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ENVELOPE AND ITS DISTRIBUTIONS

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Abstract. We review the concept of envelope of a random signal and discuss the most recent advances on the subject. In an approach to approximate the first passage probability for the underlying response the average number of envelope crossings is used to obtain an upper bound. We give a brief account of the method. The envelope field is a generalization of the envelope process for which we discuss its sampling distributions. One intrinsically multivariate problem is studying velocities of moving spatial records. Sampling properties of the envelope velocity measured at the level contours has been developed for the Gaussian model. We also discuss how the results known in the Gaussian case can be extended to non-Gaussian models that are constructed using as moving averages with respect to Laplace motion. This review paper is based on the recent contributions by the authors and their collaborators.

1. INTRODUCTION

The decomposition of traveling random waves into the envelope (low frequency varying amplitude) and the carrier (high frequency oscillations) has been introduced in [11]. The complex envelope has the underlying signal as its real component and the Hilbert transform as its imaginary part. The real envelope process which is the norm of the complex one smooths the underlying process and at high levels it generally follows its height.

The average number of envelope upcrossings gives a more precise upper bound for the probability that the maximum of the process exceeds a certain level than the averaging upcrossings of the original signal. The idea was first explored in [16] and later in [7], [6], where the Slepian model of the process at the envelope upcrossing was used to approximate more accurately the proportion of empty excursions.

One of the risks in offshore operations is the possibility of undesired responses of the vessels which can result in capsizing. This events have higher chances to occur if the speed of the vessel is comparable to the wave velocity and the time spend in a large wave group is long. It is often reported that groups of waves do more damage than waves of the same size but separated

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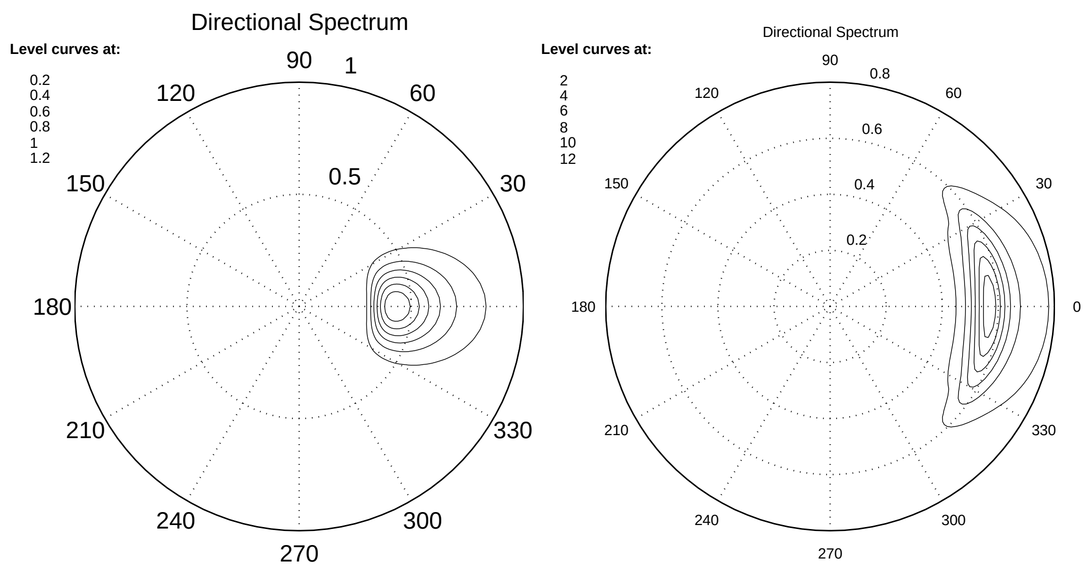

FIGURE 1. Examples of directional spectra: swell spectrum (left); JONSWAP spectrum used in examples (right).

by smaller waves (see, for example, [5]). This is partially explained by the fact that energy propagate with the rate corresponding to the speed of waves groups. For deep water waves this rate is slower than the speed of individual waves and it can be demonstrated by physical arguments that for waves having narrow band spectra it is the envelope that is responsible for the transport of energy. Therefore statistical properties of wave groups are important for ocean engineers and some aspects of these problems can be studied by analyzing the wave group velocity in the manner as it is presented in this paper.

Despite that in the original work of Longuet-Higgins the envelope was proposed for moving random surface, most of the future work was done for the envelope of univariate random processes with the notable exception of [2] and [3], where the definition and some properties of the bivariate envelope field were discussed. In [12], this original work has been extended with a focus on applications to studies of Gaussian sea surfaces. We present the summary of the obtained results.

