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with an appendix by PETER HEGARTY

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2-PILE NIM WITH A RESTRICTED NUMBER OF MOVE-SIZE IMITATIONS

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WITH AN APPENDIX BY PETER HEGARTY

ABSTRACT. We study a variation of the combinatorial game of 2-pile Nim. Move as in 2-pile Nim but with the following constraint:

Provided the previous player has just removed say $x > 0$ tokens from the pile with less tokens, the next player may remove x tokens from the pile with more tokens. But for each move, in "a strict sequence of previous player - next player moves", such an *imitation* takes place, the value of an *imitation counter* is increased by one unit. As this counter reaches a pre-determined natural number, then by the rules of this game, if the previous player once again removes a positive number of tokens from the pile with less tokens, the next player may not remove this same number of tokens from the pile with more tokens.

We show that the winning positions of this game in a sense resemble closely the winning positions of the game of Wythoff Nim - more precisely a version of Wythoff Nim with a Muller twist. In fact, we show a slightly more general result in which we have relaxed the notion of what an imitation is.

1. INTRODUCTION

A finite impartial game is usually a game where

- there are 2 players and a starting position,
- there is a finite set of possible positions of the game,
- there is no hidden information,
- there is no chance-device affecting how the players move,
- the players move alternately and obey the same game rules,
- there is at least one final position, from which a player cannot move, which determines the winner of the game and
- the game ends in a finite number of moves, no matter how it is played.

If the winner of the game is the player who makes the last move, then we play under normal play rules, otherwise we play a misére version of the game.

In this paper *a game* is always a finite impartial game played under normal rules. A position from which the player who made the last move, the previous player, can always win given perfect play, is called a *P-position.* A

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position where the next player can always win given best play is called an N*-position*.

In the classical game of Nim the players alternately remove a positive number of tokens from the top of any of a predetermined number of piles. The winning strategy of Nim is to, whenever possible, move so that the "Nim-sum" of the pile-heights equals zero, see for example $[Bou]$ or $[SmSt, page 3]$. When played on only two piles, the pile-heights should be equal to ensure victory for the previous player. When played on only one pile there are only next player winning positions. These two settings for Nim are simple, but there are many interesting and non-trivial extensions if we adjoin "move-size" or "pile-size" dynamic rules to Nim on just one or two piles. For the purpose of this paper we are especially interested in move-size dynamic games. The game of "Fibonacci Nim" in $[BeGuCo, page 483]$ is a beautiful example of a move-size dynamic game on just one pile. This game has been generalised in for example $[HoReRu]$. Treatments of two-pile move-size dynamic games can be found in $[Co]$, extending the (pile-size dynamic) "Euclid game", and in $[HoRe]$.

1.1. **Adjoin the P-positions as moves.** Another type of extension to a game is (*) *to adjoin the P-positions of the original game as moves in the new game.* If we adjoin the P-positions of the game of 2-pile Nim as moves, then we get the famous game of Wythoff Nim (a.k.a Corner the queen), see $[Wy]$. Namely, the set of moves are: Remove any number of tokens from one of the piles, or remove the same number of tokens from both piles. The P-positions of this game are well-known. Let $\phi = \frac{1+\sqrt{5}}{2}$ denote the golden ratio. Then (x, y) is a P-position if and only if

$$(x, y) \text{ or } (y, x) \in \{(\lfloor n\phi \rfloor, \lfloor n\phi^2 \rfloor) \mid n \in \mathbb{N}_0\}.$$

These P-positions exhibit many beautiful properties that, in generalised forms, will be revisited often throughout this paper. See for example (1), Definition 1 and Proposition 1.

Other examples of (*) are some extensions to the game of 2-pile Wythoff Nim in $[FraOz]$, where the authors adjoin subsets of the Wythoff Nim P-positions as moves in new games. We discuss briefly extensions of Nim on several piles in Section 4.

1.2. **Remove a game's winning strategy.** In this paper we discuss another approach for expanding well-known games, namely (**) *from the original game, remove the next-player winning strategy.* We will put our main focus on the setting of 2-pile take-away games, but in section 4 we give some examples of how our idea can be generalised for $n \geq 2$ piles as well. For 2-pile Nim this means that we remove the possibility to *imitate* the previous player's move, in the following sense:

Given two piles of tokens, A and B, one of the piles, say A, is a *leading* pile (relative to B) if the number of tokens in A (before removal of tokens) is less than or equal to the number of tokens in B. If a pile is not a leading pile it is a *non-leading* pile. Remark: For each move the "label" of the piles is renewed according to the new pile-heights.

The (initial version of the) game of *Imitation Nim* is then defined as follows: Given two piles, A and B, the players move as in 2-pile Nim but a player may *not imitate* the other player's move, where by *imitate* we mean the following 2-move sequence:

> *The previous player removed (a positive) number of tokens from a leading pile and the next player removes the same number of tokens from the non-leading pile.*

This game is a "one-sided" move-size dynamic restriction of the game of Nim.

The options for a move depend on the pile from which the previous player removed tokens, and how many tokens he removed. So, how can we then talk at all about P- and N-positions? To clarify matters, one might want to partition the positions of a (move-size dynamic) game into,

(1) *dynamic positions*: it is impossible to tell the "winning nature" of a dynamic position without knowledge about the previous players' move(s), and
(2) *non-dynamic positions*:
   (a) *non-dynamic P-positions*: these positions are P-positions regardless of previous move(s), and
   (b) *non-dynamic N-positions*: ditto, but N-positions.

Notice that, by these definitions, an initial position of a move-size dynamic game is always non-dynamic.

For our game Imitation Nim, a move from an initial position follows the rules of 2-pile Nim with no extra constraint whatsoever. In the light of the above definitions, we will now characterize the winning positions of a game of Imitation Nim via the winning positions of Wythoff Nim (this is a somewhat simplified variant of our main theorem in Section 3, notice for example the absence of Wythoff Nim P-positions that are dynamic, regarded as positions of Imitation Nim):

Proposition 0 *Let a and b be non-negative integers. Then*

(1) *if (a, b) is a P-position of Wythoff Nim, it is a non-dynamic P-position of Imitation Nim;*
(2) *if (a, b) is an N-position of Wythoff Nim, it is a non-dynamic N-position of Imitation Nim, if*
   (a) *it is the initial position of a game, or*
   (b) *if there is a P-position of Wythoff Nim, say (c, d), with $c < a$ such that $b - a = d - c$, or*
   (c) *if $a < b$ implies that there is a P-position of Wythoff Nim, (a, c) with $c < a$;*
(3) *otherwise a Wythoff Nim N-position is a dynamic position of Imitation Nim.*

Remark 1: For a given position, the rules of Wythoff Nim give more options than the rules of Nim, whereas the rules of Imitation Nim give less options than Nim.

