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ABSTRACT. Non-Gaussian stochastic fields are introduced by means of integrals with respect to independently scattered stochastic measures that have generalized Laplace distributions. In particular, we discuss stationary second order processes that, as opposed to their Gaussian counterpart, have a possibility of accounting for asymmetry and heavier tails. Additionally to this greater flexibility the discussed models continue to share most spectral properties with Gaussian processes. The models extend directly to random fields and thus can be suitable for modeling empirical data that are used in environmental and engineering sciences. Distributions of spatio-temporal characteristics can be obtained using the generalized Rice formula and effectively computed by numerical methods. The potential for stochastic modeling of real life phenomena that deviate from the Gaussian paradigm is exemplified by a stochastic field model with Matérn covariances.

1. INTRODUCTION

Spectral theory or frequency domain analysis is at the center of stochastic modeling in engineering sciences. Because of elegant mathematical properties, the Gaussian processes are the most popular second order models for which the relation between the frequency and time domain is well understood. The ability to model spatio-temporal phenomena through essentially the same framework as for time only dependent data contributed significantly to the popularity of Gaussian fields in geostatistics. Over the years however, there has gathered increasing empirical evidence that the Gaussian models often do not properly fit the phenomena they are intended to describe. These discrepancies are amplified additionally by non-linearity of deterministic models that describe the physics behind the observed data. Among the most often quoted features that are observed in the data but cannot be modeled by Gaussian distributions are asymmetry and heavy tails. For example, the skewness of sea level data is well documented and is a result of the non-linearity of the governing equations for water surface elevations as discussed in [1]. The need for non-Gaussian models for mechanical

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loads has also been acknowledged on many occasions. An extensive critical discussion of various approaches to handling non-Gaussian loads and their simulations has been presented in [11]. It has been pointed out that non-Gaussian models are necessary either if the input loads are non-Gaussian or it is dealt with a non-linear system and thus the response even to the Gaussian input is non-Gaussian. This interest in alternative stochastic models extends beyond mechanical engineering. In [14], it was observed that measurements of soil properties in geotechnical engineering problems and seismic ground motion are highly skewed data (see also references therein). The heavier than Gaussian tails were reported from such spatial phenomena as topographic data, temperature (see [16]), or well log data in petroleum applications (see [19]).

All this exemplifies the growing need for models featuring asymmetry and heavy-tails. As a result, in probability theory, considerable efforts have been put toward studies of such models. There is a well-developed theory of distributions and variables that are so heavy tailed that their second moment does not exist. Prominent examples hereof are stable and other related infinitely divisible processes. However, the non-existence of finite second moments makes these processes difficult to adopt in an engineering context where the spectral theory and the frequency domain is a well-established tool for data analysis. Therefore, the lack of simple and conveniently parameterized second order non-Gaussian models appears to be a void in the modern stochastic modelling that asks to be filled. Among interesting candidates that can serve this purpose are processes linked to the Laplace distributions. Here we propose a general model of non-Gaussian signals that is based on these distributions. The model is relatively easy to simulate from and thus enables analysis through Monte Carlo studies. It is also advantageous over the static methods and their modifications, as proposed in [11], in particular because they allow for simultaneous matching of both spectra and higher order moments of the data.

These properties make generalized Laplace distributions interesting alternatives to the Gaussian and stable distributions. In a sense they combine good properties of both, they have all moments, but still the tails are much heavier than in the Gaussian case and, additionally, they allow for asymmetry. Since they belong to infinitely divisible processes, there exists a stochastic Lévy motion with independent and homogenous increments that has Laplace distributions as marginals. Morover, the general second order properties are shared with Gaussian counterparts although some analytical formulas have to be replaced by numerical approximations.

This paper gives methodological foundations to applied work using non-Gaussian models by providing a systematic presentation of the theory, giving fundamentals of statistical inference, and presenting tools for modeling studies. By the standard extension procedure we define stochastic integrals with respect to a random measure that relates to this motion. After establishing basic distributional properties of such integrals we follow the general theory of stationary processes and the same path as in the Gaussian case to develop the theory of second order stationary Laplace processes. The main model is introduced by considering a continuous time moving average process obtained by integration with respect to the Lévy processes that arise from the asymmetric Laplace distributions. Further we propose statistical methods of estimation and fitting Laplace processes to real life data. Extention to stochastic fields and multivariate processes is fairly straightforward as the above mentioned properties of th Laplace distributions are shared by their multivariate counterparts. After accounting for the fundamental properties of the so introduced stochastic fields, we present two specific topics that may be of interest for an applied researcher. In the first one, it is presented how sample distributions of characteristics at the level crossings can be evaluated based on Rice's formula. Then we show implementation of a class of spatial models based on Matérn covariances. For them effective simulation techniques are presented illustrating asymmetry and high extremes. The proposed processes and their studies could have a substantial impact for analyzing stochastic phenomena in engineering and other areas of applied research. Here we focus more on their fundamental properties while their modeling potential has already been explored in work oriented toward applications in mechanical engineering and environmental sciences, see [5], [15], [2].

For example in [2], Laplace moving averages are effectively applied to analysis of the fatigue damage caused by sea waves on an offshore structure. While wave heights are typically modeled by means of a stationary Gaussian model, real ocean waves are often asymmetric with higher crests and shallower troughs than predicted by the Gaussian model. Therefore the Gaussian model is sometimes "corrected" by introducing a quadratic component in the model, allowing for interaction between different frequencies. In this context a non-central chi-square process is frequently considered as a more accurate model of the sea surface. We discuss an alternative description of the asymetric sea surface by means of the Laplace moving averages. By doing so, in analogy with the Gaussian model, one still has the interpretation of the sea surface as a sum of cosine waves with uncorrelated amplitudes and random phases, which is very attractive from a physical point of view. At the same time the moving average process allows for a greater flexibility when it comes to describing the marginal destribution of the sea since there is a possibility of fitting both the right covariance structure and also the correct skewness and kurtosis. However, the advantages of the model goes beyond this.

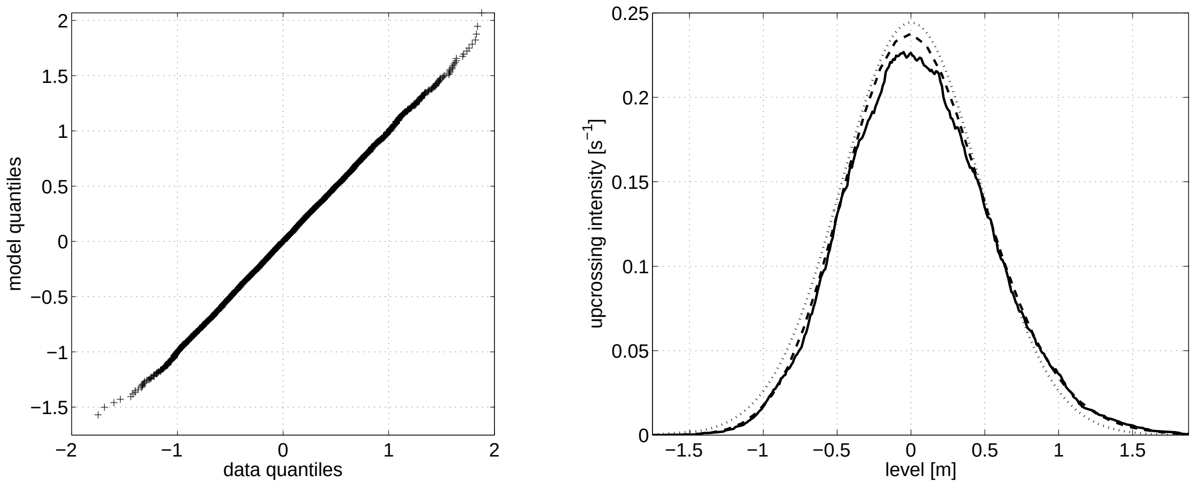

FIGURE 1. *Left:* Quantile-quantile plots for the sea elevation data and the fitted Laplace moving average process; *Right:* Empirical (solid irregular curve) crossing intensity in comparison with the one obtained by the Gaussian (dotted) and moving average model (dashed).

In this particular application the intensity of level crossings plays an important role. It is given by Rice's formula and takes a simple form for Gaussian processes whereas for the quadratic model or the Laplace driven moving average model it must be approximated and/or computed numerically. In the quoted paper, it has been demonstrated that the Laplace moving averages models hold several advantages over the Gaussian or quadratic methods. In particular, they allow for more accurate estimation of the level crossing intensities, provide with a useful tool of studying the influence of kurtosis and skewness for the fatigue damage, and are very convienient if the loads are passing through linear filters such as a stiff structure or a linear oscilator. For example, for the sea elevation measurements recorded at an oil platform off the west Africa coast, the marginal distributions and the crossing intensities are presented in Figure 1. It can be clearly seen that the Laplace driven moving average gives a good fit both the marginal disstribution and to the observed crossing intensity. It is also observed that the damage, although sometimes not so different on average, has more variability for the models with heavier tails. Additionally, it has also been demonstrated that the damage is more severe for non-Gaussian models when it is represented as a high power of rainflow cycles.

The above is one of many potential utilizations of stochastic models based on Laplace moving averages and this work gives an account of methodological tools that should make it also possible in other future applied research.

2. Integrals with respect to Laplace random measures

The focus of this paper is on stationary stochastic processes and fields but in this section we overview random variables and distributions that correspond to marginal distribution of the models discussed in the following sections. These random variables arise from integrating independently scattered stochastic measures with *generalized Laplace distributions.* The latter are also known as the *Bessel function distributions* and for completeness we provide with their definition and basic properties. The monograph [13] is a good reference for further properties of this class.

Definition 1 (Generalized Laplace Law)**.** *The generalized Laplace laws are best described by their characteristic functions that in the one dimensional case are given by*

$$\phi(u) = \left(1 - i\mu u + \frac{\sigma^2 u^2}{2}\right)^{-1/\nu},$$

where $\mu \in \mathbb{R}$ *and* $\sigma > 0$. *We use* $\mathcal{L}(\mu, \sigma, \nu)$ *for the above distribution, with the standard values of the parameters:* $\mu = 0$ *(symmetric case), scale* $\sigma = 1$*, and shape* $\nu = 1$*. By default, if any of the parameters is dropped from the notation it is assumed to be set to its standard value.*

There are two representations of $\mathcal{L}(\mu, \sigma, \nu)$ that are worth to mention in the present context. The first one is by factorization of the characteristic function

$$\phi(u) = \left(1 - i\frac{\sqrt{2}}{2} \cdot \frac{\sigma}{\kappa} \cdot u\right)^{-1/\nu} \left(1 + i\frac{\sqrt{2}}{2} \cdot \sigma\kappa \cdot u\right)^{-1/\nu}, \tag{1}$$

where $\kappa = \frac{\sqrt{2}}{2}(\sqrt{2 + \mu^2/\sigma^2} - \mu/\sigma)$, which is the distribution of a difference of two independent gamma random variables with shape parameter $1/\nu$ and scale parameters $\frac{\sqrt{2}}{2}\sigma/\kappa$ and $\frac{\sqrt{2}}{2}\sigma\kappa$, respectively. It is also possible to represent the Laplace distribution as a normal random variable with stochastic mean and variance. More precisely, if Z is a standard normal variable and Γ is an independent gamma variable with shape $1/\nu$ (and scale equal to one), then $\mathcal{L}(\mu, \sigma, \nu)$ is the distribution of $\sigma\sqrt{\Gamma}Z + \mu\Gamma$.

In the symmetric case ($\mu = 0$), it follows that the Laplace distributions belong to the more general class of type G distributions. These are discussed for example in [4] and many properties of the symmetric case could be deduced from the properties presented in that work. Instead, we opted here for the direct derivations in order to cover simultaneously the asymmetric case and thus to have a more coherent presentation. This derivations can be found in the Appendix.

The Laplace distributions are infinitely divisible and one can construct stochastic measures with values distributed according to generalized Laplace distributions. This work is a study of stochastic models and processes that arise from stochastic integrals obtained from such measures. Stochastic Laplace measures and integrals are defined through a generic method that was used on various occasions in the literature, whenever particular classes of distributions have been considered: for example, stable in [12], infinite divisible in [17], and general Hilbert space based approach for weakly stationary processes in [9]. In the appendix we provide with some further details of the method, while here we list basic properties of integrals of a deterministic function with respect to Laplace stochastic measures.

Let us consider a measure space $(\mathcal{X}, \mathcal{B}, m)$ and a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. By $L_2 = L_2(\Omega, \mathcal{F}, \mathbb{P})$, we denote the Hilbert space of random variables X on Ω for which $\mathbb{E}X^2$ is finite.