In truly two dimensional set-up where even individual waves are hard to describe in a formal manner, the notion of wave groups escapes a precise definition. On the other hand the envelope field is defined in an arbitrary dimension and its properties naturally extend from the one-dimensional case. Take for example the sea surface given by the swell spectrum shown in Figure 1. The difference between dynamics of surface and envelopes can be illustrated by recording contour movements in two time instants within $5[s]$. For each of the field, let us consider the contours crossing the significant crest height level (the significant wave height, in this case, is $2.2[m]$, thus the crossing level or the significant crest height is $1.1[m]$ above the mean sea level). The following important features are observed in Figures 2:

- the envelope field is grouping the waves,
- the envelope appears to move slower than the sea surface,
- the level crossing contours for the envelope are more stable, appearing mostly to drift with no rapid change in shape or size,
- waves entering the envelope contours are growing while these which are leaving are diminishing – expected behavior when the waves group are moving slower then the individual waves.

The above example and other numerical computations contributing to the paper have used the MATLAB toolbox WAFO – Wave Analysis in Fatigue and Oceanography – containing a comprehensive package of numerical programs for statistical analysis of random waves. This toolbox is available free of charge at `http//www.maths.lth.se/matstat/wafo`.

Extensions and analysis of the envelope beyond Gaussian processes and fields are of interest as many real signals (for example, severe sea states) show considerable non-Gaussian features. In [1], non-Gaussian random processes that are moving averages with respect to Laplace motion have been proposed as an alternative models that allow for skewness and heavier than normal tails. This class of second order processes is reviewed and the envelope process for these non-Gaussian models is discussed including possible directions of future research.

2. GLOBAL MAXIMUM AND ENVELOPE EXCURSIONS

In engineering applications such as safety analysis of offshore structures, the distribution of maximum of a signal $X(t)$ is of interest. The Rice method uses the following upper bound for the distribution of the maximum

$$\mathrm{P}(M_T > u) \le \mathrm{P}(X(0) > u) + \mathrm{P}\left(N_T^+(u) > 0\right) \le \mathrm{P}(X(0) > u) + \mu^+(u)\cdot T, \tag{1}$$

where $M_T = \max_{0\le t\le T} X(t)$, $N_T^+(u) = \#\{t \in [0,T] : X(t) = u, \dot{X}(t) > 0\}$ and the upcrossing intensity $\mu^+(u) = \mathrm{E}\left(N_T^+(u)\right)/T$. For a stationary process X the celebrated Rice formula states that

$$\mu^+(u) = \mathrm{E}\left(\dot{X}^+(0)\Big| X(0) = u\right) f_{X(0)}(u), \tag{2}$$

