

THESIS FOR THE DEGREE OF LICENTIATE OF PHILOSOPHY

Asymptotics of First-Passage Percolation on 1-dimensional graphs

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Abstract

In this thesis we consider standard first-passage percolation on certain 1-dimensional periodic graphs. One such graph of particular interest is the $\mathbb{Z} \times \{0, 1, \dots, K - 1\}^{d-1}$ nearest neighbour graph for $d, K \geq 1$, referred to as the (K, d) -tube. Our main focus lies on the random time T_n at which the first vertex at distance n from the origin is infected. We derive asymptotic results that show how the behaviour of first-passage percolation on 1-dimensional graphs differ in behaviour from what is known or expected in higher dimensions. By subadditivity $T_n/n \rightarrow \mu$ for some $\mu > 0$ as $n \rightarrow \infty$, almost surely and in L^1 . We show that for some $\sigma > 0$, $(T_n - \mu n)/\sigma\sqrt{n}$ as $n \rightarrow \infty$ converges in distribution to a standard normal, and moreover that $\limsup_{n \rightarrow \infty} |T_n - \mu n|/\sigma\sqrt{2n \log \log n} = 1$, almost surely. We will also show that the time constant $\mu = \mu(K, d)$ obtained for the (K, d) -tube converges to the time constant $\mu(\mathbb{Z}^d)$ for the \mathbb{Z}^d lattice.

We also allow different sets of initially infected vertices, and construct an exact coupling of two infections with different starting configurations. Using this coupling we prove a 0–1 law. Whether such a coupling, or 0–1 law, exists in higher dimensions is not known, but we provide a counterexample on binary trees.

Keywords: First-passage percolation, time constant, shape theorem, rate of convergence, law of large numbers, central limit theorem, law of the iterated logarithm, coupling, 0–1 law, tree of infection.

Acknowledgment

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Here I am today, thanks to family, friends, and good teachers on the way.

Daniel Ahlberg
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1 Introduction

First-passage percolation was first considered by Hammersley and Welsh (1965). It can be thought of as a model for the spread of an infection on an underlying graph with set of vertices \mathbb{V} and set of edges \mathbb{E} . Associate to the edges of the graph nonnegative i.i.d. random variables $\{\tau_e\}_{e \in \mathbb{E}}$, referred to as *passage times*. We will denote the passage time distribution by $P_\tau(\cdot) := P(\tau_e \in \cdot)$. To avoid trivialities, we assume throughout this paper that P_τ does not concentrate all mass at a single point. With the present interpretation of the model, the passage time of an edge should be thought of as the random time it takes for an infection to spread along the edge. Consider the process where we start with a finite set $I \subset \mathbb{V}$ of infected vertices. As time starts, the infection spreads to adjacent vertices with delays indicated by the passage times. Let us by a *path* refer to a set of edges $\{e_1, e_2, \dots, e_m\}$, such that $e_k \neq e_l$ for all $k \neq l$ and each edge e_k of the set shares one end vertex with the edge e_{k-1} , and the other with e_{k+1} . For a path Γ , we define the passage time of Γ as $T(\Gamma) := \sum_{e \in \Gamma} \tau_e$. The main features of the model are retained in

$$T(v) := \inf\{T(\Gamma) : \Gamma \text{ is a path from } I \text{ to } v\}, \quad (1.1)$$

interpreted as the time it takes for the infection started at the vertices of I to infect the vertex v , and

$$B_t := \{v \in \mathbb{V} : T(v) \leq t\},$$

the set of infected vertices at time t . This is first-passage percolation.

A typical choice for the underlying graph is the usual \mathbb{Z}^d *lattice*, whose vertices are the elements of \mathbb{Z}^d , and where two vertices are connected with an edge if their euclidean distance is one. It is for this choice of graph that the model have been most studied in the literature. In this paper, though, we will consider first-passage percolation on 1-dimensional graphs. Before we go on and present our main results, we begin with a presentation of some of the results for first-passage percolation on the \mathbb{Z}^d lattice. Thereafter, the motivation for considering 1-dimensional graphs, as well as our results themselves, will be better understood. A recent and more detailed survey of first-passage percolation can be found in Howard (2004).

As the tradition suggests, it is customary to consider first-passage percolation with a single initially infected vertex at the origin. However,

in our study on 1-dimensional graphs, we have reasons to be interested in different initial configurations of the infection. Since the results we are about to present regarding the \mathbb{Z}^d lattice hold for any finite initially infected set, we may as well consider such set I already at this point.

A first question already considered by Hammersley and Welsh (1965) is whether the infimum in $T(v)$ is attained. For most passage time distributions this is known to be true (for more precise conditions, see Howard (2004)). Another early and challenging task is to describe the behaviour of $T(v)$ when $|v|$ is large. Define the passage time between two vertices u and v as

$$T(u, v) := \inf\{T(\Gamma) : \Gamma \text{ is a path from } u \text{ to } v\}. \quad (1.2)$$

It follows from the definition that $T(v)$ is subadditive, i.e., $T(v) \leq T(u) + T(u, v)$ for all u and v . Thus, under mild conditions on the passage time distribution, Kingman's subadditive ergodic theorem says that there is a constant $\mu(\mathbf{e}_1)$, referred to as the *time constant*, such that

$$\lim_{n \rightarrow \infty} \frac{T(\mathbf{n})}{n} = \mu(\mathbf{e}_1), \quad \text{almost surely and in } L^1, \quad (1.3)$$

where $\mathbf{n} = (n, 0, \dots, 0)$, and \mathbf{e}_1 denotes the unit vector along the first coordinate axis. The same holds in every unit direction $\bar{x} \in \mathbb{R}^d$. So if $\lfloor n\bar{x} \rfloor$ denotes the coordinate-wise integer part of $n\bar{x}$, then there is a $\mu(\bar{x})$ such that

$$\lim_{n \rightarrow \infty} \frac{T(\lfloor n\bar{x} \rfloor)}{n} = \mu(\bar{x}), \quad \text{almost surely and in } L^1.$$

In fact, one can say more about this asymptotic growth. If we consider B_t , we can state results about the growth in all directions simultaneously. A first such result was due to Richardson (1973). For convenience, we replace B_t by the subset of \mathbb{R}^d defined as

$$\tilde{B}_t := \{x \in \mathbb{R}^d : x \in v + [0, 1]^d \text{ for some } v \in B_t\},$$

The following version of Richardson's result is due to Cox and Durrett (1981), and states that the set of infected vertices grows linearly with t and has a nonrandom asymptotic shape.

Theorem 1.1 (Shape theorem). *Consider first-passage percolation on \mathbb{Z}^d with i.i.d. passage times such that*

$$\mathbb{E} [\min(\tau_{e_1}^d, \dots, \tau_{e_{2d}}^d)] < \infty. \quad (1.4)$$

If $\mu > 0$, then there exists a nonrandom, compact, convex subset B^* in \mathbb{R}^d with nonempty interior such that for all $\epsilon > 0$, almost surely,

$$(1 - \epsilon)B^* \subset \frac{1}{t}\tilde{B}_t \subset (1 + \epsilon)B^*, \quad \text{for } t \text{ large enough.}$$

If $\mu = 0$, then for every compact set K in \mathbb{R}^d , almost surely,

$$K \subset \frac{1}{t}\tilde{B}_t, \quad \text{for } t \text{ large enough.}$$

In addition, it was shown by Kesten (1986) that

$$\mu = 0 \text{ if and only if } P_\tau(\{0\}) \geq p_c,$$

where p_c is the critical value for independent bond percolation on the \mathbb{Z}^d lattice. A simple argument shows as well that $E[\tau_e^2] < \infty$ is sufficient for (1.4) to hold.

As the Shape theorem establishes a law of large numbers for the sequence $T(\lfloor n\bar{x} \rfloor)$, one next asks about the fluctuations of the same sequence. They have turned out to be harder to understand, and depend on the dimension d . For $d = 1$, $T(n)$ reduces to a sum of i.i.d. random variables. Thus, it follows immediately that

$$\text{Var}(T(n)) = n \text{Var}(\tau_e).$$

In higher dimensions the question becomes more delicate, and the result is believed to differ in nature from the case when $d = 1$. Kesten (1993) showed that for any $d \geq 1$, if $P_\tau(\{0\}) < p_c$ and $E[\tau_e^2] < \infty$, then there are constants $C_1 > 0$ and $C_2 < \infty$ such that

$$C_1 \leq \text{Var}(T(\mathbf{n})) \leq C_2 n,$$

for all $n \geq 1$. More precise results have been few. Benjamini, Kalai and Schramm (2003) gave an example which showed that for first-passage percolation on \mathbb{Z}^d for $d \geq 2$, with positive $\{a, b\}$ -valued passage times, $T(\mathbf{n})$ has sublinear variance. More precisely they showed that there is a constant C such that

$$\text{Var}(T(\mathbf{n})) \leq C \frac{n}{\log n}, \tag{1.5}$$

for all $n \geq 2$. This result was later extended by Benaïm and Rossignol (2006, 2008) to include a wider class of passage time distributions. This is

still far from what is believed to be the precise growth rate of $\text{Var}(T(\mathbf{n}))$. For $d = 2$ it is believed that $\text{Var}(T(\mathbf{n}))$ is of the order $n^{2/3}$, and a discussion has been going on whether to expect the same behaviour in higher dimensions (see Newman and Piza (1995); Benjamini et al. (2003) for short resumés). Not much is known, but for $d = 2$ Newman and Piza (1995) have shown, under natural conditions on the passage times, that there is a constant $C > 0$ such that

$$\text{Var}(T(\mathbf{n})) \geq C \log n,$$

for all $n \geq 1$. The same lower bound was found independently by Pemantle and Peres (1994), in the case of exponential passage times.

