity parameters.

This suggests to modify the penalty term in (3) and seek $\boldsymbol{u}^h\in\boldsymbol{W}^h$ such that

$$a_h(\boldsymbol{u}^h,\boldsymbol{v}^h;\boldsymbol{g})=f_h(\boldsymbol{v}^h)\quad\forall\boldsymbol{v}^h\in\boldsymbol{W}^h\text{ and }T\in\mathcal{T}^h, \tag{11}$$

where

$$\begin{aligned}a_h(\boldsymbol{u},\boldsymbol{v};\boldsymbol{g})\quad:=\quad&\sum_T\Big(\int_T\boldsymbol{\sigma}(\boldsymbol{u}):\boldsymbol{\varepsilon}(\boldsymbol{v})\,dx+\frac{1}{2}\int_{\partial T_{\mathrm{int}}}\boldsymbol{\Sigma}([\![\boldsymbol{u}]\!]):\mathbf{E}([\![\boldsymbol{v}]\!])\,ds \qquad (12)\\&+\int_{\partial T_{\mathrm{D}}}\boldsymbol{\Sigma}(\boldsymbol{u}-\boldsymbol{g}):\mathbf{E}(\boldsymbol{v})\,ds\Big),\end{aligned}$$

and

$$f_h(\boldsymbol{v}):=\sum_T\Big(\int_T\boldsymbol{f}\cdot\boldsymbol{v}\,dx+\int_{\partial T_{\mathrm{N}}}\boldsymbol{h}\cdot\boldsymbol{v}\,ds\Big). \tag{13}$$

The reason for putting the Dirichlet data $\boldsymbol{g}$ into the bilinear form in (11) will become clear when we extend the method to elasto–plasticity.

This method is stable (in that it precludes rigid body motions) as long as $s\geq1$ and the Lamé parameter μ is bounded from below, since the jump $[\![\boldsymbol{u}^h]\!]$ is then controlled with sufficient strength, also as $h\to0$, cf. [4].

Remark 1 *Note that if $\mathbf{\Sigma}$ is interpreted as a discrete stress, then it is obvious that $s > 1$ scales incorrectly. Only by division by the meshsize (i.e., when $s = 1$) can $\mathbf{\Sigma}$ be interpreted as a discrete derivative. This scaling problem will adversely affect the conditioning of the discrete system for $s > 1$. On the other hand, the choice $s = 1$ leads to a consistency error that will degrade the accuracy of the method for higher order polynomial approximation. The method proposed here is thus best suited for low order elements. We refer to the review paper of Arnold et al. [1] for a deeper analysis.*

Remark 2 *An important feature for plasticity computations is the ability to handle incompressible behaviour, $\lambda/\mu \to \infty$, since in plasticity the load must typically be carried by the trace part of the stress. Like the discontinuous Galerkin method proposed in [8], the current method is locking free with respect to (near) incompressibility. The argument to support this assertion is briefly as follows: $\boldsymbol{W}^h$ is large enough to incorporate $H(\mathrm{div})$-conforming (or nonconforming) approximations (cf. [8, 9]), which ensures that the element can have $\nabla \cdot \boldsymbol{u}^h \equiv 0$ elementwise and still retain approximation properties. Numerical examples showing the robustness in this respect are given in Section 4.*

3 An elasto-plastic model problem

We shall consider the following isotropic von Mises model of elastoplasticity.

$$\begin{aligned}
\dot{\boldsymbol{\varepsilon}} &= \dot{\boldsymbol{\varepsilon}}^e + \dot{\boldsymbol{\varepsilon}}^p, \\
\dot{\boldsymbol{\varepsilon}}^p &= \dot{\vartheta}\boldsymbol{\sigma}^{\mathrm{D}}, \\
\dot{\boldsymbol{\sigma}} &= \lambda \,\mathrm{tr}\, \dot{\boldsymbol{\varepsilon}}^e \boldsymbol{I}, + 2\mu\, \dot{\boldsymbol{\varepsilon}}^e, \\
\phi &:= \boldsymbol{\sigma}^{\mathrm{D}} : \boldsymbol{\sigma}^{\mathrm{D}} - \frac{2}{3}\sigma_{\mathrm{Y}}^2 \leq 0.
\end{aligned} \tag{14}$$