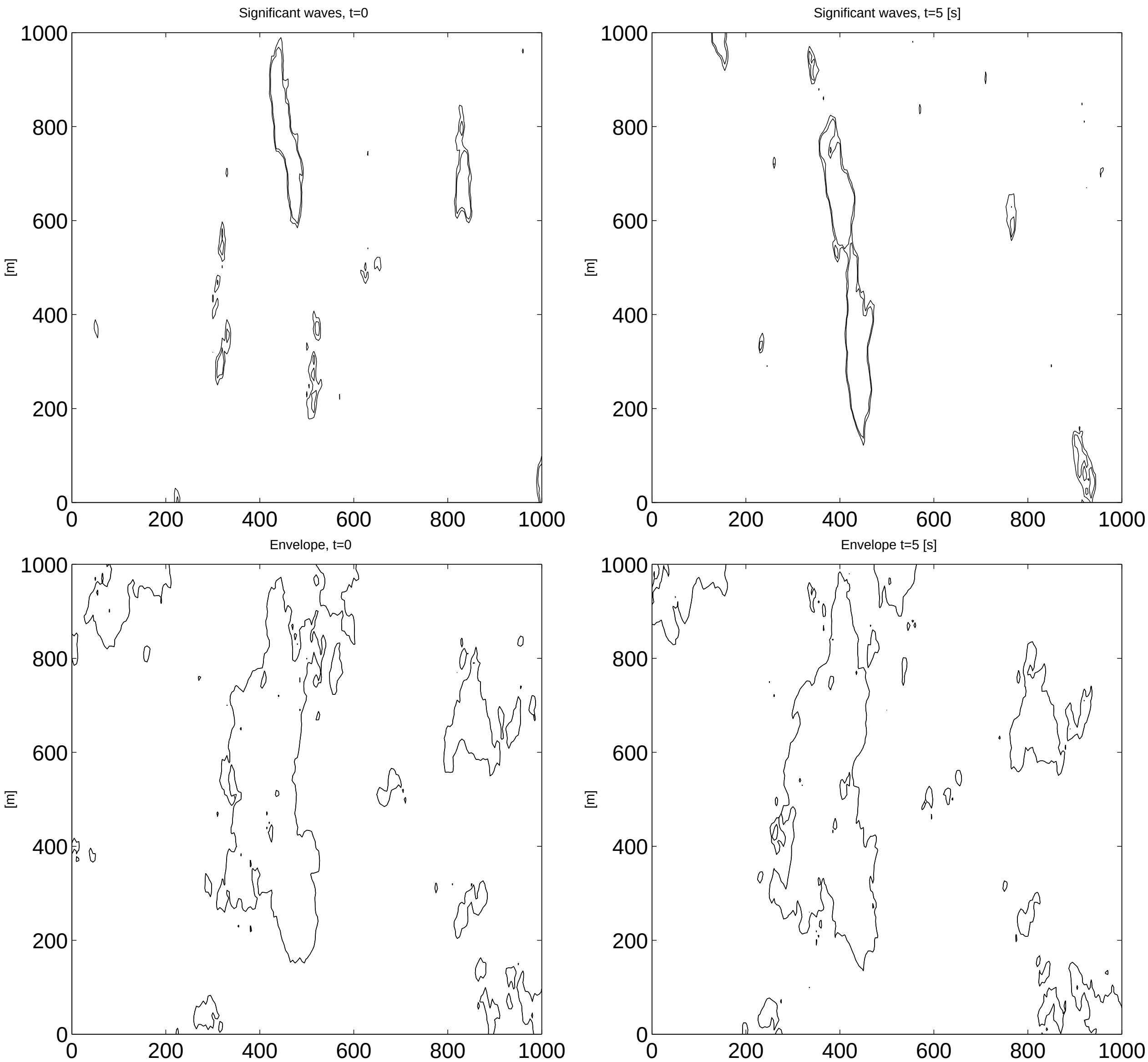

FIGURE 2. Motion of level crossing contours for sea surface (top) and envelop (bottom) – the principal direction of wave movement is from the right to the left.

where $x^+ = \max(0, x)$ and $f_{X(0)}$ is the density of $X(0)$. In the Gaussian zero mean case

$$\mu^+(u) = \frac{1}{2\pi}\sqrt{\frac{\lambda_2}{\lambda_0}}e^{-\frac{u^2}{2\lambda_0}},$$

where λ_i's are the spectral moments of the process.

An apparent wave group is difficult to define rigorously although for a narrow band process (swell) groups of waves are clearly seen in the records. They can be fairly well identified through subsequent crossings of high level by the envelope that is defined as

$$E(t) = \sqrt{X^2(t) + \hat{X}^2(t)},$$

where $\hat{X}(t)$ is Hilbert transform of $X(t)$.

The envelope $E(t)$ is always above the process $X(t)$ thus we have an obvious relation

$$\mathrm{P}(N_T^+(u) > 0) \leq \mathrm{P}(E(0) > u) + \mathrm{P}(\mathcal{N}_T^+(u) > 0),$$

where $\mathcal{N}_T^+(u)$ stands for upcrossings of the envelope. The above can be improved using the number $\mathcal{N}_{0,T}^+(u)$ of empty envelope excursions

$$\begin{aligned} \mathrm{P}\left(N_T^+(u) > 0\right) &= \mathrm{P}(E(0) > u, N_T^+(u) > 0) + \mathrm{P}\left(\mathcal{N}_T^+(u) - \mathcal{N}_{0,T}^+(u) > 0\right) \\ &\leq \mathrm{P}(E(0) > u) + \mathrm{E}\left(\mathcal{N}_T^+(u)\right) - \mathrm{E}\left(\mathcal{N}_{0,T}^+(u)\right). \end{aligned}$$

In view of the preceding comments, the following may be an improvement over (1):

$$\mathrm{P}(M_T > u) \leq \mathrm{P}(E(0) > u) + \mathrm{P}(X(0) > u) + (\nu^+(u) - \nu_0^+(u)) \cdot T, \tag{3}$$

where ν^+ and ν_0^+ are intensities of upcrossings and empty upcrossings, respectively.