Definition 2 (Stochastic Laplace Measure). *A stochastic Laplace measure Λ, with parameters $\mu \in \mathbb{R}$, $\sigma > 0$ and controlled by a measure m, is a function that maps $A \in \mathcal{B}$, $m(A) < \infty$ into L_2 such that $\Lambda(A)$, $A \in \mathcal{B}$, has generalized Laplace distribution given by the ch.f.*

$$\phi_{\Lambda(A)}(u) = \left(1 - i\mu u + \frac{\sigma^2 u^2}{2}\right)^{-m(A)}.$$

and for disjoint $A_i \in \mathcal{B}$, $\Lambda(A_i)$ are independent and with probability one

$$\Lambda\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \Lambda(A_i).$$

By the standard measure extension arguments, it is obvious that by taking sufficiently 'rich' $(\Omega, \mathcal{F}, P)$ one can define Λ for an arbitrary measure space $(\mathcal{X}, \mathcal{B}, m)$.

A special and important case of the stochastic Laplace measure can be associated with a Lévy motion corresponding to the Laplace and generalized Laplace distributions. Namely, the symmetric Laplace motion defined next can be identified with the Laplace stochastic measure on the Borel sets of the halfline $[0, \infty)$ with the control measure being the Lebesgue measure.

Definition 3 (Laplace Motion)**.** *A Laplace motion $L(t)$ with the asymmetry parameter μ, the space scale parameter σ and the time scale parameter ν, $\mathcal{LM}(\mu,\sigma,\nu)$ is defined by the following conditions*

(i) *it starts at the origin, i.e., $L(0)=0$;*
(ii) *it has independent and stationary increments;*
(iii) *the increments by the time scale unit have a symmetric Laplace distribution with the parameter σ, i.e.,*

$$L(t+\nu)-L(t) \stackrel{\mathrm{d}}{=} \mathcal{L}(\mu,\sigma).$$

If $\mu=0$, $\sigma=1$ and $\nu=1$ the process $L(t)$ is called the standard Laplace motion.

The relation between motion and the measure is given for an interval $(a,b]\subseteq[0,\infty)$ by $\Lambda(a,b]=L(b)-L(a)$. We also note that

$$\mathbb{E}\,L(t)=\mu\cdot t;\qquad \mathbb{V}ar\,L(t)=\frac{\sigma^2}{\nu}\cdot t.$$

The following proposition tells that the Laplace motion can be represented by Brownian motion subordinated to a gamma process. Recall that a stochastic process $\Gamma(t)$ is called a gamma process if it starts at zero, has independent and homogeneous increments and the distribution of the increment $\Gamma(t+s)-\Gamma(t)$ is given by a gamma distribution with shape parameter s/ν and the scale β. The case $\beta=\nu=1$ is referred to as the standard gamma process and below we always assume that $\beta=1$.

Proposition 1 (Representation of Laplace Motion)**.** *Let B(t) be a Brownian motion with scale σ and drift μ. Further assume that $\Gamma(t)$ is a gamma process with parameter ν, independent of $B(t)$. Then $\mathcal{LM}(\mu,\sigma,\nu)$ can be represented as*

$$L(t)=B(\Gamma(t)),\quad t>0.$$

Due to this representation the Laplace motion is often called variance-gamma process, since it can be viewed as a Brownian motion with the variance randomized by a gamma process. A slightly more general set-up for gamma variance models is presented in Subsection 4.1 of the Appendix.

We also have the representation of the random Laplace measures as a difference of the gamma measures (see the Appendix, Subsection 4.1 for the definitions).

Proposition 2 (Difference of Random Gamma Measures)**.** *Let Γ_1 and Γ_2 be independent gamma measures with parameters $\frac{\sqrt{2}}{2}\cdot\sigma/\kappa$ and $\frac{\sqrt{2}}{2}\cdot\sigma\kappa$, respectively and both controlled* *by a measure m on an arbitrary measurable space $(\mathcal{X}, \mathcal{B})$. Then the Laplace measure Λ with parameters $\mu = \frac{\sigma}{\sqrt{2}}(\frac{1}{\kappa} - \kappa)$ and σ that is controlled by m can be represented as*

$$\Lambda(A) = \Gamma_1(A) - \Gamma_2(A).$$

Proof. Since both sides of the equation obviously represent random measures, it is enough to check the equality of distributions of $\Lambda(A)$ and $\Gamma_1(A) - \Gamma_2(A)$ which follows easily from the factorization (1). □

The models discussed in this work are based on the standard constructions of integrals of deterministic functions with respect to random measures. We skip any detail of this standard material and limit ourselves only to the basic properties. Most of them and some derivations are located in the Appendix, Subsection 4.2.

For a Laplace measure Λ we note

$$\mathbb{V}ar\, \Lambda(A) = (\sigma^2 + \mu^2) m(A),$$

and thus we can utilize the standard Hilbert space approach with control measure m to define integrals of $f \in L_2(\mathcal{X}, \mathcal{B}, m)$ with respect to Λ.

The stochastic integral of f with respect to Λ is defined through the isometry of $L_2(\mathcal{X}, \mathcal{B}, m)$ into $L_2(\Omega, \mathcal{F}, \mathbb{P})$ that relates the indicator functions $\mathbf{1}_A(x)$ with the variables $\Lambda(A)$. This isometry is denoted by

$$X = \int_{\mathcal{X}} f(x)\; d\Lambda(x),$$

that is often shortened to $\int f\; d\Lambda$, and is defined as the value of this isometry at f. In the Appendix, Subsection 4.2, the argument is given for the following form of the characteristic function (see also Proposition 2.1 in [3]).

Proposition 3 (Characteristic Function)**.** *Let both $\int f dm$ and $\int f^2 dm$ be finite and Λ be a stochastic Laplace measure. Then the integral $X = \int f d\Lambda$ has the characteristic function*

$$\phi_X(u) = \exp\left(- \int_{\mathcal{X}} \log\left(1 - i\mu u f(x) + \frac{\sigma^2 f^2(x) u^2}{2} \right) dm(x) \right). \tag{2}$$

The random variable X defined by the above integral can be considered as a semi-parametric with two numerical parameters $\mu \in \mathbb{R}$ and $\sigma \geq 0$ and two "semi-parameters" f and m that run through infinitely dimensional spaces. Such a semi-parametric distribution, as well as any random variable that has it, will be referred to by $\mathcal{LSI}(\mu, \sigma; m, f)$. The standard values of the parameters are $\mu = 0$, $\sigma = 1$ and these are default if the parameters are dropped from the notation.

Using the characteristic function from Proposition 3, we obtain a recurrence for the moments.

Proposition 4 (Moments). *Let* $X = \int f d\Lambda$ *and assume that* $f^N \in L_2(\mathcal{X}, \mathcal{B}, m)$. *Then the following recurrence formula for the moments holds*

$$\mathbb{E}\, X^N = (N-1)! \sum_{k=1}^{N} \frac{\mathbb{E}\, X^{N-k}}{(N-k)!} \int f^k dm\, S_{k-1},$$

where

$$S_r = \begin{cases} \sum_{k=0}^{\frac{r}{2}-\frac{1}{2}} s_{r,k}, & r \text{ odd}, \\ \mu \left(\frac{\sigma^2}{2}\right)^{\frac{r}{2}} + \sum_{k=0}^{\frac{r}{2}-1} s_{r,k}, & r \text{ even}, \end{cases}$$

and

$$s_{r,k} = \mu^{r-2k-1} \left(\frac{\sigma^2}{2}\right)^k \left(\binom{r-k-1}{k} \sigma^2 + \binom{r-k}{k} \mu^2 \right).$$

In the Appendix, Subsection 4.2, the argument for the above proposition is given together with the simplified version of the formulas for the symmetric case – Propositions 8. From them the first four central moments can be expressed to yield the skewness coefficient s and excess kurtosis k_e as follows

$$\begin{aligned} s &= \operatorname{sgn}(\mu) \frac{2\mu^2 + 3\sigma^2}{(\mu^2+\sigma^2)^{\frac{3}{2}}} \cdot \frac{\int f^3 dm}{\left(\int f^2 dm\right)^{3/2}} \\ k_e &= 3\left(2 - \frac{\sigma^4}{(\mu^2+\sigma^2)^2}\right) \cdot \frac{\int f^4 dm}{\left(\int f^2 dm\right)^2}. \end{aligned}$$

Example 1. If m is the Lebesgue measure in $\mathbb{R}^d$ divided by $\nu > 0$, we obtain an extra numerical parameter (with the standard value $\nu = 1$) so we can write $\mathcal{LSI}(\mu, \sigma, \nu; f)$. We can also fully parametrize the distribution by taking a family of parametrized kernels. The case of $f(x) = \mathbf{1}_{[0,1]}(x)$ corresponds to the generalized Laplace distributions as defined in [13]. An interesting family of kernels $f(x) \sim \exp(-\beta|x|^\alpha)$, where $|x|$ is the Euclidean norm in $\mathbb{R}^d$, leads to the fully parametrical model $\mathcal{LSI}(\mu, \sigma, \nu, \alpha, \beta)$. The proportionality constant for the kernel f is chosen so that $\mathbb{V}arX = (\mu^2 + \sigma^2)/\nu$ for all members of the family, i.e. $\int f^2 = 1$. For the one dimensional case we have

$$\int_{\mathbb{R}} \exp(-\beta|x|^\alpha)\, dx = 2\beta^{-1/\alpha}\Gamma\left(\frac{\alpha+1}{\alpha}\right)$$

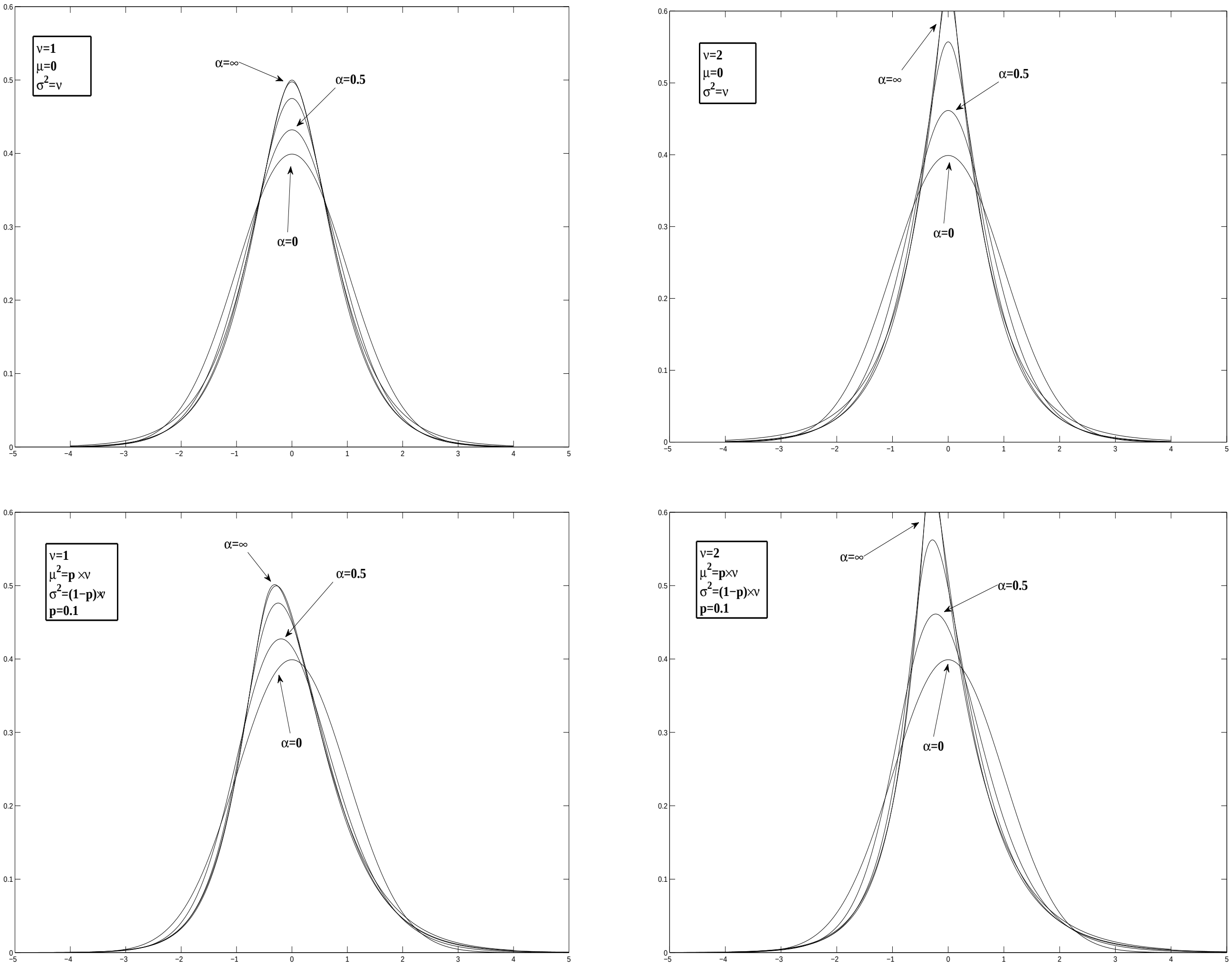

FIGURE 2. Densities and their dependence on the parameter $\alpha = 0, 0.5, 1, 2, \infty$. *Top:* Symmetric case ($\mu = 0$); *Bottom:* Asymmetric case with $\mu = \sqrt{p * \nu}$, with $p = 0.1$. From left to right $\nu = 1, 2$, respectively.

and we obtain $f(x) = K(\alpha, \beta) \cdot e^{-\beta |x|^\alpha}$, where

$$K^2(\alpha, \beta) = \frac{2^{1/\alpha - 1} \beta^{1/\alpha}}{\Gamma\left(\frac{\alpha+1}{\alpha}\right)}.$$

Thus using an explicit form of the integral of f^k with respect to the Lebesgue measure that is given by

$$\int f^k = \left(\frac{2}{k}\right)^{1/\alpha} \left(\frac{(2\beta)^{1/\alpha}}{2\Gamma\left(\frac{\alpha+1}{\alpha}\right)}\right)^{k/2-1},$$

we obtain explicit formulas for the moments, skeweness and kurtosis in terms of the gamma function. We also observe that for large α, the kernel is converging to $1/\sqrt{2}$ on $[-1, 1]$, and thus the distribution of integral becomes $\mathcal{L}(\mu/\sqrt{2}, \sigma/\sqrt{2}, \nu/2)$. On the other hand, for small α the kernel will be more like a constant on the increasing support and thus by the Central limit theorem it converges to the Gaussian distribution. The shapes of the densities in comparison to these two limiting cases are shown in Figure 2. The parameters have been normalized so the variances of all presented distributions are equal to one and the means are zero. Since except for a few special cases, there is no explicit formula for the densities the fast Fourier transform (FFT) has been used for approximation. We observe that α has a similar influence on the shape as the parameter ν. Moreover, large values of α affect very little the shape of the distribution (the densities for $\alpha = 2$ and $\alpha = \infty$ almost coincide).

