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Abstract Suppose E is a real, separable Banach space and for each $x \in E$ denote by $\mathrm{co}\{0,(1,x)\}$ the line segment joining the two points 0 and $(1,x)$ in $\mathbf{R} \times E$. The aim of this paper is to discuss the strong law of large numbers and the central limit theorem for the random line segment $\mathrm{co}\{0,(1,a+X)\}$ when X is a centred Gaussian random vector in E and $a \in E$. Finally, an application to mathematical finance is given.

1 Introduction

In this paper we will draw attention to a class of zonoids connected with Gauss measure on Banach space. Here the Mosler book [8] and the Koshevoy and Mosler article [5] treat the finite-dimensional case and give the corresponding credits. Moreover, we will show how a certain zonoid induced by Brownian motion can be used in the sensitivity analysis of the average excess return with respect to the risk aversion parameter in the mean-variance approach to optimal portfolio selections. As far as we know this is the first instance of such an application of zonoids.

Let $I = \varphi \circ \Phi^{-1}$ be the Gaussian isoperimetric function, where

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, -\infty \leq x \leq \infty$$

is the density function of a standard Gaussian random variable with the cumulative distribution function

$$\Phi(x) = \int_{-\infty}^{x} \varphi(y)dy, -\infty \leq x \leq \infty.$$

Moreover, throughout the paper, if not stated otherwise, $E=(E,\|\cdot\|_E)$ is a real, separable Banach space, a a fixed vector in E, and γ a centred Gaussian measure on E with the reproducing kernel Hilbert space $H_\gamma=(H_\gamma,\|\cdot\|_{H_\gamma})$. Set $F=\mathbf{R}\times E$ and define

$$Z(a,\gamma)=\left\{(r,x)\in F;\ 0\leq r\leq 1,\ x\in ra+H_\gamma,\ \text{and}\ \|\ x-ra\ \|_{H_\gamma}\leq I(r)\right\}.$$

The function $x=I(r+\frac{1}{2})$, $-\frac{1}{2}\leq r\leq\frac{1}{2}$, is concave and even and it follows that the set $Z(0,\gamma)$ is convex and symmetric with respect to the hyperplane $r=\frac{1}{2}$ in F. The book [8] submits 20 different figures of the set $Z(a,\gamma)$ for various choices of a and γ when E is of dimension 1 or 2. If $\gamma_{\mathbf{R}}$ is the standard Gaussian measure on $\mathbf{R}$ we write $Z(0,\gamma_{\mathbf{R}})=Z$ and have

$$Z=\left\{(r,x)\in\mathbf{R}^2;\ \ |\ x\ |\leq I(r)\,,\ 0\leq r\leq 1\ \right\}.$$

In the following, if B is a real, separable Banach space and $u,v\in B$ we denote by $\mathrm{co}\{u,v\}$ the line segment joining u and v, and by δ_B the Hausdorff metric on the class of all non-empty compact convex subsets of B. A finite (Minkowski) sum of line segments in B is called a zonotope and a limit of such sets in the metric δ_B is called a zonoid. A zonoid in a Banach space is necessarily compact.

Throughout this paper X denotes a random vector in E with probability law γ. In Section 4 we show that

$$E\left[\mathrm{co}\left\{0,(1,a+X)\right\}\right]=\int_E \mathrm{co}\left\{0,(1,a+x)\right\}d\gamma(x)=Z(a,\gamma). \tag{1.1}$$

The expectation and integral in (1.1) may be evaluated either in the sense of Aumann or Debreu (see Molchanov [7] , Chapter 2).

Next suppose $(X_i)_{i\in\mathbb{N}_+}$ is an i.i.d. in E where the probability law of each X_i equals γ for each $i\in\mathbb{N}_+$. By applying the strong law of large numbers (SLLN) for random sets in Banach space it follows that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\mathrm{co}\left\{0,(1,a+X_i)\right\}=Z(a,\gamma)\ \text{a.s. } P, \tag{1.2}$$

where the convergence is with respect to the Hausdorff metric δ_F. The relation (1.2) implies that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\mathrm{co}\left\{0,X_i\right\}=\frac{1}{\sqrt{2\pi}}\ O_\gamma\ \text{a.s. } P,$$

in the Hausdorff metric δ_E, where O_γ denotes the closed unit ball in H_γ. Thus $Z(a,\gamma)$ and O_γ are zonoids in F and E, respectively. As in the finite-dimensional case the set $Z(a,\gamma)$ will be called a lift zonoid induced by γ or X.

In Section 6 it will be proved that the sequence

$$\sqrt{n}\delta_F\left(\frac{1}{n}\sum_{i=1}^{n}\mathrm{co}\left\{0,(1,a+X_i)\right\},Z(a,\gamma)\right),\ n\in\mathbf{N}_+,$$

converges in distribution to the supremum of the absolute value of an appropriate centred Gaussian process. The proof of this result depends on a very deep regularity theorem for Gaussian processes due to Talagrand (see Ledoux and Talagrand [6] , Theorem 12.9).

In the last section of this paper an application of the relation (1.2) to mathematical finance is given. More specifically, we will derive different bounds on the average excess return in the classical Markowitz mean-variance approach to optimal portfolio selections when the utility function is time dependent. There is only one single stock (or fund) in the model and we suppose the stock and bond price processes are Itô processes with simple integrands and, in addition, the instantaneous Sharpe ratio of the stock price is constant. In the end of the paper, a numerical illustration of the inequalities obtained so far shows that fairly small variations of the risk aversion parameter from one time period to another may cause rather great changes in the average excess return over different periods of time.

Acknowledgement I am most indepted to Professor Bo Berntsson for introducing me to the Gini index of an income distribution, which was the gateway to this article.

