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Closed form of time-dependant probabilities for SIR epidemics with generalized infection mechanisms

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Abstract
Gani and Perdue outlined a matrix-geometric method for determining the total size distribution of an epidemic in a recursive manner. In this paper, we explore how this method can be used to study an SIR epidemic model with a generalized mechanism of infection. We are able to obtain an explicit formula for the Laplace transform of the transition probabilities. Using this we derive various other quantities explicitly. Examples of such quantities are the transition probabilities and the expectation of the duration of the epidemic.
Keywords: SIR model; Generalized infection mechanism; Matrix-geometric; Transition probabilities; Duration of an epidemic.

1 Introduction

In this note, we are concerned with the problem of solving the Kolmogorov forward equations of the SIR stochastic epidemic model with a generalized infection mechanism. In such a process, each individual can be in one of three possible states: susceptible, infected or removed. Billard [3] and Krysco [10] considered this problem in the case of the so called general stochastic epidemic model (cf. [1]). Billard's solution is obtained by explicitly generating the components of the off-diagonal elements of Severo's matrix solution (cf. [17]), while the diagonal elements of the matrix are recursively defined. Krysco's solution is obtained by enumeration of all possible paths that the process can follow until absorption. Ball and O'Neill [2] adapted Krysco's method to the so called modified stochastic epidemic.
The method proposed by Gani and Purdue [8] was applied by Booth [4] to an SIR model with one homogeneous population. That model was later generalized by El Maroufy in [7] to the case of an epidemic in a population consisting of two interacting subpopulations. The method in question is the matrix-geometric technique introduced by Neuts in [11] in a different context. We notice that the focus of these authors was recursively on the total size distribution of the epidemic. In this work, we investigate how this method applies to obtain an explicit time-dependent solution to the forward

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Kolmogorov equations of the stochastic model described in section 2. It turns out that the resulting solution is similar in form to those proposed by Krysco [10] in the case of the general epidemic and by Ball and O'Neill [2] in the case of the modified epidemic. However, our solution is explicitly defined and algebraically simpler. Moreover, the approach we use, in contrast to the methods proposed the the previous works, generates explicitly all components of the useful matrices.
The paper is structured as follows: The model under consideration is described in section 2; the proposed solution is presented in section 3; and the distribution and mean of the epidemic duration are derived in section 4. In section 5, we discuss some possible applications. Finally some of the derivations call for tedious algebraic manipulations that are presented in the appendix.

2 The model

The stochastic epidemic model that we consider here was proposed by Gani and Purdue [8] and was generalized and discussed in detail by Elmaroufy [7]. In this model we assume that at time $t \geq 0$ there are $X(t)$ susceptibles, $Y(t)$ infectives and $n-X(t)-Y(t)$ removed individuals where $X(0) = n$ and $Y(0) = a$. The epidemic process is thus completely determined by $\{(X(t), Y(t)); t \geq 0\}$, which is supposed to be a continuous-time Markov chain on the state space:

$$\mathbf{E}_{n,a} = \{(x, y), 0 \leq x \leq n, 0 \leq y \leq n + a - x\},$$

with the following transition probabilities :

$$\begin{cases} \Pr\{(X(t+\delta t), Y(t+\delta t)) = (x-1, y+1)|(X(t), Y(t)) = (x, y)\} \\ \qquad\qquad\qquad\qquad\qquad = f_{xy,x-1y+1}\delta t + o(\delta t), \\ \Pr\{(X(t+\delta t), Y(t+\delta t)) = (x, y-1)|(X(t), Y(t)) = (x, y)\} \\ \qquad\qquad\qquad\qquad\qquad = \mu y \delta t + o(\delta t) \end{cases} \tag{1}$$

all other transitions having probability $o(\delta t)$, and the parameter μ, being known as the removal rate. The process terminates when the number of infectives becomes zero, which will almost surely happen in finite time.
The infection rate considered above comprises different infection rates mentioned in epidemic literature. For instance, Severo [18] took $f_{xy,x-1y+1} = \beta x^{-b} y^{a-1}$, with β, a and b constants. Another possibility would be to take $f_{xy,x-1y+1} = \beta_{xy} xy$ (see. [?]), where $\beta_{xy} = \frac{\beta}{(x+y)^\alpha}$ with β is defined in [6] as the product of the contacte rate and the probability that a successive number of contacts lead to infection. In this case, $\alpha = 1$, will give the infection mechanism considered by Gleissner [9], Ball and O'Neill [2], O'Neill [13] and Sani and all [15], while when $\alpha = 0$, the model is reduced to the general stochastic epidemic model. The case $\alpha = \frac{1}{2}$ was studied by Saunders in [16].
Throughout this paper, we use the notation $f_{x,y} = f_{xy,x-1y+1}$ and adopt the convention that

$$f_{0,y} = f_{x,0} = 0 \text{ and } f_{x,y} > 0 \text{ for } x > 0 \text{ and } y > 0. \tag{2}$$