Remark 1. We note that the distribution of X is leptokurtic (positive excess kurtosis), i.e. it has a more acute "peak" around the mean and "fatter" tails than a normally distributed variable (for which excess kurtosis is zero). For example in the symmetric case, if we consider m to be the Lebesgue measure on $\mathbb{R}$ multiplied by $1/\nu$, then

$$k_e = 3\nu \frac{\int f^4}{\int^2 f^2}$$

and we see that by varying parameter ν we can make excess kurtosis either very large (very fat tails) or close to normal (ν converging to zero). In this respect, the distribution of the integral with respect Laplace measure behaves analogously to the Laplace distribution that stands behind the random Laplace measure.

Simulation of the Laplace integrals should be considered in conection with stochastic fields that are defined in the following section. In fact, there are many different ways of approaching to this problem and effectiveness of resulting simulations deserves a separate treatment. Here let us just mention that a straightforward way is by simulations of (independent) increments of the Laplace motion over an equally spaced grid and then summating them weighted by the kernel values at the grid points – the Riemman sum approach. Simulation of increments themselves can be based on Proposition 7 where the asymmetric Laplace motion is represented as a Brownian motion subordinated to Gamma process so that increments can be obtained as independent normal variables with variances equal to the simulated gamma increments.

3. Second order Laplace processes and fields

The interest in non-Gaussian modeling resulted in many attempts to provide with a general class of second order processes that would account for non-Gaussian features in the data. A majority of the models proposed so far take Gaussian processes as the starting point. In this sense, our approach differs fundamentally as it distances from the Gaussian processes from the very begining. Namely, we use the stochastic integral as defined in the previous section to define stationary stochastic fields with the marginals distributed as the integrals with respect to Laplace motion. The terminology for this class of processes and corresponding distributions is not well-established and names such as generalized Laplace convolutions or generalized Bessel function distributions can be justified either by historical reasons or by mathematical properties. The first considered class defines through the standard moving average construction a family of stationary random fields that we have decided to term as continuous time *Laplace moving averages (LMA)*. The second class is defined only for univariate argument and here is referred to as *Laplace harmonizable (LH) processes* in the tradition of similar concepts introduced for other classes of infinitely divisible processes. We believe that this terminology is a good compromise between historical reasons, descriptive value, and compactness of the names.

3.1. **Laplace moving average.** Let us assume that $\mathcal{X}$ is a Hilbert space and the measure m is shift invariant on $\mathcal{X}$. Our main focus is on Euclidean spaces, i.e. $\mathcal{X} = \mathbb{R}^n$ although most of the properties are valid without this restriction. For a Laplace measure Λ with parameters μ and σ controlled by the Lebesgue measure on $\mathcal{X}$ that is divided by ν and f as described in Section 2, the following process is referred to as a moving average

$$X(t) = \int_{\mathcal{X}} f(t-x)\ d\Lambda(x). \tag{3}$$

Since the scaling of f can be equivalently expressed by the corresponding scaling of the parameter ν, we always assume that f is scaled so that $\int f^2 = 1$. The next result lists basic facts about this class of second order processes. Here and in what follows the Fourier transform is defined by

$$\mathcal{F}f(\omega) = \int_{\mathcal{X}} \exp(-i\omega \cdot t) f(t) dt,$$

where $\cdot$ stands for the inner product in $\mathcal{X}$.

Theorem 1. *Let Λ be a stochastic Laplace measure with parameters μ and σ controlled by the Lebesgue measure on $\mathcal{X}$ that is divided by ν. Further, let $X(t)$ be the moving average process defined by* (3). *Then* *a) the marginal distribution of $X(t)$ is given by the characteristic function*

$$\phi_{X(t)}(u) = \exp\left(-\frac{1}{\nu}\int_{\mathcal{X}} \log\left(1 - i\mu u f(x) + \frac{\sigma^2 u^2 f^2(x)}{2}\right)\, dx\right),$$

and more generally its finite dimensional distribution of $X(\mathbf{t}) = (X(t_1), \ldots, X(t_n)$ is given by the characteristic function at $\mathbf{u} = (u_1, \ldots, u_n)$:

$$\phi_{X(\mathbf{t})}(\mathbf{u}) = \exp\left(-\frac{1}{\nu}\int_{\mathcal{X}} \log\left(1 - i\mu \mathbf{u}^T f_{\mathbf{t}}(x) + \frac{\sigma^2}{2}\left(\mathbf{u}^T f_{\mathbf{t}}(x) f_{\mathbf{t}}^T \mathbf{u}\right)\right)\, dx\right),$$

where $f_{\mathbf{t}}(x) = (f_{t_1}(x), \ldots, f_{t_n}(x))$.

b) the autocorrelation function $\rho(\tau)$ of $X(t)$ is given by

$$\rho(\tau) = \int_{-\infty}^{\infty} f(x-\tau) f(x)\, dx = (f * \tilde{f})(\tau),$$

where $\tilde{f}(x) = f(-x)$ and $*$ denotes the convolution operator,

c) if $\mathcal{X} = \mathbb{R}^d$, then the spectral density $R(\omega)$ of $X(t)$ is given by

$$R = \frac{\sigma^2 + \mu^2}{\nu} \cdot \frac{|\mathcal{F} f|^2}{(2\pi)^d},$$

where $\mathcal{F}$ denotes the Fourier transform. In particular, if f is symmetric and non-negative definite, then

$$f = (2\pi)^{d/2} \sqrt{\frac{\nu}{\sigma^2 + \mu^2}} \cdot \mathcal{F}^{-1}\sqrt{R}.$$

Proof. Part a) is a consequence of Proposition 3. The autocovariance function for Part b) is given by

$$\begin{aligned}\rho(\tau) = \mathbb{E}[X(0)X(\tau)] &= \int_{\mathcal{X}}\int_{\mathcal{X}} f(\tau - x) f(-y) \mathbb{E}[d\Lambda(x) d\Lambda(y)] \\ &= \int_{\mathcal{X}} f(\tau - x) f(-x)\, dx.\end{aligned}$$

Finally, Part c) follows immediately since the Fourier transform of a convolution is the product of Fourier transforms. □

In Figure 2, we have already seen how the marginal densities of moving averages depend on the kernel function as well as on the values of parameters of Laplace measure. An interesting case occurs when $(\sigma^2 + \mu^2)/\nu$ is kept constant equal to 1 say so $\nu = \sigma^2 + \mu^2$. Then, for a fixed kernel $f(x)$ the covariance function is fixed although the marginal distribution of the process is allowed to vary. In the limiting case when $\nu \to 0$ one will get the Gaussian distribution as a marginal. Take for example the case $\mu = 0$, then

$$\phi_{\Lambda(t)}(u) = \left(\frac{1}{1 + \sigma^2 u^2/2}\right)^{t/\sigma^2} = \left(1 - \frac{u^2/2}{1/\sigma^2 + u^2/2}\right)^{t/\sigma^2} \to e^{-tu^2/2}, \quad \sigma^2 \to 0,$$

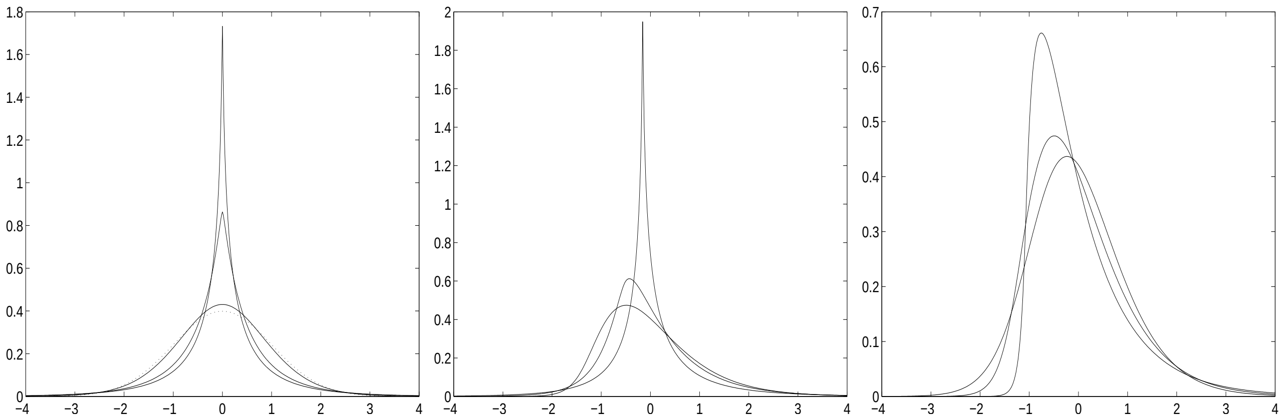

FIGURE 3. Marginal densities with the covariance e^{-t^2} symmetric and asymmetric cases. *Left:* $\mu = 0$, $\nu = 4, 2.25, 0.25$, the dashed curve is the standard normal density ($\nu = 0$); *Center:* Fixed $\mu = 0.5$ and variable $\sigma = 2, 1, 0.5$, so $\nu = 4.25, 1.25, 0.5$. *Right:* Fixed $\sigma = 0.5$ and variable $\mu = 1, 0.5, 0.2$, so $\nu = 1.25, 0.5, 0.29$.

i.e., the Laplace motion converges in distribution to the Brownian motion. This is illustrated in Figure 3 where the marginal symmetric densities are computed for a specific kernel, namely $f(x) = \sqrt{2}\pi^{-1/4}e^{-2x^2}$. For this particular choice of kernel, the covariance and autocorrelation functions simply become $\rho(\tau) = e^{-\tau^2}$.

Example 2. For illustrating purposes, we have simulated samples of X_0 for different spectral densities. In Figure 4, histograms based on 10000 samples and theoretical densities are shown in the second row for two completely different spectral densities namely a uniform spectrum and a Pierson-Moskowitz spectrum with $H_s = 7$m, $T_p = 11$s. (For a survey of spectra used in ocean engineering see [10].) In both cases we let $\nu = \sigma = 1$. The agreement between the histogram and the theoretical density is perfect so the simulation approach seems to work due to the ergodic property of moving averages that follows from the general theory of stationary processes (see [7]). The third row represents simulations for harmonizable Laplace processes that will be discussed later.