2 A more detailed description of some basic concepts

Suppose $B=(B,\|\cdot\|_B)$ is a real, separable Banach space with the topological dual B' and metric $d_B(u,v)=\| u-v \|_B$. The class of all non-empty compact convex subsets of B is denoted by $\mathcal{K}(B)$. Given $K, L \in \mathcal{K}(B)$ the Hausdorff distance from K to L is defined by

$$\delta_B(K,L) = \max(\max_{u\in K}\min_{v\in L} d_B(u,v), \max_{v\in L}\min_{u\in K} d_B(u,v))$$

and, as usual, the Minkowski sum $K+L$ of K and L is given by

$$K + L = \{x + y;\ x \in K \text{ and } y \in L\}.$$

Moreover, $\alpha K = \{\alpha x;\ x \in K\}$ if α is a real number. The support function of K is denoted by $p(K,\cdot)$, that is for every $\tau \in B'$,

$$p(K,\tau) = \max_{u\in K} < u, \tau >$$

and from this

$$p(\alpha K + \beta L;\tau) = \alpha p(K;\tau) + \beta p(L;\tau) \text{ if } \alpha, \beta \geq 0.$$

Let T be the closed unit ball in B' equipped with the weak*-topology on T. A particular case of the Hörmander embedding theorem says that the map

$$K \to p(K,\cdot)_{|T}$$

of $\mathcal{K}(B)$ into the separable Banach space $C(T)$ is isometric

$$\delta_B(K,L) = \| p(K,\cdot)_{|T} - p(L,\cdot)_{|T} \|_{C(T)}$$

(see Giné, Hahn, and Zinn [4] for the case considered here).

Next let $K : \Omega \to \mathcal{K}(B)$ be a random set such that

$$\int_\Omega \sup_{u\in K} \| u \|_B \, dP < \infty.$$

The expectation

$$E[K] = \int_\Omega K dP$$

is the member of $\mathcal{K}(B)$ characterized by the equation

$$p(E[K], \tau) = E[p(K,\tau)], \text{ all } \tau \in B'$$

(see [7], p 157). Thus, if C is another real, separable Banach space and $T{:}B \to C$ a bounded linear operator, then $T(E[K]) = E[T(K)]$. Moreover, for any sequence $(K_i)_{i\in\mathbf{N}_+}$ of independent observations on K, the SLLN for the class $\mathcal{K}(B)$ says that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}K_i = E\left[K\right] \text{ a.s. } P,$$

where the convergence is with respect to the Hausdorff metric δ_B (see Puri and Ralescu [9] or [4]; the SLLN for the class of all non-empty compact sets in B due to Taylor and Inoue [11] is not needed in this paper).

3 The support function of Z

A familiar theorem by Bolker [1] states that any compact and centrally symmetric convex subset of the plane is a zonoid (for more information see the Schneider book [10]). The relation (1.2) in the real-valued case also shows that Z is a zonoid and the support function of Z is immediate. Here we will give an independent straight-forward derivation of the support function of Z.

As said above the function $x = I(r+\frac{1}{2})$, $-\frac{1}{2}\le r\le\frac{1}{2}$, is even and to simplify the notation it is natural to introduce the set

$$Z^0 = Z - (\frac{1}{2},0) = \left\{(r,x)\in\mathbf{R}^2;\ \mid x\mid\le I(r+\frac{1}{2}),\ \ \mid r\mid\le\frac{1}{2}\ \right\}$$

which is symmetric about both coordinate axes.

Theorem 3.1 *For any* $(\varrho,\xi)\in\mathbf{R}^2$,

$$p_{Z^0}(\xi,\eta) = \frac{1}{2}\int_{-\infty}^{\infty}\mid\varrho + x\xi\mid\varphi(x)dx \tag{3.1}$$

and

$$p_Z(\xi,\eta) = \int_{-\infty}^{\infty}(\varrho + x\xi)^{+}\varphi(x)dx. \tag{3.2}$$

Moreover, if $(\varrho,\xi)\neq 0$,

$$p_{Z^0}(\varrho,\xi)=\mid\xi\mid\varphi(\frac{\varrho}{\xi})+\mid\varrho\mid\Phi(\mid\frac{\varrho}{\xi}\mid)-\frac{1}{2}\mid\varrho\mid \tag{3.3}$$

a*nd*

$$p_Z(\varrho,\xi)=\mid\xi\mid\varphi(\frac{\varrho}{\xi})+\mid\varrho\mid\Phi(\mid\frac{\varrho}{\xi}\mid)+\min(0,\varrho). \tag{3.4}$$

Proof. First (3.4) follows from (3.3) since $p_Z(\varrho,\xi)=p_{Z^0}(\varrho,\xi)+\frac{1}{2}\varrho$. Using the same formula, (3.1) implies (3.2) since φ is an even function and $a^+=\frac{1}{2}(\mid a\mid+a)$ for all real numbers a. Thus we only have to prove the equations (3.1) and (3.3).

To prove (3.3), since $p_{Z^0}(\varrho,\xi)=p_{Z^0}(\pm\varrho,\pm\xi)$ and p_{Z^0} is positively homogeneous of degree one, there is no loss of generality in assuming $\varrho,\xi>0$, and $\varrho^2+\xi^2=1$.