1.3. **Extensions and their reversals.** Our extension of Imitation Nim is to, given a fixed positive integer p, allow $p-1$ imitations (as defined on page 2) in a sequence and by one and the same player, but to prohibit the p:th imitation. In other words, an imitation counter is reset to 0, once a player stops imitating the other player. In the next section we give the precise setting of these games. We denote this game by $(p, 1)$-Imitation Nim.

Remark 2: This rule removes the winning strategy from 2-pile Nim if and only if the number of tokens in the leading pile is $\geq p$.

The winning positions of this game correspond, in a way we shall make precise, to the winning positions of a version of Wythoff Nim with a so-called *Muller twist* or *blocking manoeuver*. This is a variation on the rules of a game which states that: The previous player may, before the next player moves, point out a (predetermined) number of *options* (positions) and declare that the next player may not move there.

For a nice introduction to the concept of a Muller twist, see for example [$SmSt$]. Variations on Nim with a Muller twist can also be found in, for example, [$GaSt$] (which generalises a result in [$SmSt$]), [$HoRe1$] and [Zh].

In section 3, we also expand on the following generalisation of Imitation Nim: If the previous player removed x tokens from a leading pile, then the next player may not remove $y \in [x, x+m-1]$ tokens from the non-leading pile. We denote this game by $(1, m)$-Imitation Nim.

There is another generalisation of the original game of Wythoff Nim, which (as we will show in section 3) has a natural P-position "correspondance" with $(1, m)$-Imitation Nim. Fix a positive integer m. The rules for this game, that we denote by $(1, m)$-Wythoff Nim, are: remove tokens from one of the piles, or remove tokens from both piles, say x and y tokens respectively, with the restriction that $\mid x-y \mid < m$. In [Fra], the author shows that the P-positions of $(1, m)$-Wythoff Nim are so-called "Beatty pairs" (view for example the appendix, the original paper(s) in [Bea] or [Fra, page355]) of the form $\{n\alpha, n\beta\}$[1] where $\beta = \alpha + m$, n is a non-negative integer and

$$\alpha = \frac{2-m+\sqrt{m^2+4}}{2}. \tag{1}$$

In [$HeLa$] we put a Muller twist on the game of $(1, m)$-Wythoff Nim. Fix two positive integers p and m. Suppose the current pile-position is (a, b). The rules are: Before the N-player removes any tokens, the P-player is allowed to announce $j \in [1, p-1]$ positions, say $(a_1, b_1), \dots, (a_j, b_j)$ where $b_i - a_i = b - a$, to which the N-player may not move. Otherwise move as in

[1]We use the notation $\{x, y\}$ for unordered pairs of integers, that is, whenever (x, y) and (y, x) are considered the same.

$(1,m)$-Wythoff Nim. In this paper we denote this game by (p,m)-Wythoff Nim, or simply $W_{p,m}$.

The P-positions of this game can easily be calculated by a minimal exclusive algorithm (but with exponential complexity in succint input size) as follows: Let X be a set of non-negative integers. Define $\mathrm{mex}(X)$ as the least non-negative integer not in X, formally $\mathrm{mex}(X) := \min\{x \mid x \in \mathbb{N}_0 \setminus X\}$.

Definition 1 Given positive integers p and m, the integer sequences (a_i) and (b_i) are: $a_0 = b_0 = 0$ and for $i > 0$

$$\begin{aligned} a_i &= \mathrm{mex}\{a_j, b_j \mid 0 \le j < i\}; \\ b_i &= a_i + \delta(i), \end{aligned}$$

where $\delta(i) = \delta_{m,p}(i) := \lfloor \frac{i}{p} \rfloor m$.

From $[HeLa]$, Proposition 3.1 (see also Remark 1) we can derive the following results.

Proposition 1 *Let p and m be positive integers.*
a) *The P-positions of (p,m)-Wythoff Nim are the pairs $\{a_i, b_i\}$, $i \in \mathbb{N}_0$, as in Definition 1;*
b) *Together these pairs partition the natural numbers;*
c) *Suppose (a,b) and (c,d) are two distinct P-positions of (p,m)-Wythoff Nim with $a \le b$ and $c \le d$. Then $a < c$ implies $b - a \le d - c$ (and $b < d$);*
d) *For each $\delta \in \mathbb{N}$, if $m \mid \delta$ then $\#\{i \in \mathbb{N}_0 \mid b_i - a_i = \delta\} = p$, otherwise $\#\{i \in \mathbb{N}_0 \mid b_i - a_i = \delta\} = 0$.*

The "(p,m)-Wythoff pairs" from Proposition 1 may be expressed via Beatty pairs if $p \mid m$. Then the P-positions of (p,m)-Wythoff Nim are of the form

$$(pa_n, pb_n), (pa_n + 1, pb_n + 1), \ldots, (pa_n + p - 1, pb_n + p - 1),$$

where (a_n, b_n) are the P-positions for the game $(1, m/p)$-Wythoff Nim.

For any other p and m we do not have a polynomial time algorithm for telling wether a given position is an N- or a P-position. But we do have a conjecture in $[HeLa]$ section 5, saying in a specific sense that the (p,m)-Wythoff pairs are "close to" certain Beatty pairs. This has recently been settled for the case $m = 1$, see the appendix.

As we have already hinted, the winning positions for $(p,1)$-Imitation Nim correspond to the winning positions of $(p,1)$-Wythoff Nim whereas the winning positions for $(1,m)$-Imitation Nim correspond to the winning positions of $(1,m)$-Wythoff Nim. For our main theorem in section 3, we show how our two extensions above can thrive in one and the same game, namely for fixed positive integers p and m, we define a game that we denote by (p,m)-Imitation Nim and show how the winning positions of this game correlate to (p,m)-Wythoff Nim. Section 2 is devoted to some preliminary terminology for the definition of the rules of our games.

Suppose $p > 1$. The analogous partitioning of the winning positions for (p, m)-Imitation Nim as we did for $(1, 1)$-Imitation Nim in Proposition 0, will essentially be done in the beginning of section 3. For this generalisation there are "more" dynamic positions. Namely, we will see that some of the P-positions of (p, m)-Wythoff Nim are dynamic.

Remark 3: The number of permitted imitations in $(p, 1)$-Imitation Nim may be viewed as "a reversal" of the number of positions the previous player may block from the next player's options in the game of $(p, 1)$-Wythoff Nim. A greater p in the game of $(p, 1)$-Wythoff Nim makes "life a little bit harder" for the next player - less options to choose from. A greater p in the game of $(p, 1)$-Imitation Nim gives more leeway to imitate a move, hence less imposed constraint and "game-life is easier". For $m \geq 1$, the rules of $(1, m + 1)$-Imitation Nim impose more restrictions on a move than the rules of $(1, m)$-Imitation Nim. On the other hand, the rules of $(1, m+1)$-Wythoff Nim are less restrictive than those of $(1, m)$-Wythoff Nim.

In section 2 we give a formal notation to our previous discussions to be used in the proof of the main theorem in Section 3.