Next, we present how to obtain the level crossing sampling distributions that are based on the Rice formula. Here we limit ourselves to the expected number of upcrossings in any finite interval although the approach applies to the fields as well. In its simplest setting, Rice's formula considers a stationary, zero mean Gaussian process $X(t)$, $t \in \mathbb{R}$, possessing a quadratic mean derivative $X'(t)$, say. Under these conditions the expected number of upcrossings of the level v in the interval $[0, t]$, here denoted by $\mathbb{E}[N_v(t)]$, can be computed

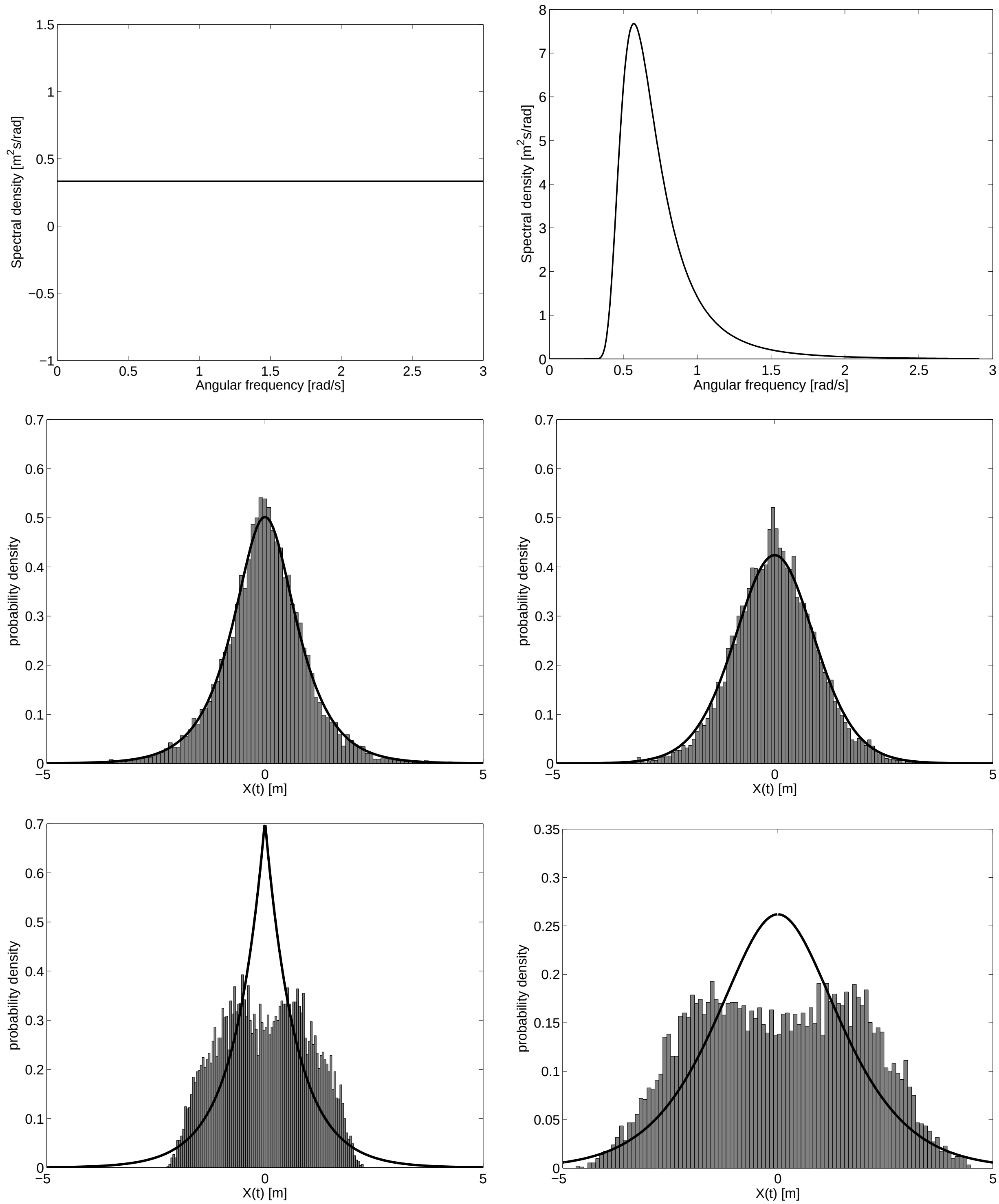

FIGURE 4. Theoretical densities and histograms of $X(0)$ based on 10000 simulations of LMA (middle row) and HL (bottom row) for a uniform spectrum (left) and a Pierson Moskowitz spectrum ($H_s = 7\text{m}$, $T_p = 11\text{s}$) (right), the top row depicts the spectra themselves.

by

$$\mathbb{E}[N_v(t)] = \frac{t}{2\pi}\sqrt{\frac{\lambda_2}{\lambda_0}}e^{-\frac{v^2}{2\lambda_0}} = t\int_0^\infty z f_{X(0),X'(0)}(v,z)\,dz,$$

where λ_j's equal spectral moments defined as $\int w^j R(\omega)\,d\omega$. A proof of this statement can be found in [8]. For non-Gaussian processes it is hard to show a general Rice formula. However, one can under fairly mild conditions on the process show a weaker, almost everywhere, version of it, see [21] for a proof. Namely, for a stationary process $X(t)$ having a.s. continuously differentiable sample paths and such that $X(t), X'(t)$ have a joint density $f_{X(t),X'(t)}(x,y)$, for almost every v,

$$\mathbb{E}[N_v(t)] = t\int_0^\infty z f_{X(0),X'(0)}(v,z)\,dz.$$

In order to find the intensity of upcrossings, $\mathbb{E}[N_v(1)]$, it is thus important to fing the joint density of the process and its derivative. For the *LMA*, the joint distribution of process and derivative can be expressed in terms of its characteristic function given next.

Proposition 5. *Let $X(t)$ denote the Laplace driven moving average process and assume that its derivative exists in a sample path or quadratic mean sense and is given by $X'(t) = \int_{-\infty}^{\infty} f'(t-x)\,d\Lambda(x)$. Then the characteristic function of $(X(0), X'(0))$ is given by*

$$\phi(u_1,u_2) =$$
$$\exp\left(-\frac{1}{\nu}\int_{-\infty}^{\infty}\log\left(1 - i\mu(u_1 f(x) + u_2 f'(x)) + \frac{\sigma^2}{2}(u_1 f(x) + u_2 f'(x))^2\right)\,dx\right),$$

where ν, σ and μ are parameters of the asymmetric Laplace measure.

Proof. Note that $u_1X(0) + u_2X'(0) = \int (u_1 f(-x) + u_2 f'(-x))\,d\Lambda(x)$ so that the proposition follows by applying Proposition 3 with the kernel function equal to $u_1 f(-x) + u_2 f'(-x)$. □

With this characteristic function at hand the intensity of level crossings can be expressed as

$$\mathbb{E}[N_v(1)] = \frac{1}{(2\pi)^2}\int_0^\infty\int_{-\infty}^\infty\int_{-\infty}^\infty z\,e^{-i(u_1v+u_2z)}\phi_{X(0),X'(0)}(u_1,u_2)\,du_1\,du_2\,dz.$$

The crossing intensity cannot be written on a closed form, but must be integrated with some numerical method.

Remark 2. An alternative method to compute Rice's formula for the *LMA* is to use its relation to Gaussian processes. Recall that the *LMA* is obtained as an integral with respect to Laplace motion which in its turn, according to Proposition 1, is a superposition of Brownian motion with Gamma process. Consequently, conditional on a realization of the Gamma process, the

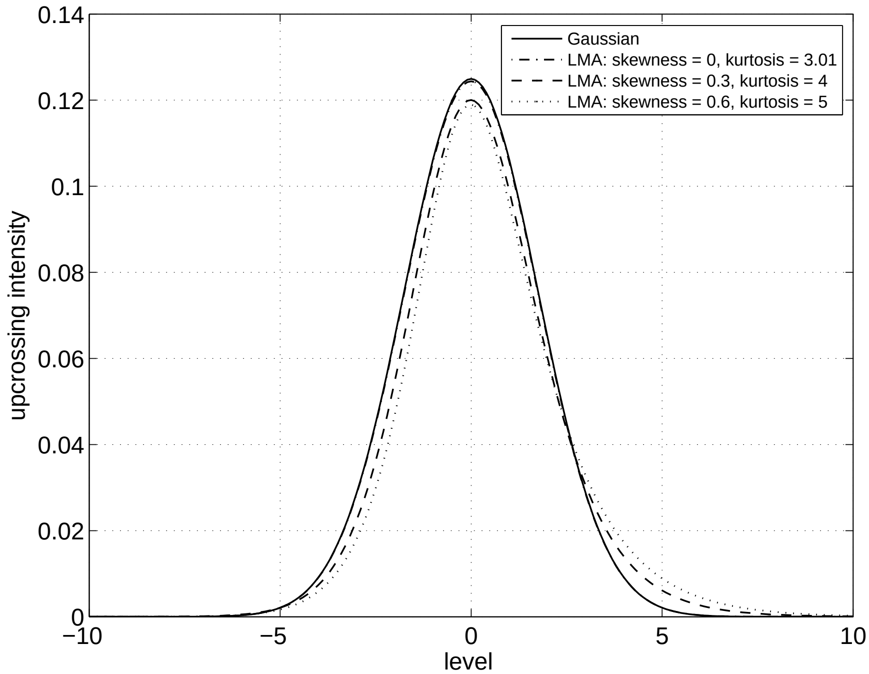

FIGURE 5. Crossing intensity for the *LMA* for different values of skewness and kurtosis of the marginal distribution. As a reference the crossing intensity for the corresponding Gaussian process is shown.

LMA becomes a non-stationary Gaussian process for which the crossing intensity is easier to find. Thus the crossing intensity may be computed in a Monte-Carlo fashion according to

$$\mathbb{E}[N_v(t)] \approx \frac{1}{n}\sum_{k=1}^{n} \mathbb{E}[N_v(t) \mid \Gamma = \gamma_k],$$

where γ_k are realizations of the appropriate Gamma-process and the terms in the sum can be efficiently computed using theory for Gaussian processes.

Example 3 (RICE'S FORMULA). This example is supposed to exemplify Rice's formula. In Figure 5 the crossing intensity for a Gaussian process and the *LMA* is shown for different values of skewness and kurtosis of the marginal distribution of the *LMA*. The spectrum used is a Pierson-Moskowitz spectrum commonly used in sea surface modelling. For the *LMA* the crossing intensity is computed using the Monte-Carlo approach previously described. For skewness equal to zero and kurtosis almost 3 the *LMA* is essentially equivalent to the Gaussian model, which is also seen on the crossing intensity. As skewness and kurtosis increases the crossing intensity becomes asymmetric with a heavier right tail.

The problem of linear filtering which is very important in practical applications has been studied in the full detail in [2]. Below we just summarize the result that have been applied there to analyze the fatigue damage of a structure subject to non-Gaussian loads.

Proposition 6 (Linear Filtering). *Let LMA with kernel $f(x)$ be input to a linear time invariant system with impulse response $h(x)$. Then the output $Y(t)$ is also LMA with kernel $h * f$, viz.*

$$Y(t) = \int_{-\infty}^{\infty} (h * f)(t - x)\, d\Lambda(x).$$

Before turning to our main example of spatial modelling, we would like briefly discuss *LMA* model fitting. This can be done in two steps. First, the kernel f can be estimated from the estimated spectral density function and in a second step the parameters of the Laplace motion can be fitted using the method of moments based on Proposition 4. Estimation methods for the parameters of the Laplace motion and their accuracy is a seperate issue that will be investigate in some future research. For the kernel estimation, if the kernel is from a parametric family one can fit the parameters using the correlation function. This parametric approach depends much on the assumed family and will not be discussed here. In a non-parametric approach and under the assumption that the kernel satisfies $f(x) = f(-x)$ and $\int f^2 = 1$, a kernel estimate $\widehat{f}$ can be taken as

$$\widehat{f}(x) = \mathcal{F}^{-1}\sqrt{\widehat{R}(\omega)},$$

where $\widehat{R}(\omega)$ is an estimate of the (twosided) spectral density function. In fact, the estimate of f is determined up to a non-identifiable constant. Thus one can always pick an estimate such that $\int \widehat{f}^2(x)\, dx = 1$, which means that $f * f$ becomes a correlation function. This is obtained by letting

$$\widehat{f}(x) = (2\pi)^{d/2} \frac{\mathcal{F}^{-1}\sqrt{\widehat{R}(\omega)}}{\sqrt{\int_{-\infty}^{\infty} \widehat{R}(\omega)\, d\omega}}.$$

For illustration consider a data set *sea.dat* that can be found in the WAFO-toolbox for matlab (available for download, free of charge, at `www.maths.lth.se/matstat/wafo`). This set contains 40 minutes of sea elevation measurements sampled at 4Hz at a fixed location in the North Sea. Before starting the analysis the mean is removed from the data and then, as a first step, the spectral density function is estimated, see Figure 6 (left). The estimated spectrum has two main peeks indicating that the waves are composed of both swell and wind waves. Next, some simple statistics of the data are computed and it was found that the sample variance of the data is 0.23 m^2, the sample skewness 0.25 and sample excess kurtosis 0.17. The latter two values show some evidence that the data has a skewed distribution and possesses tails that are heavier than the Gaussian ones (skewness and excess kurtosis equal zero for the Gaussian distribution). The next step in the model fitting procedure is to estimate a symmetric kernel f, satisfying $\int f^2 dx = 1$, see Figure 6 (right). Having this kernel one can then find the parameters μ, ν and σ of the asymmetric Laplace motion by fitting skewness, kurtosis and variance. The parameters were in this case estimated to $\mu = 0.076$, $\nu = 0.059$, $\sigma^2 = 0.0076$ and then to use the sample mean to obtain the shift (location) parameter $d = -1.27$. The fitted process has now the same covariance structure, skewness and kurtosis as the sea elevation data.

