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On Solutions to the Linear Boltzmann Equation with Inelastic Granular Collisions and Infinite Range Forces

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Abstract. This paper considers the time- and space-dependent linear Boltzmann equation with general boundary conditions in the case of inelastic (granular) collisions. First, in the angular cut-off case, mild L^1-solutions are constructed as limits of iterate functions, and boundedness of higher velocity moments are studied in the case of inverse power collision forces. Then the problem with inelastic collisions in the infinite range case (without cut-off) will be studied in an integral weak form, combining methods from our earlier papers, and using an H-theorem for a relative entropy functional.

Keywords: linear Boltzmann equation, granular collisions, infinite range forces.
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INTRODUCTION

The linear Boltzmann equation is frequently used for mathematical modelling in physics, (e.g. for describing the neutron distribution in reactor physics, cf. [1]–[3]). In our earlier papers [4]–[6] we have studied the linear Boltzmann equation, both in the angular cut-off case and the infinite range case, for a function $f(\mathbf{x},\mathbf{v},t)$ representing the distribution of particles with mass m colliding elastically and binary with other particles with mass m_* and with a given (known) distribution function $Y(\mathbf{x},\mathbf{v}_*)$. In recent years a significant interest has been focused on the study of kinetic models for granular flows, mostly with the non-linear Boltzmann equation; see ref. [7] for an overview, with many further references, and also [8]–[9]. Our papers [10] and [11] consider the time-dependent respectively the stationary linear Boltzmann equation for inelastic (granular) collisions, both papers in the angular cut-off case. The purpose of this paper is to combine the methods from [4]–[6] and [10]–[11] to get results for granular collisions in the infinite range case.

So we will study collisions between particles with mass m and particles with mass m_*, such that momentum is conserved,

$$m\mathbf{v}+m_*\mathbf{v}_* = m\mathbf{v}'+m_*\mathbf{v}'_*, \tag{1}$$

where $\mathbf{v},\mathbf{v}_*$ are velocities before and $\mathbf{v}',\mathbf{v}'_*$ are velocities after a collision.

In the elastic case, where also kinetic energy is conserved, one finds that the velocities after a binary collision terminate on two concentric spheres, so all velocities $\mathbf{v}'$ lie on a sphere around the center of mass, $\bar{\mathbf{v}} = (m\mathbf{v}+m_*\mathbf{v}_*)/(m+m_*)$, with radius $\frac{m_* w}{m+m_*}$, where $w = |\mathbf{v}-\mathbf{v}_*|$, and all velocities $\mathbf{v}'_*$ lie on a sphere with the same center $\bar{\mathbf{v}}$ and with radius $\frac{mw}{m+m_*}$, cf Figure 1 in [4].

In the granular, inelastic case we assume the following relation between the relative velocity components normal to the plane of contact of the two particles,

$$\mathbf{w}'\mathbf{u} = -a(\mathbf{w}\mathbf{u}), \tag{2}$$

where a is a constant, $0 < a \le 1$, and $\mathbf{w} = \mathbf{v}-\mathbf{v}_*, \mathbf{w}' = \mathbf{v}'-\mathbf{v}'_*$ are the relative velocities before and after the collision, and $\mathbf{u}$ is a unit vector in the direction of impact, $\mathbf{u} = (\mathbf{v}-\mathbf{v}')/|\mathbf{v}-\mathbf{v}'|$. Then we find that $\mathbf{v}' = \mathbf{v}'_a$ lies on the line between $\mathbf{v}$ and $\mathbf{v}'_1$, where $\mathbf{v}'_1$ is the postvelocity in the case of elastic collision, i.e. with $a = 1$, and $\mathbf{v}'_{*a}$ lies on the (parallel) line between $\mathbf{v}_*$ and $\mathbf{v}'_{*1}$.

Now the following relations hold for the velocities in a granular, inelastic collision

$$\mathbf{v}' = \mathbf{v} - (a+1)\frac{m_*}{m+m_*}(\mathbf{w}\mathbf{u})\mathbf{u}, \quad \mathbf{v}'_* = \mathbf{v}_* + (a+1)\frac{m}{m+m_*}(\mathbf{w}\mathbf{u})\mathbf{u}, \tag{3}$$

where $\mathbf{w}\mathbf{u} = w\cos\theta,\ w = |\mathbf{v}-\mathbf{v}_*|$, if the unit vector $\mathbf{u}$ is given in spherical coordinates,

$$\mathbf{u} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta),\ \ 0 \le \theta \le \pi/2,\ 0 \le \phi < 2\pi. \tag{4}$$