For $(i,l) \in \mathbf{E}_{n,a}$, we define $P_{il}(t) = \Pr\{X(t) = i, Y(t) = l\}$. It follows directly from (1) that these transition probabilities satisfy the following set of Kolmogorov equations:

$$\frac{\partial P_{il}(t)}{\partial t} = f_{i+1l-1}P_{i+1l-1}(t) + \mu(l+1)P_{il+1}(t) - (\mu l + f_{il})P_{il}(t), \tag{3}$$

for $(i,l) \in \mathbf{E}_{n,a}$, with $P_{il}(t) \equiv 0$ if $(i,l) \notin \mathbf{E}_{n,a}$ and $P_{na}(0) = 1$.

3 The time-dependent solution

The Kolomogorov equations (3) can be solved by using the matrix geometric method. For $i = 0, ..., n$, let A_i and D_i be the diagonal matrices with l-th diagonal elements equal to μl and f_{il} respectively for $l = 0, .., n+a-i$. Let further C_i be the matrix of the same dimension with $(l, l+1) - th$ entries equal to $\mu(l+1)$, $l = 0, .., n+a-i-1$ and all other entries equal to 0. In addition, for $i = 0, 1, ..., n$, take the column vector

$$P_i(t) = (P_{i0}(t), P_{i1}(t), ..., P_{in+a-i}(t))^T.$$

Furthermore, for each matrix M of order $n+a-i-p+1$ for $0 \leq i \leq n$ and $0 \leq p \leq n-i$ we define an augmented matrix

$$M_{i+1}(p) = \begin{pmatrix} \Theta_i^p & 0 \\ 0 & M \end{pmatrix},$$

where Θ_i^p is the zero matrix of order p; and for each vector $U(t)$ of dimension $(n + a - i - p + 1)$, we also define $U(t,p) = ((\theta_i^p)^T, U^T(t))^T$, where θ_i^p is the p zero column vector.

Using the previous notations and arguments, equation (3) takes now the following form:

$$\frac{\partial P_i(t)}{\partial t} = (C_i - A_i - D_i)P_i(t) + D_{i+1}(1)P_{i+1}(t,1), \quad \text{for } i = 0, 1, ..., n-1 \tag{4}$$

and

$$\frac{\partial P_n(t)}{\partial t} = (C_n - A_n - D_n)P_n(t). \tag{5}$$

The transition probabilities can now be studied using the Laplace Transformation $\hat{P}_i(v) = \int_0^{+\infty} e^{-vt} P_i(t)\, dt$. Using (4) and (5), we can formulate the following result :

Theorem 1
For $l = 0, ..., a$ we can show that

$$\hat{P}_{nl}(v) = \mu_1^{a-l} \frac{a!}{l!} \prod_{k=0}^{a-l} (v + \mu(l+k) + f_{n,l+k})^{-1} \tag{6}$$

and for $(i,l) \in \mathbf{E}_{n,a}$ *and* $v \geq 0$,

$$\hat{P}_{il}(v) = \mu^{n+a-i-l} \frac{a!}{l!} \sum_{L \in \hat{D}^{n-i}_{ln+a-i}} \frac{\prod_{w=1}^{n-i} (l_w - w + 1) f_{i+w, l_w - w}}{\prod_{(w,k) \in D_{n-i}} (v + \mu(l_w - w + k) + f_{i+w, l_w - w + k})} \tag{7}$$

where

$$D^j_{lh} = \{(i_1, i_2, \ldots, i_j) /\ l \leq i_1 \leq i_2 \leq \ldots \leq i_j \leq h\} \tag{8}$$

$$\hat{D}^j_{lh} = D^j_{lh} \cap \{(i_1, i_2, ..., i_j)/i_1 > 1, ..., i_j > j\} \tag{9}$$

and

$$D_i = \{(w,k)/w = 0, ..., i, k = 0, ..., l_{w+1} - l_w\}. \tag{10}$$

with $L = (l_1, l_2, ..., l_{n-i})$ *and* $(l_0, l_{n-i+1}) = (l, n+a-i)$.