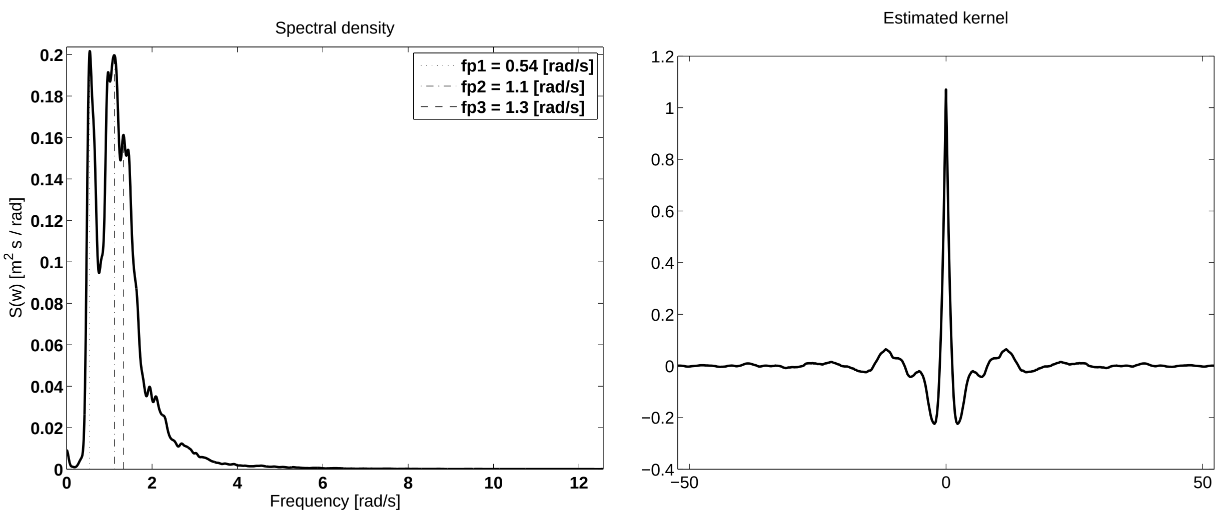

FIGURE 6. Kernel estimation for *sea.dat*. *Left:* Estimated one sided spectral density; *Right:* Symmetric kernel.

Example 4 (MATÉRN COVARIANCES). In the definition of the integral and thus also *LMA*, the argument of the kernel can be of an arbitrary dimension. Consequently, most of the presented results are valid for stochastic fields. Here, as an example of application, we summarize the *LMA* model of two dimensional fields with covariance function from the Matérn family – a popular class of covariances that is commonly used in geostatistics. The covariance is given by

$$r(x) = \frac{\phi}{2^{\beta-1}\Gamma(\beta)}(\alpha|x|)^{\beta} K_{\beta}(\alpha|x|), \tag{4}$$

where ϕ is the variance, β a smoothing parameter, $1/\alpha$ is related to the range of the covariance function and K is a modified Bessel function of the second kind (occasionally referred to as of the third kind) of order β. The family includes the exponential covariance and has the

Gaussian covariance as a limit. The exponential covariance is obtained if $\beta = 1/2$ and if $\beta = m + 1/2$ for some non-negative integer m, $r(x)$ is the product of a polynomial of degree m in $(\alpha|x|)$ and $\exp(-\alpha|x|)$. For example

$$\begin{aligned}
\beta = 1/2, \quad & r(x) = \phi \exp(-\alpha|x|), \\
\beta = 3/2, \quad & r(x) = \phi\,(\alpha|x| + 1) \exp(-\alpha|x|), \\
\beta = 5/2, \quad & r(x) = \phi\,\big((\alpha|x|)^2 + 3\alpha|x| + 3\big) \exp(-\alpha|x|).
\end{aligned}$$

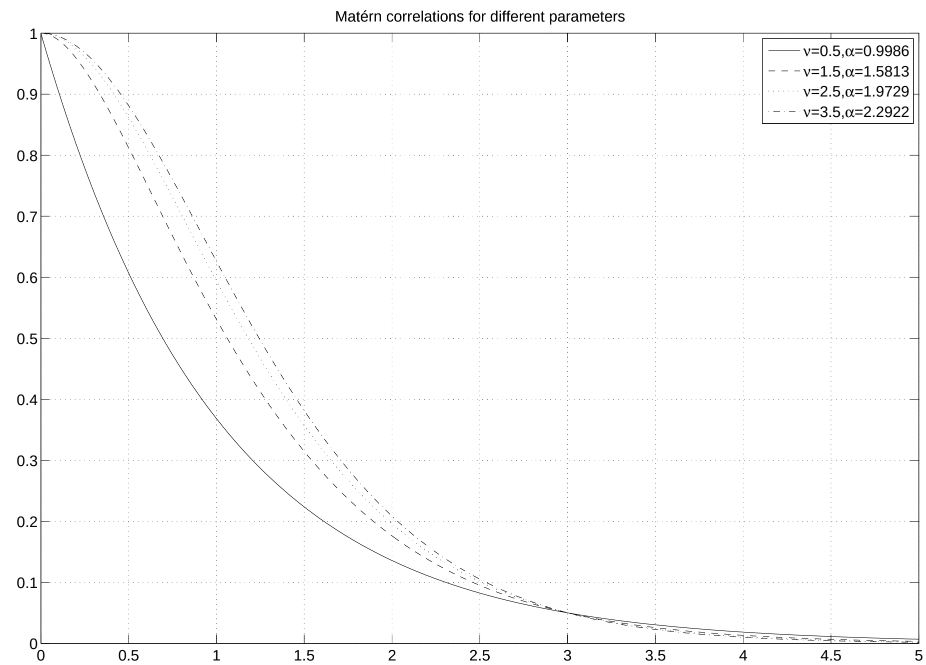

FIGURE 7. Examples of Matérn correlation functions.

The corresponding family of spectral densities is given by

$$R(\omega) = \pi^{-d/2} \frac{\Gamma(\beta + \frac{d}{2})}{\Gamma(\beta)} \frac{\phi \alpha^{2\beta}}{(\alpha^2 + |\omega|^2)^{\beta + \frac{d}{2}}}, \tag{5}$$

where d is the dimension of the space on which the process is defined. We recognize that incidentally they have the form of symmetric Laplace densities.

In order to find a symmetric kernel f in the Laplace moving average that corresponds to a Matérn covariance structure we use the fact that $r \propto f * f$ and thereby $\mathcal{F}f(\omega) \propto \sqrt{R(\omega)}$. Due to the form of $R(\omega)$ one gets a new member of the Matérn family by taking the square root. More precisely

$$\sqrt{R(\omega)} = \frac{\Gamma(\frac{\beta}{2}+\frac{d}{4})\alpha^{\beta-\frac{d}{2}}}{\Gamma(\frac{\beta}{2}-\frac{d}{4})\pi^{d/2}} \frac{1}{(\alpha^2+|\omega|^2)^{(\frac{\beta}{2}-\frac{d}{4})+\frac{d}{2}}}$$
$$\times \phi^{1/2}\pi^{d/4}\alpha^{d/2}\sqrt{\frac{\Gamma(\beta+\frac{d}{2})}{\Gamma(\beta)}\frac{\Gamma(\frac{\beta}{2}-\frac{d}{4})}{\Gamma(\frac{\beta}{2}+\frac{d}{4})}}.$$

Since this expression is written in the same form as (5) and since (5) and (4) constitute a Fourier transform pair, it holds, with $g = \mathcal{F}^{-1}\sqrt{R}$, that

$$f(x) \propto g(x) = \frac{1}{(2\pi)^d}\frac{\phi^{1/2}\pi^{d/4}\alpha^{d/2}}{2^{\frac{\beta}{2}-\frac{d}{4}-1}\Gamma(\frac{\beta}{2}+\frac{d}{4})}\sqrt{\frac{\Gamma(\beta+\frac{d}{2})}{\Gamma(\beta)}}(\alpha|x|)^{\frac{\beta}{2}-\frac{d}{4}}K_{\frac{\beta}{2}-\frac{d}{4}}(\alpha|x|).$$

The kernel scaling is determined by assuming that $\int f^2(x)\,dx = 1$. We note that by Parseval's theorem $\int g^2(x)\,dx = \int R(\omega)\,d\omega/(2\pi)^d = \phi/(2\pi)^d$. Thus the normalized kernel is given by

$$f(x) = \frac{(2\pi)^{d/2}}{\phi^{1/2}}g(x) = \frac{\alpha^{\frac{d}{2}}}{\Gamma(\frac{\beta}{2}+\frac{d}{4})\,2^{\frac{\beta}{2}+\frac{d}{4}-1}\pi^{d/4}}\sqrt{\frac{\Gamma(\beta+\frac{d}{2})}{\Gamma(\beta)}}(\alpha|x|)^{\frac{\beta}{2}-\frac{d}{4}}K_{\frac{\beta}{2}-\frac{d}{4}}(\alpha|x|),$$

having Fourier representation

$$\mathcal{F}f(\omega) = 2^{d/2}\pi^{d/4}\sqrt{\frac{\Gamma(\beta+\frac{d}{2})}{\Gamma(\beta)}}\frac{1}{(\alpha^2+|\omega|^2)^{\frac{\beta}{2}+\frac{d}{4}}}. \tag{6}$$

The smoothness of the sample paths is determined by the smoothness of the kernel. In order for the sample paths to be differentiable n times, the kernel has to be differentiable n times. This condition can also be expressed in terms of the parameter β of the Matérn covariance function. If the smoothing parameter $\beta > m$, then the kernel with the corresponding Matérn covariance is m times differentiable. Therefore, if $\beta > s/2$ the covariance function is differentiable s times, i.e. expressing this for the kernel we get that if $\beta > s + d/2$,then the kernel is s times differentiable. This means for example that in one dimension ($d = 1$) for the sample paths to be differentiable one needs $\beta > 3/2$ while in two dimensions ($d = 2$) one needs to require $\beta > 2$.

LMA may be thought of as a convolution of the increments of the Laplace motion with the kernel f. In a computer the convolution operation can be quite slow, in particular if the dimension is larger than one. Therefore it is more efficient to work in the frequency plane. Besides, as is the case with the Matérn kernels for some values of β, the convolution may not work for unbounded kernels. A simulation is thus preferably done in the frequency plane according to the following scheme:

(1) Simulate independent increments of the Laplace motion,

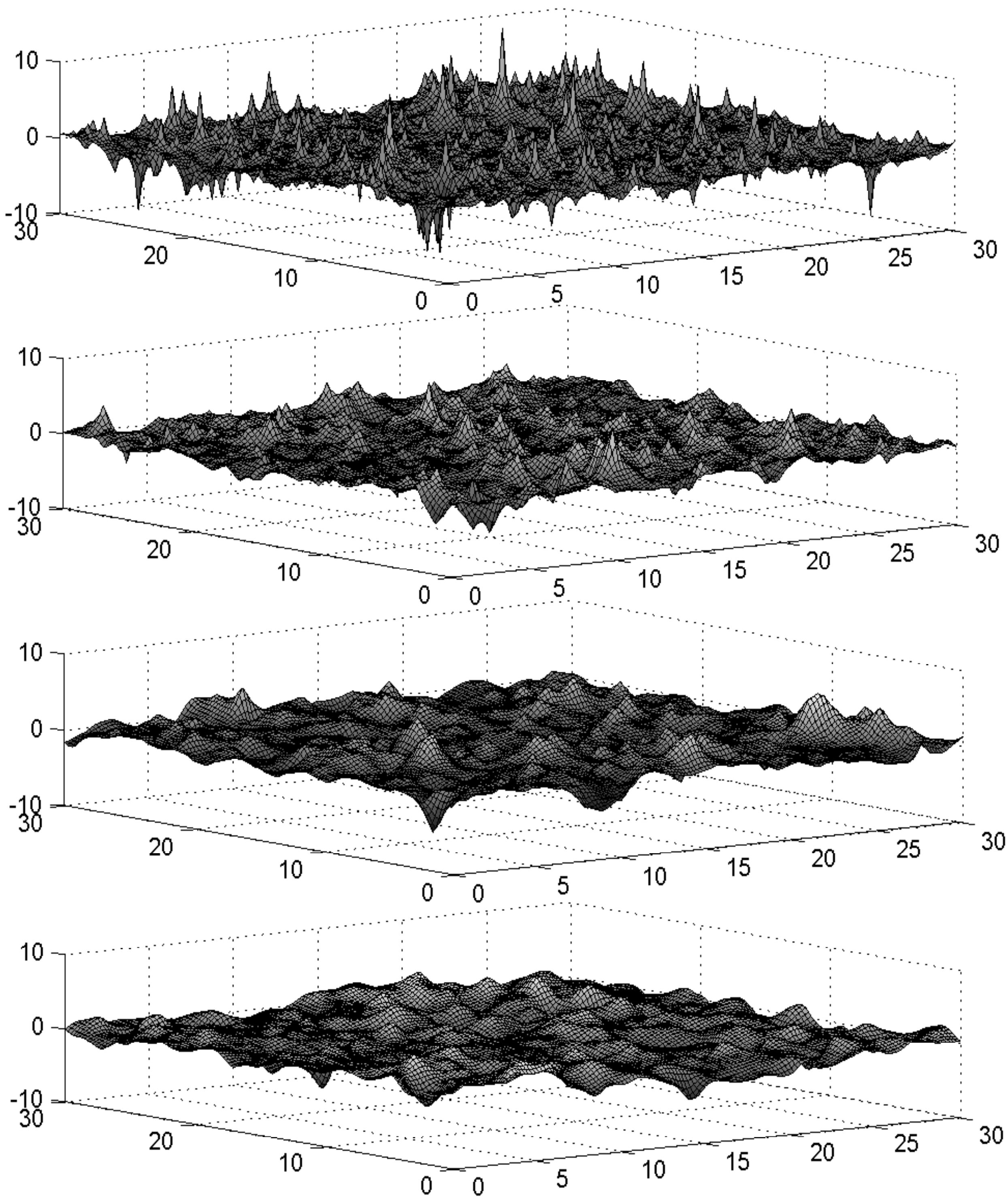

FIGURE 8. Laplace MA fields with Matérn correlations. The parameters from top to bottom are $\beta = 0.5, 1.5, 2.5, 3.5$, $\alpha = 1.0, 1.6, 2.0, 2.3$ and for the generating Laplace process $[\nu, \sigma, \mu, \gamma]$ =$[1, 1, 0, 0]$ yielding mean zero, variance one and a symmetric marginal distribution.