Moreover, if we change notations, and let $'\mathbf{v}, '\mathbf{v}_*$ be the velocities before, and $\mathbf{v}, \mathbf{v}_*$ the velocities after a binary inelastic collision, then by (2) and (3), cf. [7]–[11],

$$'\mathbf{v} = \mathbf{v} - \frac{(a+1)m_*}{a(m+m_*)}(\mathbf{wu})\mathbf{u}, \quad '\mathbf{v}_* = \mathbf{v}_* + \frac{(a+1)m}{a(m+m_*)}(\mathbf{wu})\mathbf{u}. \tag{5}$$

PRELIMINARIES

We consider the time-dependent transport equation for a distribution function $f(\mathbf{x},\mathbf{v},t)$, depending on a space variable $\mathbf{x} = (x_1,x_2,x_3)$ in a bounded convex body D with (piecewise) C^1-boundary $\Gamma = \partial D$, and depending on a velocity variable $\mathbf{v} = (v_1,v_2,v_3) \in V = \mathbb{R}^3$ and a time variable $t \in \mathbb{R}_+$. Then the linear Boltzmann equation is in the strong form

$$\frac{\partial f}{\partial t}(\mathbf{x},\mathbf{v},t) + \mathbf{v}\,\mathrm{grad}_{\mathbf{x}} f(\mathbf{x},\mathbf{v},t) = (Qf)(\mathbf{x},\mathbf{v},t), \tag{6}$$

$$\mathbf{x} \in D, \quad \mathbf{v} \in V = \mathbb{R}^3, \quad t \in \mathbb{R}_+,$$

supplemented by initial data

$$f(\mathbf{x},\mathbf{v},0) = f_0(\mathbf{x},\mathbf{v}), \quad \mathbf{x} \in D, \quad \mathbf{v} \in V. \tag{7}$$

The collision term can, in the case of inelastic (granular) collision, be written, cf. [7]–[11],

$$(Qf)(\mathbf{x},\mathbf{v},t) = \int_V\int_\Omega \left[J_a(\theta,w)Y(\mathbf{x},'\mathbf{v}_*)f(\mathbf{x},'\mathbf{v},t) - Y(\mathbf{x},\mathbf{v}_*)f(\mathbf{x},\mathbf{v},t)\right]B(\theta,w)\, d\mathbf{v}_* d\theta d\phi \tag{8}$$

with $w = |\mathbf{v}-\mathbf{v}_*|$, where $Y \geq 0$ is a known distribution, $B \geq 0$ is given by the collision process, and finally J_a is a factor depending on the granular process (and giving mass conservation, if the gain and the loss integrals converge separately). Furthermore, $'\mathbf{v}$, $'\mathbf{v}_*$ in (8) are the velocities before and $\mathbf{v}$, $\mathbf{v}_*$ the velocities after the binary collision, cf. (5), and $\Omega = \{(\theta,\phi) : 0 \leq \theta < \hat{\theta},\ 0 \leq \phi < 2\pi\}$ represents the impact plane, where $\hat{\theta} < \frac{\pi}{2}$ in the angular cut-off case, and $\hat{\theta} = \frac{\pi}{2}$ in the infinite range case. The collision function $B(\theta,w)$ is in the physically interesting case with inverse k:th power collision forces given by

$$B(\theta,w) = b(\theta)w^\gamma, \quad \gamma = \frac{k-5}{k-1}, \quad w = |\mathbf{v}-\mathbf{v}_*|, \tag{9}$$

with hard forces for $k > 5$, Maxwellian for $k = 5$, and soft forces for $3 < k < 5$, where $b(\theta)$ has a non-integrable singularity for $\theta = \frac{\pi}{2}$. So in the angular cut-off case one can choose $\hat{\theta} < \frac{\pi}{2}$, and then the gain and the loss terms in (8) can be separated

$$(Qf)(\mathbf{x},\mathbf{v},t) = (Q^+f)(\mathbf{x},\mathbf{v},t) - (Q^-f)(\mathbf{x},\mathbf{v},t), \tag{10}$$

where the gain term can be written (with a kernel K_a)

$$(Q^+f)(\mathbf{x},\mathbf{v},t) = \int_V K_a(\mathbf{x},\, '\mathbf{v} \to \mathbf{v})f(\mathbf{x},'\mathbf{v},t)\, d\,'\mathbf{v}, \tag{11}$$

and the loss term is written with the collision frequency $L(\mathbf{x},\mathbf{v})$ as

$$(Q^-f)(\mathbf{x},\mathbf{v},t) = L(\mathbf{x},\mathbf{v})f(\mathbf{x},\mathbf{v},t). \tag{12}$$