In Section 4, where the paper again has a more informal style, we give a couple of suggestions for future work. In the first part we discuss a different setup of an impartial game, where one of the players, say the first player knows how to play optimally, and the second does not. In our setting, the first player is not aware of the second player's lack of knowledge of the strategy. Given that the same game is going to be played several times, this setting gives us an interpretation of games as primitive forms of learning devices.

In the last part of Section 4 we discuss extensions of Nim to several piles. We generalise in both directions, from 2-pile Wythoff Nim and from 2-pile Imitation Nim. At the end some questions are formulated.

2. Terminology and rules of the game

Let $\mathbb{N}$ denote the natural numbers $\{1, 2, \ldots\}$ and let $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

We will now give a more formal setting to our discussion in Section 1. Let G denote a 2-pile take-away game. Then $V(G)$ denotes the set of all positions of G. Suppose $X \in V(G)$. Then $F(X)$ denotes the set of *followers* or *options* of X, the set of positions that the next player may move to from X. Let $X_2 \geq X_1 \geq 0$ denote the respective pile-heights of X and let $\Delta(X) := X_2 - X_1 \geq 0$.

Suppose A and B are (in this order) the two piles of a 2-pile take-away game, and they contain a and b tokens respectively. As before, $\{a, b\}$ denotes the unordered pair of the pile-heights. The *pile-position* is (a, b) and a *move* is denoted by $(a, b) \to (c, d)$, where $a - c \geq 0$ and $b - d \geq 0$ but not both $a = c$ and $b = d$. For notational purposes, we allow a move to be either *legal* or *illegal*. A move is legal if, by removing tokens, the player ends up in a pile-position such that none of the rules of game have been violated. Otherwise a suggested move is illegal and may not be carried out.

2.1. **Positions with memory.** By the move-size dynamic nature of our games, to simplify the statement and proof of our main theorem in section 3, we have felt encouraged to introduce some non-standard terminology.

Suppose X is a position of a 2-pile take-away game and suppose that the players have this far removed tokens a combined number of n times. Let X^{-i} denote the i:th last pile-position of this specific game if $n \geq i$, otherwise let X^{-i} denote the initial position. Notice that with this notation, $X = X^{-1}$ if and only if X is the initial position of a game. Whenever we refer to a *(game-)position* or a position with *memory*, we mean not only the current position but also the two players' preceeding pile-positions (if any) - although, as we will soon see, the players never need to remember more than a predetermined fixed number of moves for our specific setting.

Let us analyse a few examples of this notation:

$$X^{-1}{}_1 = X_1, \tag{2}$$

$$X^{-2}{}_2 = X^{-1}{}_2 \tag{3}$$

and

$$X^{-2}{}_1 - X_1^{-1} \geq X^{-1}{}_2 - X_2 > 0. \tag{4}$$

By the condition (2) one understands that the previous player (by the most recent move) only removed tokens from the heap with more tokens. The condition (3) means that in the "next" player's last move, the pile with more tokens was left untouched. Then it is not hard to see that (4) implies (2) and (3) for any game where

($\mathcal{O}$) a move consists of removing a positive number of tokens from precisely one of the piles.

2.2. **An imitation rule.** Let G be a take-away game on 2 piles of tokens where ($\mathcal{O}$) holds. Define $\mu : V(G) \to \mathbb{N} \cup \{\infty\}$ as follows. Suppose $X \in V(G)$. Then,

$\mu(X) := X^{-1}{}_2 - X_2 - (X^{-2}{}_1 - X^{-1}{}_1)$, if (4) holds.
$\mu(X) := \infty$, otherwise.

Notice that for example $X_2 = X_2^{-1}$ or $X_1^{-1} = X_1^{-2}$ implies $\mu(X) = \infty$. Let m be a positive integer. Then we say that $X^{-1} \to X$ *is an m-imitation of* $X^{-2} \to X^{-1}$ if $0 \leq \mu(X) < m$. When m is fixed we say simply that

$X^{-1} \to X$ *is an imitation*. Note that if $m = 1$ then $X^{-1} \to X$ is an imitation if and only if $\mu(X) = X^{-1}{}_2 - X_2 - (X^{-2}{}_1 - X^{-1}{}_1) = 0$.

Let $L = L_m : V(G) \to \mathbb{N}$ be a recursively defined *imitation counter*:

$L(X) := L(X^{-2}) + 1$, if $X^{-1} \to X$ is an m-imitation of $X^{-2} \to X^{-1}$.

$L(X) := 0$, otherwise.

In particular $L(X) = 0$ if X is the initial position of a game.

Fix two positive integers p and m and a game-position X. We will now define the rules of our 2-pile take-away game which will be denoted by (p, m)-*Imitation Nim*, or simply $I_{p,m}$.

The players move as in 2-pile Nim, but with the constraint: The next player may move $X \to Y$ only if

$$L_m(Y) < p. \tag{5}$$

2.3. **Sets of winning positions.** Let a and b be non-negative integers and define

$$\xi\big((a,b)\big) := \#\big\{\, (i,j) \in \mathcal{P}_{W_{p,m}} \mid\ j - i = b - a,\ i \geq a \,\big\}.$$

Then according to Proposition 1d, $1 \leq \xi((a,b)) \leq p$.

Given that a position is treated as a game-position (each position contains information about the last $2p - 1$ moves at least) we denote with

$$\mathcal{N}_{I_{p,m}} := \{X \mid X \text{ is an } N\text{-position of } (p,m)\text{-imitation Nim } \}$$

and

$$\mathcal{P}_{I_{p,m}} := \{X \mid X \text{ is a } P\text{-position of } (p,m)\text{-imitation Nim } \}.$$

Whenever positive integers p and m are fixed we may abbreviate (p, m)-Wythoff Nim simply by *Wythoff Nim*.

3. A winning strategy

Let p and m be positive integers. Suppose X is a game-position of (p, m)-imitation Nim. Then we define the sets $\mathcal{P}' = \mathcal{P}'(p, m)$ and $\mathcal{N}' = \mathcal{N}'(p, m)$ as:

$X \in \mathcal{P}'$ if:

- ($\mathcal{P}1$) X is a P-position of (p, m)-Wythoff Nim and $L_m(X) < \xi(X)$, or
- ($\mathcal{P}2$) X is an N-position of (p, m)-Wythoff Nim and there is a $X_1 \leq j < X_2$ such that $\{X_1, j\}$ is a P-position of (p, m)-Wythoff Nim and $(X_1, X_2) \to (X_1, j)$ implies $L_m((X_1, j)) \geq \xi((X_1, j))$.