Proof By letting $F_i = C_i - A_i - D_i$, equations (4)-(5) become

$$\hat{P}_n(v) = (vI_n - F_n)^{-1} P_n(0), \tag{11}$$

and

$$\hat{P}_i(v) = (vI_i - F_i)^{-1} D_{i+1}(1) \hat{P}_{i+1}(v, 1) \tag{12}$$

for $0 \leq i \leq n-1$, where I_i denotes the identity matrix of order $n + a - i + 1$.
In order to determine the Laplace transform of $P_{il}(t)$, we have to determine the entries of $(vI_i - F_i)^{-1}$ explicitly. First we show that $(vI_i - F_i)^{-1}$ has an upper triangular form (cf. The Appendix) i.e.

$$[(vI_i - F_i)^{-1}]_{lh} = \begin{cases} C_i(v,l,h) & \text{if } 0 \leq l \leq h \leq n+a-i \\ 0 & \text{otherwise,} \end{cases} \tag{13}$$

where

$$C_i(v,l,h) = \mu_1^{h-l} \frac{h!}{l!} \prod_{k=l}^{h} (v + \mu l + f_{ik})^{-1}, \quad \text{for } \ i = 0, ..., n. \tag{14}$$

Using (14) for $h = a, i = n$ and since $P_n(0) = (0, ..., 1)^T$ we deduce directly from (11) that $\hat{P}_{nl}(v) = \mu_1^{a-l} \frac{a!}{l!} \prod_{k=l}^{a} (v + \mu k + f_{n,k})^{-1}$ which implies (6). To determine $\hat{P}_{il}(v)$ for $i = 0, ..., n-1$, we need the following quantities:

$$\mathcal{X}_i(v, w) = (vI_{i+w} - F_{i+w})^{-1}$$

and

$$\mathcal{Y}_i = \begin{pmatrix} \Theta_w & 0 \\ 0 & \mathcal{X}_i(v,w) D_{i+w+1}(1) \end{pmatrix} \tag{15}$$

where $(v,w) \in I\!R_+^* \times \{0,...,n-i\}$, Θ_w is the null matrix of rank w and where $D_{i+w+1} = I_n$ if $w = n-i$ if $w = n-i$ and for $w \neq n-i$

$$[\mathcal{Y}_i(v,w)]_{mk} = \begin{cases} [\mathcal{X}_i(v,w)]_{m'k'} f_{i+w+1,h-w-1} & \text{if } m = w+m',\ k = w+1+k' \\ & \text{ with } m' = l-w, k' = h-w \\ & w \leq l \leq h \leq n+a-i \\ & \text{and } w+1 \leq h, \\ 0 & \text{otherwise.} \end{cases} \tag{16}$$

As for $w = n-1$

$$[\mathcal{Y}_i(v,w)]_{mk} \quad = \quad \begin{cases} [\mathcal{X}_i(v,w)]_{m'k'} & \text{if } m = n-i+m',\ k = n-i+1+k' \\ 0 & \text{otherwise.} \end{cases}$$

For $i = 0,...,n$ and $v > 0$, we deduce from equations (11), (12) and (15) that

$$\hat{P}_i(v) = \mathcal{Y}_i(v,0)\hat{P}_{i+1}(v,1) = \left[\prod_{w=0}^{n_1-i} \mathcal{Y}_i(v,w)\right] E_n \tag{17}$$

where $E_n = (0,...,0,1)^T$ is the column vector of dimension $n+a-i+1$.
Since $\hat{P}_{il}(v) = \int_0^{+\infty} e^{-vt} P_{il}(t)dt$, for $(i,l) \in \mathbf{E}_{n,a}$, it corresponds to the l-th (after zero) element of the vector $\hat{P}_i(v)$. Then, using (16) and (17) we obtain