In the case of non-absorbing body we have that

$$L(\mathbf{x},\mathbf{v}) = \int_V K_a(\mathbf{x},\mathbf{v} \to \mathbf{v}')\, d\mathbf{v}'. \tag{13}$$

Furthermore, equation (8) is in the space-dependent case supplemented by (general) boundary conditions

$$f_-(\mathbf{x},\mathbf{v},t) = \int \frac{|\mathbf{n}\tilde{\mathbf{v}}|}{|\mathbf{n}\mathbf{v}|}R(\mathbf{x},\tilde{\mathbf{v}} \to \mathbf{v})f_+(\mathbf{x},\tilde{\mathbf{v}},t)d\tilde{\mathbf{v}}, \tag{14}$$

$$\mathbf{n}\mathbf{v} < 0,\ \mathbf{n}\tilde{\mathbf{v}} > 0,\ \mathbf{x} \in \Gamma = \partial D,\ t \in \mathbb{R}_+,$$

where $\mathbf{n} = \mathbf{n}(\mathbf{x})$ is the unit outward normal at $\mathbf{x} \in \Gamma = \partial D$. The function $R \geq 0$ satisfies (in the non-absorbing boundary case)

$$\int_V R(\mathbf{x}, \tilde{\mathbf{v}} \to \mathbf{v}) d\mathbf{v} \equiv 1, \tag{15}$$

and f_- and f_+ represent the ingoing and outgoing trace functions corresponding to f. In the specular reflection case the function R is represented by a Dirac measure $R(\mathbf{x}, \tilde{\mathbf{v}} \to \mathbf{v}) = \delta(\mathbf{v} - \tilde{\mathbf{v}} + 2(\mathbf{n}\tilde{\mathbf{v}})\mathbf{n})$, and in the diffuse reflection case $R(\mathbf{x}, \tilde{\mathbf{v}} \to \mathbf{v}) = |\mathbf{n}\mathbf{v}| W(\mathbf{x}, \mathbf{v})$ with some given function $W \geq 0$, (e.g. Maxwellian function).

Let $t_b \equiv t_b(\mathbf{x}, \mathbf{v}) = \inf_{\tau \in \mathbb{R}_+} \{\tau : \mathbf{x} - \tau\mathbf{v} \notin D\}$, and $\mathbf{x}_b \equiv \mathbf{x}_b(\mathbf{x}, \mathbf{v}) = \mathbf{x} - t_b\mathbf{v}$, where t_b represents the time for a particle going with velocity $\mathbf{v}$ from the boundary point $\mathbf{x}_b$ to the point $\mathbf{x}$.

Then, using differentiation along the characteristics, equation (6) can formally be transformed to a *mild* equation, and also to an *exponential* form of equation in the angular cut-off case, cf. [10] and also [4]–[6].

CONSTRUCTION OF SOLUTIONS IN THE CUT-OFF CASE

We construct L^1-solutions to our problems as limits of iterate functions f^n, when $n \to \infty$. Let first $f^{-1}(\mathbf{x}, \mathbf{v}, t) \equiv 0$. Then define for given f^{n-1} the next iterate f^n, first at the ingoing boundary (using the appropriate boundary condition), and then inside D and at the outgoing boundary (using the exponential form of the equation),

$$f^n_-(\mathbf{x}, \mathbf{v}, t) = \int_V \frac{|\mathbf{n}\tilde{\mathbf{v}}|}{|\mathbf{n}\mathbf{v}|} R(\mathbf{x}, \tilde{\mathbf{v}} \to \mathbf{v}) f^{n-1}_+(\mathbf{x}, \tilde{\mathbf{v}}, t) d\tilde{\mathbf{v}}, \tag{16}$$

$$f^n(\mathbf{x}, \mathbf{v}, t) = \bar{f}^n(\mathbf{x}, \mathbf{v}, t) \exp\Big[-\int_0^t L(\mathbf{x} - s\mathbf{v}, \mathbf{v}) ds\Big] + \tag{17}$$

$$+ \int_0^t \exp\Big[-\int_0^\tau L(\mathbf{x} - s\mathbf{v}, \mathbf{v}) ds\Big] \int_V K_a(\mathbf{x} - \tau\mathbf{v}, {}'\mathbf{v} \to \mathbf{v}) f^{n-1}(\mathbf{x} - \tau\mathbf{v}, {}'\mathbf{v}, t - \tau)\, d'\mathbf{v} d\tau,$$

where

$$\bar{f}^n(\mathbf{x}, \mathbf{v}, t) = \begin{cases} f_0(\mathbf{x} - t\mathbf{v}, \mathbf{v}), & 0 \leq t \leq t_b, \\ f^n_-(\mathbf{x}_b, \mathbf{v}, t - t_b), & t > t_b. \end{cases} \tag{18}$$