The intensities of upcrossings of the envelope can be evaluate from Rice's formula and is given by

$$\nu^+(u) \;=\; \sqrt{\frac{2\pi(1-\rho^2)}{\lambda_0}} \cdot u \cdot \frac{e^{-\frac{u^2}{2\lambda_0}}}{T_2},$$

where $T_2 = 2\pi\sqrt{\frac{\lambda_0}{\lambda_2}}$ and $\rho^2 = \lambda_1^2/(\lambda_0\lambda_2)$.

Evaluating $\nu_0^+(u)$ that is used in (3) is a more challenging task. An approach that allows for simplification of this computationally difficult problem has been successfully implemented in [7]. The method is based on the Slepian model at envelope upcrossing. In [7] using Taylor expansions of this process for the Slepian model at the exit form the circle by the complex envelope the following approximation has been obtained

$$\frac{\nu_0^+(u)}{\nu^+(u)} \approx 2\int_0^u \phi(\eta)\left(1 - \sqrt{2\pi}\frac{\Phi\left(\gamma\pi\frac{u^2-\eta^2}{u} - \frac{1}{2}\right)}{\gamma\pi\frac{u^2-\eta^2}{u}}\right)\, d\eta. \tag{4}$$

where ϕ and Φ are the standard normal density and distribution function, respectively.

3. ENVELOPE FOR NON-GAUSSIAN PROCESSES

Generalized Rice formulas remain valid for quite arbitrary processes – for example (2) can be computed as long as joint distribution of $\dot{X}(0)$ and $X(0)$ is known. Thus in principle the sampling distribution and crossing intensity problems for envelope can be studied in a fairly general situation. However, the envelope distributions has not been considered except for models based on Gaussian processes. In the sea surface elevetion it is partially due to the fact that until recently very few essentially non-Gaussian models has been found adequate. In [1], the model based on Laplace distributions has been proposed to model such non-Gaussian features observed in the sea elevation as asymmetry and heavy tails. Using this model a successful analysis of the fatigue caused by such non-Gaussian loads to an offshore structure has been performed. This motivates studies of the envelope for this model and here after a brief introduction to the model, the envelope for such processes is described and an approach to computing sampling distributions is discussed.

Let us consider independently scattered random measure Λ such that $\Lambda(A)$ has the generalized asymmetric Laplace distribution (see [9]). This random measure is parametrized by ν, μ, σ γ, controlling shape, asymmetry, scale, and location, respectively. It follows from the general theory that if f and f^2 are integrable the following process

$$X(t) = \int_{-\infty}^{\infty} f(t-x)\, d\Lambda(x), \tag{5}$$

is strictly stationary, second order and ergodic. We refer to it as the *Laplace driven moving average* (LMA).

One can show that the spectrum $S(\omega)$ of LMA is given by

$$S(\omega) = \frac{\sigma^2 + \mu^2}{\nu} |\mathcal{F}f(\omega)|^2, \tag{6}$$

where $\mathcal{F}$ denotes the Fourier transform. This means that by choosing f one can in principle model any spectrum. Two extra parameters as compared with its Gaussian counterpart can be used to fit skewness s and excess kurtosis κ of the marginal distribution of $X(t)$. Note that for a Gaussian process both skewness and excess kurtosis equal zero, i.e. $s = \kappa = 0$. In fact, a Gaussian process can be obtained from the Laplace driven MA as a limiting case as $s = 0$ and $\kappa \to 0$, see [9] (page 183).

In order to compute the upper bound (1) for LMA one can evaluate the upcrossing intensity using the inverse Fourier transform

$$\mu^+(u) = \frac{1}{(2\pi)^2}\int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty z\, e^{-i(v_1 u + v_2 z)} \phi_{X(0),\dot{X}(0)}(t_1, t_2)\, dv_1\, dv_2\, dz.$$

Thus the upcrossing intensity is not given by an explicit formula and must be evaluated numerically. Nevertheless, it can be done quite effectively by the means of the FFT and thus in many practically important situations the sampling distributions for LMA can be effectively obtained.

The envelope process, if intended to follow the original signal along both maxima and minima, applies only to symmetric records. If we define $h(s) = 1/(\pi s)$, then the Hilbert transform $\hat{X}(t)$ of the signal $X(t)$ is defined through the convolution

$$\hat{X}(t) = X * h(t) = \int_{-\infty}^\infty \frac{X(t-s)}{\pi s} ds.$$

For a moving average process it leads to

$$\hat{X}(t) = \int_{-\infty}^\infty \hat{f}(t-x) d\Lambda(x),$$

where $\hat{f}$ stand for the Hilbert transform of f. The Hilbert transform has the same spectrum as the original one and at a fixed point t the original signal $X(t)$ is uncorrelated with $\hat{X}(t)$. However, $X(t)$ is no longer independent of $\hat{X}(t)$. but Nevertheless, since the Hilbert transform of a LMA is again a LMA and that their joint distributions can be obtained through inverting characteristic functions. Therefore the distributions of envelope that is defined by $E(t) = \sqrt{X^2(t) + \hat{X}^2(t)}$ can be also obtained and its crossing distributions studied as in the Gaussian case, although one should not expect explicit formulas for the crossing intensities. For a symmetric LMA, i.e. when $\mu = \gamma = 0$, this usual definition of envelope should lead to similar properties as in the Gaussian case.