In this paper we consider first-passage percolation on *essentially 1-dimensional periodic graphs* \mathcal{G} . The class of essentially 1-dimensional periodic graphs consists of all connected graphs \mathcal{G} that can be constructed from a sequence $\{\mathcal{G}_n\}_{n \in \mathbb{Z}}$ of identical copies of any finite connected graph \mathcal{G}_0 , by connecting vertices of \mathcal{G}_n to vertices of \mathcal{G}_{n+1} in an identical way for each n . For a precise definition, see Definition 2.4 in Section 2.3. Denote by $\mathbb{V}_{\mathcal{G}_n}$ the set of vertices of \mathcal{G}_n . Since we have not specified any particular structure of the graph \mathcal{G} , we will as a measure of distance refer to the copy \mathcal{G}_n to which a vertex belongs. That a vertex v of \mathcal{G} is at *level* n therefore means that $v \in \mathbb{V}_{\mathcal{G}_n}$. An essentially 1-dimensional graph of particular interest is the $\mathbb{Z} \times \{0, 1, \dots, K-1\}^{d-1}$ nearest neighbour graph, i.e., the subgraph of the \mathbb{Z}^d lattice which has set of vertices $\mathbb{Z} \times \{0, 1, \dots, K-1\}^{d-1}$ for some $d, K \geq 1$, and where any two vertices are connected by an edge if their euclidean distance is 1. We will refer to this graph as the (K, d) -*tube* (cf. Figure 1). For this graph, the vertices at level n are the ones

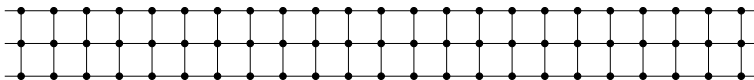


Figure 1: A part of the $(3, 2)$ -tube.

with first coordinate equal to n . Even though this graph looks more and more like the \mathbb{Z}^d lattice as K increases, we shall see that the asymptotic behaviour encountered in one dimension is present still.

Because of the unspecified structure of the underlying graph, it is

convenient to consider

$$T_n := \inf\{T(\Gamma) : \Gamma \text{ is a path from } I \text{ to level } n\}, \quad (1.6)$$

i.e., the time until a vertex at level n is infected. Our main results include that there are nonnegative, finite constants μ and σ such that

Theorem 1.2 (Law of large numbers). *If $E[\tau_e] < \infty$, then*

$$\lim_{n \rightarrow \infty} \frac{T_n}{n} = \mu, \quad \text{almost surely and in } L^1.$$

Theorem 1.3 (Central limit theorem). *If $E[\tau_e^2] < \infty$, then*

$$\frac{T_n - \mu n}{\sigma \sqrt{n}} \xrightarrow{d} \chi, \quad \text{in distribution,}$$

where χ has a standard normal distribution.

Theorem 1.4 (Law of the iterated logarithm). *If $E[\tau_e^2] < \infty$, then*

$$\limsup_{n \rightarrow \infty} \frac{|T_n - \mu n|}{\sigma \sqrt{2n \log \log n}} = 1, \quad \text{almost surely.}$$

Note that Theorem 1.2 actually follows from Kingman's subadditive ergodic theorem. However, it is natural and instructive to include and reprove it as we do. It will become clear that the constants μ and σ are nonnegative, finite, and depend on the underlying graph \mathcal{G} and the passage time distribution, but do not depend on the set of initially infected vertices I . Theorems 1.2, 1.3 and 1.4 are stated for the sequence $\{T_n\}_{n \geq 1}$. The same results hold with the same constants μ and σ also for the sequence $\{\max_{v \in \mathcal{V}_{\sigma_n}} T(v)\}$ and, if you wish, for the sequence $\{T(v_n)\}_{n \geq 1}$, where $\{v_n\}_{n \geq 1}$ is any sequence of vertices such that v_n is at level n (cf. Corollary 3.7). We preferred at this stage to give simple moment conditions in Theorems 1.2, 1.3 and 1.4. But we will later point out that they may in fact be relaxed somewhat (cf. Theorems 3.4, 3.5 and 3.6).

At a comparison with the asymptotic results in higher dimensions, Theorem 1.2 is the 1-dimensional analogue to the Shape theorem. In the case of (K, d) -tubes, replace B_t with the set \tilde{B}_t as we did for the \mathbb{Z}^d lattice. Let μ_K denote the time constant associated with the (K, d) -tube, and set

$$B^* = [-\mu_K^{-1}, \mu_K^{-1}] \times [0, K/t]^{d-1}.$$

It then follows that for all $\epsilon > 0$, almost surely,

$$(1 - \epsilon)B^* \subset \frac{1}{t}\tilde{B}_t \subset (1 + \epsilon)B^*, \quad (1.7)$$

for t large enough (will follow from Corollary 1.5, to be stated next). Theorems 1.3 and 1.4 on the other hand, point out a 1-dimensional behaviour that is not expected in higher dimensions. In particular, we see from Theorem 1.3 that $\text{Var}(T_n)$ grows linearly in n , in contrast to the higher dimensional subdiffusive behaviour in (1.5), pointed out by Benjamini, Kalai and Schramm. Theorem 1.4 also allows us to make the rate of convergence in (1.7) more precise. We formulate this in the following corollary.

Corollary 1.5. *Consider first-passage percolation on a (K, d) -tube with $\mathbb{E}[\tau_e^2] < \infty$. We have for all $\lambda > \sigma\sqrt{2/\mu}$, almost surely, that*

$$\left(1 - \lambda\sqrt{t^{-1}\log\log t}\right) B^* \subset \frac{1}{t}\tilde{B}_t \subset \left(1 + \lambda\sqrt{t^{-1}\log\log t}\right) B^*, \quad (1.8)$$

for all t large enough. Moreover, for all $\lambda < \sigma\sqrt{2/\mu}$ and $s \geq 0$, there exists a $t \geq s$ such that (1.8) does not hold, almost surely.

It is a trivial fact that the sequence $\{\mu_K\}_{K \geq 1}$ of time constants associated with the (K, d) -tube (for fixed d) is decreasing and bounded below by $\mu(\mathbf{e}_1)$. Thus, the sequence is convergent, and it is natural to ask whether or not

$$\lim_{K \rightarrow \infty} \mu_K = \mu(\mathbf{e}_1).$$

The answer is positive, as seen by the following result.

Proposition 1.6. *For $\mathbb{E}[\tau_e] < \infty$, we have $\mu_K \rightarrow \mu(\mathbf{e}_1)$, as $K \rightarrow \infty$.*

This shows that the rate of growth of an infection on the (K, d) -tube approaches the rate of growth of an infection on the \mathbb{Z}^d lattice, as K increases. Proposition 1.6 is similar to a result for a Markovian continuous growth model considered by Deijfen, Häggström and Bagley (2004, Lemma 4.4).

As another of our main results we construct a coupling of two first-passage percolation infections on 1-dimensional graphs. As an application of the coupling we prove a 0–1 law. Define the σ -algebra $\mathcal{T}_t := \sigma(\{B_s\}_{s \geq t})$

and the tail σ -algebra $\mathcal{T} := \cap_{t \geq 0} \mathcal{T}_t$. We may think of \mathcal{T}_t as the σ -algebra of events that do not depend on the times at which vertices are infected before time t . The 0–1 law we provide takes the form:

Theorem 1.7 (0–1 law). *Consider first-passage percolation on an essentially 1-dimensional periodic graph \mathcal{G} , with a finite set of initially infected vertices. Assume that the passage time distribution has an absolutely continuous component (with respect to Lebesgue measure). Then $P(A) \in \{0, 1\}$, for any event $A \in \mathcal{T}$.*

The 0–1 law follows from an application of the following coupling.

Proposition 1.8 (Coupling). *Let I and I' be finite subsets of the set of vertices of an essentially 1-dimensional periodic graph \mathcal{G} . Assume that the passage time distribution P_τ has an absolutely continuous component (with respect to Lebesgue measure). Then, there exists a coupling of $\{\tau_e\}_{e \in \mathbb{E}}$ and $\{\tau'_e\}_{e \in \mathbb{E}}$ such that $\{\tau_e\}_{e \in \mathbb{E}}$ and $\{\tau'_e\}_{e \in \mathbb{E}}$ form sequences of i.i.d. random variables with distribution P_τ , and if first-passage percolation is performed with $(I, \{\tau_e\}_{e \in \mathbb{E}})$ and $(I', \{\tau'_e\}_{e \in \mathbb{E}})$, respectively, then with probability one there exists $T_c < \infty$, such that*

$$B_t = B'_t,$$

for all $t \geq T_c$.

The same coupling is presented for discrete passage time distribution, but then on the more restrictive class of (K, d) -tubes (cf. Proposition 5.2). Theorem 1.7 is extended to include this case as well. Motivating examples are given to show why it is not possible to make the coupling as general as Proposition 1.8 also in the discrete case (cf. Remark 5.6 and 5.7).

The mild condition of an absolutely continuous component to be sufficient for the 0–1 law on essentially 1-dimensional periodic graphs, opens up for a discussion. We do not know on which other graphs this condition is sufficient. But, we give a counterexample showing that a 0–1 law analogous to Theorem 1.7 cannot hold on the binary tree \mathbb{T}^2 . An interesting and challenging case to settle would be on the \mathbb{Z}^d lattice. We will not argue for either possibility.

Another model closely related to first-passage percolation is the *two-type Richardson model*. In this model, each vertex of the \mathbb{Z}^d lattice can

be in either of the states 0, 1, or 2. State 0 is to be considered as healthy, and the states 1 and 2 as infected by one of two infections. An infected vertex stays in its infected state forever. Let one vertex x and another y be initially infected by infection 1 and 2, respectively. The infections evolve and a vertex in state 1 or 2 infects adjacent vertices in state 0 after an exponentially distributed time with rate λ_1 or λ_2 , respectively. This generalizes the one-type Richardson model, which is equivalent to first-passage percolation with exponentially distributed passage times.

The two-type Richardson model was introduced by Häggström and Pemantle (1998). The question they considered was whether both infections will grow to infect infinitely many vertices with positive probability. Denote by A the event that both infections grow indefinitely. They conjectured that

$$P(A) > 0 \quad \text{if and only if} \quad \lambda_1 = \lambda_2.$$

Häggström and Pemantle (1998) proved that $\lambda_1 = \lambda_2$ implies $P(A) > 0$ for $d = 2$. This was later generalized to higher dimensions (each of Garet and Marchand (2005); Hoffman (2005a); Deijfen and Häggström (2007) presents a different proof). The “only if” part is not yet fully proved (see Deijfen and Häggström (2008) for a survey of the field).