(2) Fourier transform the array of increments using the fast Fourier transform (FFT),

(3) Evaluate the Fourier transform of the kernel using (6),

(4) Take the product of the two Fourier transforms,

(5) Make the inverse FFT to obtain the simulation.

This scheme has been used to produce simulations of the Laplace MA fields shown in the following example.

In this example we will show simulations of Laplace moving average fields having a Matérn correlation with different degrees of smoothness and different marginal distributions. The parameters investigated are:

β	0.5	1.5	2.5	3.5
α	0.9986	1.5813	1.9729	2.2922

The values of α are chosen such that the correlation equals 0.05 at distance 3, as shown in Figure 7.

A fragment of a simulation of the different processes is shown in Figure 8 for Laplace process parameter values $[\nu, \sigma, \mu, \gamma]$ $=[1, 1, 0, 0]$. As it is apparent from the figures, the process gets smoother and smoother with increasing value of β. Recall that the sample paths are once differentiable if $\beta > 2$ and twice if $\beta > 3$, which corresponds to the two simulations on the second row. The corresponding marginal densities are shown in Figure 9 (left).

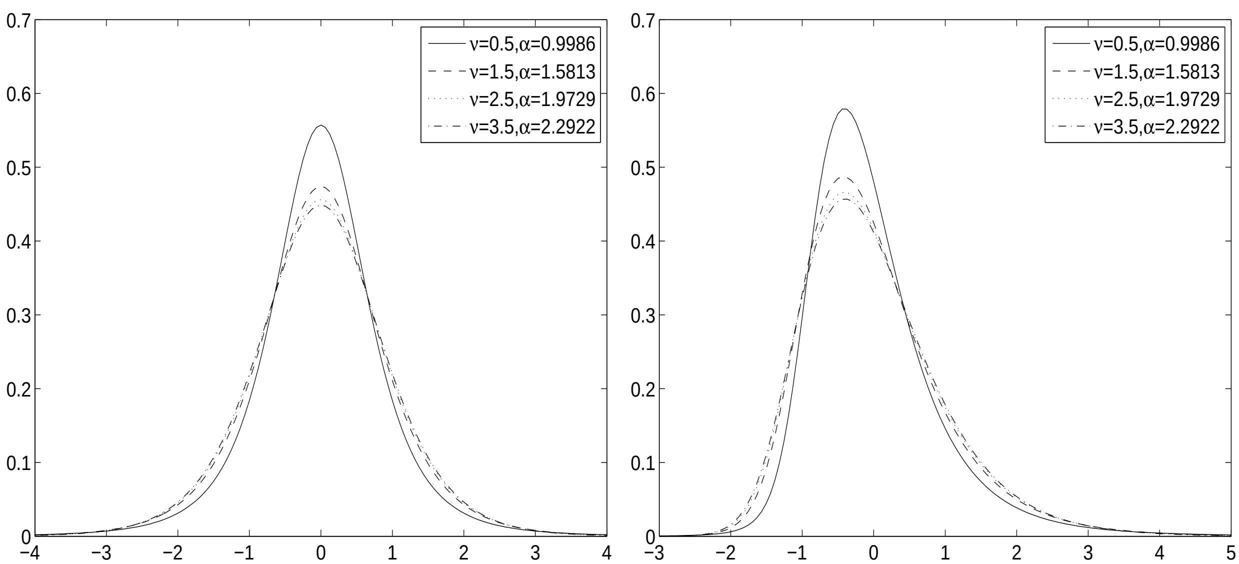

FIGURE 9. Marginal densities of *LMA* field for different correlations. The parameters of the driving Laplace process are $[\nu, \sigma, \mu, \gamma]$ $=[1, 1, 0, 0]$ – left (symmetric case) and $[\nu, \sigma, \mu, \gamma]$ $=[1, 1/\sqrt{2}, 1/\sqrt{2}, -1/\sqrt{2}]$ – right (asymmetric case).

Next the same correlation functions are kept but the parameters of the Laplace process are changed to $[\nu, \sigma, \mu, \gamma]$ $=[1, 1/\sqrt{2}, 1/\sqrt{2}, -1/\sqrt{2}]$. In this way the marginal density will be right skewed, having mean equal to zero and variance equal to one. In Figure 10 some simulations are shown and in Figure 9 (right) the corresponding marginal densities are shown.

3.2. **Laplace harmonizable processes.** Let $X(t)$ be a real-valued weakly stationary process having zero mean with covariance function

$$r(t) = \int_0^\infty \cos(\lambda t)\, dF(\lambda),$$

where $F : \mathbb{R}^+ \mapsto \mathbb{R}$ is a never-decreasing bounded function which is continuous to the right. The function $F(\lambda)$ is called the one-sided spectral distribution function of the process $X(t)$. In the following we assume that $F(\lambda)$ has no jump at zero and that $F(0) = 0$. Then $X(t)$ has the following spectral representation

$$X(t) = \int_0^\infty \cos(\lambda t)\, du(\lambda) + \int_0^\infty \sin(\lambda t)\, dv(\lambda), \tag{7}$$

where real-valued processes $u(\lambda)$ and $v(\lambda)$ with uncorrelated increments satisfy

$$\begin{aligned}
&\mathbb{E}[u(\lambda)] = \mathbb{E}[v(\lambda)] = 0,\\
&\mathbb{E}[(u(\lambda))^2] = \mathbb{E}[(v(\lambda))^2] = F(\lambda),\\
&\mathbb{E}[(du(\lambda))^2] = \mathbb{E}[(dv(\lambda))^2] = dF(\lambda),\\
&\mathbb{E}[du(\lambda)dv(\lambda')] = 0.
\end{aligned}$$

Conversly, if one starts up with processes $u(\lambda)$ and $v(\lambda)$ having the above properties, then the process $X(t)$ defined by (7) is weakly stationary.

Here, we consider the processes for which the uncorrelated increments processes u and v are given by Laplace motions. More precisely, let $B(\lambda)$ be a standard Brownian motion and $\Gamma(\lambda)$ a standard gamma process. Then, due to the representation of Laplace motion, we know that $L(\lambda) = B(\Gamma(\lambda))$ is standard Laplace motion. Further, let $F : \mathbb{R}^+ \mapsto \mathbb{R}$ be a never-decreasing, right continuous and bounded function and define a time-shifted Laplace motion, $L_F(\lambda)$, say, by $L_F(\lambda) = B(\Gamma(F(\lambda))$. Note that $L_F(\lambda)$ is a process with independent but non-stationary increments and that $\mathbb{V}ar[L_F(\lambda)] = F(\lambda)$.

Next, let $B(\lambda)$ and $\widetilde{B}(\lambda)$ be two independent standard Brownian motions and let $\Gamma(\lambda)$ be a standard gamma process independent of both $B(\lambda)$ and $\widetilde{B}(\lambda)$. Using these processes define two time-shifted Laplace motions by $L_F(\lambda) = B(\Gamma(F(\lambda))$ and $\widetilde{L}_F(\lambda) = \widetilde{B}(\Gamma(F(\lambda))$.

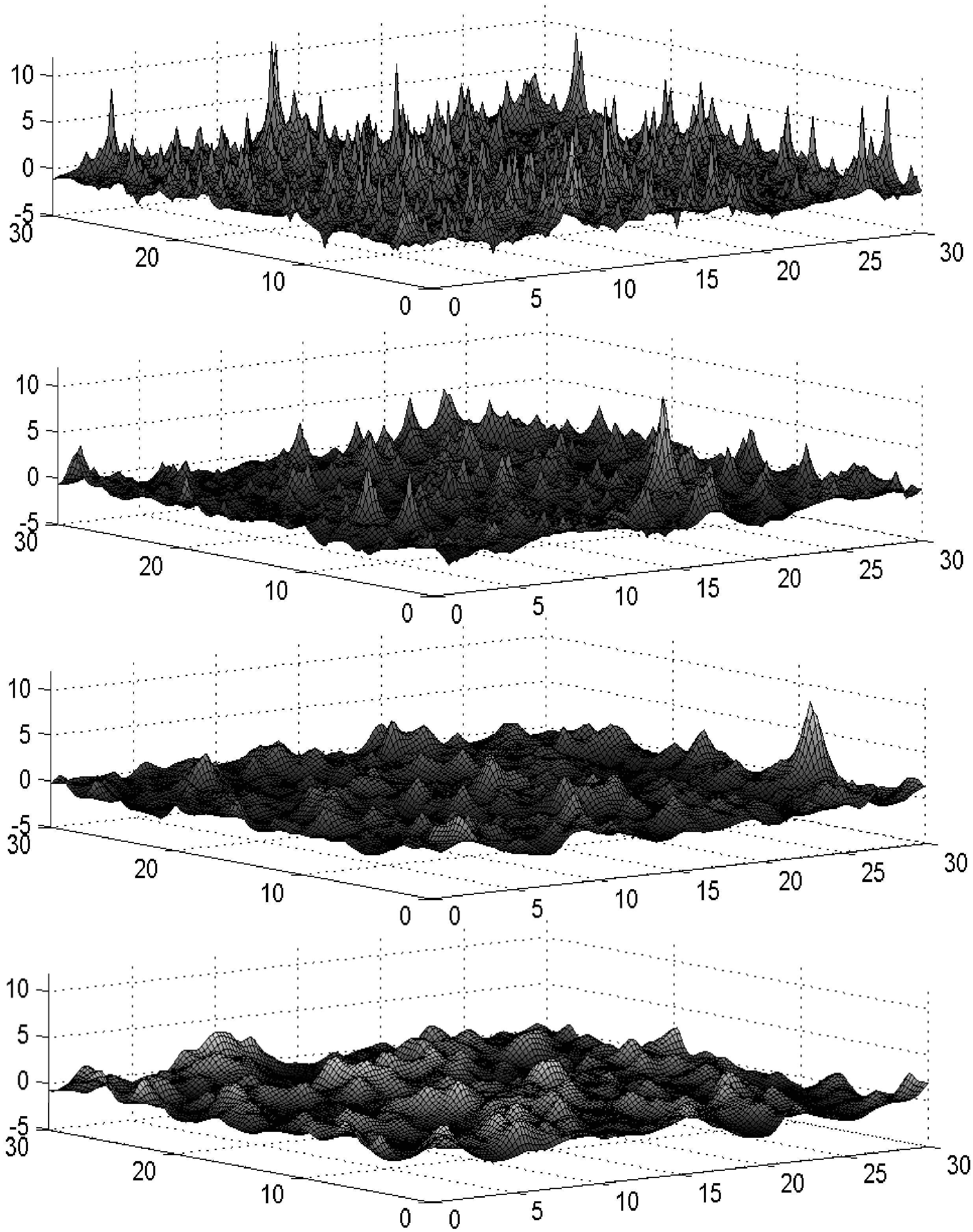

FIGURE 10. Laplace MA fields with Matérn correlations. The parameters from top to bottom are $\beta = 0.5, 1.5, 2.5, 3.5$, $\alpha = 1.0, 1.6, 2.0, 2.3$ and for the generating Laplace process $[\nu, \sigma, \mu, \gamma] = [1, 1/\sqrt{2}, 1/\sqrt{2}, -1/\sqrt{2}]$ yielding mean zero, variance one and a right skewed marginal distribution.

Obviously, since $L_F(\lambda)$ and $\widetilde{L}_F(\lambda)$ are constructed using the same gamma process, they are not independent. However, it easy to check by conditioning on Γ that they are uncorrelated. Thus the processes $L_F(\lambda)$ and $\widetilde{L}_F(\lambda)$ can serve as the processes $u(\lambda)$ and $v(\lambda)$ in the spectral representation to define a real-valued weakly stationary process by

$$X(t) = \int_0^\infty \cos(\lambda t)\, dL_F(\lambda) + \int_0^\infty \sin(\lambda t) d\widetilde{L}_F(\lambda). \tag{8}$$

The process $X(t)$ defined in this way will be called a *harmonizable Laplace (HL)* process.