For an asymmetric signal, it would more appropriate to use upper and lower envelope that correspond to maxima and minima of the signal, respectively. One can do it by considering the positive and negative part of the centered signal and their envelopes. In the case of LMA, one can utilize for this purpose the following representation of the Laplace motion $\Lambda(u) = B(\Gamma(u)) + \mu\Gamma(u) + \gamma u$, where B is the Brownian motion with the scale parameter σ and $\Gamma(u)$ is the gamma process. Then one can consider the envelope of symmetric signal $\int f(t-u)\, dB(\Gamma(u))$ which corresponds to the symmetric LMA and treat separately the signal $\int f(t-u)\, d\Gamma(u)$. An alternative approach can utilize the representation $\Lambda(u) = \Gamma_1(u) - \Gamma_2(u)$, where Γ_i are independent gamma processes with different scale parameters. Here one can consider $\Lambda_1(u) = \Gamma_1(u) - \tilde{\Gamma}_1(u)$ and $\Lambda_2(u) = \Gamma_2(U) - \tilde{\Gamma}_2(u)$, where $\tilde{\Gamma}_1$ and $\tilde{\Gamma}_2$ are independent versions of Γ_1 and Γ_2, respectively. Then the envelope for $\int f(t-u)\, d\Lambda_1(u)$ could be consider the upper envelope while the envelope for $\int f(t-u)\, d\Lambda_2(u)$ as the lower envelope. Which of this approaches is more appropriate may depend on the application in hand and the formal approach to the envelope of asymmetric signals requires some further studies.

4. ENVELOPE FIELD FOR GAUSSIAN SEA SURFACE

Let $\{X(\boldsymbol{\tau})\}_{\boldsymbol{\tau}\in\mathbb{R}^3}$ be a stationary Gaussian field, where $\boldsymbol{\tau} = (\mathbf{p}, t) = (x, y, t)$ is a point in $\mathbb{R}^3$. The covariance function $R(\boldsymbol{\tau}) = \mathrm{Cov}(X_{\boldsymbol{\tau}_0+\boldsymbol{\tau}}, X_{\boldsymbol{\tau}_0})$ can be written in the form

$$R(\boldsymbol{\tau}) = \int_{\mathbb{R}^3} \exp(i\boldsymbol{\lambda}^T\boldsymbol{\tau})\sigma(\boldsymbol{\lambda})d\boldsymbol{\lambda} = 2\int_{\Lambda^+} \cos(\boldsymbol{\lambda}^T\boldsymbol{\tau})\sigma(\boldsymbol{\lambda})d\boldsymbol{\lambda}, \tag{7}$$

where $\sigma(\boldsymbol{\lambda})$ is the *spectral density* of $X(\boldsymbol{\tau})$.

The process $X(\boldsymbol{\tau})$ has the following spectral representation

$$X(\boldsymbol{\tau}) = \int_{\mathbb{R}^3} \exp(i\boldsymbol{\lambda}^T\boldsymbol{\tau})d\zeta(\boldsymbol{\lambda}) = 2\Re\left(\int_{\Lambda^+} \cos(\boldsymbol{\lambda}^T\boldsymbol{\tau})d\zeta(\boldsymbol{\lambda})\right), \tag{8}$$

where the process $\zeta(\boldsymbol{\lambda})$ is complex valued with zero mean, orthogonal increments, and such that $E(|\zeta(A)|^2) = \int_A \sigma(\boldsymbol{\lambda})d\boldsymbol{\lambda}$, Λ^+ is any set in $\mathbb{R}^3$ such that $-\Lambda^+ \cap \Lambda^+$ has measure zero and $-\Lambda^+ \cup \Lambda^+ = \mathbb{R}^3$. An example of such a set is $\Lambda^+ = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 \geq 0\}$.