Suppose the point (r_0,x_0) is located on the graph of the curve $x=I(r+\frac{1}{2})$ and that $(-\xi,\varrho)$ is a tangent vector at this point. Then

$$I'(r_0+\frac{1}{2})=-\frac{\varrho}{\xi}$$

and, since $I'(r)=-\Phi^{-1}(r)$, we have

$$r_0=-\frac{1}{2}+\Phi(\frac{\varrho}{\xi})$$

and

$$x_0=I(r_0+\frac{1}{2})=\varphi(\frac{\varrho}{\xi}).$$

Accordingly from these equations, the tangent line of the graph of the curve $x=I(r+\frac{1}{2})$ at the point (r_0,x_0) equals

$$x=\varphi(\frac{\varrho}{\xi})-\frac{\varrho}{\xi}(r+\frac{1}{2}-\Phi(\frac{\varrho}{\xi}))$$

and it intersects the line

$$x=\frac{\xi}{\varrho}r$$

at the point (r_1, x_1), where

$$(\frac{\xi}{\varrho}+\frac{\varrho}{\xi})r_1=\varphi(\frac{\varrho}{\xi})-\frac{\varrho}{\xi}(\frac{1}{2}-\Phi(\frac{\varrho}{\xi})).$$

Thus

$$r_1=\varrho\left\{\xi\varphi(\frac{\varrho}{\xi})+\varrho\Phi(\frac{\varrho}{\xi})-\frac{1}{2}\varrho\right\}$$

and, since $r_1>0$,

$$g_{Z^0}(\varrho,\xi)=\sqrt{r_1^2+x_1^2}=\frac{r_1}{\varrho}=\xi\varphi(\frac{\varrho}{\xi})+\varrho\Phi(\frac{\varrho}{\xi})-\frac{1}{2}\varrho.$$

This proves (3.3).

For completeness we also check (3.1) for $\xi,\eta>0$. Now

$$\frac{1}{2}\int_{-\infty}^{\infty}\mid\varrho+x\xi\mid\varphi(x)dx$$

$$=\tfrac{1}{2}\left\{\int_{x>-\frac{\varrho}{\xi}}(\varrho+x\xi)\varphi(x)dx-\int_{x\le-\frac{\varrho}{\xi}}(\varrho+x\xi)\varphi(x)dx\right\}$$

$$=\frac{1}{2}\left\{\varrho(1-\Phi(-\frac{\varrho}{\xi}))+\xi\varphi(\frac{\varrho}{\xi})-\varrho\Phi(-\frac{\varrho}{\xi}))+\xi\varphi(\frac{\varrho}{\xi})\right\}$$

$$=\xi\varphi(\frac{\varrho}{\xi})+\varrho\Phi(\frac{\varrho}{\xi})-\frac{1}{2}\varrho$$

where we used the identity $1-\Phi(-a)=\Phi(a)$ to get the last equality. Now (3.3) proves (3.1) and completes the proof of Theorem 3.1.

4 The SLLN for $\mathrm{co}\{0,(1,a+X)\}$

We abide by the notation in the Introduction and let X denote an E-valued random vector with the probability law γ. The notation that follows is close to my paper [2] .

The closure of E' in $L^2(\gamma)$ is denoted by E'_γ. There is an injective map $\Gamma : E'_\gamma \to E$ defined by

$$\Gamma(\eta) = \int_E x\eta(x)d\gamma(x)$$

that is $\Gamma(\eta)$ is the unique element in E such that

$$\xi(\Gamma(\eta)) = \int_E \xi(x)\eta(x)d\gamma(x)$$

for every $\xi \in E'$. Set $H_\gamma = \Gamma(E'_\gamma)$ and $\tilde{h} = \Gamma^{-1}h$ if $h \in H_\gamma$. The vector space H_γ equipped with the inner product

$$< h, k >_{H_\gamma} = < \tilde{h}, \tilde{k} >_{L^2(\gamma)}$$

is a Hilbert space, the so called reproducing kernel Hilbert space of γ. We let $\| h \|_{H_\gamma} = \sqrt{< h, h >_{H_\gamma}}$ and denote by $O_\gamma = \left\{ h \in H_\gamma ; \| h \|_{H_\gamma} \leq 1 \right\}$ the closed unit ball of H_γ. The set O_γ is a compact subset of the Banach space E and the closure of H_γ in E equals the topological support of γ. The vector space H_γ is a γ-null set if H_γ is of infinite dimension.

For any set $A \subseteq F$ and $r \in \mathbf{R}$, the r-section A_r of A is given by $A_r = \{x \in E;\ (r, x) \in A\}$. In particular, the lift zonoid $Z(a, \gamma)$ introduced in the Introduction possesses the r-sections

$$Z(a, \gamma)_r = ra + I(r)O_\gamma,\ 0 \leq r \leq 1.$$

Example 4.1 If γ denotes the distribution measure of real-valued Brownian motion in the unit interval, that is γ is Wiener measure on $C([0, 1])$, then for each $0 \leq r \leq 1$, the r-section $Z(a, \gamma)_r$ equals the set of all absolutely continuous functions $x : [0, 1] \to \mathbf{R}^d$ such that $x(0) = ra(0)$ and

$$\sqrt{\int_0^1 (x(t) - ra(t))^2 dt} \leq I(r),\ 0 \leq r \leq 1.$$

Note that $\gamma(Z(a, \gamma)_r) = 0$ since $\gamma(Z(a, \gamma)_r) = \gamma(ra + I(r)O_\gamma) \leq \gamma(I(r)O_\gamma)$ by the Anderson inequality. □

Example 4.2 We claim that

$$E\left[\operatorname{co}\{0, X\}\right] = \frac{1}{\sqrt{2\pi}} O_\gamma. \tag{4.1}$$

Actually this formula is a corollary to Theorem 4.1 below but a direct derivation of the equation (4.1) may be of some independent interest. First for any $\xi \in E'$,

$$p(E\left[\operatorname{co}\{0, X\}\right], \xi) = E\left[p(\operatorname{co}\{0, X\}, \xi)\right]$$
$$= E\left[\xi(X)^+\right] = \parallel \xi \parallel_{L^2(\gamma)} E\left[G^+\right]$$

where $G \in N(0,1)$. But

$$E\left[G^+\right] = \int_{-\infty}^{\infty} t^+ \varphi(t) dt = \frac{1}{\sqrt{2\pi}}$$

and we get

$$p(E\left[\operatorname{co}\{0, X\}\right], \xi) = \frac{1}{\sqrt{2\pi}} \parallel \xi \parallel_{L^2(\gamma)} .$$