$$\begin{aligned} \hat{P}_{il}(v) &= \sum_{k_1,...,k_{n-i}=0}^{n+a-i-1} \prod_{w=0}^{n-i} [\mathcal{Y}_i(v,w)]_{k_w k_{w+1}} \\ &= \sum_{k_1,...,k_{n-i}=0}^{n+a-i-1} \prod_{w=0}^{n-i} [\mathcal{X}_i(v,w)]_{k_w k_{w+1}} \prod_{w=0}^{n-i-1} f_{i+w+1,l_{w+1}-w-1} \end{aligned} \tag{18}$$

with $k_w = l_w - w$, $l = l_0 \leq l_1 \leq ... \leq l_{n-i+1} = n+a-i$ and $l_w \geq w$ for $w = 1,...,n-i$.
Injecting (15) in (18) we find, by considering the sets defined in (8)-(10) and after some tedious manipulations, that for $v \geq 0$ and $(i,l) \in \mathbf{E}_{n,a}$

$$\begin{aligned} \hat{P}_{il}(v) &= \sum_{L \in \hat{D}^{n-i}_{ln+a-i}} \prod_{w=0}^{n-i} C_{i+w}(v, l_w - w, l_{w+1} - w) \prod_{w=1}^{n-i} fi + w, l_w - w \\ &\overset{(14)}{=} \sum_{L \in \hat{D}^{n-i}_{ln+a-i}} \frac{\prod_{w=0}^{n-i} \mu^{l_{w+1}-l_w} \dfrac{(l_{w+1}-w)!}{(l_w-w)!} \prod_{w=1}^{n-i} f_{i+w,l_w-w}}{\prod_{w=0}^{n-i} \prod_{k=0}^{l_{w+1}-l_w} (v + \mu(l_w - w + k) + f_{i+w,l_w-w+k})} \\ &= \mu^{n+a-i-l} \frac{a!}{l!} \sum_{L \in \hat{D}^{n-i}_{ln+a-i}} \frac{\prod_{w=1}^{n-i} (l_w - w + 1) f_{i+w,l_w-w}}{\prod_{(w,k) \in D_{n-i}} (v + \mu(l_w - w + k) + f_{i+w,l_w-w+k})} \end{aligned}$$

■

In order to derive an expression for $\hat{P}_{il}(v)$ which will be used to obtain a simple explicit expression for $P_{il}(t)$, the following formula will be useful. Let $a_i, x \in I\!R$. If $a_i \neq a_j$ for $i \neq j$, then it can be proved that

$$\prod_i \frac{1}{a_i + x} = \sum_i \frac{b_i}{a_i + x} \tag{19}$$

where $b_i = \prod_{k \neq i} (a_k - a_i)^{-1}$.

We now assume that μ and f_{xy} have been chosen to satisfy

$$\mu l + f_{il} \neq \mu h + f_{jh}, \text{ if } (i,l) \neq (j,h), \text{ for } (i,l),\ (j,h) \text{ in } \mathbf{E}_{na}, \text{ with } h \neq 0. \tag{20}$$

Notice that the above restriction is similar to the one placed on the relative removal rate by Krysco [10] and Ball and O'Neill [2]. It is a necessary and sufficient condition to ensure that the solution to (3) is a linear exponential combination of terms with constant coefficients. But by straightforward computations, one can prove that the assumption (20) is always verified for the cases $f_{xy} = \beta xy$ and $f_{xy} = \beta \frac{xy}{x+y}$ considered by the above authors when $\frac{\beta}{\mu}$ is in $I\!R^+ \backslash \mathbb{Q}$. The requirement in (20), together with (6), (7) and (19), imply that

$$\hat{P}_{il}(v) = \mu^{n+a-i-l} \frac{a!}{l!} \sum_{L \in \hat{D}^{n-i}_{ln+a-i}} \left[\sum_{(w,k) \in D_{n-i}} \frac{g(i,w,k)}{v + f(i,w,k)} \right] \times$$
$$\times \prod_{w=1}^{n-i} (l_w - w + 1) f_{i+w, l_w - w}, \tag{21}$$

with the convention that

$$\prod_{p \in B} Ap = 1 \text{ and} \sum_B 1 = 1 \text{ if B} = \emptyset \text{ and } A_p > 0. \tag{22}$$

where

$$f(i,w,k) = \mu(l_w - w + k) + f_{i+w, l_w - w + k}, \tag{23}$$

and

$$g(i,w,k) = \prod_{(w',k') \neq (w,k)} [f(i,w',k') - f(i,w,k)]^{-1}. \tag{24}$$

Finally, by applying the Laplace transform inversion formula to (21), we can readily show the following theorem:

Theorem 2
Let $i = 0, ..., n$ and $l = 0, ..., n+a-i$. Assume that (20) is satisfied. Then for $t \geq 0$

$$P_{nl}(t) = \mu^{a-l} \frac{a!}{l!} \sum_{k=l}^{a} \prod_{k' \neq k} [\mu(k - k') + f_{n,k} - f_{n,k'}]^{-1} \mathbf{e}^{-(\mu k + f_{n,k})t}, \tag{25}$$

and if $i = 0, ..., n-1$ *then*

$$P_{il}(t) = \mu^{n+a-i-l}\frac{a!}{l!}\sum_{L\in\hat{D}^{n-i}_{ln+a-i}}\sum_{(w,k)\in D_{n-i}}\left[\frac{g(i,w,k)\prod_{w=1}^{n-i}(l_w-w+1)f_{i+wl_w-w}}{\mathbf{e}^{f(i,w,k)t}}\right] \tag{26}$$

where for $w = 0, ..., n-i$ *and* $k = 0, ..., l_{w+1} - l_w$, *the quantities* $f(i, w, k)$ *and* $g(i, w, k)$ *are defined by* (23) *and* (24) *respectively with* $(l_0, l_{n+a-i}) = (l, n + a - i)$.

In particular, if we replace f_{xy} by βxy and $\beta xy/(x+y)$ in (25) and (26), we will obtain two expressions similar in form to those required by Krysco [10] and Ball and O'Neill [2]. However, the exact relationship between Krysco's or Ball and O'Neill's solutions and (26) is more difficult to determine.

4 Duration and number of infectives

Let

$$T_{na} = \inf\{t \geq 0, Y(t) = 0\}$$

be the duration of the epidemic defined as the duration of the time between the start of the epidemic and the moment at which the number of infectives becomes zero. From Theorem 2, we can obtain the distribution of T_{na} explicitly by considering he convention (22)

$$\begin{aligned}\Pr\quad\{T_{na}\leq t\} &= \Pr\{Y(t)=0\} = \sum_{i=0}^{n}\Pr\{X(t)=i,Y(t)=0\} = \sum_{i=0}^{n}P_{i0}(t)\\ &\overset{(25)-(26)}{=} a!\sum_{i=0}^{n}\sum_{L\in D^{i}_{0,a+i}}\left[\sum_{(w,k)\in\hat{D}_i}\frac{g(n-i,w,k)}{\mathbf{e}^{f(n-i,w,k)t}}\right]\prod_{w=1}^{i}(l_w-w+1)f_{n-i+w,l_w-w}.\end{aligned}$$

On another hand, if we suppose that $\mathbf{E}(T_{na}) < \infty$, then we have

$$\begin{aligned}\mathbf{E}(T_{na}) &= -\sum_{i=0}^{n}\frac{\partial(v\hat{P}_{i0}(v))}{\partial v}\mid_{v=0}\\ &= a!\sum_{i=0}^{n}\sum_{L\in\hat{D}^{i}_{0,a+i}}\left[\sum_{(w,k)\in\hat{D}_i}\frac{g(n-i,w,k)}{(f(n-i,w,k))}\right]\times\\ &\qquad\prod_{w=1}^{i}(l_w-w+1)f_{n-i+w,l_w-w}.\end{aligned}$$

Using similar, but more complicated, arguments we get the following expression for $\mathbf{E}(T^2_{na})$,

$$\mathbf{E}(T^2_{na}) = -2\times\sum_{i=0}^{n}\frac{\partial^2(v^2\bar{P}_{i0}(v))}{(\partial v)^2}\mid_{v=0}$$

$$= \quad 2 \times a! \sum_{i=0}^{n} \sum_{L \in \hat{D}^i_{0,a+i}} \left[\sum_{(w,k)\in \hat{D}_i} \frac{g(n-i,w,k)}{(f(n-i,w,k))^2} \right] \times \prod_{w=1}^{i} (l_w - w + 1) f_{n-i+w, l_w - w}.$$

where $\bar{P}_{i0}(v) = \int_0^{+\infty} e^{-vt} t P_{i0}(t)\, dt$ and $\hat{D}_i = D_i \setminus \{(0,0)\}$.
Finally we can derive the mean and variance of the number of infectives easily from (26).
To illustrate these results, we consider the modified model (see [2] and [15]) used for AIDS modelling in which the infection rate is written as $f_{il} = \frac{\beta il}{i+l}$.
Figure 1 shows how the mean and standard deviation of T_{na} vary as functions of $\rho = \frac{\mu}{\beta}$, the relative removal rate (a : number of initial infectives). In Figure 2 we plot the mean and standard deviation of T_{na} as function of the initial number of infectives. Figure 3 illustrates the expectation number and the standard deviation of the number of infectives. In this case some, quantities of interest are plotted over time.