$X \in \mathcal{N}'$ if:

- ($\mathcal{N}1$) X is an N-position of (p, m)-Wythoff Nim, and
  - (a) X is the initial position of the game, or
  - (b) there is an $i < X_1$ such that $\{X_1, i\}$ is a P-position of (p, m)-Wythoff Nim, or
  - (c) there is a $j > X_2$ such that $\{X_1, j\}$ is a P-position of (p, m)-Wythoff Nim, or

(d) there is a $X_1 \leq j < X_2$ such that $\{X_1, j\}$ is a P-position of (p, m)-Wythoff Nim, but $(X_1, X_2) \rightarrow (X_1, j)$ implies $L_m((X_1, j)) < \xi((X_1, j))$;

or

($\mathcal{N}2$) X is a P-position of (p, m)-Wythoff Nim and $p > L_m(X) \geq \xi(X)$.

Notice that $\mathcal{N}1b$, $\mathcal{N}1c$ and $\mathcal{N}1d$ are mutually exclusive. Let us now study some of the structure of the above definition and show that $\mathcal{P}' \cap \mathcal{N}' = \emptyset$.

Lemma 1 *Given a position X and the definitions of $\mathcal{P}'$ and $\mathcal{N}'$,*

(i) $\mathcal{P}1$ *holds if X is the initial position of a game,*
(ii) $\mathcal{P}1$ *holds if $X_1 \leq i$ for all P-positions $\{i, j\}$ of (p, m)-Wythoff Nim such that $X_2 - X_1 = j - i \geq 0$.*
(iii) $\mathcal{N}1$a *implies* $\mathcal{N}1$b, $\mathcal{N}1$c *or* $\mathcal{N}1$d, *and*
(iv) $X \in \mathcal{N}'$ *if and only if* $X \notin \mathcal{P}'$.

Proof. Case i: If X is the initial position then by definition $L(X) = 0$, but then since X is a P-position of (p, m)-Wythoff Nim surely $\xi(X) > 0$.

Case ii: If X is of the form as in (ii) then $\xi(X) = p$, but by the rules of game $L(X) < p$.

Case iii: If $X_1 = j \geq i \geq 0$ for some P-position $\{i, j\}$ of Wythoff Nim, then by $\mathcal{N}1$b we are done. Else, since the P-positions of Wythoff Nim partition the natural numbers, we may assume $X_1 = i \leq j$ for some P-position $\{i, j\}$. But, if $X_2 < j$ then $\{X_1, X_2\}$ is of form $\mathcal{N}1$c, so assume $X_2 > j$. Then, since X is an initial position, $X \rightarrow (X_1, j)$ is not an m-imitation, which implies $\mathcal{N}1$d.

Case iv: Viewed as a position of the game of Wythoff Nim, X is either

I. a P-position, or
II. an N-position.

Let us start with the "only if" part. If $X \in \mathcal{P}'$ then

- if I. holds, $\mathcal{N}1$ is false. By (i) and (ii), X satisfies $L(X) < \xi(N)$ so that X can not be of the form $\mathcal{N}2$.
- if II. holds, $\mathcal{N}2$ is false. By Proposition 1b, $\mathcal{N}1$b and $\mathcal{N}1$c are false. With notation as in $\mathcal{P}2$, since $L((X_1, j)) \geq \xi((X_1, j))$, $\mathcal{N}1$d is false.

For the "if" part. If $X \notin \mathcal{P}'$ then

- if II. holds, by the negotiation of $\mathcal{P}2$, if $\mathcal{N}1$b or $\mathcal{N}1$c holds, we are done, so suppose $X_2 > j$ with notation as in $\mathcal{N}1$d. Then again, the falsity of $\mathcal{P}2$, $L((X_1, j)) < \xi((X_1, j))$ gives $\mathcal{N}1$d;
- if I. holds, $\mathcal{P}1$ must be false. But then since $L(X) < \xi(X) \leq p$, $\mathcal{N}2$ gives $X \in \mathcal{N}'$.

We are done.

With terminology as in the introduction, we have given a strong hint that the non-dynamic P-positions are as in Lemma 1(ii). On the other hand, we will see in the proof of the main theorem that the non-dynamic N-positions are precisely of the forms $\mathcal{N}1$b or $\mathcal{N}1$c.

3.1. **Putting it all together.** For the proof of the main theorem, we will need some well-known facts on impartial games. For a specific game,

- a non-terminal position is a P-position if and only if each one of its followers is an N-position;
- a position is an N-position if and only if there is a P-position among its set of followers.

Main theorem *Let p and m be positive integers. Then*

$$\mathcal{N}'(p,m) = \mathcal{N}_{I_{p,m}}$$

and

$$\mathcal{P}'(p,m) = \mathcal{P}_{I_{p,m}}.$$

Proof. It suffices to show that, for a position $Y = (Y_1, Y_2)$ with $Y_1 \le Y_2$:

I. if $Y \in \mathcal{P}'$ then $F(Y) \subset \mathcal{N}'$;
II. if $Y \in \mathcal{N}'$ then $\mathcal{P}' \cap F(Y) \neq \emptyset$.

Notice that if Y is a final position, then either

$Y = (0,0)$, or
$Y \to (0,0)$ is illegal, but then $L((0,0)) = p = \xi((0,0))$, which implies $\mathcal{P}2$.

Hence, if Y is a final position, then $Y \in \mathcal{P}'$. But then $F(Y) = \emptyset \subset \mathcal{N}'$ so that I. is true. For the remainder of the proof assume that Y is not a final position.

Case I. Define $S = S(Y) := \mathcal{P}' \cap F(Y)$. There are two possibilities for $Y \in \mathcal{P}'$:

A) Y is a P-position, viewed as a position of Wythoff Nim. By $\mathcal{O}$, we get that $S(Y) \subset \mathcal{N}_{W_{p,m}}$. Then either the next player removes tokens from the pile with:
   1) less tokens, say $Y \to X = (X_1, Y_2)$, (where $X \in \mathcal{N}_{W_{p,m}}$, as in the definition of $\mathcal{N}'$). Then
      * if there is an integer $i \le X_1$ such that (X_1, i) is a P-position of Wythoff Nim, by $\mathcal{N}1$b, we get $X \in \mathcal{N}'$;
      * otherwise, by Proposition 1c, there is an integer j with $X_1 < j < Y_2 = X_2$ such that (X_1, j) is a P-position of Wythoff Nim. Then, if $j - X_1 \le Y_2 - Y_1 < j - X_1 + m$, the move $X \to (X_1, j)$ is an m-imitation and so, by $\mathcal{P}2$, we get $\xi((X_1, j)) > \xi(Y) \ge L(Y) + 1 = L((X_1, j))$. Then, by $\mathcal{N}1$d, we get $X \in \mathcal{N}'$. If on the other hand $X \to (X_1, j)$ is not an m-imitation we get $L((X_1, j)) = 0$ and again, $\mathcal{N}1$d gives $X \in \mathcal{N}'$;
   2) more tokens, say $Y \to X = (Y_1, X_2)$. Then,
      * if $X_1 = Y_1 < X_2 < Y_2 = j$, by $\mathcal{N}1$c, we get $X \in \mathcal{N}'$;

- ∗ if $X_2 \leq Y_1 = X_1$, by $\mathcal{N}1$b and $\mathcal{N}1$c we may assume that there is a $X_1 \leq j < X_2$ such that $(Y_1, j) \in \mathcal{P}_{W_{p,m}}$. But, since the previous player removed tokens from the larger pile, the move $X \to (X_1, j)$ is no imitation and so $0 = L((X_1, j)) < 1 \leq \xi((X_1, j))$, which implies $\mathcal{N}1$d.