Let also $f^n(\mathbf{x}, \mathbf{v}, t) \equiv 0$ for $\mathbf{x} \in \mathbb{R}^3 \setminus D$. Now we get a monotonicity lemma, $f^n(\mathbf{x}, \mathbf{v}, t) \geq f^{n-1}(\mathbf{x}, \mathbf{v}, t)$, which is essential and can be proved by induction.

Then, by differentiation along the characteristics and integration (with Green's formula), we find (using the equations above, cf. [10]), that

$$\int_D \int_V f^n(\mathbf{x}, \mathbf{v}, t)\, d\mathbf{x} d\mathbf{v} \leq \int_D \int_V f_0(\mathbf{x}, \mathbf{v})\, d\mathbf{x} d\mathbf{v}, \tag{19}$$

so Levi's theorem (on monotone convergence) gives existence of (mild) L^1-solutions

$$f(\mathbf{x}, \mathbf{v}, t) = \lim_{n \to \infty} f^n(\mathbf{x}, \mathbf{v}, t)$$

to our problem with granular gases (almost in the same way as for the elastic collision case). Furthermore, if $L(\mathbf{x}, \mathbf{v}) f(\mathbf{x}, \mathbf{v}, t) \in L^1(D \times V)$, then we get equality in (19) for the limit function f, giving mass conservation together with uniqueness in the relevant function space (cf [4]–[6], [10], [11], and also Proposition 3.3, chapter 11 in [3]).

Remark 1 The assumption $Lf \in L^1(D \times V)$ is, for instance, satisfied for the solution f in the case of inverse power collision forces, cf. (9), together with e.g. specular boundary reflections. This follows from a statement on global boundedness (in time) of higher velocity moments, (cf. Theorem 4.1 and Corollary 4.1 in [10]).

Remark 2 There holds also in the granular inelastic collision case an *H-theorem* for a general relative entropy functional

$$H_F^\Phi(f)(t) = \int_D \int_V \Phi\Big(\frac{f(\mathbf{x}, \mathbf{v}, t)}{F(\mathbf{x}, \mathbf{v})}\Big) F(\mathbf{x}, \mathbf{v})\, d\mathbf{x} d\mathbf{v}, \tag{20}$$

giving that this H-functional is nonincreasing in time, if $\Phi = \Phi(z)$, $\mathbb{R}_+ \to \mathbb{R}$, is a convex C^1-function, and if there exists a corresponding stationary solution $F(\mathbf{x}, \mathbf{v})$ with the same total mass as the initial data $f_0(\mathbf{x}, \mathbf{v})$ for the time-dependent solution $f(\mathbf{x}, \mathbf{v}, t)$; cf. Theorem 5.1 in [10].

ON L^1-SOLUTIONS IN THE INFINITE RANGE CASE

In this section the linear Boltzmann equation for granular inelastic collisions is considered without cut-off in the collision term, i.e. including infinite range forces. It is studied in the following weak integral form, which can formally be derived from equation (6) with (7) and (8):

$$\int_D\int_V g(\mathbf{x},\mathbf{v},t)f(\mathbf{x},\mathbf{v},t)\;d\mathbf{x}d\mathbf{v} = \int_D\int_V g(\mathbf{x},\mathbf{v},0)f_0(\mathbf{x},\mathbf{v})\;d\mathbf{x}d\mathbf{v}+ \tag{21}$$

$$\int_0^t\int_D\int_V\Big[\mathbf{v}\;\mathrm{grad}_{\mathbf{x}}g(\mathbf{x},\mathbf{v},\tau)+\frac{\partial}{\partial\tau}g(\mathbf{x},\mathbf{v},\tau)\Big]f(\mathbf{x},\mathbf{v},\tau)\;d\mathbf{x}d\mathbf{v}d\tau+$$