Furthermore, the relation

$$\xi(h) = \int_E \xi \tilde{h} d\gamma$$

shows that

$$\max_{h \in O_\gamma} \xi(h) = \parallel \xi \parallel_{L^2(\gamma)}$$

and, hence,

$$p(E\left[\operatorname{co}\{0, X\}\right], \xi) = \frac{1}{\sqrt{2\pi}} \max_{h \in O_\gamma} \xi(h) = p(\frac{1}{\sqrt{2\pi}} O_\gamma, \xi)$$

which proves (4.1). □

Next we introduce the Banach space $F = \mathbf{R} \times E$ equipped with the norm $\parallel (r, x) \parallel_F = \mid r \mid + \parallel x \parallel_E$.

Theorem 4.1 *With definitions as above,*

$$E\left[\operatorname{co}\{0, (1, a + X)\}\right] = Z(a, \gamma)$$

and if $G \in N(0,1)$,

$$p(Z(a,\gamma),(\varrho,\xi)) = E\left[(\varrho+\xi(a+X))^+\right]$$

$$= E\left[(\varrho+\xi(a)+G \parallel \xi \parallel_{L^2(\gamma)})^+\right]$$

for every $(\varrho,\xi) \in \mathbf{R}\times E'$.

Proof. For any $(\varrho,\xi) \in F' = \mathbf{R}\times E'$,

$$p(E\left[\mathrm{co}\left\{0,(1,a+X)\right\}\right],(\varrho,\xi)) = E\left[p(\mathrm{co}\left\{0,(1,a+X)\right\},(\varrho,\xi))\right]$$

$$= E\left[(\varrho+\xi(a)+\xi(X))^+\right] = E\left[(\varrho+\xi(a)+G \parallel \xi \parallel_{L^2(\gamma)})^+\right]$$

$$= p(Z,(\varrho+\xi(a),\parallel \xi \parallel_{L^2(\gamma)}))$$

where we used (3.2) in the last equality. Moreover,

$$p(Z(a,\gamma),(\varrho,\xi)) = \max_{(r,x)\in Z(a,\gamma)}(\varrho r+\xi(x))$$

$$= \max\left\{\varrho r+\xi(x);\ x\in ra+I(r)O_\gamma \text{ and } 0\le r\le 1\right\}$$

$$= \max\left\{(\varrho+\xi(a))r+I(r)\parallel \xi \parallel_{L^2(\gamma)};\ 0\le r\le 1\right\} = p(Z,(\varrho+\xi(a),\parallel \xi \parallel_{L^2(\gamma)})).$$

This proves Theorem 4.1.

Introducing the projection $\mathrm{Proj}_E(r,x) = x$, if $(r,x)\in F$, Theorem 4.1 implies that

$$E\left[\mathrm{co}\left\{0,a+X\right\}\right] = \mathrm{Proj}_E Z(a,\gamma)$$

$$= \left\{x\in E;\ \parallel x-ra \parallel_{H_\gamma}\le I(r) \text{ for some } 0\le r\le 1\right\}$$

which improves the relation (4.1).

Corollary 4.1 *Suppose* $(X_i)_{i\in\mathbb{N}_+}$ *is an i.i.d. in* E *where the law of each* X_i *equals* γ*. Then*

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\mathrm{co}\left\{0,(1,a+X_i)\right\} = Z(a,\gamma) \text{ a.s. } P,$$

and

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\mathrm{co}\,\{0,a+X_i\}=\mathrm{Proj}_E Z(a,\gamma)\ \text{a.s. } P.$$

In particular,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\mathrm{co}\,\{0,X_i\}=\frac{1}{\sqrt{2\pi}}\,O_\gamma\ \text{a.s. } P.$$

5 Affine images of $Z(a,\gamma)$

In this section, suppose $\theta^0,\theta^1\in\mathbf{R}$ and $\theta^0\le\theta^1$ and as above assume $(X_i)_{i\in\mathbb{N}_+}$ is an i.i.d. in E where the probability law of each X_i equals γ. Moreover, for each $i\in\mathbb{N}_+$ let

$$L_i^{\theta^0,\theta^1}(a,\gamma)=\left\{\theta(1,a+X_i);\ \theta^0\le\theta\le\theta^1\right\}$$

or

$$L_i^{\theta^0,\theta^1}(a,\gamma)=\theta^0(1,a+X_i)+(\theta^1-\theta^0)\mathrm{co}\,\{0,(1,a+X_i)\}$$

and set

$$Z_n^{\theta^0,\theta^1}(a,\gamma)=\frac{1}{n}\sum_{i=1}^{n}L_i^{\theta^0,\theta^1}(a,\gamma).$$

Since $\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}X_i=0$ a.s. by the SLLN in Banach space, Corollary 4.1 gives the following

Theorem 5.1 *With definitions as above,*

$$\lim_{n\to\infty}Z_n^{\theta^0,\theta^1}(a,\gamma)=\theta^0(1,a)+(\theta^1-\theta^0)Z(a,\gamma)\ \text{a.s. } P,$$

where

$$(\theta^0(1,a)+(\theta^1-\theta^0)Z(a,\gamma))_r=ra+(\theta^1-\theta^0)I(\frac{r-\theta^0}{\theta^1-\theta^0})O_\gamma,\ \theta^0\le r\le\theta^1.$$