Hence, we may conclude that $S(Y)$ does not contain any Wythoff Nim P-positions.

B) Y is an N-position viewed as a position of Wythoff Nim, and hence of the form $\mathcal{P}2$:

1) Suppose that $S(Y)$ contains a position of form $\mathcal{P}1$, say (Y_1, j), with $Y_1 \leq j < Y_2$. But this is impossible, since by $\mathcal{P}1$ and $\mathcal{P}2$, in this order, we get $\xi((Y_1, j)) > L((Y_1, j)) \geq \xi((Y_1, j))$.
2) Suppose that $S(Y)$ contains a position of form $\mathcal{P}2$, say X.
   (a) Then, if there is a $X_1 \leq j < X_2 < Y_2$ such that (Y_1, j) is a Wythoff Nim P-position, the move $(Y_1, X_2) \to (Y_1, j)$ does not imitate $Y \to (Y_1, X_2)$, a contradiction to $\mathcal{P}2$.
   (b) Suppose rather that $X_1 < Y_1$ and that there are integers $j' < j < Y_2 = X_2$ such that (X_1, j') and (Y_1, j) are (distinct) P-positions of Wythoff Nim.
   Then, by Proposition 1, we have two cases, either $j - Y_1 = j' - X_1$, or $j - Y_1 \geq j' - X_1 + m$. by $L((Y_1, j)) \geq \xi((Y_1, j))$, the definition of an m-imitation, for the move $Y^{-1} \to Y$, a player has removed tokens from the pile with less tokens. This means that the move $Y^{-1} \to Y$ did not imitate $Y^{-2} \to Y^{-1}$. Hence we get at best $L((X_1, j')) = 1$. If $j - Y_1 = j' - X_1$ then, $2 \leq \xi((X_1, j'))$ contradicting $\mathcal{P}2$, so we may assume that $j - X_1 \geq j' - X_1 + m$. Since $Y_2 = X_2 > j$ we get $\mu((X_1, j')) = X_2 - j' - Y_1 - X_1 \geq m$. So, by definition $X \to (X_1, j')$ is not an imitation of $Y \to X$ and therefore $L((X_1, j')) = 0 < \xi((X_1, j'))$, contradicting $\mathcal{P}2$.

Hence there is no Wythoff Nim N-position in $S(Y)$.

We may conclude that $S(Y) = \emptyset$ which settles Case I.

Case II. We are going to explicitly find an $X \in \mathcal{P}' \cap F(Y)$ for each form of $Y \in \mathcal{N}'$:

A) Y is a P-position viewed as a position of Wythoff Nim. Hence, by $\mathcal{N}2$, $p > L(Y) \geq \xi(Y) \geq 1$, which implies that there is an largest $i < Y_1$ with $Y_2 - Y_1 = j - i$ such that (i, j) is a P-position of Wythoff Nim. Take $X = (i, Y_2)$. Then $X \to (i, j)$ is an imitation of $Y \to X$ and so

$$L((i, j)) = L(Y) + 1 \geq \xi(Y) + 1 = \xi((i, j))$$

implies $\mathcal{P}2$.

B) Y is an N-position viewed as a position of Wythoff Nim. Due to Lemma 1(iii), we have three cases to consider:

1) Y is of the form $\mathcal{N}1$b. Here we can remove tokens from the pile with more tokens, Y_2, as to get to a Wythoff Nim P-position,

say $(Y_1, Y_2) \to (i, Y_1)$, where $Y_1 > i$. This move can not be an imitation of $Y^{-1} \to Y$ since with the notation as in (2), we may take $X^{-1}{}_1 = Y_1 > i = X_1$ which contradicts (2). Hence $0 = L((i, Y_1)) < 1 \le \xi((i, Y_))$ and so by $\mathcal{P}1$c we can take $X := (i, Y_1) \in \mathcal{P}'$.

2) Y is of the form $\mathcal{N}1c$. For this case, by Proposition 1, there is a $j > Y_2$ and a largest number $0 \le \alpha < Y_1$, such that (α, β) is a P-position of Wythoff Nim and

$$\beta - \alpha \le Y_2 - Y_1 < j - Y_1.$$

Since, by Proposition 1, $j - i = \beta - \alpha + m$, we get that $(\alpha, Y_2) \to (\alpha, \beta)$ is an m-imitation of $Y \to (\alpha, Y_2)$ and hence $1 = L(\alpha, \beta) \ge \xi(\alpha, \beta) = 1$, where the last equality comes from the particular choice of α. Hence, take $X := (\alpha, Y_2)$, a $\mathcal{P}'$-position of form $\mathcal{P}2$.

3) Y is of the form $\mathcal{N}1$d. There is a $j < Y_2$ such that $X := (Y_1, j)$ is a P-position of Wythoff Nim. Then $Y \to X$ is a legal move since, by $\mathcal{N}1$d and Proposition 1a, $L(X) < \xi(X) \le p$ and X is of the form $\mathcal{P}1$, hence a $\mathcal{P}'$ position.

We are done with case II.

We may, by Lemma 1(iv), conclude that X is a P-position of (p, m)-Imitation Nim if and only if $X \in \mathcal{P}'(p, m)$.

Remark 4: Suppose that the starting position is an N-position of (p, m)-Wythoff Nim (as it almost always is). The first player's initial winning move for (p, m)-Wythoff Nim is precisely the same as for (p, m)-Imitation Nim, except for one "class of" N-positions where, in the Wythoff setting, the first player makes a "diagonal move". As one can read out of the above proof, this is the situation where the first player has to rely on the "full power" of the imitation rule, (5).

Remark 5: If on the other hand the initial position is a (p, m)-Wythoff Nim P-position, then the first player has to move to a Wyhtoff Nim N-position which in this case, by (5), with certainty is an Imitation Nim N-position. Then, for this (very) special case, the second player can, by $\mathcal{N}2$ and the Main theorem be certain to win.

4. Suggestins for future work

In [*Ow*], Section VI.2 the author discusses "Games with incomplete information" (mostly for probabalistic games). Given that a particular game is going to be played many times, a second player's strategy will be to try and *learn* the first player's strategy. He has to not only interpret the way the first player moves, but also, by his own moves, try and "encourage" the first player to reveal the P-positions of the game. By this second player strategy, also the first player gets a new challange, to try her best to conceal the victorious path, but at the same time try to assure the final victory. This setting can be framed naturally in both probabilistic and deterministic games, as can be seen for example on page 17 of [BeCoGu], volume 1, where the authors discuss "When is a move good?" in a sense similar to the above.