$$\int_0^t\int_D\int_V\int_V\int_\Omega\left[g(\mathbf{x},\mathbf{v}',\tau)-g(\mathbf{x},\mathbf{v},\tau)\right]B(\theta,w)Y(\mathbf{x},\mathbf{v}_*)f(\mathbf{x},\mathbf{v},\tau)\;d\mathbf{x}d\mathbf{v}d\mathbf{v}_*d\theta d\phi d\tau,$$

for all test functions $g\in C_0^{1,\infty}$ (for simplicity).
Here $C_0^{1,\infty}=\{g\in C^{1,\infty}: g(\mathbf{x},\mathbf{v},t)=0,\;\mathbf{x}\in\Gamma=\partial D\}$, where $C^{1,\infty}=\{g\in C^1(D\times V\times[0,\infty)):$
$\|g\|_1=\sup|g(\mathbf{x},\mathbf{v},t)|+\sup\left|\frac{\partial}{\partial\tau}g(\mathbf{x},\mathbf{v},\tau)\right|+\sup|\mathrm{grad}_{\mathbf{x}}g(\mathbf{x},\mathbf{v},t)|+\sup|\mathrm{grad}_{\mathbf{v}}g(\mathbf{x},\mathbf{v},t)|<\infty\}$.

The mathematical problems in the non-cut-off case come from the non-integrability (when $\theta\to\frac{\pi}{2}$) of the function $B(\theta,w)=b(\theta)w^\gamma$ in the inverse power case, cf. (9), where

$$\int_0^{\frac{\pi}{2}}b(\theta)\;d\theta=\infty,\quad\int_0^{\frac{\pi}{2}}b(\theta)\cos\theta\;d\theta<\infty.$$

Now the possibility of getting a solution $f(\mathbf{x},\mathbf{v},t)$ to equation (21) depends (among others), cf. (3), on the following estimate for the test functions, cf [4]–[6],

$$|g(\mathbf{x},\mathbf{v}',\tau)-g(\mathbf{x},\mathbf{v},\tau)|\le\|g\|_1(a+1)\frac{m_*}{m+m_*}w\cos\theta. \tag{22}$$

Then we want to prove existence of L^1-solutions $f(\mathbf{x},\mathbf{v},t)$ to the integral equation (21) in the infinite range case without angular cut-off, $\hat\theta=\frac{\pi}{2}$. Therefore we start with a sequence of solutions $f_p(\mathbf{x},\mathbf{v},t)\in L^1(D\times V)$, $t\in\mathbb{R}_+$, from the cut-off case, with e.g. $\hat\theta_p=\frac{\pi}{2}-\frac{1}{p}$, $p=1,2,3,\ldots$ These solutions $f_p(\mathbf{x},\mathbf{v},t)$ satisfy equation (21).

The existence theorem in the non-cut-off case is based on a compactness lemma, which is analogous to that given by Arkeryd in [12], (cf. also Lemma 4.1 in [5]); the formulation and the proof are omitted here. Then we can formulate the following theorem on existence of L^1-solutions to equation (21) in the case of inelastic (granular) collisions with inverse power forces, cf. [4]–[6].

Theorem Let $B(\theta,w)$ satisfy (9) and let $Y(\mathbf{x},\mathbf{v}_*)=X(\mathbf{x})Z(\mathbf{v}_*)$ with $X(\mathbf{x})$ continuous on D and $Z(\mathbf{v}_*)$ measurable on V, and suppose specular reflections at the boundary. Assume alternatively:
a) that $f_0\log(\frac{f_0}{E})\in L^1(D\times V)$ holds for the initial function f_0, where $E(\mathbf{x},\mathbf{v})\in L^1(D\times V)$ satisfies a detailed balance relation $E(\mathbf{x},\mathbf{v})Y(\mathbf{x},\mathbf{v}_*)=E(\mathbf{x},\mathbf{v}')Y(\mathbf{x},\mathbf{v}_*')$, or
b) that $f_0\log(\frac{f_0}{F})\in L^1(D\times V)$ holds for f_0 and a corresponding stationary solution $F(\mathbf{x},\mathbf{v})$ (independent of the cut-offs).
Then there exists (for $t>0$) a solution $f(\mathbf{x},\mathbf{v},t)\in L^1_+(D\times V)$ to the linear Boltzmann equation in the integral form (21) for granular, inelastic collisions in the infinite range case. The solution conserves mass: $\int_D\int_V f(\mathbf{x},\mathbf{v},t)\;d\mathbf{x}d\mathbf{v}=\int_D\int_V f_0(\mathbf{x},\mathbf{v})\;d\mathbf{x}d\mathbf{v}$, and higher velocity moments are globally bounded in time for hard forces ($k\ge5$) and locally bounded for soft forces ($3<k<5$).