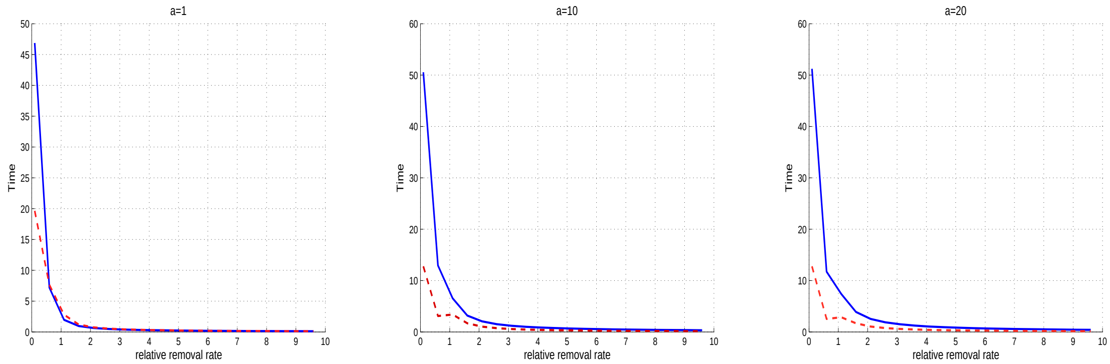

Figure 1: mean : — , standard deviation :−−, $n = 60$.

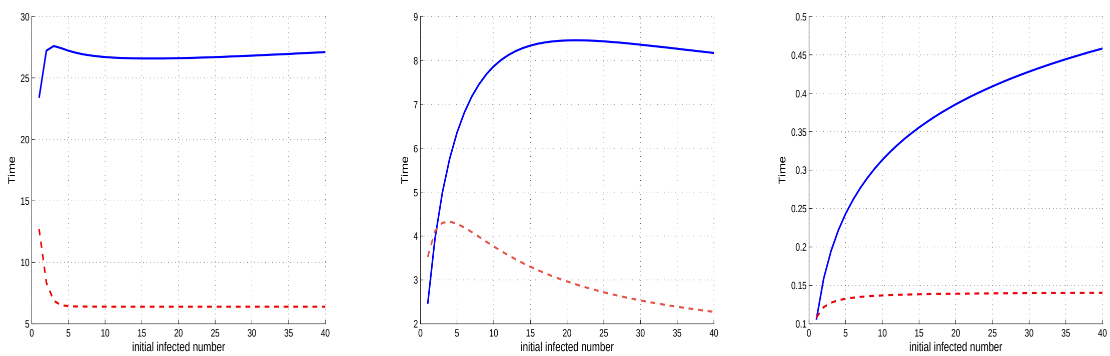

Figure 2: mean : — , standard deviation :−−, $n = 60, \rho = 0.2,$ 1 and 10.

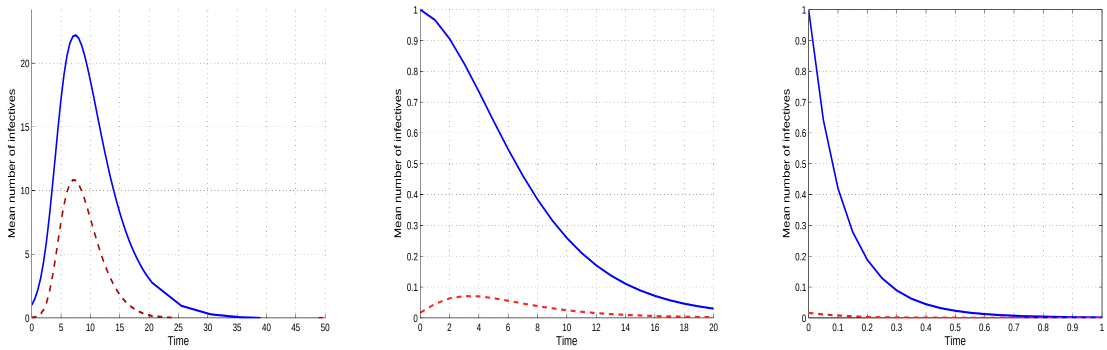

Figure 3: mean : — , standard deviation :−−, $n = 60, \rho = 0.2$, 1 and 10.