On essentially 1-dimensional periodic graphs \mathcal{G} , the question becomes trivial. It is easy to see that both infections may grow to prevent the other from making further progress, one in positive direction, and the other in negative. Therefore

$$P(A) > 0, \quad \text{for all } \lambda_1, \lambda_2,$$

unless one infection is trivially blocked by the other already from the start.

A closely related problem concerns the *tree of infection* Ψ , to be defined. Consider first-passage percolation on \mathbb{Z}^d with a single infected vertex at the origin, and a passage time distribution P_τ without atoms. Let $\Psi(v)$ be the minimizing path (assuming such exists) of $T(v)$, i.e., the path along which v got infected. Then Ψ is defined as the tree $\Psi = \bigcup_{v \in \mathbb{V}} \Psi(v)$ spanning \mathbb{Z}^d , which we refer to as the tree of infection (see Figure 2, page 12 for a realization on the $(2, 2)$ -tube). The question about Ψ that is related to the two-type Richardson model concerns the number of infinite self-avoiding paths in Ψ started at the origin, denoted by $\kappa(\Psi)$. If you wish, this equals the number of so-called topological ends of Ψ . Newman

(1995) conjectured that

$$\kappa(\Psi) = \infty, \quad \text{almost surely,}$$

and he proved this under some plausible, but still unproven, assumption on the asymptotic shape \tilde{B}_0 . A first (small) step towards this conjecture was made by Häggström and Pemantle (1998). For $d = 2$ and exponential passage time distribution, they could from the possible coexistence of two infections in the two-type Richardson model conclude that

$$P(\kappa(\Psi) \geq 2) > 0.064.$$

This was later improved to

$$\kappa(\Psi) \geq 2d, \quad \text{almost surely}$$

for $d = 2$ by Hoffman (2005b), and in higher dimensions by Gouréré (2007). Note that a 0–1 law analogous to Theorem 1.7 on \mathbb{Z}^d would in particular imply that $P(\kappa(\Psi) = k) \in \{0, 1\}$ for all $k \geq 0$.

Most of the results proved in this paper make use of a regenerative behaviour that arises for first-passage percolation on essentially 1-dimensional periodic graphs. With a regenerative behaviour we mean in this case that there is a sequence $\{\rho_k\}_{k \geq 1}$ such that

$$T_{\rho_n} = T_{\rho_1} + \sum_{k=1}^{n-1} T_{\rho_{k+1}} - T_{\rho_k},$$

and $\{T_{\rho_{k+1}} - T_{\rho_k}\}_{k \geq 1}$ forms an i.i.d. sequence. As we will see, a direct consequence of this behaviour is that the infima in (1.1), (1.2) and (1.6) are in fact minima (cf. Remark 2.8). It further follows from this behaviour that on any essentially 1-dimensional periodic graph \mathcal{G} , for the tree of infection $\Psi_{\mathcal{G}}$ we have

$$\kappa(\Psi_{\mathcal{G}}) = 2, \quad \text{almost surely.} \tag{1.9}$$

In Remark 2.9 we will give an argument for this. Again, both for the two-type Richardson model and the tree of infection we see how the behaviour of first-passage percolation on essentially 1-dimensional periodic graphs differ from what is believed to be the case in higher dimensions.

Section 2 is dedicated to exploring the regenerative behaviour and provide the results we need for the following sections. Theorems 1.2, 1.3 and 1.4, as well as Corollary 1.5 are proved in Section 3. In Section 4 we prove Proposition 1.6. In the final Section 5 the coupling of Proposition 1.8 is constructed, in its continuous and its discrete version. The 0–1 law Theorem 1.7 is also derived and the counterexample of the 0–1 law on trees is presented at the very end of this section and paper.

2 Regenerative behaviour

We will direct special interest to first-passage percolation on the (K, d) -tube. The (K, d) -tube is essentially 1-dimensional, and recall that we say that a vertex is at *level* n if its first coordinate equals n . We will also distinguish the set of vertices $\mathbb{Z} \times \{0\}^{d-1}$, and refer to each such vertex as a vertex at *height zero*. As we will see, the key idea of this paper arises naturally for first-passage percolation with exponential passage times. In Section 2.1, we demonstrate this idea on the $(2, 2)$ -tube. In Section 2.2, we see how these ideas extend to general passage times. Finally, we use Section 2.3 for the somewhat technical modifications needed to adapt these ideas to first-passage percolation on any essentially 1-dimensional periodic graph, as defined in Definition 2.4. We hope in this way that we achieve greater clarity of presentation.

2.1 Exponential passage times

Let the edges of the $(2, 2)$ -tube be equipped with i.i.d. exponential passage times $\{\tau_e\}_{e \in \mathbb{E}}$, and let both vertices at level zero be initially infected. At any fixed time t , given the infected component B_t , each edge with exactly one endpoint in the infected component is equally likely to be passed by the infection next. This is a direct consequence of the lack-of-memory property of the exponential distribution. With probability one, at some level r , both vertices will become infected before any vertex at level $r + 1$. Denote by ρ the first level for which this happens, and let τ_ρ denote the time. By the lack-of-memory property, the time it takes for the infection from this moment to reach m levels further has the same distribution as the time it would take to reach level m , i.e.,

$$T_{\rho+m} - \tau_\rho \stackrel{d}{=} T_m. \tag{2.1}$$

In fact, at infinitely many levels, both vertices will be infected before any vertex at higher levels. We see that the process holds certain regenerative properties. What we mean by this is that if we repeat the argument, we

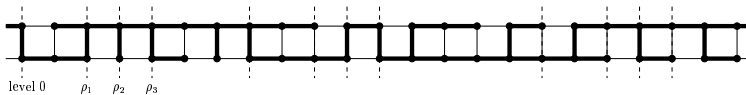


Figure 2: A realization of the spread of an infection on the $(2, 2)$ -tube. The broken lines indicate levels at which both vertices will become infected before any vertex ahead. The thick line constitutes the tree of infection.

generate a sequence of regenerative levels $\{\rho_k\}_{k \geq 1}$ (see Figure 2), with corresponding sequence of instants $\{\tau_{\rho_k}\}_{k \geq 1}$, such that (2.1) holds. This is made precise in Lemma 2.2. Refer to the part of the graph between level ρ_k and ρ_{k+1} as a *patch*. Since the passage times are i.i.d., the sequence of lengths of a patch will be i.i.d., as well as $\tau_{\rho_{k+1}} - \tau_{\rho_k}$, the sequence of times it takes to pass a patch. In particular, the n th regenerative level and the time at which it is reached may be written as sums of i.i.d. random variables, i.e.,

$$\rho_n = \sum_{k=0}^{n-1} \rho_{k+1} - \rho_k \quad \text{and} \quad \tau_{\rho_n} = \sum_{k=0}^{n-1} \tau_{\rho_{k+1}} - \tau_{\rho_k},$$

where $\rho_0 = 0$ and $\tau_{\rho_0} = 0$. Thus, it is easy to see that classical results, such as the law of large numbers (cf. Lemma 3.1), central limit theorem (cf. Lemma 3.2) and the law of the iterated logarithm (cf. Lemma 3.3), applies to τ_{ρ_n} , the passage time to the n th regeneration. As a part of our main results, we expand this further to involve T_n , the passage time to level n . This is all done in Section 3

2.2 First-passage percolation on a tube

Let us now focus on the (K, d) -tube with general passage times. When we refer to an edge *at* some level n , we mean an edge connecting two vertices at level n . When we refer to an edge *between* levels n and $n+m$, we mean an edge connecting two vertices at any of the levels (same or

different) $n, n + 1, \dots, n + m$. When we refer to an edge *at* height zero, we mean that both its endpoints are at height zero.

Let N be a positive integer and denote the set of edges between level n and $n + 2N$ by E_n . Define the subset \hat{E}_n of E_n as

$$\hat{E}_n := \{e \in E_n : e \text{ is at height zero, at level } n, \text{ or at level } n + 2N\}. \quad (2.2)$$

Define

$$\begin{aligned} m_\tau &:= \inf \{x \geq 0 : P_\tau([0, x]) > 0\}, \\ M_\tau &:= \sup \{x \geq 0 : P_\tau([x, \infty)) > 0\}. \end{aligned} \quad (2.3)$$

Note that $0 \leq m_\tau < M_\tau \leq \infty$, where the strict inequality holds for passage time distributions that do not concentrate all mass at a single point. For constants t' and t'' such that $m_\tau < t' < t'' < M_\tau$, define the regenerative event

$$A_n := \left\{ \tau_e \leq t', \forall e \in \hat{E}_n \right\} \cap \left\{ \tau_e \geq t'', \forall e \in E_n \setminus \hat{E}_n \right\}. \quad (2.4)$$

Trivially $P(A_n) > 0$.

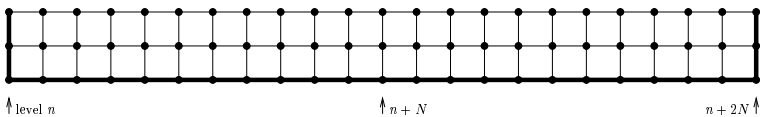


Figure 3: The $(3, 2)$ -tube between level n and $n + 2N$. If A_n occurs, the thick edges are “quick”, and the thin edges are “slow”.

Denote by $\rho_0 := \max\{n \geq 0 : \exists v \in I \text{ at level } n\}$ the furthest initially infected level, and by \hat{v}_n the vertex $(n, 0, \dots, 0)$.

Lemma 2.1. *Let t' and t'' be constants such that $m_\tau < t' < t'' < M_\tau$. Then there exists an $N \in \mathbb{N}$, such that for all $n \geq \rho_0$, if A_n occurs, then $T(\hat{v}_{n+N}) = T_{n+N}$ and*

$$T(v) = T(\hat{v}_{n+N}) + T(\hat{v}_{n+N}, v)$$

for each vertex v at level $n + 2N$.