Proof (sketch): Cf. the proof of Theorem 4.2 in [5] and also the proof of Theorem 3 in [12]. Let $\{f_p(\mathbf{x},\mathbf{v},t)\}_{p=1}^\infty$ be a sequence of (mild) solutions for equation (21) with angular cut-offs $\hat\theta_p=\frac{\pi}{2}-\frac{1}{p}$. By an H-theorem and the compactness lemma one can select a subsequence $\{f_{p_j}\}_{j=1}^\infty$ converging weakly to a function $f\in L^1_+(D\times V)$ for all rational t. But $\int_D\int_V g(\mathbf{x},\mathbf{v},t)f_p(\mathbf{x},\mathbf{v},t)\;d\mathbf{x}d\mathbf{v}$ is equicontinuous in t, and the subsequence $\{f_{p_j}\}$ converges weakly to a function $f\in L^1_+(D\times V)$ for all t. One finds that the function $f=w-\lim_{j\to\infty}f_{p_j}$ satisfies the integral equation (21), and the statements on velocity moments follow from the cut-off case.

Remark The assumption $g(\mathbf{x},\mathbf{v},t) = 0,\ \mathbf{x} \in \Gamma = \partial D$ on the test functions can be weakened, and the boundary terms

$$\int_0^t \int_\Gamma \int_V \int_V \left[g(\mathbf{x},\mathbf{v}',\tau) - g(\mathbf{x},\mathbf{v},\tau)\right] R(\mathbf{x},\mathbf{v} \to \mathbf{v}') f(\mathbf{x},\mathbf{v},\tau) |\mathbf{n}\mathbf{v}|\ d\Gamma d\mathbf{v} d\mathbf{v}' d\tau \tag{23}$$

can be included in equation (21), e.g. in the case of specular reflection, cf. [13].

Final remark Granular inelastic collisions can also be studied using transformation of masses and velocities to the problem of elastic collisions, cf, ref. [14], but our method above gives somewhat precise estimates.

REFERENCES

1. N. Bellomo, A. Palczewski, and G. Toscani,*Mathematical topics in Nonlinear Kinetic Theory*, World Scientific (1989).
2. C. Cercignani, *The Boltzmann Equation and its Applications*, Springer-Verlag (1988).
3. W. Greenberg, C. van der Mee, and V. Protopopescu, *Boundary Value Problems in Abstract Kinetic Theory*, Birkhäuser-Verlag (1987).
4. R. Pettersson, *On solutions and higher moments for the linear Boltzmann equation with infinite range forces*, IMA J. Appl. Math. **38**, pp. 151-166 (1987).
5. R. Pettersson, *On solution of the linear Boltzmann equation with general boundary conditions and infinite range forces*, J. Stat. Phys., **59**, pp. 403-440 (1990).
6. R. Pettersson, *On weak and strong convergence to equilibrium for solutions of the linear Boltzmann equation*, J. Stat. Phys., **72**, pp. 355-380 (1993).
7. C. Villani, *Mathematics of granular materials*, J. Stat. Phys. **124**, pp. 781-822 (2006).
8. J.J. Brey, J.W. Dufty, A. Santos, *Dissipative dynamics for hard spheres*, J. Stat. Phys., **87**, pp. 1051-1068 (1997).
9. J.J. Brey, J.W. Dufty, A. Santos, *Kinetic models for granular flow*, J. Stat. Phys., **97**, pp. 281-322 (1999).
10. R. Pettersson, *On solution of the linear Boltzmann equation for granular gases*, Transp. Th. Stat. Phys. **33**, pp. 527-543 (2004).
11. R. Pettersson, *On stationary solution of the linear Boltzmann equation with inelastic granular collisions*, A I P, Conf. Proc. **762**, RGD24, Bari, pp. 252-257 (2005).
12. L. Arkeryd *Intermolecular forces of infinite range and the Boltzmann equation*, Arch. Rat. Mech. Anal. **77**, pp. 11-21 (1981).
13. R. Pettersson, *On the linear Boltzmann equation with sources, external forces, boundary conditions and infinite range collisions*, Math. Mod. Meth. Appl. Sci. **1**, pp. 259-291 (1991).
14. A. Santos, J. W. Dufty, *Dynamics of hard sphere granular impurity*, Phys. Rev. Lett. **97**, 058001 (2006).