Proof. To prove the last part in Theorem 5.1, the special case $\theta^0 = \theta^1$ is trivial so we may assume $\theta^0 < \theta^1$. If $(r,x) \in \theta^0(1,a) + (\theta^1 - \theta^0)Z(a,\gamma)$ there exists an s such that $(s,y) \in Z(a,\gamma)$ and

$$\begin{cases} r = \theta^0 + (\theta^1 - \theta^0)s \\ x = \theta^0 a + (\theta^1 - \theta^0)y. \end{cases}$$

Now

$$s = \frac{r - \theta^0}{\theta^1 - \theta^0}$$

and

$$\theta^0(1,a) + (\theta^1 - \theta^0)(s,y) \in \theta^0(1,a) + (\theta^1 - \theta^0)(s, sa + I(s)O_\gamma)$$

$$= (r, ra + (\theta^1 - \theta^0)I(\frac{r - \theta^0}{\theta^1 - \theta^0})O_\gamma).$$

Hence

$$(\theta^0(1,a) + (\theta^1 - \theta^0)Z(a,\gamma))_r \subseteq ra + (\theta^1 - \theta^0)I(\frac{r - \theta^0}{\theta^1 - \theta^0})O_\gamma$$

and the converse inclusion is proved in a similar way. This completes the proof of Theorem 5.1.

6 CLT for $\mathrm{co}\{0,(1,X)\}$

A general version of the central limit theorem (CLT) for the class $\mathcal{K}(B)$ based on a certain entropy condition is submitted in ([7] , Theorem 2.7, p 218). The purpose of this section is to show that the sequence

$$\sqrt{n}\delta_F\left(\frac{1}{n}\sum_{i=1}^{n} \mathrm{co}\left\{0,(1,a+X_i)\right\}, Z(a,\gamma)\right), \; n \in \mathbf{N}_+,$$

converges in distribution to the supremum of the absolute value of an appropriate centred Gaussian process. In this very special case, it turns out that the entropy condition in [7] can be dispensed with.

To prove this, without loss of generality, we will assume that

$$E = \mathrm{span}(a) + \mathrm{supp}(\gamma).$$

Let S and $T = [-1, 1] \times S$ be the unit balls in the topological duals of E and F, respectively, and introduce

$$U(t) = (\varrho + \xi(a + X))^{+} - E\left[(\varrho + \xi(a + X))^{+}\right], \ t = (\varrho, \xi) \in T,$$

and

$$\begin{cases} d_U(t, t') = \max(\mid \varrho - \varrho' \mid, \mid \xi(a) - \xi'(a) \mid, \parallel \xi - \xi' \parallel_{L^2(\gamma)}), \\ \qquad\qquad t = (\varrho, \xi), \ t' = (\varrho', \xi') \in T. \end{cases}$$

Let d be a metric for the weak*-topology on T and recall that the metric space (T, d) is compact. Note also that (T, d_U) is a metric space. In fact, if $t = (\varrho, \xi)$, $t' = (\varrho', \xi') \in T$ and $d_U(t, t') = 0$ then $\varrho = \varrho'$, $\xi(a) = \xi'(a)$, and $\xi = \xi'$ on H_γ. But then $\xi = \xi'$ on $\text{supp}(\gamma)$ and it follows that $t = t'$. Furthermore, the canonical injection mapping of (T, d) onto (T, d_U) is continuous by standard integrability theorems for Gaussian semi-norms. Hence d and d_U induce the same topology on T.

Theorem 6.1 *The process $U = (U(t))_{t \in T}$ is subgaussian with respect to the metric cd_U for an appropriate $c > 0$, that is, there exists an $c > 0$ such that*

$$E\left[e^{\lambda(U(t) - U(t'))}\right] \leq e^{\frac{\lambda^2}{2} c^2 d_U^2(t, t')}$$

for every $\lambda \in \mathbf{R}$ and t, $t' \in T$.

Proof. Suppose

$$\psi(x) = e^{x^2} - 1, \ x \geq 0.$$

If V is a random variable and the Orlicz norm

$$\parallel V \parallel_\psi = \inf\left\{\alpha > 0; \ E\left[\psi(\frac{V}{\alpha})\right] \leq 1\right\}$$

does not exceed 1, then there exists a numerical constant $C > 0$ such that

$$E\left[e^{\lambda V}\right] \leq e^{C^2 \lambda^2}, \ \lambda \in \mathbf{R}$$

(see [6] pp. 322-323).

Now, if $t = (\varrho, \xi)$, $t' = (\varrho', \xi') \in T$, $t \neq t'$, and $c > 0$,

$$E\left[\psi(\frac{U(t) - U(t')}{c d_U(t, t')})\right]$$

$$= E\left[\exp(\frac{(U(t)-U(t'))^2}{c^2 d_U^2(t,t')})\right] - 1.$$

But for any real numbers α and β, $\mid \alpha^+ - \beta^+ \mid \leq \mid \alpha - \beta \mid$, and, hence,

$$\mid U(t) - U(t') \mid \leq \mid (\varrho + \xi(a+X))^+ - (\varrho' + \xi'(a+X))^+ \mid$$

$$+ \mid E\left[(\varrho + \xi(a+X))^+ - (\varrho' + \xi'(a+X))^+\right] \mid$$

$$\leq 2 \mid \varrho - \varrho' \mid + 2 \mid \xi - \xi' \mid (a) + \sqrt{\frac{2}{\pi}} \parallel \xi - \xi' \parallel_{L^2(\gamma)} + \mid \xi - \xi' \mid (X).$$

Now, defining $\eta = \xi - \xi'$ and applying the Cauchy-Schwarz inequality,

$$E\left[\psi(\frac{U(t)-U(t')}{c d_U(t,t')})\right]$$

$$\leq E\left[\exp(\frac{4}{c^2 d_U^2(t,t')}(4 \mid \varrho - \varrho' \mid^2 + 4\eta^2(a) + \parallel \eta \parallel^2_{L^2(\gamma)} + \eta^2(X))\right] - 1$$

$$\leq \frac{1}{\sqrt{1 - \frac{8\|\eta\|^2_{L^2(\gamma)}}{c^2 d_U^2(t,t')}}} \exp(\frac{4}{c^2 d_U^2(t,t')}(4 \mid \varrho - \varrho' \mid^2 + + 4\eta^2(a) + \parallel \eta \parallel^2_{L^2(\gamma)})) - 1$$

$$\leq \frac{1}{\sqrt{1 - \frac{8}{c^2}}} \exp(\frac{36}{c^2}) - 1$$

where the member in the last line does not exceed 1 if c is large enough. This proves Theorem 6.1.