In the first part of this section we make a couple of statements regarding, which on the one hand, of our games are good for the first player who wishes to conceal (vital parts of) her winning strategy, and on the other, for which games there is an optimal second player learning strategy. And indeed, we show that "imitation" is a means for learning a strategy and "blocking" is a means for concealing information, maybe not all that surprising!

In the last part of this final section we define 4 extensions of Nim on several piles, of which two are generalisations of Imitation Nim (sample-games A and B)and the others of Wythoff Nim. There is literature available on the latter.

Let $n \geq 2$ be an integer. Denote n-pile Nim with N_n. In $[Fra1]$, the idea (*) from Section 1 is used in the context of finding the "correct" extension of n-pile Nim to *n-pile Wythoff Nim*, denoted here by WN_n. Namely, if we adjoin every P-position of N-pile Nim, as a move in n-pile Nim, we clearly get a new game, generalising Wythoff Nim. The rules are to move as in n-pile Nim *or* to remove a positive number of tokens from any positive number of piles as long as the Nim-sum of the number of removed tokens from each pile equals zero.

We have not elaborated on this version of n-pile Wythoff Nim for $n > 2$, but rather give references for further information to the interested reader. Two conjectures were phrased in $[Fra1, section5]$ on the winning positions of n-pile Wythoff Nim. These conjectures have been further investigated in the articles $[SuZe]$, $[Su]$ and $[FrKr]$. In the two latter, the authors independently prove that conjecture 1 implies conjecture 2.

If one adjoins a subset S of $\mathcal{P}_{N_n}$ as moves to N_n one may arrive at games with P-positions distinct from Nim. For a complete answer on what subsets S changes Nim's winning strategy, see $[BlFr]$. For two specific examples, see our sample-games 1) and 2) below.

4.1. **A second player learning strategy.** Suppose X is a position of a 2-pile take-away game, then X is

- P-stable, if X is of form $X_1 \leq i$ for all P-positions $\{i, j\}$ of (p, m)-Wythoff Nim such that $X_2 - X_1 = j - i$;
- P-free, if X is a P-position of (p, m)-Wythoff Nim, but not P-stable;
- N-stable, if there is an $i < X_1$ such that $\{i, X_1\}$ is a P-position of (p, m)-Wythoff Nim, or if there is a $j > X_2$ such that $\{X_1, j\}$ is a P-position of (p, m)-Wythoff Nim;
- N-free, if there is a $j < X_2$ such that $\{X_1, j\}$ is a P-position of (p, m)-Wythoff Nim.

Let us for the purpose of this section change the setting of a game so that

- the second player does not have perfect information about the winning strategy, but the first player is not aware of this fact;
- the game is going to be played several times.

Then the question is, given that the first player will not take any risks, can the second player,

- force the first player to move in such a way that she in order to be certain to win the game reaveals the winning strategy;
- use the information of how the first player moves, in such a manner as to get full control of the winning strategy for a future play of the game?

We will hint that for some games this is possible, but for other games the first player can by moving intelligently "conceal" most of the P-positions of the game, still assuring herself of the final victory. Of course, the second player can find out the P-position by some other intelligent means, by a minimal exclusive algorithm or otherwise, but our emphasis here is the actual "learning situation", the interaction between the two players, as they move in a game-graph towards a sink of the game.

4.1.1. *Wythoff Nim's second player strategy.* For the classical game of Wythoff Nim there is a successful learning strategy for the second player. There is no question of what the first player should do, since almost always a "random" starting position is an N-position. Then, for $(1, 1)$-Wythoff Nim, the second player can by using the correct strategy *learn* the winning structure for the game:

Suppose the first player has moved to the P-position (a, b). If the second player moves to $(a, b-1)$, then by Proposition 1 (with $p = m = 1$), the first player is encouraged to move to the P-position (c, d), where $d-c = b-a-1$. Succesively every P-position will be revealed.

4.1.2. *When learning how to win is blocked.* However for (p, m)-Wythoff Nim in general this strategy will not work for the simple reason that, by Proposition 1, the first player may immediately grab a P-stable position and then the second player will not have a clue where the corresponding P-free positions are, precisely:

Fix the integers $p > 1$ and $m > 0$. Suppose the first player moved to a (stable) P-position (a, b). If the second player moves to $(a, b-1)$, then by Proposition 1, the first player is encouraged to move to the unique P-stable position (c, d) where $d - c = b - a - m$ (existence is clear by Proposition 1). But by Proposition 1 there is at least one more P-position, say (e, f), such that $c < e < a$. The existence is clear but the precise location of (e, f) will remain hidden for the second player. There is no other learning strategy for the second player.

4.1.3. *When imitation is learning.* There is a second player strategy for the game of (p, m)-Imitation Nim for the purpose of learning the P-positions for (p, m)-Wythoff Nim. By our main theorem it suffices to know these positions to win (p, m)-Imitation Nim. What the second player should do is to try and actively encourage the first player to remove tokens from a leading pile as to get to a least number in a P-position of (p, m)-Wythoff Nim.

Fix the integers $p > 0$ and $m > 0$. Suppose the starting position, say X, of (p, m)-Imitation Nim is an N-position. Then X is either

- N-free, or
- N-stable.

If X is N-free, then the first player will move to a P-position as in the game of (p, m)-Wythoff Nim. The second player strategy is to remove the smallest possible number of tokens (given the previous player's move) from the non-leading pile.

If X is N-stable, then if there is a P-position, say Y, of (p, m)-Wythoff Nim with $X_1 > Y_1$ and $\Delta(X) \in [\Delta(Y), \Delta(Y) + m - 1]$, the first player will move $X \to (Y_1, X_2)$. Then the second player should move to Y and he can do this without any knowledge of the P-positions except that $\Delta(Y) \equiv 0 \pmod{m}$. Otherwise the idea is similar to the N-free case.

4.2. **Variations of Nim on several piles.** Given $n \geq 2$ piles of tokens, as before we denote n-pile Nim with N_n.

4.2.1. *Adjoin P-positions as moves.* We will now define two extensions of N_n that also generalises 2-pile Wythoff Nim:

Sample-game 1: Adjoin as moves to N_n the positions $(a_1, \ldots, a_n)$ such that $a_i = a_j > 0$ for exactly one pair of indices i, j and $a_k = 0$ for all other indices. By this we mean that the next player, in addition to the ordinary Nim rules may remove precisely the same number of tokens from precisely two piles. Let us denote this game by W_2N_n. For example the "diagonal" move from the position $(1, 1, 1)$ is to $\{0, 0, 1\}$ and from $(1, 2, 3)$ the set of "diagonal" moves are to $(0, 1, 3), (1, 1, 2), (0, 2, 2)$ and $(0, 1, 1)$.