The marginal distributions of this process can be obtained in a similar fashion as for *LMA*. Interestingly, the density distribution can be explicitely written using the Bessel functions.

Theorem 2 (One-dimensional distribution). *The HL process defined by* (8) *has a generalized Laplace mariginal distribution, defined by the characteristic function*

$$\mathbb{E}\left[e^{i\xi X_t}\right] = \left(\frac{1}{1+\frac{\xi^2}{2}}\right)^{\lambda_0},$$

where $\lambda_0 = F(\infty) - F(0)$. *Moreover, the density of* $X(t)$ *is given by*

$$f_{X_t}(x) = \frac{\sqrt{2}}{\Gamma(\lambda_0)\sqrt{\pi}} \left(\frac{|x|}{\sqrt{2}}\right)^{\lambda_0 - 1/2} K_{\lambda_0 - 1/2}(\sqrt{2}|x|), \quad x \neq 0,$$

where Γ *is the gamma function and* $K_\nu(x)$ *is the modified Bessel function of the second kind with index* ν.

Proof. The process $X(t)$ conditional on Γ can be written as

$$X(t)|\Gamma = \int_0^\infty \cos(\lambda t)\, dB_{\Gamma\circ F}(\lambda) + \int_0^\infty \sin(\lambda t)\, d\widetilde{B}_{\Gamma\circ F}(\lambda),$$

i.e., conditional on Γ, $X(t)$ is a stationary Gaussian process having control measure $\Gamma \circ F$. Thus, since $\mathbb{E}[X(t)|\Gamma] = 0$ and $\mathbf{V}ar[X(t)|\Gamma] = \Gamma(F(\infty)) - \Gamma(F(0))$ we have

$$\mathbb{E}\left[e^{i\xi X_t}\right] = \mathbb{E}\left[\mathbb{E}\left[e^{i\xi X_t}|\Gamma\right]\right] = \mathbb{E}\left[e^{-\frac{\xi^2}{2}(\Gamma(F(\infty))-\Gamma(F(0)))}\right].$$

This can be recognized as the moment generating function, evaluated at $-\xi^2/2$, for the random variable $\Gamma(F(\infty)) - \Gamma(F(0))$ which has a standard gamma distribution with shape parameter $F(\infty) - F(0)$. Consequently

$$\mathbb{E}\left[e^{i\xi X_t}\right] = \left(\frac{1}{1+\frac{\xi^2}{2}}\right)^{F(\infty)-F(0)},$$

so that $X(t)$ has a generalized Laplace distribution. □

The finite dimensional distributions have a less explicit form and the joint density have to be obtained by numerical approximation.

Theorem 3 (Finite-dimensional distributions). *The finite-dimensional distributions of the HL process are defined by the following characteristic function*

$$\mathbb{E}\left[e^{i\sum_{j=1}^{n}\xi_j X(t_j)}\right] = \exp\left\{-\int_0^\infty \ln\left(1+\frac{1}{2}\xi^T A\xi\right) dF(\lambda)\right\},$$

where $\xi = (\xi_1, \xi_2, \ldots, \xi_n)^T$ *and* A *is a matrix with entries* $A_{jk} = \cos(\lambda(t_k - t_j))$.

Proof. Introduce the random variable $Y = \sum_{j=1}^n \xi_j X(t_j)$ then, by definition of the Laplace stationary process,

$$Y = \int_0^\infty f(\lambda)\, dL_F(\lambda) + \int_0^\infty \widetilde{f}(\lambda)\, d\widetilde{L}_F(\lambda),$$

where $f(\lambda) = \sum_{j=1}^n \xi_j \cos(\lambda t_j)$ and $\widetilde{f}(\lambda) = \sum_{j=1}^n \xi_j \sin(\lambda t_j)$. Then, conditional on Γ, we have

$$Y|\Gamma = \int_0^\infty f(\lambda)\, dB_{\Gamma\circ F}(\lambda) + \int_0^\infty \widetilde{f}(\lambda)\, d\widetilde{B}_{\Gamma\circ F}(\lambda).$$

Since this is just a sum of two independent Gaussian variables we get that

$$\mathbb{E}\left[e^{i\tau Y}|\Gamma\right] = e^{-\frac{\tau^2}{2}\int_0^\infty\left(f^2(\lambda)+\widetilde{f}^2(\lambda)\right) d\Gamma_F(\lambda)}.$$

Thus, by the properties of generalized Gamma convolutions, it holds that

$$\begin{aligned}\mathbb{E}[e^{iY}] &= \mathbb{E}\left[\mathbb{E}[e^{iY}|\Gamma]\right] = \mathbb{E}\left[e^{-\frac{1}{2}\int_0^\infty\left(f^2(\lambda)+\widetilde{f}^2(\lambda)\right) d\Gamma_F(\lambda)}\right] \\ &= \exp\left(-\int_0^\infty \ln\left(1+\frac{1}{2}\left(f^2(\lambda)+\widetilde{f}^2(\lambda)\right)\right) dF(\lambda)\right).\end{aligned}$$

Now, the theorem follows by observing that

$$f^2(\lambda) + \widetilde{f}^2(\lambda) = \sum_{j=1}^n\sum_{k=1}^n \xi_j\xi_k \cos(\lambda(t_k - t_j)) = \xi^T A\xi.$$

□

Since, by theorem 3, the finite dimensional distributions are invariant with respect to a translation in time the following corollary is true.

Corollary 1. *The harmonizable Laplace process is stationary in the strict sense.*

Despite stationarity, the *HL* processes are not ergodic which follows, for example, from the results in [7] and [20]. This can be also seen from the simulations of this process as the sampling distribution does not coincide with the theoretical marginal, see Figure 4. There we see a simulated long trajectory (5000 samples) of the stationary Laplace process and look at a histogram of the values in the trajectory. We use the same spectra as in Example 2. As it can be clearly seen the histogram and the theoretical density of $X(0)$ do not match at all, neither for the uniform nor the Pierson-Moskowitz spectrum. This lack of ergodicity could be considered as a serious drawback of the model unless the σ-field of invariant sets for the process is explicitely characterized. In the latter case, the model could be used in applications where the sample-to-sample variability can not be explained by too scarce records so some stochastic models of it are needed. Within non-ergodic stationary processes this requires distributional identification of the conditional expectation with respect to shift invariant sets. The result of [20] suggests that sample to sample variability which is represented by the conditional expectation with respect to the invariant sets can be expressed in terms of jumps of the Gamma process. This line of study as well as the extension of *HL* to the asymmetric case is left for some future work.

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4. Appendix

4.1. **Gamma variance model.** For a random, possibly asymmetric, Laplace measure Λ on the real line and controlled by a measure m on Borel sets, one can obtain an analogue of the gamma variance model. To this end, we start with the definition of a gamma stochastic measure on $(\mathcal{X}, \mathcal{B}, m)$ in the standard manner, i.e. we call a σ-additive function Γ on $\mathcal{B}$ into L_2 an independently scattered gamma measure if for each $A \in \mathcal{B}$ the random variable $\Gamma(A)$ has a gamma distribution with shape parameter $m(A)$ (and scale equal to one). This correspond to the totally skewed case of the Laplace measure, i.e. the case of $\sigma^2 = 0$. More generally, the Laplace measure with $\sigma = 0$ and $\mu > 0$ is referred to as the gamma measure controlled by m. Typically, the letter Γ instead of Λ will be used to denote such a measure.

For a Brownian motion B with variance σ^2 and drift μ and any measurable non-decreasing function $F : [0, \infty) \mapsto [0, \infty)$, the process $B_F(t) = B(F(t))$ is Gaussian with independent although typically non-homogenous increments. Consequently, it corresponds to the randomly scattered measure (controlled by dF) that will be also denoted by B_F.

Proposition 7 (General Gamma Variance Model). *If B is a Brownian motion with variance σ^2 and drift μ and Γ is a stochastic measure controlled by a measure m on Borel sets of the real half-line, then the Laplace measure Λ with parameters μ and σ that is controlled by m is represented with probability one as*

$$\Lambda(A) = B_\Gamma(A),$$

where B and Γ are mutually independent.

Proof. The trajectories of Γ are non-decreasing with probability one thus B_Γ conditionally on such trajectories is well defined. Since both $\Lambda(t) = \Lambda([0,t])$ and $B_\Gamma(t) = B_\Gamma([0,t])$ are processes with independent increments, it is sufficient to verify that the distributions of their increments are identical. We have

$$\phi_{\Lambda(t+h)-\Lambda(t)}(u) = \left(1 - i\mu u + \frac{\sigma^2 u^2}{2}\right)^{m(t)-m(t+h)},$$

where $m(t) = m([0,t])$. Conditionally on $\Gamma(t+h) = \gamma(t+h)$ and $\Gamma(t) = \gamma(t)$ we have

$$\begin{aligned} \mathbb{E}\, e^{iu(B(\gamma(t+h))-B(\gamma(t))} &= \phi_{B(\gamma(t+h)-\gamma(t))}(u) \\ &= e^{(i\mu u - \sigma^2 u^2/2)(\gamma(t+h)-\gamma(t))}. \end{aligned}$$

Since for each complex z such that $\Im z > 0$ we have

$$\mathbb{E}\, e^{iz(\Gamma(t+h)-\Gamma(t))} = (1 - iz)^{m(t)-m(t+h)},$$

we obtain

$$\phi_{B_\Gamma(t+h)-B_\Gamma(t)}(u) = (1 - i(\mu u + i\frac{\sigma^2 u^2}{2}))^{m(t)-m(t+h)}$$

which concludes the proof. □

4.2. **Moments of Laplace stochastic integrals.** Consider first a symmetric Laplace measure Λ, i.e. assume that $\mu = 0$. Formally, the definition of $X = \int f d\Lambda$ extends in a standard way from simple functions to an arbitrary $f \in L_2(\mathcal{X}, \mathcal{B}, m)$ so that

$$\phi_X(u) = \exp\left(-\int_{\mathcal{X}} \ln(1 + \sigma^2 f^2(x) u^2/2)\, dm(x)\right).$$

From this characteristic function it is possible to obtain a recurrent formula for the even moments as shown in the following result.

Proposition 8 (Moments – Symmetric Case). *Let $X = \int f d\Lambda$ and assume that $f^{2N} \in L_2(\mathcal{X}, \mathcal{B}, m)$. Then the following recurrent formula for the even moments holds*

$$\mathbb{E}\, X^{2N} = (2N-1)! \sum_{k=1}^{N} \frac{\sigma^{2k} \int f^{2k} dm}{2^{k-1}} \frac{\mathbb{E}\, X^{2N-2k}}{(2N-2k)!}.$$

Proof. First note that

$$\phi_X'(u) = \phi_X(u)\nu_X(u),$$

where

$$\nu_X(u) = -\int_{\mathcal{X}} \frac{\sigma^2 f^2(x) u}{1 + \sigma^2 f^2(x) u^2/2} dm(x).$$

The above is justified by Lemma 1, Subsection 4.3 of this Appendix, as the derivative of the integrand is bounded by $\sigma^2 |u + \delta| f^2(x)$.