Let Y be a random vector in a real, separable Banach space B and suppose V_i, $i \in \mathbf{N}_+$, are independent observations on Y. We say that Y satisfies the CLT if the sequence

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} Y_i, \; n \in \mathbf{N}_+,$$

converges weakly in B. If Y satisfies the CLT, $Y \in L^p(P)$ for each $0 < p < 2$, $E[Y] = 0$, and $\tau(Y) \in L^2(P)$ for each $\tau \in B'$ ([6], pp. 273-274). In particular, Y is pregaussian, that is the covariance of Y is the covariance of a centred Gaussian random vector in B.

Theorem 6.2 *U satisfies the* CLT*.*

Proof. Let $G \in N(0,1)$ be such that G and X are independent. The process $V = (\xi(aG + X))_{\xi \in S}$ is continuous on (S, d_S) where d_S is a metric of the weak* topology on S. Moreover, as above we conclude that the metric

$$d_V(\xi, \xi') =_{def} \sqrt{E\left[(V(\xi) - V(\xi'))^2\right]}$$

$$= \sqrt{\| \xi - \xi' \|^2_{L^2(\gamma)} + (\xi - \xi')^2(a)}$$

on S induces the same topology on S as d_S. Hence V is bounded and continuous on (S, d_V) and the Talagrand regularity theorem ([6] , Theorem 12.9) gives us a probability measure m on S such that

$$\lim_{b \to 0+} \int_0^b \sqrt{\ln \frac{1}{m(B_{d_V}(\xi, \varepsilon))}} d\varepsilon = 0$$

where $B_{d_V}(\xi, \varepsilon)$ is the open ball of centre ξ and radius ε in (S, d_V). Here m is termed a majorizing measure.

Now let μ be $\frac{1}{2}$ times Lebesgue measure on the interval $[-1, 1]$ and denote by $B_{d_U}(t, \varepsilon)$ the open ball in (T, d_U) with centre t and radius ε. Then, if $(\varrho, \xi) \in T$ and $0 < \varepsilon < 1$,

$$(\mu \times m)(B_{d_U}(t, \varepsilon))$$

$$\geq \frac{\varepsilon}{2} m(\{\xi' \in S;\ \max(\| \xi - \xi' \|_{L^2(\gamma)}, | \xi(a) - \xi'(a) |) < \varepsilon\}$$

$$\geq \frac{\varepsilon}{2} m(B_{d_V}(\xi, \varepsilon)).$$

Hence,

$$\lim_{b \to 0+} \int_0^b \sqrt{\ln \frac{1}{(\mu \times m)(B_{d_U}(\xi, \varepsilon))}} d\varepsilon = 0$$

and since U is subgaussian by Theorem 6.1, a theorem due to Jain and Marcus states that U satisfies the CLT ([6] , Theorem 14.1). This completes the proof of Theorem 6.2.

Next we introduce the covariance

$$R(\mu, \mu') = E\left[\mu(U)\mu'(U)\right], \ \ \mu, \mu' \in C(T)',$$

and let Y be a Gaussian random vector in the Banach space $C(T)$ with covariance R. Then, in particular, the sequence

$$\max_{(\varrho,\xi)\in T} \mid \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left\{ (\varrho + \xi(a+X_i))^{+} - E\left[(\varrho + \xi(a+X_i))^{+} \right] \right\} \mid, \ \ n \in \mathbf{N}_{+},$$

converges to the random variable

$$\max_{(\varrho,\xi)\in T} \mid Y(\varrho,\xi) \mid$$

in distribution as n goes to infinity. The explicit form of the support function of $Z(a,\gamma)$ given in Theorem 4.1 combined with the Hörmander embedding theorem now yield the following

Theorem 6.3 *The sequence*

$$\sqrt{n}\delta_F \left(\frac{1}{n} \sum_{i=1}^{n} \mathrm{co}\left\{ 0, (1, a+X_i) \right\}, Z(a,\gamma) \right), \ \ n \in \mathbf{N}_{+},$$

converges to $\max_{t\in T} \mid Y(t) \mid$ *in distribution.*

7 On the excess return in investments when utility changes

In this section we are going to study the excess return in the classical mean-variance approach to portfolio selections when utility is time dependent. There is only one single stock (or fund) in the model and we suppose the stock and bond price processes are Itô processes with simple integrands and, in addition, the instantaneous Sharpe ratio of the stock price is constant. Since our main interest here is a sensitivity analysis of the average excess return of an individual investor with changeable risk aversion, we find it natural to keep the instantaneous Sharpe ratio constant since it reflects a constant market price of risk.