Here,

$$\boldsymbol{\sigma}^{\mathrm{D}} := \boldsymbol{\sigma} - \frac{1}{3}\mathrm{tr}\, \boldsymbol{\sigma}\, \boldsymbol{I}$$

is the stress deviator, σ_{Y} is the yield stress, $\dot{\boldsymbol{\varepsilon}}^e$ and $\dot{\boldsymbol{\varepsilon}}^p$ are the elastic and plastic strain rates, respectively, $\dot{\vartheta}$ is the plastic multiplier, and ϕ is the yield function. In this model it is thus assumed that the stresses must reside in a convex elastic domain E in stress space defined by ϕ:

$$E = \{\boldsymbol{\sigma} : \quad \phi(\boldsymbol{\sigma}) \leq 0\}.$$

Since we are now dealing with a time-dependent problem, we will in the following let $\boldsymbol{u}$ denote the *displacement velocity* and thus we have

$$\dot{\varepsilon}_{ij} := \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right).$$

Discretizing in space, and in time by use of the backward Euler method, by letting $\boldsymbol{u}_n^h \approx \boldsymbol{u}(t_n, \cdot)$ and

$$\boldsymbol{\sigma}_n := \boldsymbol{\sigma}(\boldsymbol{u}_n^h), \quad \frac{\boldsymbol{\sigma}_n - \boldsymbol{\sigma}_{n-1}}{k_n} = \lambda \,\mathrm{tr}\, \dot{\boldsymbol{\varepsilon}}_n^e \boldsymbol{I}, + 2\mu\, \dot{\boldsymbol{\varepsilon}}_n^e,$$

where $k_n = t_n - t_{n-1}$, and denoting $|\boldsymbol{\tau}| := (\boldsymbol{\tau} : \boldsymbol{\tau})^{1/2}$, we can ensure the fulfillment of the yield condition by defining an elastic trial stress

$$\tilde{\boldsymbol{\sigma}}_n := \boldsymbol{\sigma}_{n-1} + k_n \left(\lambda \,\mathrm{tr}\, \boldsymbol{\varepsilon}(\boldsymbol{u}_n^h)\, \boldsymbol{I} + 2\mu \boldsymbol{\varepsilon}(\boldsymbol{u}_n^h)\right) \tag{15}$$

and computing the plastically admissible stress by

$$\boldsymbol{\sigma}(\boldsymbol{u}_n^h) = \begin{cases} \tilde{\boldsymbol{\sigma}}_n^{\mathrm{D}} + \frac{1}{3}\mathrm{tr}\,\tilde{\boldsymbol{\sigma}}_n\,\boldsymbol{I}, & \text{if } |\tilde{\boldsymbol{\sigma}}_n^{\mathrm{D}}| < \sqrt{\frac{2}{3}}\sigma_{\mathrm{y}}, \\ \frac{\sqrt{\frac{2}{3}}\sigma_{\mathrm{y}}}{|\tilde{\boldsymbol{\sigma}}_n^{\mathrm{D}}|}\tilde{\boldsymbol{\sigma}}_n^{\mathrm{D}} + \frac{1}{3}\mathrm{tr}\,\tilde{\boldsymbol{\sigma}}_n\,\boldsymbol{I}, & \text{if } |\tilde{\boldsymbol{\sigma}}_n^{\mathrm{D}}| \geq \sqrt{\frac{2}{3}}\sigma_{\mathrm{y}}. \end{cases} \tag{16}$$

We write this more compactly as

$$\boldsymbol{\sigma}(\boldsymbol{u}_n^h) = \Pi\tilde{\boldsymbol{\sigma}}_n^{\mathrm{D}} + \frac{1}{3}\mathrm{tr}\,\tilde{\boldsymbol{\sigma}}_n \tag{17}$$

where

$$\Pi\boldsymbol{\sigma}^{\mathrm{D}} = \begin{cases} \boldsymbol{\sigma}^{\mathrm{D}}, & \text{if } |\boldsymbol{\sigma}^{\mathrm{D}}| < \sqrt{\frac{2}{3}}\sigma_{\mathrm{y}}, \\ \frac{\sqrt{\frac{2}{3}}\sigma_{\mathrm{y}}}{|\boldsymbol{\sigma}^{\mathrm{D}}|}\boldsymbol{\sigma}^{\mathrm{D}}, & \text{if } |\boldsymbol{\sigma}^{\mathrm{D}}| \geq \sqrt{\frac{2}{3}}\sigma_{\mathrm{y}}. \end{cases} \tag{18}$$