5 Discussion

This note is devoted to the problem of evaluating the time-dependent transition probabilities of the SIR stochastic model in the case of a generalized infection mechanism. The method considered here can be generalized to include the SIR epidemic in a population consisting of two interacting groups of individuals. Also the quantities derived here can be used to solve algebraically many problems related to threshold behavior of the epidemic as it is the case, under some conditions, with the convergence in distribution to birth-death processes. One can also show the threshold theorem (see, for instance, [19] and [14]). Other quantities of interest in the model are often expressible in terms of these functions. For example, Billard ([3], section 4) indicates how the transition probabilities might be used to evaluate the factorial moment of the number of susceptibles. As another example, suppose that $f_{xy} = \beta f(x, y)$ and that we are interested in estimating the contact rate β and the removal rate μ when the epidemic is observed on a finite number of fixed time points. Then Severo [18] and Omari [12] show that the likelihood function of the unknown parameters may be expressed in terms of the transition probabilities of the process. Thus, at least in principle, the search for the maximum likelihood estimates of the unknown parameters β and μ is just a matter of computation.

Appendix

In what follows we give a proof of (13). We define $\triangle_i$ as the off-diagonal matrix of the rank $n + a - i + 1$, with the $(k, k + 1 = 1)$ element is equal to 1 for $k = 1, ..., n + a - i + 1$. By using the matrices defined in section (3) we see that $C_i = \triangle_i A_i$ and $F_i = -(D_i + A_i) + \triangle_i A_i$ so it follows that

$$\begin{aligned}(vI_i - F_i)^{-1} &= (vI_i + D_i + A_i - \triangle_i A_i)^{-1} \\ &= (I_i - (vI_i + D_i + A_i)^{-1} \triangle_i A_i)^{-1} \times (vI_i + D_i + A_i)^{-1}. \qquad (A.1)\end{aligned}$$

The off diagonal of $\triangle_i$ and the diagonal form of $M_i(v) = (vI_i + D_i + A_i)^{-1}$ imply that $[vI_i + D_i + A_i)^{-1} \triangle_i A_i]^l = 0$ for all $l > n + a - i$. Hence,

$$(I_i - M_i(v) \triangle_i A_i)^{-1} = \sum_{l=0}^{n+a-i} [M_i(v) \triangle_i A_i]^l = R_i(v).$$

Let $[R_i(v)]_{lh}$ and$[M_i(v)]_{lh}$ be respectively, the $(l,h) - th$ entries of the matrix $R_i(v)$ and $M_i(v)$ of rank $n + a - i + 1$. Since for $k = 0, ..., n + a - i$, the $(l,h) - th$ elements of $(M_i(v) \triangle_i A_i)^k$ are equal to

$$[M_i(v)]_{ll+1}(A_i)_{l+1,l+1}[M_i(v)]_{ll+1}(A_i)_{l+1,l+1}...[M_i(v)]_{l+k-1l+k+1}(A_i)_{l+k,l+k}$$

if $h = l + k$ and are equal to zero otherwise, then for $0 \le l \le h \le n + a - i$ we obtain

$$[R_i(v)]_{lh} = \prod_{k=l}^{h-1} [M_i(v)]_{kk}(A_i)_{k+1,k+1}. \qquad (A.2)$$

Finally, by injecting (A.1) in (A.2), we have for $l \le h \le n + a - i$

$$\begin{aligned}
[(vI_i - F_i)^{-1}]_{lh} &= [R_i(v)]_{lh}(v + \mu h + f_{ih})^{-1} \\
&= \prod_{k=l}^{h-1} (v + \mu k + f_{ik})^{-1} \mu(k+1).(v + f_{ih} + \mu h)^{-1} \\
&= \mu^{h-l} \frac{h!}{l!} \prod_{k=l}^{h} (v + \mu k + f_{ik})^{-1} \\
&= C_i(v, l, h).
\end{aligned}$$

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