In words the lemma says: If A_n occurs, then \hat{v}_{n+N} is the vertex first infected at level $n + N$, and for each vertex v at level $n + 2N$, the path Γ between I and v that minimizes $T(v)$ passes \hat{v}_{n+N} .

Proof. For given t' and t'' , choose $N \geq \frac{(d-1)(K-1)t'}{t''-t'}$ and assume that $n \geq \rho_0$. The time it takes to reach level $n + N$ from level n using edges of \hat{E}_n only is at most $((d-1)(K-1) + N)t'$. If there is a path using fewer edges to reach level $n + N$, than any path using only edges $e \in \hat{E}_n$, it inevitably has to pass N edges whose passage time are $\geq t''$. But,

$$((d-1)(K-1) + N)t' \leq Nt'', \quad (2.5)$$

by the choice of N . Thus, it is easy to see that $T(\hat{v}_{n+N}) = T_{n+N}$, and level $n + N$ is first reached at height zero.

In an analogous manner, the path from the vertex at level $n + N$, height zero, to a vertex v at level $n + 2N$, using only edges of \hat{E}_n , has shorter passage time than any other path from a vertex at level $n + N$ to the vertex v , again, because of (2.5). \square

We will assume from now on that t' , t'' and N are chosen according to Lemma 2.1. Let $n_k := k(2N + 1)$, and note that the sequence of events $\{A_{n_k}\}_{k \geq 0}$ is readily seen to be i.i.d. Recall that ρ_0 denotes the furthest initially infected level, and define

$$\rho_k := N + \min\{n_m : n_m > \rho_{k-1} \text{ and } A_{n_m} \text{ occurs}\}$$

for $k \geq 1$. If A_{n_k} occurs for infinitely many k , this generates a sequence $\{\rho_k\}_{k \geq 1}$, where $\rho_k < \infty$ almost surely. Since $\{A_{n_k}\}_{k \geq 0}$ is i.i.d. and $P(A_{n_k}) > 0$, the second Borel-Cantelli lemma gives that

$$P(A_{n_k} \text{ occurs for infinitely many } k) = 1.$$

From Lemma 2.1 we see that a path along which any vertex at level $\rho_k + N$ and beyond is infected has to pass the vertex \hat{v}_{ρ_k} . We will refer to this vertex as a *regeneration point*. For $k \geq 1$, we introduce the notation S_k for the part of the graph between regeneration point ρ_k and ρ_{k+1} , and refer to this as a *patch*. Let $|S_k| := \rho_{k+1} - \rho_k$ denote the length of the patch, and $\tau_{S_k} := T_{\rho_{k+1}} - T_{\rho_k}$ denote the passage time of the patch. The part of the graph between the origin and level ρ_1 is denoted by S_0 . Thus, $|S_0| = \rho_1$ and $\tau_{S_0} = T_{\rho_1}$. With this notation, we write the

time it takes for the infection to reach the level of the n th regeneration, $\rho_n = \sum_{k=0}^{n-1} |S_k|$, as

$$T_{\rho_n} = T(\hat{v}_{\rho_n}) = \tau_{S_0} + \sum_{k=1}^{n-1} \tau_{S_k}. \quad (2.6)$$

It is immediate from the construction that the distributions of $|S_k|$ and τ_{S_k} for $k \geq 1$ are independent of the set of initially infected vertices I .

Lemma 2.2. *Consider first-passage percolation on the (K, d) -tube with a finite initially infected set of vertices. The sequences $\{|S_k|\}_{k \geq 1}$ and $\{\tau_{S_k}\}_{k \geq 1}$ are i.i.d. Moreover,*

$$T_{\rho_{k+m}} - T_{\rho_k} \stackrel{d}{=} T_{\rho_{1+m}} - T_{\rho_1}, \quad \text{for all } m \geq 0, k \geq 1.$$

Proof. Since the events A_{ρ_n} are independent from each other, with equal success probability, it follows that $\{|S_k|\}_{k \geq 1}$ is an i.i.d. sequence of geometrically distributed random variables, times a factor $2N + 1$. Let \mathbb{E}_n denotes the set of edges up to, but not including, level n . By Lemma 2.1, $T_{\rho_{k+m}} - T_{\rho_k}$ is independent of $\{\tau_e\}_{e \in \mathbb{E}_{\rho_k}}$. It follows that

$$T_{\rho_{k+m}} - T_{\rho_k} \stackrel{d}{=} T_{\rho_{1+m}} - T_{\rho_1}.$$

Thus, since $|S_k| = \rho_{k+1} - \rho_k$, for $k \geq 1$ are identically distributed, we conclude that

$$T_{\rho_{k+1}} - T_{\rho_k} \stackrel{d}{=} T_{\rho_2} - T_{\rho_1}.$$

By the fact that $\tau_{S_k} = T_{\rho_{k+1}} - T_{\rho_k}$ only depends on the passage times of edges between level ρ_k and ρ_{k+1} , excluding the ones at level ρ_k and ρ_{k+1} , the sequence $\{\tau_{S_k}\}_{k \geq 1}$ is i.i.d. \square

In order to state results with simple moment conditions, it will be important to know how moments of patches behave given moments of the passage time distribution. More precisely, the following lemma is what we need.

Lemma 2.3. *For a passage time distribution P_τ that does not concentrate all mass in a single point,*

- a) *there exists an $\alpha > 0$ such that $\mathbb{E}[e^{\alpha|S_k|}] < \infty$.*
- b) *if $\mathbb{E}[\tau_e^\alpha] < \infty$, for some $\alpha > 0$, then $0 < \mathbb{E}[\tau_{S_k}^\alpha] < \infty$.*

Proof. It follows from its definition that the length $|S_k| = \rho_{k+1} - \rho_k$ of a patch is geometrically distributed with parameter $\epsilon = P(A_0)$, times a factor $2N + 1$. Thus, if θ is geometrically distributed with parameter ϵ , then $|S_k| \stackrel{d}{=} (2N + 1)\theta$, and we deduce that

$$\begin{aligned} \mathbb{E}[e^{\alpha|S_k|}] &= \sum_{k=1}^{\infty} e^{\alpha(2N+1)k} (1 - \epsilon)^{k-1} \epsilon \\ &= e^{\alpha(2N+1)} \epsilon \sum_{k=0}^{\infty} \left(e^{\alpha(2N+1)} (1 - \epsilon) \right)^k < \infty, \end{aligned}$$

if $e^{\alpha(2N+1)}(1 - \epsilon) < 1$.

For the second part of the lemma, set $\lambda_l := \rho_k - N + (2N + 1)l$, for integers $l \geq 0$. Written as events, for $n \geq 1$,

$$\{|S_k| = (2N + 1)n\} = A_{\lambda_0} \cap A_{\lambda_n} \bigcap_{l=1}^{n-1} A_{\lambda_l}^c. \quad (2.7)$$

Let $\Lambda_n := \{|S_k| = (2N + 1)n\}$, and let D denote the set of edges between level ρ_k and $\rho_k + |S_k|$ at height zero. To see that $\mathbb{E}[\tau_{S_k}^\alpha] > 0$, it is enough to note that the passage time of each edge between a vertex at level $\lambda_0 - 1$ and a vertex at λ_0 is independent of Λ_n . In order to see that $\mathbb{E}[\tau_{S_k}^\alpha]$ is finite, note that τ_{S_k} is bounded by the sum of passage times for the edges in D , i.e.,

$$\tau_{S_k} = T_{\rho_k + |S_k|} - T_{\rho_k} \leq \sum_{e \in D} \tau_e. \quad (2.8)$$

Given the event Λ_n , there is a constant $C < \infty$ such that, for each edge $e \in D$ we have

$$\mathbb{E}[\tau_e^\alpha | \Lambda_n] \leq C \mathbb{E}[\tau_e^\alpha]. \quad (2.9)$$

For any $e \in D$ between level λ_l and λ_n , this follows since for some $l \geq 1$

$$\mathbb{E}[\tau_e^\alpha | \Lambda_n] = \mathbb{E}[\tau_e^\alpha | A_{\lambda_l}^c] \leq P(A_{\lambda_l}^c)^{-1} \mathbb{E}[\tau_e^\alpha],$$

and $P(A_{\lambda_l}^c) > 0$. For any other $e \in D$, the bound in (2.9) holds trivially.

Thus, combining (2.7), (2.8) and (2.9) we deduce that

$$\begin{aligned}
\mathbb{E}[\tau_{S_k}^\alpha] &\leq \mathbb{E} \left[\left(\sum_{e \in D} \tau_e \right)^\alpha \right] \leq \mathbb{E} \left[|S_k|^\alpha \sum_{e \in D} \tau_e^\alpha \right] \\
&= (2N + 1)^\alpha \sum_{n=1}^{\infty} n^\alpha \mathbb{E} \left[\sum_{e \in D} \tau_e^\alpha \middle| \Lambda_n \right] P(\Lambda_n) \\
&\leq (2N + 1)^{\alpha+1} \sum_{n=1}^{\infty} n^{\alpha+1} C \mathbb{E}[\tau_e^\alpha] P(\Lambda_n) \\
&\leq C \mathbb{E}[\tau_e^\alpha] \mathbb{E}[|S_k|^{\alpha+1}],
\end{aligned}$$

where the second inequality follows since for any nonnegative numbers a_j we have

$$\left(\sum_{j=1}^n a_j \right)^\alpha \leq (n \max_j a_j)^\alpha \leq n^\alpha \sum_{j=1}^n a_j^\alpha. \quad (2.10)$$

Thus, $\mathbb{E}[\tau_{S_k}^\alpha] < \infty$ by assumption, and the first part of the lemma. \square

2.3 First-passage percolation in one dimension

All arguments presented in Section 2.2 extend to any essentially 1-dimensional graph that is periodic. The adaptation of the arguments will turn out to be mostly straightforward.