Throughout, transaction costs and taxes are neglected and we suppose the information structure $(\mathcal{F}(t))_{t\geq 0}$ is given by the filtration generated by a standard Wiener process $W = (W(t))_{t\geq 0}$ in $\mathbf{R}$. Furthermore, suppose T is a given positive number and $t_n = nT$, $n \in \mathbf{N}$. In the following, $(S(t))_{t\geq 0}$ and $(B(t))_{t\geq 0}$ denote a stock price and bond price process, respectively, and these are governed by the stochastic differential equations

$$\begin{cases} \frac{dS(t)}{S(t_{n-1})} = \mu(t)dt + \sigma(t)dW(t), \ t_{n-1} \leq t < t_n, \\ \frac{dB(t)}{B(t_{n-1})} = r(t)dt, \ t_{n-1} \leq t < t_n, \end{cases}$$

where $(\mu(t))_{t\geq 0}$ and $(\sigma(t))_{t\geq 0}$ are adapted processes satisfying the conditions

$$\begin{cases} \mu(t) = \mu(t_{n-1}) \text{ in } [t_{n-1}, t_n[\\ \sigma(t) = \sigma(t_{n-1}) \text{ in } [t_{n-1}, t_n[\\ r(t) = r(t_{n-1}) \text{ in } [t_{n-1}, t_n[\\ \sigma(t_{n-1}) > 0 \end{cases}$$

for each $n \in \mathbf{N}_+$. In addition, it will be assumed that the instantaneous Sharpe ratio λ of the stock price is non-zero and independent of time, that is

$$\lambda = \frac{\mu(t) - r(t)}{\sigma(t)} \neq 0, \ t \geq 0.$$

Next for each $n \in \mathbf{N}_+$, define

$$\chi_{n-1}(\omega) = \sigma(t_{n-1}, \omega)/\sigma(0)$$

and note that $\chi_0 = 1$ and

$$\mu(t_{n-1}, \omega) - r(t_{n-1}, \omega) = \chi_{n-1}(\omega)(\mu(0) - r(0))$$

since the stock price has constant instantaneous Sharpe ratio.

Now, at time 0, consider a portfolio of wealth K_0, where the fraction α is put into the stock and the fraction $1 - \alpha$ into the bond. The (simple) return at time T of this portfolio equals

$$R(\alpha) = r(0)T + \alpha(S(T)/S(0) - B(T)/B(0)).$$

Next we follow a standard mean-variance approach and suppose the investor at time 0 maximizes a linear combination of mean and variance of $R(\alpha)$ with a positive weight on mean and a negative on variance:

$$\max_{\alpha} \left\{ E\left[R(\alpha)\right] - \frac{\kappa_0}{2} \text{Var}(R(\alpha)) \right\}$$

where κ_0 is the risk aversion parameter at time zero. The solution to this maximization problem is

$$\alpha_0 = \frac{1}{\kappa_0}\frac{\mu(0) - r(0)}{\sigma(0)^2}$$

(see e.g. Campell and Viciera [3]).

The above optimal portfolio is not rebalanced during the time interval $[0, T[$ and the excess return at time $t \in [0, T]$ equals

$$E_1(t) = \alpha_0(S(t)/S(0)) - B(t)/B(0)).$$

From this we get

$$E_1 = \alpha_0(e_0 + Y_1)$$

where

$$Y_1 = (\sigma(0)W(t))_{0 \leq t \leq T}$$

and

$$e_0 = ((\mu(0) - r(0))t)_{0 \leq t \leq T}.$$

At time T the investor chooses an optimal portfolio with the new parameters available at time T and continues successively in this way. The investor's risk aversion parameter equals κ_{n-1} at time t_{n-1}, $n \in \mathbf{N}_+$. Thus for each $n \in \mathbf{N}_+$ it is assumed that the fraction α_{n-1} of the investor's wealth is invested in the stock and the fraction $1 - \alpha_{n-1}$ in the bond at time t_{n-1}, where

$$\alpha_{n-1} = \frac{1}{\kappa_{n-1}}\frac{\mu(t_{n-1}) - r(t_{n-1})}{\sigma(t_{n-1})^2} = \frac{\lambda}{\kappa_{n-1}\sigma(0)}\frac{1}{\chi_{n-1}}$$

and the portfolio is not rebalanced during the time interval $[t_{n-1}, t_n[$. We introduce

$$Y_n = \sigma(t_{n-1})(W(t_{n-1} + t) - W(t_{n-1}))_{0 \leq t \leq T}$$

and denote the excess return during the interval $[t_{n-1}, t_{n-1} + t]$ by

$$E_n(t) = \alpha_{n-1}(e_{n-1}(t) + Y_n(t))$$

where

$$e_{n-1}(t) = (\mu(t_{n-1}) - r(t_{n-1}))t = \chi_{n-1}e_0(t),\ \ 0 \leq t \leq T.$$

Thus defining

$$X_n = (\sigma(0)(W(t_{n-1} + t) - W(t_{n-1}))_{0 \leq t \leq T}$$

it follows that

$$E_n = \alpha_{n-1}\chi_{n-1}(e_0 + X_n) = \frac{\lambda}{\kappa_{n-1}\sigma(0)}(e_0 + X_n)$$

and introducing

$$\theta_{n-1} = \frac{\lambda}{\kappa_{n-1}\sigma(0)}$$

the excess return process during period n equals

$$E_n = \theta_{n-1}(e_0 + X_n).$$

Note that θ_{n-1} is $\mathcal{F}(t_{n-1})$-measurable for every $n \in \mathbf{N}_+$.

Below we will discuss how the above time-dependent aversion towards risk will effect the average $\bar{E}_n = \frac{1}{n}\Sigma_1^n E_i$ of the excess return processes over the first n periods. Clearly, if the risk aversion parameter is constant and equal to κ at each point of time, by the SLLN in Banach space the average $\bar{E}_n$ converges to $\frac{\lambda}{\kappa\sigma(0)}e_0$ as the number of periods goes to infinity.