Therefore as long as all involved derivatives exist

$$\phi_X^{(n+1)}(u) = \sum_{k=0}^{n} \binom{n}{k} \phi_X^{(n-k)}(u) \nu_X^{(k)}(u). \tag{9}$$

Since by the assumptions, $f^k \in L_2(\mathcal{X}, \mathcal{B}, m)$, for $k = 1, \dots, 2N$ it follows from Lemma 2 of the Appendix that for $r < 2N$:

$$\nu_X^{(r)}(0) = \begin{cases} 2(-1)^{\frac{r+1}{2}} \left(\frac{\sigma^2}{2}\right)^{\frac{r+1}{2}} r! \int f^{r+1} dm & ; \quad r \text{ odd}, \\ 0 & ; \quad r \text{ even}. \end{cases}$$

Substituting this and $\phi_X^k(0) = i^k \mathbb{E}\, X^k$ into (9) leads after some simple algebra to the recurrent relation for the moments. □

In order to extend the definition of integrals to the asymmetric Laplace measures we start with the integrals with respect to the gamma stochastic measures. The class of distributions that are obtained through the integration of the positive functions with respect to the gamma measure is well-known in the literature under the name of *generalized gamma convolutions* (see [6] for a monographic treatment of this class). Throughout this subsection we assume without further mention that f is an integrable function on $\mathcal{X}$ with respect to m. For the definition of the integral for $f \geq 0$ it is more natural to consider the Laplace transform rather than characteristic functions. Let us consider a random Gamma measure Γ controlled by m and with $\mu > 0$ as the scale parameter. Then

$$\psi_{\Gamma(A)}(z) = e^{-\ln(1+\mu z) m(A)}.$$

Thus if $f \geq 0$, then for $X = \int f \, d\Gamma$:

$$\ln \psi_X(z) = -\int_{\mathcal{X}} \ln(1 + \mu z f(x)) dm(x).$$

To extend the above defintion to a not necessarily non-negative function f we switch back to the characteristic functions and notice that

$$\log \phi_{\Gamma(A)}(u) = \int_0^\infty (e^{iu\xi} - 1)\frac{e^{-\xi/\mu}}{\xi} d\xi \cdot m(A).$$

For an arbitrary $f \in L_2(\mathcal{X}, \mathcal{B}, m)$, we obtain

$$\begin{aligned} \log \phi_X(z) &= \int_0^\infty \int_{\mathcal{X}} (e^{iu\xi f(x)} - 1) \; dm(x) \; \frac{e^{-\xi/\mu}}{\xi} \; d\xi \\ &= \int_{\mathcal{X}} k(uf(x); \mu) \; dm(x), \end{aligned}$$

where the complex function $k(y;\mu)$ is defined as

$$k(y;\mu) = \int_0^\infty \left(e^{iy\xi} - 1\right) \frac{e^{-\xi/\mu}}{\xi} \; d\xi.$$

Proof of Proposition 3. We can now turn to the most general case of $\mu \in \mathbb{R}$, $\sigma > 0$, and $f \in L_2(\mathcal{X}, \mathcal{B}, m)$. It follows from Proposition 2 that the characteristic function of $X = \int f \, d\Lambda$ is given by

$$\log \phi_X(u) \;\; = \textstyle\int_{\mathcal{X}} k(uf(x); \sigma/(\sqrt{2}\kappa)) \; dm(x) + \int_{\mathcal{X}} k(-uf(x); \sigma\kappa/\sqrt{2}) \; dm(x),$$

where $\mu = \frac{\sigma}{\sqrt{2}}\left(\frac{1}{\kappa} - \kappa\right)$, or by introducing

$$K(y;\mu,\sigma) = \int_{-\infty}^\infty \left(e^{iy\xi} - 1\right) \frac{\exp\left(-\frac{\sqrt{2}}{\sigma}\kappa^{\mathrm{sign}(\xi)}|\xi|\right)}{|\xi|} \; d\xi,$$

where sign(ξ) is the sign of ξ, we can simply write

$$\log \phi_X(u) = \int_{\mathcal{X}} K(uf(x); \mu, \sigma) dm(x). \tag{10}$$

The function $K(y;\mu,\sigma)$ can be more explicitly written using the complex logarithm. Due to the representation in Proposition 2 one can identify the function $K(y;\mu,\sigma)$ as being the characteristic exponent of $\Lambda([0,1])$. However, since the characteristic function for $\Lambda([0,1])$ is known on closed form we have that

$$\frac{1}{1 - i\mu y + \frac{\sigma^2 y^2}{2}} = \exp\left(K(y;\mu,\sigma)\right)$$

and consequently

$$-\log\left(1 - i\mu y + \frac{\sigma^2 y^2}{2}\right) = K(y;\mu,\sigma) + i2\pi n(y;\mu,\sigma), \tag{11}$$

where $n(y;\mu,\sigma)$ is an integer which possibly depends on y and the parameters μ and σ. Denoting the left hand side of this equation by $L(y;\mu,\sigma)$, the imaginary parts of $L(y;\mu,\sigma)$ and of $K(y;\mu,\sigma)$ are given by

$$\Im L(y;\mu,\sigma) = \arctan\left(\frac{\mu y}{1+\frac{\sigma^2 y^2}{2}}\right)$$

and

$$\Im K(y;\mu,\sigma) = \int_{-\infty}^{\infty} \sin(y\xi)\frac{\exp\left(-\frac{\sqrt{2}}{\sigma}\kappa^{\mathrm{sign}(\xi)}|\xi|\right)}{|\xi|}\,d\xi,$$

respectively. Since both these functions are continuous functions of y and so is the logarithmic function on the left hand side of (11) (note that the real part of the argument under logarithm is positive), it follows that $n(y;\mu,\sigma) = n(0;\mu,\sigma)$. Moreover, since $\Im L(0;\mu,\sigma) = \Im K(0;\mu,\sigma) = 0$, it holds that $n(0;\mu,\sigma) = 0$. Thus $K(y;\mu,\sigma) = L(y;\mu,\sigma)$ which concludes the proof. □

The following is a proof of the recurrent formula for moments for the general not necessarily symmetric case.

Proof of Proposition 4. The characteristic function for the integral with respect to asymmetric Laplace motion is given by (2). Using Lemma 1 in the Appendix (the integrand is bounded in absolute value by $\sqrt{6(\sigma^2+\mu^2)}|f(x)| + \sqrt{3\sigma^2\mu^2}\sqrt{|u+\delta|}f^2(x)$), we have

$$\phi_X^{(1)}(u) = \phi_X(u)\nu_X(u),$$

where

$$\nu_X(u) = -\int_{\mathcal{X}} \frac{\sigma^2 f^2(x)u - i\mu f(x)}{1 - i\mu f(x)u + \frac{\sigma^2 f^2(x)u^2}{2}}\,dm(x).$$

The recursion for the moments follows easily from Lemma 3 in the Appendix and

$$\begin{aligned}\mathbb{E}\,X^N &= (-i)^N\phi_X^{(N)}(0) = (-i)^N\sum_{k=1}^{N}\binom{N-1}{k-1}\phi_X^{N-k}(0)\nu_X^{(k-1)}(0)\\ &= \sum_{k=1}^{N}\binom{N-1}{k-1}(-i)^k\mathbb{E}\,X^{N-k}\nu_X^{(k-1)}(0).\end{aligned}$$

□

4.3. **Lemmas.** For completeness, we include here some technical facts that allows to carry out some formal argument at certain parts of the proofs. Due to its very technical character and for clarity of the main ideas of the proofs, we gathered them in this Appendix.

Lemma 1. *Consider* $I(u) = \int g(x,u)dm(x)$ *with* g *differentiable with respect to* u*. Then*

$$\frac{dI}{du}(u) = \int \frac{\partial g}{\partial u}(x,u)\ dm(x)$$

if there exists $G(x) \geq 0$ *integrable with respect to* m *and* $\delta > 0$ *such that for each* $h \in [0,\delta]$*:*

$$\left|\frac{\partial g}{\partial u}(x,u+h)\right| \leq G(x).$$

Proof. By the first order Taylor expansion

$$\frac{I(u+h)-I(u)}{h} = \int \frac{\partial g}{\partial u}(x,u+o_x(h))\ dm(x).$$

The term under the integral is uniformly bounded by $G(x)$ for each $0 < h < \delta$ that allows for direct application exchange the limit with integration. □

Lemma 2. *Let* $g_{k,j}(u;a) = u^k(a^2+u^2)^{-j}$, $k \geq 0$, $j > 0$ *and for* h *such that* $h^j \in L_2(\mathcal{X},\mathcal{B},m)$ *let*

$$G_{k,j}(u) = \int_{\mathcal{X}} g_{k,j}\left(u;1/h(x)\right) dm(x).$$

Then if both h^j *and* h^{j+1} *are in* $L_2(\mathcal{X},\mathcal{B},m)$*:*

$$G'_{k,j}(u) = kG_{k-1,j}(u) - 2jG_{k+1,j+1}(u). \tag{12}$$

If $h, h^{n+1} \in L_2(\mathcal{X},\mathcal{B},m)$*, then* $G_{1,1}^{(n)}(u)$ *exists and*

$$G_{1,1}^{(r)}(0) = \begin{cases} (-1)^{\frac{r-1}{2}} r! \cdot \int h^{r+1} dm & , \quad r \leq n \text{ odd} \\ 0 & , \quad r \leq n \text{ even} \end{cases}$$

Proof. The recurence relations between derivatives follows from Lemma 1 if we notice that

$$|g_{k,j}(u;a)| \leq a^{-2j}|u|^k.$$

For h that is bounded and such that h^k is integrable for each $k \in \mathbb{N}$ the formula follows by the expansion of

$$\frac{u}{a^2+u^2} = \sum_{k=0}^{\infty}(-1)^k \frac{u^{2k+1}}{a^{2k+2}},$$

for $u^2/a^2 < 1$. So for $u^2 < M^{-2}$, where M is an upper bound for $|h(x)|$ we have

$$\begin{aligned} G_{1,1}(u) &= \int_{\mathcal{X}} \sum_{k=1}^{\infty} (-1)^{k+1} h^{2k}(x) u^{2k-1}\, dm(x) \\ &= \sum_{k=1}^{\infty} (-1)^{k+1} \int h^{2k} dm u^{2k-1}. \end{aligned}$$

This proves the formula for the derivatives at zero for functions h that are bounded and integrable in any power.

For a general $h, h^{n+1} \in L_2(\mathcal{X}, \mathcal{B}, m)$, we notice from relation (12) that there is a formula expressing $G^n_{1,1}(u)$ as a linear combination of $G_{k,j}(u)$ for $0 \le k \le n+1$ and $0 \le j \le n+1$. The same relation obviously holds for functions that are bounded and integrable in any power. One can take a sequence of such functions that approximates h and the norm of their powers approximate $\int h^{2k} dm$. We note that

$$\begin{aligned} G_{0,j}(0) &= \frac{\sigma^{2j}}{2^j} \int h^{2j} dm, \\ G_{k,j}(0) &= 0,\ k > 0, \end{aligned}$$

Thus the formula for $G^{(n)}_{1,1}(0)$ has to be expressed as a linear combination of $G_{0,j}(0)$, $1 \le j \le n+1$ and must agree with the corresponding formula for the sequence of approximating functions and this concludes the proof. □

The above result is extended to a more general case as follows.

Lemma 3. *Let for $k, j \ge 0$, $j \ge k$:*

$$g_{k,j}(u; a, b) = (a^2 u - ib)^k (1 - ibu + a^2 u^2/2)^{-j}.$$

Then for h such that $h^k \in L_2(\mathcal{X}, \mathcal{B}, m)$ the following integral is well-defined

$$G_{k,j}(u; a) = \int_{\mathcal{X}} g_{k,j}\left(u; a^2 h^2(x), h(x)\right) dm(x).$$

If h^{k-1} (if $k > 0$), h^k, h^{k+1} all are in $L_2(\mathcal{X}, \mathcal{B}, m)$, then

$$G'_{k,j}(u; a) = a^2 k \cdot G_{k-1,j}(u; a) - j \cdot G_{k+1,j+1}(u; a). \tag{13}$$

If $h, h^{r+1} \in L_2(\mathcal{X}, \mathcal{B}, m)$, then $G^{(r)}_{1,1}(u)$ exists and

$$G^{(r)}_{1,1}(0; a) = \begin{cases} i^{r-1} r! \int f^{r+1} dm \sum_{k=0}^{\frac{r}{2}-\frac{1}{2}} s_{r,k}, & r \text{ odd}, \\ i^{r-1} r! \int f^{r+1} dm \left(\mu \left(\frac{\sigma^2}{2}\right)^{\frac{r}{2}} + \sum_{k=0}^{\frac{r}{2}-1} s_{r,k} \right), & r \text{ even}, \end{cases}, \tag{14}$$

where

$$s_{r,k} = \mu^{r-2k-1}\left(\frac{\sigma^2}{2}\right)^k \left(\binom{r-k-1}{k}\sigma^2 + \binom{r-k}{k}\mu^2\right).$$

Proof. The main idea of the proof is identical to the one in the previous result, therefore we just highlight the main steps.

We note that there exists a constant K independent of u, a, and b such that

$$|g_{k,j}(u;a,b)|^2 \le K\left(a^{2k} + b^{2k}\right). \tag{15}$$

From this it follows that $G_{k,j}(u;a)$ is properly defined if h is such that $h^k \in L_2(\mathcal{X}, \mathcal{B}, m)$. The same inequality (15), Lemma 1, and the following recurrence relation

$$\frac{d}{du} g_{k,j}(u;a,b) = a^2 k \cdot g_{k-1,j}(u;a,b) - j \cdot g_{k+1,j+1}(u;a,b)$$

give (13) as long as the conditions on h are satisfied.

By using the fact that $(1-x)^{-1} = \sum_{n=0}^{\infty} x^n$ if $|x| < 1$, it holds for bounded and integrable in any positive integer power h that

$$\frac{a^2h^2u - ih}{1 - ihu + \frac{a^2h^2u^2}{2}} = (a^2h^2u - ih)\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}(ihu)^k\left(-\frac{a^2h^2u^2}{2}\right)^{n-k}$$

By reorganizing this expression, taking the r-th derivative and evaluating at zero some tedious but straightforward calculations leads to (14).

For a general $h, h^{r+1} \in L_2(\mathcal{X}, \mathcal{B}, m)$, we notice from relation (12) that there is a formula expressing $G_{1,1}^r(u)$ as a linear combination of $G_{k,j}(u)$ for $0 \le k \le r+1$ and $0 \le j \le r+1$. The same relation obviously holds for functions that are bounded and integrable in any power and the rest of the proof follows the same as in Lemma 2. □

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