Since we use $\boldsymbol{\Sigma}$ as an independent model of the stress on the interfaces between elements, $\boldsymbol{\Sigma}$ must be projected onto the yield surface in the same way. To this end, we define a trial stress

$$\tilde{\boldsymbol{\Sigma}}_n = \boldsymbol{\Sigma}_{n-1} + k_n\frac{\gamma}{h^s}\left(\lambda\,\mathrm{tr}\,\mathbf{E}(\boldsymbol{u}_n^h), \boldsymbol{I} + 2\mu\,\mathbf{E}(\boldsymbol{u}_n^h)\right) \tag{19}$$

followed by projection so that

$$\boldsymbol{\Sigma}(\boldsymbol{u}_n^h) := \Pi\tilde{\boldsymbol{\Sigma}}_n^{\mathrm{D}} + \frac{1}{3}\mathrm{tr}\,\tilde{\boldsymbol{\Sigma}}_n \in E. \tag{20}$$

We note in particular that since the trace of $\tilde{\boldsymbol{\Sigma}}_n$ is not affected by the projection, normal continuity is always enforced by penalty (due to (10)), whereas tangential sliding will not be penalized beyond the plastic limit.

We can now define the following radial return method: for $n = 1, 2, \ldots$, seek $\boldsymbol{u}_n^h \in \boldsymbol{W}^h$ such that

$$\begin{aligned} \sum_T \Big(\int_T \boldsymbol{\sigma}(\boldsymbol{u}_n^h) : \boldsymbol{\varepsilon}(\boldsymbol{v}^h)\,dx + \frac{1}{2}\int_{\partial T_{\mathrm{int}}} \boldsymbol{\Sigma}([\![\boldsymbol{u}_n^h]\!]) : \mathbf{E}([\![\boldsymbol{v}^h]\!])\,ds \\ + \int_{\partial T_{\mathrm{D}}} \boldsymbol{\Sigma}(\boldsymbol{u}_n^h - \boldsymbol{g}_n) : \mathbf{E}(\boldsymbol{v}^h)\,ds\Big) = \sum_T \Big(\int_T \boldsymbol{f}_n \cdot \boldsymbol{v}^h\,dx + \int_{\partial T_{\mathrm{N}}} \boldsymbol{h}_n \cdot \boldsymbol{v}^h\,ds\Big) \end{aligned} \tag{21}$$

for all $\boldsymbol{v}^h \in \boldsymbol{W}^h$, where $\boldsymbol{\sigma}(\cdot)$ and $\boldsymbol{\Sigma}(\cdot)$ are given by (17) and (20), respectively.

Remark 3 *There is a close relation between the penalty formulation used here and previous mathematical work on discontinuous FEM for plasticity [3, 10, 12, 13]. In these papers, minimization of the complementary energy was considered, with the position of a line of discontinuity either given or as a part of the minimization problem. In the complementary energy functional, the jump terms in (21) are then also present, but in the continuous formulation and thus without mesh dependence (corresponding to letting $\gamma \to \infty$). This relation to interior penalty methods is also explicitly pointed out in the concluding remarks of [3].*

4 Numerical example

We consider a domain $(0, 3/4) \times (0, 3/4) \setminus (0, 1/4) \times (0, 1/4)$ meters in a state of plane strain. The material data are $E = 100$ GPa, $\nu = 0.3$, $\sigma_{\mathrm{y}} = 700$ MPa. The domain is fixed at the bottom and fixed horizontally at the top and pulled a distance $\delta = 1$ cm upwards. The penalty parameter was set to $\gamma = 10$. The problem was solved using one timestep (i.e., as a Hencky problem).

In Figure 1 we show a sequence of adapted meshes (the adaptive algorithm being based on a stress projection scheme, cf. [5]). Note the pronounced slip at the lower right corner and at the inward pointing corner. In Figures 2 and 3 we show the plastic zone and the edges where plastic slip may occur. Note the absence of plastic edges in a part of the plastic zone; we interpret this as an effect of the orientation of the mesh in that the edges are not well aligned with slip lines (cf. the top right edge in Figure 3 where slip lines should go from top right to bottom left but can not due to mesh orientation). This type of information is not achieved directly from a standard finite element simulation. We also remark that no slip line failure seems to occur in this plane strain problem (cf. the results of [11]).