Sample-game 2: Adjoin as moves to N_n the positions $(a_1, \ldots, a_n)$. By this we mean that the next player in addition to the ordinary Nim rules may remove precisely the same number of tokens from any number of piles. Let us denote these games by $W_{2,\ldots,n}N_n$. For example the "diagonal" moves from the position $(1, 1, 1)$ are to $\{0, 0, 1\}$ and $(0, 0, 0)$. The "diagonal" moves from $(1, 2, 3)$ are to $(0, 1, 3), (1, 1, 2), (0, 2, 2), (1, 0, 1)$ and $(0, 1, 2)$.

4.2.2. *Remove the winning strategy from N_n.* We are going to define two games that comes to mind when applying the idea: via "an imitation-rule", remove the winning strategy from n-pile Nim.

Sample-game A: Move as in n-pile Nim, but with the restriction

- given that the previous player removed say x tokens from a leading pile
- before the next player moves, the previous player points at one non-leading pile and declares that the next player may not remove x tokens from this pile.

If the previous player removed tokens from a non-leading pile, then pointing at another pile does not impose any restriction to the next players move. Let us denote this game with I_1N_n. One can check that this game removes the winning strategy from Nim. It has a curious nature of giving a Muller twist to a move-size dynamic variation of Nim.

Sample-game B: Move as in n-pile Nim, but with the restriction

- given that the previous player removed say x tokens from a leading pile
- the next player may not remove x tokens from any one of the non-leading piles.

Let us denote this game with I_nN_n. One can check that this game removes the winning strategy from Nim.

4.2.3. *A negative result.* The sample-games 1. and A. are attempts to find "closely related" variations of the 2-pile Nim setting, following the idea (*) and (**) in Section 1. Analogously for sample game 2. and B. But, by running computer-simulations for the three-dimensional case, we have found the following:

For $n > 2$ (regarded as starting positions),

- $\mathcal{P}_{I_1N_n} \cap \mathcal{N}_{W_2N_n} \neq \emptyset$;
- $\mathcal{P}_{I_nN_n} \cap \mathcal{N}_{W_{2,\ldots,n}N_n} \neq \emptyset$;
- $\mathcal{N}_{WN_n}$ is neither disjoint from $\mathcal{P}_{I_1N_n}$ nor from $\mathcal{P}_{I_nN_n}$.

We have choosen not to include these simulations, since the ambition of this section is merely to suggest possible directions for further work.

Questions:

- Is there a (non-trivial) generalisation of 2-pile Wythoff Nim to $n \geq 2$ piles of tokens, say WN'_n, together with a generalisation, say IN'_n, of 2-pile Imitation Nim, such that for some (each) $n > 2$, $\mathcal{P}_{WN'_n} = \mathcal{P}_{IN'_n}$?
- Can one generate the P-positions for the 4 sample-games in a polynomial time (as in succint input-size), look first at $n = 3$?
- Are there other combinatorial games where an imitation rule correspond in a natural way to a blocking manuever?
- Can one formulate a general rule as to when such correspondances can be found and when not?

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Appendix

Peter Hegarty

The purpose of this appendix is to provide a proof of Conjecture 5.1 of $[HeLa]$ in the case $m = 1$, which is the most natural case to consider. Notation concerning 'multisets' and 'greedy permutations' is consistent with Section 2 of $[HeLa]$. We begin by recalling

Definition : Let r, s be positive irrational numbers with $r < s$. Then (r, s) is said to be a *Beatty pair* if

$$\frac{1}{r} + \frac{1}{s} = 1. \tag{6}$$

Theorem *Let (r, s) be a Beatty pair. Then the map $\tau : \mathbb{N} \to \mathbb{N}$ given by*

$$\tau([nr]) = [ns], \ \forall\, n \in \mathbb{N}, \qquad \tau = \tau^{-1},$$

is a well-defined involution of $\mathbb{N}$. If M is the multiset of differences $\pm\{[ns] - [nr] : n \in \mathbb{N}\}$, then $\tau = \pi_g^M$. M has asymptotic density equal to $(s - r)^{-1}$.

Proof : That τ is a well-defined permutation of $\mathbb{N}$ is Beatty's theorem. The second and third assertions are then obvious.

Proposition *Let $r < s$ be positive real numbers satisfying (7), and let $d := (s - r)^{-1}$. Then the following are equivalent*

(i) r is rational

(ii) s is rational

(iii) d is rational of the form $\frac{mn}{m^2 - n^2}$ for some positive rational m, n with $m > n$.

Proof : Straightforward algebra exercise.

Notation : Let (r, s) be a Beatty pair, $d := (s - r)^{-1}$. We denote by M_d the multisubset of $\mathbb{N}$ consisting of all differences $[ns] - [nr]$, for $n \in \mathbb{N}$. We denote $\tau_d := \pi_g^{\pm M_d}$.

As usual, for any positive integers m and p, we denote by $\mathcal{M}_{m,p}$ the multisubset of $\mathbb{Z}$ consisting of p copies of each multiple of m and $\pi_{m,p} := \pi_g^{\mathcal{M}_{m,p}}$. We now denote by $M_{m,p}$ the submultiset consisting of all the positive integers in $\mathcal{M}_{m,p}$ and $\overline{\pi}_{m,p} := \pi_g^{\pm M_{m,p}}$. Thus

$$\overline{\pi}_{m,p}(n) + p = \pi_{m,p}(n + p) \quad \text{for all } n \in \mathbb{N}. \tag{7}$$

Since $\mathcal{M}_{m,p}$ has density p/m, there is obviously a close relation between $M_{m,p}$ and $M_{p/m}$, and thus between the permutations $\pi_{m,p}$ and $\tau_{p/m}$. The precise nature of this relationship is, however, a lot less obvious on the level of permutations. It is the purpose of the present note to explore this matter.

We henceforth assume that $m = 1$.

To simplify notation we fix a value of p. We set $\pi := \overline{\pi}_{1,p}$, Note that

$$r = r_p = \frac{(2p-1) + \sqrt{4p^2+1}}{2p}, \quad s = s_p = r_p + \frac{1}{p} = \frac{(2p+1) + \sqrt{4p^2+1}}{2p}.$$

FURTHER NOTATION : If X is an infinite multisubset of $\mathbb{N}$ we write $X = (x_k)$ to denote the elements of X listed in increasing order, thus strictly increasing order when X is an ordinary subset of $\mathbb{N}$. The following four subsets of $\mathbb{N}$ will be of special interest :

$$\begin{aligned} A_\pi &:= \{n : \pi(n) > n\} := (a_k), \\ B_\pi &:= \mathbb{N}\backslash A_\pi := (b_k), \\ A_\tau &:= \{n : \tau(n) > n\} := (a_k^*), \\ B_\tau &:= \mathbb{N}\backslash A_\tau := (b_k^*). \end{aligned}$$

Note that $b_k = \pi(a_k)$, $b_k^* = \tau(a_k^*)$ for all k. We set

$$\epsilon_k := (b_k - a_k) - (b_k^* - a_k^*) = (b_k - b_k^*) - (a_k - a_k^*).$$

Lemma 1 *(i) For every $n > 0$,*

$$|M_p \cap [1,n]| = |M_{1,p} \cap [1,n]| + \epsilon,$$

where $\epsilon \in \{0, 1, ..., p-1\}$.
(ii) $\epsilon_k \in \{0,1\}$ for all k and if $\epsilon_k = 1$ then $k \not\equiv 0$ (mod p).
(iii) $a_{k+1}^* - a_k^* \in \{1,2\}$ for all $k > 0$ and cannot equal one for any two consecutive values of k.
(iv) $b_{k+1}^* - b_k^* \in \{2,3\}$ for all $k > 0$.