The *spectral moments* λ_{ijk}, if they are finite, are defined as

$$\lambda_{ijk} = 2\int_{\Lambda^+} \lambda_1^i \lambda_2^j \lambda_3^k \, d\sigma(\boldsymbol{\lambda}), \tag{9}$$

where $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3)$. The variance of the field can be expressed in terms of spectral moments as the zero moment λ_{000}. If $i + j + k$ is even, then λ_{ijk} does not depend on a choice of Λ^+ because of the symmetry of σ. However for odd $i + j + k$, different Λ^+ can, in general, lead to different λ_{ijk} which is important when the distribution of the envelope field is discussed.

In oceanography coordinates of $\boldsymbol{\tau}$ corresponds to spatial and temporal coordinates, i.e. $\boldsymbol{\tau} = (x, y, t)$, while for the sea surface instead of $X(\boldsymbol{\tau})$ we use the notation $W(x, y, t)$.

The problem with spectra for the sea surface is that they are degenerated in the full three dimensional space. This is due to the dispersion relation which reduce dimension of the spectral domain by one and its spectrum is defined by the unitary directional spectrum $S(\omega, \theta)$.

The symmetry of σ translates to $S(T(\omega,\theta)) = S(\omega,\theta)$, where

$$T(\omega,\theta) = \begin{cases} (-\omega, \theta-\pi) & \text{if } \theta \in [0,\pi], \\ (-\omega, \theta+\pi) & \text{if } \theta \in [-\pi, 0). \end{cases} \tag{10}$$

The set Λ^+ corresponds in the reduced domain to $\Gamma^+ = \boldsymbol{\lambda}^{-1}(\Lambda^+)$ and a choice of $\Lambda^+ \subseteq \mathbb{R}^3$ is equivalent to a choice of $\Gamma^+ \subset \mathbb{R} \times (-\pi,\pi]$ such that the intersection $T(\Gamma^+) \cap \Gamma^+$ has zero Lebesgue measure and $T(\Gamma^+) \cup \Gamma^+ = \mathbb{R} \times (-\pi, \pi]$.

Define $\hat{h}(\boldsymbol{\lambda}) = i(\mathbf{1}_{\Lambda^+}(-\boldsymbol{\lambda}) - \mathbf{1}_{\Lambda^+}(\boldsymbol{\lambda}))$ and the Hilbert transform of $X(\boldsymbol{\tau})$ by

$$\hat{X}(\boldsymbol{\tau}) = \int_{\mathbb{R}^3} \exp\left(i\boldsymbol{\lambda}^T\boldsymbol{\tau}\right) \hat{h}(\boldsymbol{\lambda}) d\zeta(\boldsymbol{\lambda}).$$

The (real) envelope process is defined by

$$E(\boldsymbol{\tau}) = |\mathcal{E}(\boldsymbol{\tau})| = \sqrt{X^2(\boldsymbol{\tau}) + \hat{X}^2(\boldsymbol{\tau})}.$$

Let us turn to the special case of the process $X(\boldsymbol{\tau}) = W(x,y,t)$ which represent the sea surface. Note that because of the symmetry of σ the statistical distributions of W (or its Hilbert transform when considered separately) do not depend on the choice of Λ^+. However, one has to bear in mind that the choice of Λ^+ affects distributional properties of the envelope. For example, the values of spectral moments λ_{ijk} are affected by Λ^+ and, as a result, also the joint distributions of W, $\hat{W}$. In a given application it maybe important to choose Λ^+ in such a way that the envelope process will possess natural or desirable properties. While in general there are infinitely many such choices some additional symmetries of random sea surface suggest the following "natural" one $\Lambda^+ = \{(x_1,x_2,x_3) \in \mathbb{R}^3 : x_3 \geq 0\}$.

The Hilbert transform has the same unitary spectrum and thus the same distribution as the original field W. Also evaluated at the same fixed point $(\mathbf{p},t)$ the two random variables $W(\mathbf{p},t)$ and $\hat{W}(\mathbf{p},t)$ are independent. However stochastic fields W and $\hat{W}$ are dependent. For example, the covariances between derivatives of W and $\hat{W}$ are given through the spectral moments of W. We have already remarked that these covariances are affected by a choice of Λ^+ and thus so is the dependence structure of W and $\hat{W}$.

Note that the *real envelope field* $E(\mathbf{p},t)$ is positive and always stays above the sea surface. It is also depending on a choice of Λ^+ because of the dependence between W and $\hat{W}$. The following subsection illustrate the importance of a choice of Λ^+ for modelling the sea-surface elevation.