5 Concluding remarks

The method proposed has two important properties: firstly, it allows for (near) incompressibility which can be important in elastoplastic simulations; secondly, it allows for interelement slip. Allowing for slip seems completely natural in the context of plasticity simulations, but in order to take full advantage of this property the elements have to be aligned along preferred directions of slip, which may not be so easy to achieve in practice. Another possibility is to allow for discontinuities independent of the mesh, as for example in [7]. In the work on discontinuous FEM for slip lines by Stephan and Temam [12] (using a remeshing technique), the position of a slip line was found through global minimization arguments. This can hardly be the way an actual slip line forms, and an approach similar to that of tracking crack paths (as in [7]), for instance based on the eigenvalues of the acoustic tensor as in [11], seems more natural.

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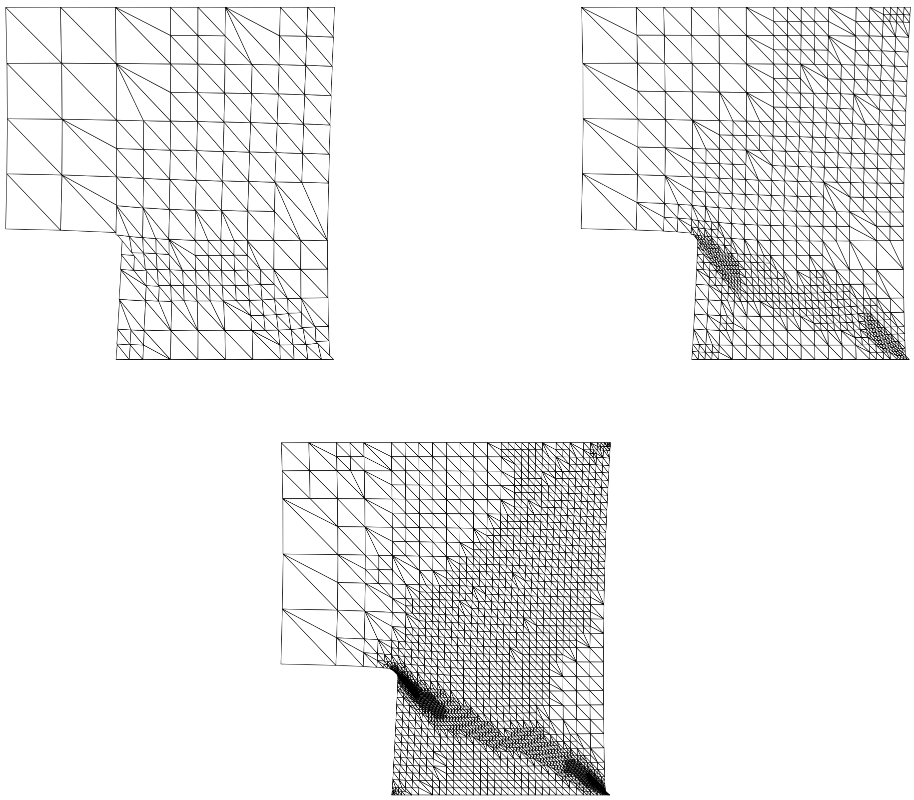