Definition 2.4. The class of *essentially 1-dimensional periodic graphs* consists of all connected graphs \mathcal{G} that can be constructed in the following manner. Let $\{\mathcal{G}_n\}_{n \in \mathbb{Z}}$ be a sequence of identical copies of some finite connected deterministic graph, each with set of vertices $\mathbb{V}_{\mathcal{G}_n} = \{v_{n,1}, \dots, v_{n,k}\}$ and set of edges $\mathbb{E}_{\mathcal{G}_n} = \{e_{n,1}, \dots, e_{n,l}\}$. Fix a nonempty set $J \subseteq \{(i, j) : 1 \leq i, j \leq k\}$, and connect \mathcal{G}_n to \mathcal{G}_{n+1} for each n by adding an edge $e(v_{n,i}, v_{n+1,j})$ between $v_{n,i}$ and $v_{n+1,j}$, for each $(i, j) \in J$. Let $\mathcal{G} = (\mathbb{V}, \mathbb{E})$ denote the resulting graph where

$$\mathbb{V} = \bigcup_{n \in \mathbb{Z}} \mathbb{V}_{\mathcal{G}_n} \quad \text{and} \quad \mathbb{E} = \bigcup_{n \in \mathbb{Z}} (\mathbb{E}_{\mathcal{G}_n} \cup \{e(v_{n,i}, v_{n+1,j}) : (i, j) \in J\}).$$

If $v \in \mathbb{V}_{\mathcal{G}_n}$ we say that v is at *level* n .

Note that the (K, d) -tube is an essentially 1-dimensional periodic graph, since it can be constructed from the sequence $\{\mathcal{G}_n\}_{n \in \mathbb{Z}}$, where \mathcal{G}_n has vertex set $\{n\} \times \{0, 1, \dots, K-1\}^{d-1}$ and where any two vertices at euclidean distance one are joined by an edge. As for first-passage percolation on a (K, d) -tube, we shall see that on \mathcal{G} , there is a regenerative behaviour. Our main concern will be to define the event A_{n_k} such that Lemma 2.1 extends easily. The extension of Lemma 2.2 and 2.3 turn out to be straightforward. We will reuse much of the notation for sets and constants from Section 2.2, since they have practically the same meaning.

As before, for some nonnegative integer N , let E_n denote the set of edges between level n and $n+2N$. Fix a path γ_n of shortest length between $\mathbb{V}_{\mathcal{G}_n}$ and $\mathbb{V}_{\mathcal{G}_{n+2N}}$, i.e., between two vertices at level n and $n+2N$, respectively. In accordance to (2.2), define

$$\hat{E}_n := \gamma_n \cup \mathbb{E}_{\mathcal{G}_n} \cup \mathbb{E}_{\mathcal{G}_{n+2N}}. \quad (2.11)$$

For constants t' and t'' such that $m_\tau < t' < t'' < M_\tau$, recall that we defined

$$A_n = \left\{ \tau_e \leq t', \forall e \in \hat{E}_n \right\} \cap \left\{ \tau_e \geq t'', \forall e \in E_n \setminus \hat{E}_n \right\}. \quad (2.12)$$

Trivially, $P(A_n) > 0$. Denote by \hat{v}_{n+N} the vertex at level $n+N$ first reached via γ_n . Set $\beta := \text{dist}(\hat{v}_{n+N}, \mathbb{V}_{\mathcal{G}_{n+2N}})$, where $\text{dist}(v, V)$ denotes the smallest number of edges one has to pass in order to reach a vertex of V from v . Define

$$\mathcal{V}_n := \left\{ v \in \bigcup_{j=n}^{n+2N} \mathbb{V}_{\mathcal{G}_j} : \text{dist}(v, \mathbb{V}_{\mathcal{G}_{n+2N}}) = \beta \right\}.$$

Note that $\gamma_n \cap \mathcal{V}_n = \{\hat{v}_{n+N}\}$.

Let $\rho_0 = \max\{n \geq 0 : \mathbb{V}_{\mathcal{G}_n} \cap I \neq \emptyset\}$ denote the furthest initially infected level. Lemma 2.1 extends in the present setting to

Lemma 2.5. *Let t' and t'' be constants such that $m_\tau < t' < t'' < M_\tau$. Then there exists $N \in \mathbb{N}$, such that for all $n \geq \rho_0$, if A_n occurs, then $T(\hat{v}_{n+N}) = \min_{v \in \mathcal{V}_n} T(v)$ and*

$$T(v) = T(\hat{v}_{n+N}) + T(\hat{v}_{n+N}, v) \quad (2.13)$$

for each vertex $v \in \mathbb{V}_{\mathcal{G}_{n+2N}}$.

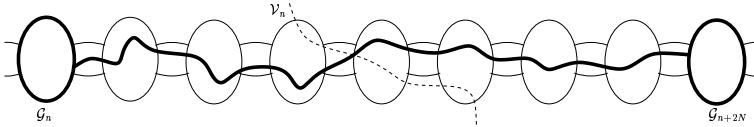


Figure 4: The graph \mathcal{G} between level n and $n + 2N$. If A_n occurs, the thick edges at level n , level $n + 2N$ and of the path γ_n are “quick”.

In words the lemma says: If A_n occurs, then \hat{v}_{n+N} is the first vertex in \mathcal{V}_n to become infected, and for each vertex v at level $n + 2N$, the path Γ between I and v that minimizes $T(v)$ passes \hat{v}_{n+N} .

Proof. By assumption $n \geq \rho_0$. For given t' and t'' , choose

$$N \geq \frac{|\mathbb{E}_{\mathcal{G}_n}| t'}{t'' - t'}.$$

Let γ_n^- denote the part of the path γ_n between $\mathbb{V}_{\mathcal{G}_n}$ and \hat{v}_{n+N} , and γ_n^+ the path between \hat{v}_{n+N} and $\mathbb{V}_{\mathcal{G}_{n+2N}}$.

Condition on A_n . The time it takes to reach the vertex \hat{v}_{n+N} from any vertex at level n is bounded by $(|\mathbb{E}_{\mathcal{G}_n}| + |\gamma_n^-|) t'$. By definition, γ_n^- is the shortest path between $\mathbb{V}_{\mathcal{G}_n}$ and \mathcal{V}_n . Thus, \hat{v}_{n+N} is the first vertex in \mathcal{V}_n to become infected, unless there is a path Γ between $\mathbb{V}_{\mathcal{G}_n}$ and \mathcal{V}_n , disjoint with γ_n^- and with passage time $T(\Gamma) < (|\mathbb{E}_{\mathcal{G}_n}| + |\gamma_n^-|) t'$. Such a path Γ would have to pass at least $|\gamma_n^-|$ edges with passage time at least t'' . By the choice of N

$$|\mathbb{E}_{\mathcal{G}_n}| t' \leq N(t'' - t') \leq |\gamma_n^-|(t'' - t'),$$

since $|\gamma_n^-| \geq N$. Thus,

$$(|\mathbb{E}_{\mathcal{G}_n}| + |\gamma_n^-|) t' \leq |\gamma_n^-| t'' \leq T(\Gamma) \quad (2.14)$$

and it follows that $T(\hat{v}_{n+N}) = \min_{v \in \mathcal{V}_n} T(v)$.

It follows that (2.13) holds for every vertex v passed by the path γ_n^+ . A similar argument to the above one shows that no vertex at level $n + 2N$ can be infected along a path between \mathcal{V}_n and level $n + 2N$ disjoint with γ_n^+ , again because of (2.14). Hence, (2.13) holds for each v at level $n + 2N$. \square

Assume from now on that N is chosen according to Lemma 2.5. Recall that we defined $n_k = k(2N + 1)$ for $k \geq 0$, and

$$\rho_k = N + \min\{n_m : n_m > \rho_{k-1} \text{ and } A_{n_m} \text{ occurs}\}$$

for $k \geq 1$. As in the previous section, this generates a sequence $\{\rho_k\}_{k \geq 1}$, where $\rho_k < \infty$ almost surely, by the second Borel-Cantelli lemma.

Lemma 2.5 says that the vertex \hat{v}_{ρ_k} is “responsible” for the infection of every vertex at level $\rho_k + N$ and beyond. Refer to this vertex as a regeneration point. As before, let $|S_k| := \rho_{k+1} - \rho_k$ and refer to this as the length (measured in levels) of a patch. Let $\tau_{S_k} := T(\hat{v}_{\rho_{k+1}}) - T(\hat{v}_{\rho_k})$ be its passage time. Also, we set $|S_0| = \rho_1$ levels, and $\tau_{S_0} = T(\hat{v}_{\rho_1})$.

The vertex \hat{v}_{ρ_k} is not necessarily the first vertex to be reached at level ρ_k , in this setting. But, it is the first to be reached among the set $\mathcal{V}_{\rho_k - N}$. Therefore, $T(\hat{v}_{\rho_n}) \neq T_{\rho_n}$ in general, and (2.6) takes the appearance

$$T(\hat{v}_{\rho_n}) = \tau_{S_0} + \sum_{k=1}^{n-1} \tau_{S_k}.$$

Lemma 2.2 generalizes to the following. Its proof is analogous in the present setting, except for an occasional change from T_{ρ_k} to $T(\hat{v}_{\rho_k})$. The proof will be left out.

Lemma 2.6. *Consider first-passage percolation on \mathcal{G} with a finite initially infected set of vertices. The sequences $\{|S_k|\}_{k \geq 1}$ and $\{\tau_{S_k}\}_{k \geq 1}$ are i.i.d. Moreover,*

$$T_{\rho_k+m} - T(\hat{v}_{\rho_k}) \stackrel{d}{=} T_{\rho_1+m} - T(\hat{v}_{\rho_1}), \quad \text{for all } m \geq N, k \geq 1.$$

In order to adapt the proof of Lemma 2.3, only minor adjustments are necessary. In particular, let ψ_k denote the path of shortest length between \hat{v}_{ρ_k} and $\hat{v}_{\rho_k+|S_k|}$. Since \mathcal{G} is connected and periodic, the length of ψ_k is bounded by $C|S_k|$, for some constant C . Then, similar to (2.8), the passage time of a patch τ_{S_k} can be bounded by

$$\tau_{S_k} = T(\hat{v}_{\rho_k+|S_k|}) - T(\hat{v}_{\rho_k}) \leq \sum_{e \in \psi_k} \tau_e.$$

Continue as in the proof of Lemma 2.3.