Let us consider the intensity of up-crossings of a u-level by the envelope in the direction $y = 0$. We note that the intensity crossing is dependent on a chosen direction on the plane

TABLE 1. VERSIONS OF ENVELOPE IN TERMS OF Λ^+ and Γ^+.

Λ^+	Γ^+	Comment
$x_1 \geq 0$	$\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$	Maximizes ρ^2, minimizes crossing intensity
$x_2 \geq 0$	$\theta \in [0, \pi]$	Maximizes crossing intensity, $\rho^2 = 0$
$x_3 \geq 0$	$\omega \geq 0$	Natural choice for the sea surface.

although we will not indicate it in our notation and thus we will "reuse" the notation $\nu^+(u)$ for this intensity. Thus

$$\begin{aligned} \nu^+(u) &= \mathrm{E}(E_x^+(\mathbf{0})|E(\mathbf{0}) = u) \cdot \frac{u}{\lambda_{000}} e^{-\frac{u^2}{2\lambda_{000}}}, \\ &= \sqrt{\frac{\lambda_{200}}{2\pi}} \sqrt{1-\rho^2} \cdot \frac{u}{\lambda_{000}} e^{-\frac{u^2}{2\lambda_{000}}},\ u > 0, \end{aligned}$$

where $\rho^2 = \lambda_{100}^2/(\lambda_{000}\lambda_{200})$ (see also [10]). The highest intensity is reached for the level $u = \sqrt{\lambda_{000}}$ often called the reference level for the envelope, and is equal

$$\nu^+_{max} = \sqrt{\frac{\lambda_{200}}{2\pi \cdot e \cdot \lambda_{000}}} \sqrt{1-\rho^2}.$$

We observe that the intensity of envelope crossing in the direction $y = 0$ depends on the choice of Λ^+ only through λ_{100} and in such a way that larger $|\lambda_{100}|$ (larger the squared correlation ρ^2) corresponds to lower crossing intensity. If one is interested in an envelope that is smoother than the sea surface and following closer to the local extremes, then a reasonable choice of Λ^+ is the one that minimizes crossing intensity and thus maximizes λ_{100}. Clearly, the choice will depend on the form of a spectrum in hand. Some properties of diferent choices for JONSWAP spectrum of Figure 1 are presented in Table 1.

5. DISTRIBUTIONS OF WAVE AND ENVELOPE VELOCITIES

There is a variety of ways to introduce velocity for dynamically changing surface (see [4]). Here for simplicity we focus on the velocity describing the motion of a contour level in the specified direction given by an azimuth α. We define this velocity by the equations

$$\begin{bmatrix} E_x & E_y \\ -\sin\alpha & \cos\alpha \end{bmatrix} \mathbf{V}_\alpha = -\begin{bmatrix} E_t \\ 0 \end{bmatrix}, \tag{11}$$

where the first equation in the system guarantees that the motion following $\mathbf{V}_\alpha$ stays on the same envelope level and in this sense describes motion of the constant level contours, while the second equation implies that the velocity points always in the direction α. Further assume that $\alpha = 0$, i.e. that we are interested in the constant direction coinciding with the principle direction of waves.

Here we obtain the statistical sampling distributions of the velocity V for general spectra and discuss how they are influenced by different sampling schemes. Three cases are considered:

a) unbiased sampling,

b) sampling at the points of u-level crossings of the envelope field cross-section in the principal wave direction, i.e. at the crossings of $E(x, 0, 0)$,

c) sampling at the u-level crossings contours of the envelope field.

It follows from the generalization of Rice's formula that the distributions in b) and c) are different and given by

$$-\frac{1}{a}\left(b + \sqrt{a \cdot c - b^2} \cdot \frac{X}{Y}\right), \tag{12}$$

where the constants are given by $a = \lambda_{200}(1 - \rho^2)$, $b = \lambda_{101} - \lambda_{100}\lambda_{001}/\lambda_{000}$, $c = \lambda_{002} - \lambda_{001}^2/\lambda_{000}S$ and variables X and Y are independent, X having the standard normal distribution while the distribution of Y is

a) the standard normal,

b) the Rayleigh distribution,

c) a complex but explicit density expressed by the means of Bessel functions (see [12]).

In the above, λ_{ijk} are spectral moments of $W(\mathbf{p}, t)$ as defined by (9). Additionally for the part c) it assumed (as in all our examples) that the directional spectrum is symmetric with respect to the principal wave direction and thus $\lambda_{010} = \lambda_{011} = \lambda_{110} = 0$.