In the following, for each $n \in \mathbf{N}_+$ it is assumed that $0 < \kappa^0 \leq \kappa_{n-1} \leq \kappa^1$, and we introduce

$$\theta^0 = \min(\frac{\lambda}{\kappa^0\sigma(0)}, \frac{\lambda}{\kappa^1\sigma(0)})$$

and

$$\theta^1 = \max(\frac{\lambda}{\kappa^0\sigma(0)}, \frac{\lambda}{\kappa^1\sigma(0)}).$$

By definition the sequence $(X_n)_{n\in\mathbf{N}}$, is an i.i.d. but the excess return processes E_n, $n \in \mathbf{N}_+$, need not even be independent. Setting $\gamma = P_X$ and using the same notation as in Theorem 5.1,

$$\frac{1}{n}\sum_{i=1}^{n}\theta_{i-1}(1, e_0 + X_i) \in Z_n^{\theta^0,\theta^1}(e_0, \gamma)$$

for all $n \in \mathbf{N}_+$. Define

$$\bar{\theta}_n = \frac{1}{n}(\theta_0 + ... + \theta_{n-1})$$

and note that

$$(\bar{\theta}_n, \bar{E}_n) \in Z_n^{\theta^0,\theta^1}(e_0, \gamma).$$

By applying Theorem 5.1 we conclude that there exists an appropriate event Ω_0 of probability one, such that for every $\varepsilon > 0$ and $\omega \in \Omega_0$ the random function $\bar{E}_n(\omega)$ is contained in the set

$$\bar{\theta}_n e_0 + (\theta^1 - \theta^0) I((\frac{\bar{\theta}_n - \theta^0}{\theta^1 - \theta^0}) O_\gamma + B_{C[0,T]}(0,\varepsilon)$$

for large n, where $B_{C[0,T]}(0,\varepsilon)$ denotes the open ball of centre 0 and radius ε in the Banach space $C\,[0,T]$. Recall that O_γ is the set of all absolutely continuous functions $h : [0,T] \to \mathbf{R}$ such that

$$h(t) = \int_0^t f(s)ds$$

and

$$\sqrt{\int_0^T f^2(t)dt} \leq \sigma(0).$$

In particular, if $h \in O_\gamma$, $\mid h(t) \mid \leq \sigma(0)\sqrt{t}$, $0 \leq t \leq T$.

Next we will give a more detailed analysis of the above bound on the process $(\bar{E}_n(t))_{0\leq t \leq T}$ at the time point $t = T$. If

$$\delta =_{def} \sigma(0)\sqrt{T},$$

$$M(\bar{\theta}_n) =_{def} (\bar{\theta}_n e_0(T) + \delta(\theta^1 - \theta^0) I((\frac{\bar{\theta}_n - \theta^0}{\theta^1 - \theta^0})),$$

and

$$m(\bar{\theta}_n) =_{def} (\bar{\theta}_n e_0(T) - \delta(\theta^1 - \theta^0) I((\frac{\bar{\theta}_n - \theta^0}{\theta^1 - \theta^0})),$$

it follows that for every $\omega \in \Omega_0$ and $\varepsilon > 0$ the average excess return $\bar{E}_n(T,\omega)$ over the first n time periods is smaller than or equal to $M(\bar{\theta}_n) + \varepsilon$ and greater than or equal to $m(\bar{\theta}_n) - \varepsilon$ for large n. The interval $J(\bar{\theta}_n) = \left[m(\bar{\theta}_n), M(\bar{\theta}_n\right]$ is of maximal length $\delta(\theta^1 - \theta^0)\sqrt{2/\pi}$ for $\bar{\theta}_n = \frac{1}{2}(\theta^0 + \theta^1)$. Moreover, since $I'(\theta) = -\Phi^{-1}(\theta)$ it follows that the function $M(\bar{\theta}_n)$, $\theta_0 \leq \bar{\theta}_n \leq \theta_1$, has a maximum for

$$\theta_{\max} = \theta^0 + (\theta^1 - \theta^0)\Phi(\frac{e_0(T)}{\delta})$$

and

$$M(\theta_{\max}) = (\theta^0 + (\theta^1 - \theta^0)\Phi(\frac{e_0(T)}{\delta}))e_0(T) + \delta(\theta^1 - \theta^0)\varphi(\frac{e_0(T)}{\delta}).$$

In a similar way, the function $m(\bar{\theta}_n)$, $\theta^0 \leq \bar{\theta}_n \leq \theta^1$ has a minimum at the point

$$\theta_{\min} = \theta^0 + (\theta^1 - \theta^0)\Phi(-\frac{e_0(T)}{\delta})$$

and

$$m(\theta_{\min}) = (\theta^0 + (\theta^1 - \theta^0)\Phi(-\frac{e_0(T)}{\delta}))e_0(T) - \delta(\theta^1 - \theta^0)\varphi(\frac{e_0(T)}{\delta}).$$

Note that $M(\theta_{\max})$ is the smallest number b such that limes superior of the sequence $(\bar{E}_n(T))_{n=1}^{\infty}$ is smaller or equal to b with probability one and $m(\theta_{\min})$ the largest number a such that limes inferior of the sequence $(\bar{E}_n(T))_{n=1}^{\infty}$ is larger or equal to a with probability one.

Example 7.1 Assume the time unit is year and consider the following data

$$\begin{cases} T = 1/12 \\ \mu(t) = 0.1 \\ \sigma(t) = 0.2 \\ r(t) = 0.05 \end{cases}$$

and

$$\begin{cases} \kappa^0 = 2 \\ \kappa^1 = 3. \end{cases}$$

In this special case the interval $J(\bar{\theta}_n)/T$ is of maximal length 0.1152, that is 11.52% on the annual basis. Furthermore, $M(\theta_{\max})/T = 0.084$ and $m(\theta_{\min})/T = -0.032$. Recall that for a constant risk aversion parameter $(\kappa^0 = \kappa^1)$ the interval $J(\bar{\theta}_n)/T$ reduces to a singleton set. □

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