Lemma 2.7. *For a passage time distribution P_τ that does not concentrate all mass in a single point,*

a) there exists an $\alpha > 0$ such that $\mathbb{E} [e^{\alpha|S_k|}] < \infty$.

b) if $\mathbb{E}[\tau_e^\alpha] < \infty$, for some $\alpha > 0$, then $0 < \mathbb{E} [\tau_{S_k}^\alpha] < \infty$.

Remark 2.8. On any essentially 1-dimensional periodic graph, the infima in T_n , $T(v)$ and $T(u, v)$ are in fact minima. This follows directly from Lemma 2.5. Focus on $T(u, v)$, where $u \in \mathbb{V}_{\mathcal{G}_{n_u}}$ and $v \in \mathbb{V}_{\mathcal{G}_{n_v}}$. Let $m = \max(n_u, n_v)$. With probability one the event A_n will occur for infinitely many $n \geq 0$. Let l be the smallest integer such that $l \geq m$ and A_l occurs. It follows from Lemma 2.5 that for any path Γ_1 between u and v that reach beyond level $l + 2N$, there is another path Γ_2 that does not, and $T(\Gamma_1) \geq T(\Gamma_2)$. The same argument applies in the negative direction. We conclude that the infimum in $T(u, v)$ is in fact a minimum among an almost surely finite number of paths. The other minima follows in a similar way. \square

Remark 2.9. On any essentially 1-dimensional periodic graph, as stated in (1.9), the number of self-avoiding paths $\kappa(\Psi_{\mathcal{G}})$ in the tree of infection $\Psi_{\mathcal{G}}$ equals 2, almost surely. To see this, for any $N \geq 1$, let κ_N denote the number of self-avoiding paths in $\Psi_{\mathcal{G}}$ that reach level N . With probability one, for some $n \geq N$ the event A_n will occur. It follows from Lemma 2.5 that the minimizing path of $T(v)$, for all vertices v at level $n + 2N$, have all to pass a certain vertex at level $n + N$. Thus, only one of the κ_N self-avoiding paths in $\Psi_{\mathcal{G}}$ will survive beyond level $n + 2N$. This implies that precisely one self-avoiding path will reach infinitely far in positive direction. The same applies in negative direction. From this we conclude that $\kappa(\Psi_{\mathcal{G}}) = 2$ almost surely. \square

Remark 2.10. It is worth pointing out that the initially infected component does not need to be finite. But, there needs to be a level ν beyond which no vertex is initially infected. The results we have shown hold also in this case. \square

3 Asymptotics for first-passage percolation

In this section we will present some of our main results for first-passage percolation considered on an essentially 1-dimensional periodic graph \mathcal{G} , with a finite initially infected component. We will assume that t' , t'' and N are chosen in accordance with Lemma 2.5. Recall the notation $|S_k|$

for the length of the k th patch, and τ_{S_k} for its passage time. Introduce

$$X_n := \sum_{k=0}^n |S_k|, \quad \text{and} \quad \tilde{T}_{X_n} := \sum_{k=0}^n \tau_{S_k}.$$

In view of Section 2, $X_n = \rho_{n+1}$, while $\tilde{T}_{X_n} = T(\hat{v}_{\rho_{n+1}})$. When the underlying graph is a (K, d) -tube, then $\tilde{T}_{X_n} = T_{\rho_{n+1}} = T_{X_n}$. In general we only know for sure that $\tilde{T}_{X_n} \geq T_{X_n}$. From Lemma 2.5 we also know that $\tilde{T}_{X_{n-1}} \leq T_{X_n}$. Hence, we conclude that if $X_{m-1} \leq n < X_m$, then

$$\tilde{T}_{X_{m-2}} \leq T_n \leq \tilde{T}_{X_m}. \quad (3.1)$$

By Lemma 2.6, $\{\tau_{S_k}\}_{k \geq 1}$ and $\{|S_k|\}_{k \geq 1}$ are both i.i.d. sequences. Let $\mu_\tau = \mathbb{E}[\tau_{S_k}]$ and $\mu_{|S|} = \mathbb{E}[|S_k|]$ denote the expected passage time and length of a patch, respectively. Trivially, $\mathbb{E}[|S_k|] \geq 1$. Under the additional condition that $\mathbb{E}[\tau_e] < \infty$, we have by Lemma 2.7 that

$$1 \leq \mu_{|S|} < \infty \quad \text{and} \quad 0 < \mu_\tau < \infty.$$

We define the time constant μ and the asymptotic variance σ^2 as

$$\mu := \frac{\mu_\tau}{\mu_{|S|}}, \quad \text{and} \quad \sigma^2 := \frac{\text{Var}(\tau_{S_k} - \mu|S_k|)}{\mu_{|S|}}.$$

Thus, again by Lemma 2.7, assuming $\mathbb{E}[\tau_e^2] < \infty$, we have

$$0 < \mu < \infty \quad \text{and} \quad 0 < \sigma^2 < \infty.$$

The regenerative behaviour explored in Section 2 allows us to easily derive asymptotic results for the sequence $\{\tilde{T}_{X_n}\}_{n \geq 1}$ from classical ones.

Lemma 3.1. *If $\mathbb{E}[\tau_{S_k}] < \infty$, then*

$$\lim_{n \rightarrow \infty} \frac{\tilde{T}_{X_n}}{X_n} = \mu, \quad \text{almost surely.}$$

Proof. Lemma 2.6 assures that $\{|S_k|\}_{k \geq 1}$ and $\{\tau_{S_k}\}_{k \geq 1}$ are i.i.d. sequences with finite first-moments. By the classical strong law of large numbers we have, almost surely,

$$\frac{\tilde{T}_{X_n}}{X_n} = \frac{\tau_{S_0} + \sum_1^n \tau_{S_k}}{n} \frac{n}{|S_0| + \sum_1^n |S_k|} \rightarrow \mathbb{E}[\tau_{S_k}] \frac{1}{\mathbb{E}[|S_k|]}.$$

□

Lemma 3.2. *If $E[\tau_{S_k}^2] < \infty$, then*

$$\frac{\tilde{T}_{X_n} - \mu X_n}{\sigma \sqrt{X_n}} \xrightarrow{d} \chi, \quad \text{in distribution,}$$

where χ has a standard normal distribution.

Proof. Lemma 2.6 assures that $\{\tau_{S_k} - \mu|S_k|\}_{k \geq 1}$ is an i.i.d. sequence with zero mean and finite variance. Let $\tilde{\sigma}^2 = \text{Var}(\tau_{S_k} - \mu|S_k|) = \sigma^2 \mu_{|S|}$. By the classical central limit theorem, we have that

$$\frac{\tilde{T}_{X_n} - \mu X_n}{\sigma \sqrt{X_n}} = \frac{\tau_{S_0} - \mu|S_0|}{\sigma \sqrt{X_n}} + \frac{\sum_{k=1}^n (\tau_{S_k} - \mu|S_k|)}{\tilde{\sigma} \sqrt{n}} \sqrt{\mu_{|S|}} \sqrt{\frac{n}{X_n}}$$

converges weakly to a standard normal, since the first term converges to zero, and the rightmost factor to $\mu_{|S|}^{-1/2}$. \square

Lemma 3.3. *If $E[\tau_{S_k}^2] < \infty$, then*

$$\limsup_{n \rightarrow \infty} \frac{|\tilde{T}_{X_n} - \mu X_n|}{\sigma \sqrt{2X_n} \log \log X_n} = 1, \quad \text{almost surely.}$$

Proof. Recall that $\{\tau_{S_k} - \mu|S_k|\}_{k \geq 1}$ are i.i.d. with zero mean and finite variance, due to Lemma 2.6. Rewrite

$$\frac{|\tilde{T}_{X_n} - \mu X_n|}{\sigma \sqrt{2X_n} \log \log X_n} = \frac{|\tilde{T}_{X_n} - \mu X_n|}{\tilde{\sigma} \sqrt{2n} \log \log n} \sqrt{\mu_{|S|}} \sqrt{\frac{n}{X_n}} \sqrt{\frac{\log \log n}{\log \log X_n}}.$$

By the strong law of large numbers $\frac{n}{X_n} \rightarrow \mu_{|S|}^{-1}$, almost surely, as $n \rightarrow \infty$. We further conclude that

$$\lim_{n \rightarrow \infty} \frac{\log \log n}{\log \log X_n} = \lim_{n \rightarrow \infty} \frac{\log \log n}{\log(\log n + \log \frac{X_n}{n})} = 1, \quad (3.2)$$

almost surely. We may bound $|\tilde{T}_{X_n} - \mu X_n|$ from above and below by

$$\left| \sum_1^n (\tau_{S_k} - \mu|S_k|) \right| \pm |\tau_{S_0} - \mu|S_0|. \quad (3.3)$$

Then, the lim sup convergence follows by an application the classical law of the iterated logarithm on (3.3). \square

3.1 Proof of Theorems 1.2, 1.3 and 1.4

We are now ready to prove some of our main theorems. This will be done under slightly weaker assumptions than those stated in Theorems 1.2, 1.3 and 1.4. Instead of assuming $E[\tau_e^\alpha] < \infty$ we assume only $E[\tau_{S_k}^\alpha] < \infty$. That the former implies the latter follows from Lemma 2.7. That the latter is indeed weaker follows from Example 3.8, given later in this section. Theorems 1.2, 1.3 and 1.4 follow immediately from the theorems we are about to state and prove.

Theorem 3.4. *If $E[\tau_{S_k}] < \infty$, then*

$$\lim_{n \rightarrow \infty} \frac{T_n}{n} = \mu, \quad \text{almost surely.}$$

If in addition $E[\tau_{S_0}] < \infty$, the convergence holds also in L^1 .