PROOF : (i) and (ii) are easy consequences of the various definitions. (iii) follows from the fact that $r_p \in (3/2, 2)$ and (iv) from the fact that $s_p \in (2,3)$.

Main Theorem *For all $k > 0$, $|a_k - a_k^*| \le p - 1$.*

REMARK : We suspect, but have not yet been able to prove, that $p-1$ is best-possible in this theorem.

PROOF OF THEOREM : The proof is an induction on k, which is most easily phrased as an argument by contradiction. Note that $a_1 = a_1^* = 1$. Suppose the theorem is false and consider the smallest k for which $|a_k^* - a_k| \ge p$. Thus $k > 1$.

Case I : $a_k - a_k^* \ge p$.

Let $a_k - a_k^* := p' \ge p$. Let b_l be the largest element of B_π in $[1, a_k)$. Then $b_{l-p'+1}^* > a_k^*$ and Lemma 1(iv) implies that $b_l^* - b_l \ge p'$. But Lemma 1(ii) then implies that also $a_l^* - a_l \ge p' \ge p$. Since obviously $l < k$, this contradicts the minimality of k.

Case II : $a_k^* - a_k \ge p$.

Let $a_k^* - a_k := p' \ge p$. Let b_l^* be the largest element of B_τ in $[1, a_k^*)$.

Then $b_{l-p'+1} > a_k$. Lemma 1(iv) implies that $b_{l-p'+1} - b^*_{l-p'+1} \geq p'$ and then Lemma 1(ii) implies that $a_{l-p'+1} - a^*_{l-p'+1} \geq p' - 1$. The only way we can avoid a contradiction already to the minimality of k is if all of the following hold :

(a) $p' = p$.
(b) $b^*_i - b^*_{i-1} = 2$ for $i = l, l-1, ..., l-p+2$.
(c) $l \not\equiv -1 \pmod{p}$ and $\epsilon_{l-p+1} = 1$.

To simplify notation a little, set $j := l-p+1$. Now $\epsilon_j = 1$ but parts (i) and (ii) of Lemma 1 imply that we must have $\epsilon_{j+t} = 0$ for some $t \in \{1, ..., p-1\}$. Choose the smallest t for which $\epsilon_{j+t} = 0$. Thus

$$b^*_j - a^*_j = b^*_{j+1} - a^*_{j+1} = \cdots = b^*_{j+t-1} - a^*_{j+t-1} = (b^*_{j+t} - a^*_{j+t}) - 1.$$

From (b) it follows that

$$a^*_{j+t} - a^*_{j+t-1} = 1, \;\; a^*_{j+\xi} - a^*_{j+\xi-1} = 2, \; \xi = 1, ..., t-1. \tag{8}$$

Let b^*_r be the largest element of B_τ in $[1, a^*_j)$. Then from (9) it follows that

$$b^*_{r+t} - b^*_{r+t-1} = 3, \;\; b^*_{r+\xi} - b^*_{r+\xi-1} = 2, \; \xi = 2, ..., t-1. \tag{9}$$

Together with Lemma 1(iv) this implies that

$$b^*_{r+p-1} - b^*_{r+1} \geq 2p - 3. \tag{10}$$

But since $a^*_j = a_j - (p-1)$ we have that $b_{r+p-1} < a_j$. Together with (11) this forces $b^*_{r+p-1} - b_{r+p-1} \geq p$, and then by Lemma 1(ii) we also have $a^*_{r+p-1} - a_{r+p-1} \geq p$. Since it is easily checked that $r + p - 1 < k$, we again have a contradiction to the minimality of k, and the proof of the theorem is complete.

This theorem implies Conjecture 5.1 of $[HeLa]$. Recall that the P-positions of $(p, 1)$-Wythoff Nim are the pairs $(n-1, \pi_{1,p}(n) - 1)$ for $n \geq 1$.

Corollary *With*

$$L = L_p = \frac{s_p}{r_p} = \frac{1 + \sqrt{4p^2 + 1}}{2p}, \quad l = l_p = \frac{1}{L_p},$$

we have that, for every $n \geq 1$,

$$\pi_{1,p}(n) \in \{\lfloor nL \rfloor + \epsilon, \lfloor nl \rfloor + \epsilon : \epsilon \in \{-1, 0, 1, 2\}\}. \tag{11}$$

Proof : We have $\pi_{1,p}(n) = n$ for $n = 1, ..., p$, and one checks that (12) thus holds for these n. For $n > p$ we have by (8) that

$$\pi_{1,p}(n) = \pi(n-p) + p, \tag{12}$$

where $\pi = \overline{\pi}_{1,p}$. There are two cases to consider, according as to whether $n - p \in A_\pi$ or B_π. We will show in the former case that $\pi_{1,p}(n) = \lfloor nL \rfloor + \epsilon$ for some $\epsilon \in \{-1, 0, 1, 2\}$. The proof in the latter case is similar and will be omitted.

So suppose $n - p \in A_\pi$, say $n - p = a_k$. Then

$$\pi(a_k) = b_k = a_k + (b^*_k - a^*_k) + \epsilon_k. \tag{13}$$

Moreover $a_k^* = \lfloor kr_p \rfloor$ and $b_k^* = \lfloor ks_p \rfloor$, from which it is easy to check that

$$b_k^* = a_k^* L + \delta, \quad \text{where } \delta \in (-1, 1).$$

Substituting into (14) and rewriting slightly, we find that

$$\pi(a_k) = a_k L + (a_k^* - a_k)(L - 1) + \delta + \epsilon_k,$$

and hence by (13) that $\pi_{1,p}(n) = nL + \gamma$ where

$$\gamma = (a_k^* - a_k - p)(L - 1) + \delta + \epsilon_k.$$

By Lemma 1, $\epsilon_k \in \{0, 1\}$. By the Main Theorem, $|a_k^* - a_k| \le p - 1$. It is easy to check that $(2p - 1)(L - 1) < 1$. Hence $\gamma \in (-2, 2)$, from which it follows immediately that $\pi_{1,p}(n) - \lfloor nL \rfloor \in \{-1, 0, 1, 2\}$. This completes the proof.

Remark : As stated in Section 5 of $[HeLa]$, computer calculations seem to suggest that, in fact, (12) holds with just $\epsilon \in \{0, 1\}$. So once again, the results presented here may be possible to improve upon.

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