For comparison, the velocity of the sea surface has the same template (12) but with the constants $a = \lambda_{200}$, $b = \lambda_{101}$, and $c = \lambda_{002}$. Notice that for JONSWAP spectra and for the second choice in Table 1 ($\lambda_{100} = 0$) the template constants a and b for the waves coincide with the ones for the envelope. Thus statistically velocities of the envelope and of individual waves are centered at the same values which again demonstrates how counter intuitive things can go for certain choices of the envelope.

It is interesting that the considered velocity under the both biased sampling distributions on u-level contours does not depend on the level u, i.e. they are the same independently of the elevation at which the velocity is measured.

Consider the directional Gaussian sea surface obtained from the JONSWAP spectrum $\tilde{S}(\omega, \theta) = S(\omega)D(\omega, \theta)$, where

$$S(\omega) = g^2 \frac{\alpha}{\omega^5} e^{-1.25\omega_p^4/\omega^4} \rho^{\psi(\omega)},$$

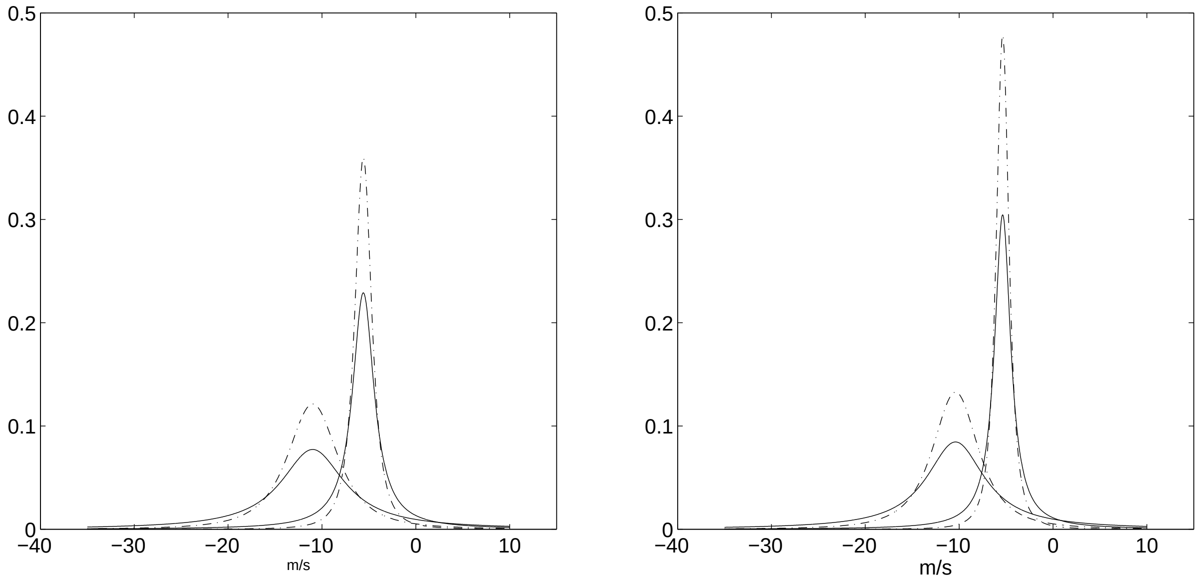

FIGURE 3. Distributions of velocities in the direction $y = 0$ for envelope field and sea surface sampled at: the contours (left); and the crossing points along $y = 0$ (right).

with $\psi(\omega) = e^{-(\omega-\omega_p)^2/(2\sigma^2\omega_p^2)}$, where σ is a jump function of ω:

$$\sigma = \begin{cases} 0.07 & \text{if} \quad \omega/\omega_p \le 1, \\ 0.09 & \text{if} \quad \omega/\omega_p > 1. \end{cases}$$

and α is a scale, ρ controls the shape, and ω_p is the peak frequency. The spreading function is given by $D(\omega, \theta) = G_0 \cos^{2c}(\theta/2)$. The spectrum is shown in Figure 1 (right). For the envelope we have chosen Λ^+ that corresponds to $\Gamma^+ = [0, \infty) \times (-\pi, \pi]$.

In Figure 3 (left), we present the unbiased and biased sampling distributions of velocities both for the envelope and for the sea surface. The solid lines represent the unbiased densities and the dashed-dotted ones corresponds to the biased sampling densities. We see that the biased sampling distribution which are more important for applications, are more concentrated around its center. The group velocity is smaller than that of individual waves as it is observed in the real life records. The peaks are at $-5.58[m/s]$ and $-10.98[m/s]$. Thus waves are roughly twice as fast as groups, the result in agreement with conclusions of the "narrow banded" example.

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