Proof of almost sure convergence. Define $M_n = \min\{m \in \mathbb{N} : X_m > n\}$. It is clear that $M_n \uparrow \infty$ as $n \rightarrow \infty$. Moreover, by (3.1) we get that

$$\frac{X_{M_n-2}}{n} \frac{\tilde{T}_{X_{M_n-2}}}{X_{M_n-2}} \leq \frac{T_n}{n} \leq \frac{\tilde{T}_{X_{M_n}}}{X_{M_n}} \frac{X_{M_n}}{n}.$$

By Lemma 3.1, the almost sure convergence of the statement follows if we conclude that the leftmost and rightmost factors in the above expression converge to 1. This is not hard to see. Note that for any i.i.d. sequence $\{\xi_n\}_{n \geq 1}$, a simple application of the Borel-Cantelli lemmas shows that

$$\lim_{n \rightarrow \infty} \frac{\xi_n}{n} = 0, \quad \text{almost surely} \quad \Leftrightarrow \quad E[|\xi_n|] < \infty. \quad (3.4)$$

To see this, note that $E[|\xi_n|] < \infty$ is equivalent to

$$\sum_{n=1}^{\infty} P(|\xi_n| > \epsilon n) < \infty$$

for any $\epsilon > 0$. This is by the Borel-Cantelli lemmas equivalent to $\lim_{n \rightarrow \infty} \frac{|\xi_n|}{n} < \epsilon$, and (3.4) follows. From (3.4) and the definition of M_n , we have as $n \rightarrow \infty$, almost surely,

$$1 \leq \frac{X_{M_n}}{n} = \frac{X_{M_n-1}}{n} + \frac{|S_{M_n}|}{n} \leq 1 + \frac{|S_{M_n}|}{n} \rightarrow 1,$$

and

$$\begin{aligned} 1 &\geq \frac{X_{M_n-2}}{n} = \frac{X_{M_n}}{n} - \frac{|S_{M_n}| + |S_{M_n-1}|}{n} \\ &\geq 1 - \frac{|S_{M_n}| + |S_{M_n-1}|}{n} \rightarrow 1, \end{aligned}$$

since $\mathbb{E}[|S_k|] < \infty$.

Proof of L^1 -convergence. It is a trivial fact that $n \leq X_n$. Thus $T_n \leq T_{X_n} \leq \tilde{T}_{X_n}$, and it follows that

$$\frac{\tilde{T}_{X_n}}{n} + \mu - \left| \frac{T_n}{n} - \mu \right| \geq 0.$$

If we apply Fatou's lemma on this expression, then, together with the almost sure convergence of T_n/n , we find that

$$\begin{aligned} \mu_\tau + \mu &= \mathbb{E} \left[\liminf_{n \rightarrow \infty} \left(\frac{\tilde{T}_{X_n}}{n} + \mu - \left| \frac{T_n}{n} - \mu \right| \right) \right] \\ &\leq \liminf_{n \rightarrow \infty} \mathbb{E} \left[\frac{\tilde{T}_{X_n}}{n} + \mu - \left| \frac{T_n}{n} - \mu \right| \right] \\ &= \mu_\tau + \mu + \liminf_{n \rightarrow \infty} \frac{\mathbb{E}[\tau_{S_0}]}{n} - \limsup_{n \rightarrow \infty} \mathbb{E} \left[\left| \frac{T_n}{n} - \mu \right| \right], \end{aligned}$$

since $\{\tau_{S_k}\}_{k \geq 1}$ is an i.i.d. sequence, by Lemma 2.6. Under given moment condition μ_τ and μ are finite. So, as desired

$$\lim_{n \rightarrow \infty} \mathbb{E} \left[\left| \frac{T_n}{n} - \mu \right| \right] = 0.$$

□

For future reference, we note that

$$\lim_{n \rightarrow \infty} \frac{X_{M_n}}{n} = 1, \quad \text{almost surely.} \quad (3.5)$$

Since $X_{M_n-1} \leq n$ implies $M_n - 1 \leq n$, and

$$\frac{X_{M_n}}{M_n} - \frac{|S_{M_n}|}{M_n} \leq \frac{n}{M_n} \leq \frac{X_{M_n}}{M_n},$$

we have by (3.4) and the strong law of large numbers that

$$\lim_{n \rightarrow \infty} \frac{n}{M_n} = \mu_{|S|}, \quad \text{almost surely.} \quad (3.6)$$

Theorem 3.5. *If $E[\tau_{S_k}^2] < \infty$, then*

$$\frac{T_n - \mu n}{\sigma \sqrt{n}} \xrightarrow{d} \chi, \quad \text{in distribution,}$$

where χ has a standard normal distribution.

Proof. As before, let $M_n = \min\{m \in \mathbb{N} : X_m > n\}$. Since M_n is random, we cannot apply Lemma 3.2 to the sequence $\{\tilde{T}_{X_{M_n}}\}_{n \geq 1}$, in contrast to what we did in the proof of Theorem 1.2. According to (3.6), $\frac{M_n}{n} \rightarrow a := \mu_{|S|}^{-1}$, almost surely. Introduce the quantitatively similar, nonrandom sequence $m_n := \lfloor a(n + n^{3/4}) \rfloor$. We shall see that this is the right form to define m_n , in particular to treat the last term in the expression

$$\begin{aligned} \frac{T_n - \mu n}{\sigma \sqrt{n}} &= \frac{\tilde{T}_{X_{m_n}} - \mu X_{m_n}}{\sigma \sqrt{X_{m_n}}} \sqrt{\frac{X_{m_n}}{n}} \\ &\quad + \frac{T_n - \tilde{T}_{X_{M_n}} - \mu(n - X_{M_n})}{\sigma \sqrt{n}} \\ &\quad + \frac{\tilde{T}_{X_{M_n}} - \tilde{T}_{X_{m_n}} - \mu(X_{M_n} - X_{m_n})}{\sigma \sqrt{n}}. \end{aligned} \quad (3.7)$$

Lemma 3.2 assures weak convergence of the first term of (3.7) to a standard normal, since by the choice of a

$$\frac{X_{m_n}}{n} = \frac{X_{m_n}}{m_n} \frac{m_n}{n} \rightarrow \mu_{|S|} a = 1, \quad \text{almost surely.}$$

From (3.1) and the definition of M_n , we conclude that

$$\frac{|T_n - T_{X_{M_n}} - \mu(n - X_{M_n})|}{\sigma \sqrt{n}} \leq \frac{\tau_{S_{M_n}}}{\sigma \sqrt{n}} + \frac{\mu |S_{M_n}|}{\sigma \sqrt{n}}.$$

Therefore, the second term of (3.7) vanishes because it is bounded by two terms that vanishes almost surely by (3.4), since $E[\tau_{S_k}^2] < \infty$ and $E[|S_k|^2] < \infty$.

The last term of (3.7) will need some further attention. The convergence of the first term of (3.7) motivates the choice of m_n . We did not choose m_n to be $\lfloor an \rfloor$, because which of M_n and $\lfloor an \rfloor$ that is greater varies with n . However, we shall see that

$$P(m_n - M_n \geq \sqrt{n}) \rightarrow 1, \quad \text{as } n \rightarrow \infty. \quad (3.8)$$

For integers a we have that $M_n \leq a$ if and only if $X_a > n$, by definition of M_n . Thus,

$$\begin{aligned} P(m_n - M_n \geq \sqrt{n}) &= P(M_n \leq m_n - \sqrt{n}) \\ &= P(X_{\lfloor m_n - \sqrt{n} \rfloor} > n). \end{aligned}$$

To see that (3.8) holds, note that an application of the classical central limit theorem gives as, $n \rightarrow \infty$

$$\frac{X_{\lfloor m_n - \sqrt{n} \rfloor} - \mu_{|S|}(m_n - \sqrt{n})}{\sigma_{|S|} \sqrt{m_n - \sqrt{n}}} \xrightarrow{d} \text{standard normal}, \quad (3.9)$$

where $\sigma_{|S|}^2 = \text{Var}(|S_k|)$. From this we conclude that (3.8) holds since

$$\frac{n - \mu_{|S|}(m_n - \sqrt{n})}{\sigma_{|S|} \sqrt{m_n - \sqrt{n}}} \leq \frac{(n - \mu_{|S|} \lfloor an \rfloor) - (\lfloor an^{3/4} \rfloor - \sqrt{n})}{\sigma_{|S|} \sqrt{\lfloor a(n + n^{3/4}) \rfloor - \sqrt{n}}},$$

which tend to $-\infty$ as $n \rightarrow \infty$.

We will now attack the last term of (3.7). In order to keep what follows less cumbersome, introduce

$$\begin{aligned} Y_n &:= \frac{\tilde{T}_{X_{m_n}} - \tilde{T}_{X_{M_n}} - \mu(X_{m_n} - X_{M_n})}{\sigma \sqrt{X_{m_n} - X_{M_n}}}, \\ Z_n &:= \sqrt{\frac{X_{m_n} - X_{M_n}}{m_n - M_n}}. \end{aligned}$$

Then the last term of (3.7) may be rewritten as

$$Y_n Z_n \sqrt{\frac{m_n - M_n}{n}}. \quad (3.10)$$

Let $C_n = \{X_{m_n} - X_{M_n} \geq \sqrt{n}\}$ and $D_n = \{m_n - M_n \geq \sqrt{n}\}$. It is clear that $D_n \subseteq C_n$. Since $\{|S_k|\}_{k > M_n}$ and $\{\tau_{S_k}\}_{k > M_n}$ are i.i.d. sequences

independent of M_n , another application of the classical central limit theorem together with (3.8) gives, as $n \rightarrow \infty$

$$\begin{aligned} P(Y_n \leq x) &= P(Y_n \leq x | C_n)P(C_n) + P(Y_n \leq x | C_n^c)P(C_n^c) \\ &\rightarrow \Phi(x). \end{aligned} \quad (3.11)$$

For the same reason, an application of the classical law of large numbers together with (3.8) gives, for any $\epsilon > 0$, as $n \rightarrow \infty$

$$\begin{aligned} P(Z_n \geq \epsilon) &= P(Z_n \geq \epsilon | D_n)P(D_n) + P(Z_n \geq \epsilon | D_n^c)P(D_n^c) \\ &\rightarrow 0. \end{aligned} \quad (3.12)$$