Figure 1: Adapted meshes after 3, 5 and 7 refinement levels

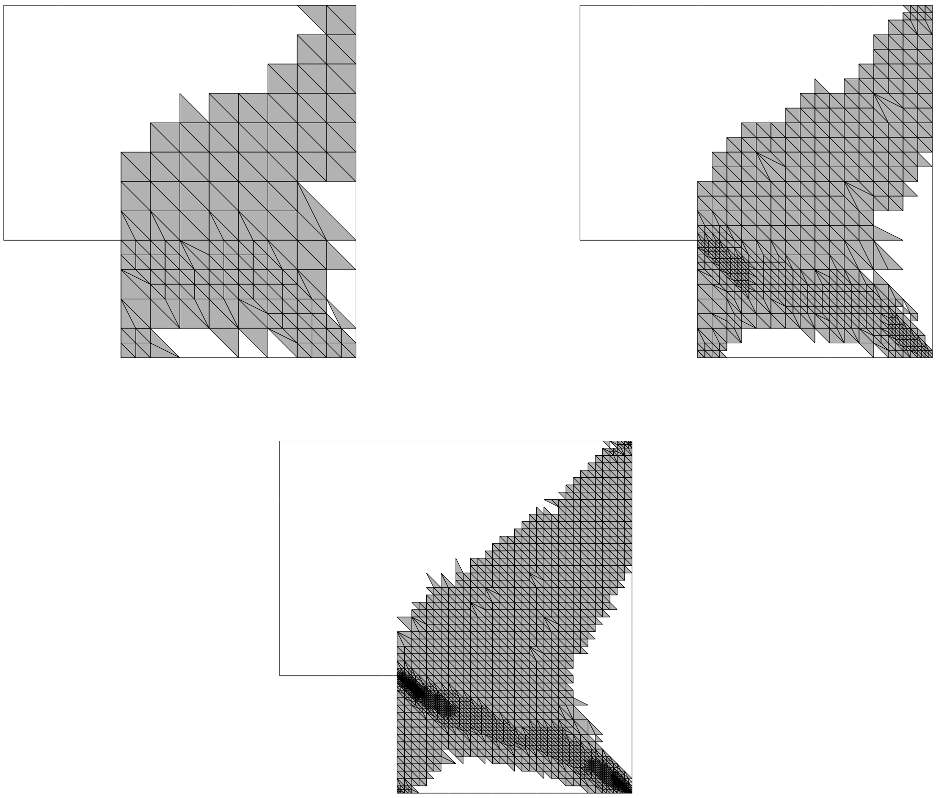

Figure 2: Plastic elements after 3, 5 and 7 refinement levels

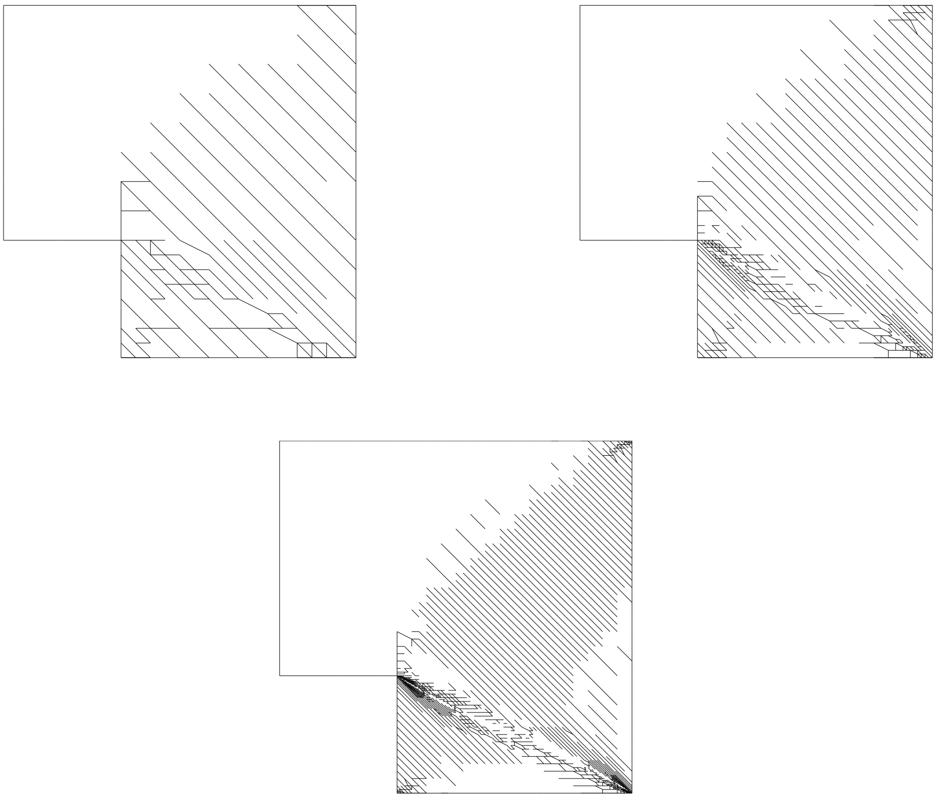

Figure 3: Plastic edges after 3, 5 and 7 refinement levels