According to (3.6) and the definition of m_n , we also conclude that as $n \rightarrow \infty$

$$\frac{m_n - M_n}{n} = \frac{m_n}{n} - \frac{M_n}{n} \rightarrow a - a = 0, \quad \text{almost surely.} \quad (3.13)$$

Finally, we may combine the convergence in (3.11), (3.12) and (3.13) to conclude that the expression in (3.10) converges to zero in distribution. This observation completes the proof. \square

Theorem 3.6. *If $E[\tau_{S_k}^2] < \infty$, then*

$$\limsup_{n \rightarrow \infty} \frac{|T_n - \mu n|}{\sigma \sqrt{2n \log \log n}} = 1, \quad \text{almost surely.}$$

Proof. As before, let $M_n = \min\{m \in \mathbb{N} : X_m > n\}$. By the triangle inequality we have

$$\begin{aligned} \frac{|T_n - \mu n|}{\sigma \sqrt{2n \log \log n}} &\leq \frac{|\tilde{T}_{X_{M_n}} - \mu X_{M_n}|}{\sigma \sqrt{2X_{M_n} \log \log X_{M_n}}} \left(\frac{X_{M_n} \log \log X_{M_n}}{n \log \log n} \right)^{1/2} \\ &\quad + \frac{|T_n - \tilde{T}_{X_{M_n}}| + \mu |n - X_{M_n}|}{\sigma \sqrt{2n \log \log n}}. \end{aligned}$$

By definition of M_n , we have $|T_n - \tilde{T}_{X_{M_n}}| \leq \tau_{S_{M_n}}$ and $|n - X_{M_n}| \leq |S_{M_n}|$. Since $E[\tau_{S_{M_n}}^2] < \infty$ and $E[|S_{M_n}|^2] < \infty$ by Lemma 2.7, we may use (3.4) to conclude that the latter term of the above expression vanishes as $n \rightarrow \infty$. We now apply Lemma 3.3 and together with (3.5) conclude that

$$\limsup_{n \rightarrow \infty} \frac{|T_n - \mu n|}{\sigma \sqrt{2n \log \log n}} \leq 1.$$

The reverse inequality is shown in the same fashion, using the inverse triangle inequality and the fact that M_n goes through all positive integers. \square

Theorems 1.2, 1.3 and 1.4 follow immediately from Theorems 3.4, 3.5 and 3.6, respectively, since $\mathbb{E}[\tau_e^\alpha] < \infty$ implies $\mathbb{E}[\tau_{S_k}^\alpha] < \infty$ according to Lemma 2.7.

3.2 Proof of Corollary 1.5

Before we go on and prove Corollary 1.5, we will first state and prove another corollary. Theorems 1.2, 1.3 and 1.4 are stated for T_n , the time it takes for an infection to reach level n . We prefer sometimes to consider $\max_{v \in \mathbb{V}_{\mathcal{G}_n}} T(v)$, or even $T(v_n)$, the time it takes to infect a certain vertex v_n at level n . The asymptotic results hold in these cases too, with the same asymptotic constants μ and σ .

Corollary 3.7. *Let $\{v_n\}_{n \geq 1}$ be a sequence of vertices such that $v_n \in \mathbb{V}_{\mathcal{G}_n}$. The statements of Theorems 1.2, 1.3 and 1.4, under respective moment condition, hold for the sequence $\{T(v_n)\}_{n \geq 1}$. In particular, they hold for the sequence $\{\max_{v \in \mathbb{V}_{\mathcal{G}_n}} T(v)\}_{n \geq 1}$.*

Proof. First note that

$$T_n \leq T(v_n) \leq \max_{v \in \mathbb{V}_{\mathcal{G}_n}} T(v) \leq T_n + \sum_{e \in \mathbb{E}_{\mathcal{G}_n}} \tau_e, \quad (3.14)$$

where the last inequality follows since \mathcal{G}_n is connected. Let for the rest of the proof \hat{T} denote either $T(v_n)$ or $\max_{v \in \mathbb{V}_{\mathcal{G}_n}} T(v)$. From (3.14) we have

$$\frac{T_n}{n} \leq \frac{\hat{T}}{n} \leq \frac{T_n}{n} + \frac{1}{n} \sum_{e \in \mathbb{E}_{\mathcal{G}_n}} \tau_e,$$

and

$$\frac{T_n - \mu n}{\sigma \sqrt{n}} \leq \frac{\hat{T} - \mu n}{\sigma \sqrt{n}} \leq \frac{T_n - \mu n}{\sigma \sqrt{n}} + \sum_{e \in \mathbb{E}_{\mathcal{G}_n}} \frac{\tau_e}{\sigma \sqrt{n}}.$$

Under appropriate moment conditions on τ_e , i.e., finite first and second moment respectively, we may use (3.4), and Theorems 1.2 and 1.3 to conclude that

$$\frac{\hat{T}}{n} \rightarrow \mu, \quad \text{almost surely and in } L^1,$$

and

$$\frac{\hat{T} - \mu n}{\sigma\sqrt{n}} \xrightarrow{d} \text{standard normal.}$$

From (3.14) we may observe that

$$|T_n - \mu n| - \sum_{e \in \mathbb{E}_{\mathcal{G}_n}} \tau_e \leq |\hat{T} - \mu n| \leq |T_n - \mu n| + \sum_{e \in \mathbb{E}_{\mathcal{G}_n}} \tau_e.$$

Hence

$$\begin{aligned} \frac{|T_n - \mu n|}{\sigma\sqrt{2n \log \log n}} - \frac{\sum_{e \in \mathbb{E}_{\mathcal{G}_n}} \tau_e}{\sigma\sqrt{2n \log \log n}} &\leq \frac{|\hat{T} - \mu n|}{\sigma\sqrt{2n \log \log n}} \\ &\leq \frac{|T_n - \mu n|}{\sigma\sqrt{2n \log \log n}} + \frac{\sum_{e \in \mathbb{E}_{\mathcal{G}_n}} \tau_e}{\sigma\sqrt{2n \log \log n}}, \end{aligned}$$

and if $E[\tau_e^2] < \infty$, we use (3.4) and Theorem 1.4 to conclude that

$$\limsup_{n \rightarrow \infty} \frac{|\hat{T} - \mu n|}{\sigma\sqrt{2n \log \log n}} = 1, \quad \text{almost surely.}$$

□

We are now ready to prove Corollary 1.5.

Proof of Corollary 1.5. By Theorem 1.4, for all $\epsilon > 0$ there exists an almost surely finite $N = N(\epsilon)$ such that

$$\mu n - (1 + \epsilon)\sigma\sqrt{n \log \log n} \leq T_n, \quad (3.15)$$

for all $n \geq N$. From (3.15) and the definition of \tilde{B}_t , we see that

$$\tilde{B}_t \subseteq [-n, n + 1] \times [0, K]^{d-1}, \quad (3.16)$$

for all $n \geq N$ and $t \geq 0$ such that

$$\mu n - (1 + \epsilon)\sigma\sqrt{2n \log \log n} \geq t. \quad (3.17)$$

In particular, when n is large (3.16) holds for all t such that

$$\mu n(1 - \epsilon) \leq t,$$

and also satisfies (3.17). It follows that for n large enough, the inclusion in (3.16) holds for all t such that

$$n \geq \frac{1}{\mu} \left(t + (1 + \epsilon)\sigma \sqrt{\frac{2t}{\mu(1 - \epsilon)} \log \log \frac{t}{\mu(1 - \epsilon)}} \right).$$

Since $\epsilon > 0$ was arbitrary, we have shown that for all $\epsilon > 0$ there exists an almost surely finite $T = T(\epsilon)$ such that

$$\tilde{B}_t \subseteq \left(t + (1 + \epsilon)\sigma \sqrt{\frac{2t}{\mu} \log \log t} \right) B^*$$

for all $t \geq T$. The proof of the lower inclusion in (1.8) follows in a similar way from Theorem 1.4 applied to $\max_{v \in \mathbb{V}_{g_n}} T(v)$.

It remains to prove the second statement of the theorem. It follows from Theorem 1.4 that almost surely, for all $\epsilon > 0$ and $n \geq 1$ there exists an $N = N(\epsilon) \geq n$ such that either

$$T_N \geq \mu N + (1 - \epsilon)\sigma \sqrt{2N \log \log N} \quad (3.18)$$

or

$$T_N \leq \mu N - (1 - \epsilon)\sigma \sqrt{2N \log \log N}. \quad (3.19)$$

Assume that (3.19) is the case. That is, there is an N such that

$$\tilde{B}_t \not\subseteq [-N, N + 1] \times [0, K]^{d-1}$$

for all t such that

$$\mu N - (1 - \epsilon)\sigma \sqrt{2N \log \log N} \leq t.$$

In particular, there exists a $T = T(\epsilon, N) \leq \mu N$ such that

$$\tilde{B}_T \not\subseteq \left(T + (1 - \epsilon)\sigma \sqrt{\frac{2T}{\mu} \log \log \frac{T}{\mu}} \right) B^*.$$

The case (3.18) is treated by a similar argument. In conclusion, since $\epsilon > 0$ was arbitrary, we have shown that for all $\epsilon > 0$ and $s > 0$, there exists an almost surely finite $t = t(\epsilon)$ such that (1.8) cannot hold. \square

To see that there are passage time distributions such that $E[\tau_e^\alpha] = \infty$ but $E[\tau_{S_k}^\alpha] < \infty$, consider the following example.

Example 3.8. Let the edges of the (K, d) -tube, $d, K \geq 2$ be equipped with passage times with distribution

$$P(\tau_e > x) = x^{-\alpha}, \text{ for } x > 1, \quad (3.20)$$

for some $\alpha > 0$. Note first that $P(\tau_e^\alpha > x) = x^{-1}$, for $x > 1$, so

$$E[\tau_e^\alpha] = \int_1^\infty P(\tau_e^\alpha > x) dx = \int_1^\infty x^{-1} dx = \infty.$$

Consider the passage time τ_{S_k} of the k th patch. We shall see that $E[\tau_{S_k}^\alpha] < \infty$. Because of translation invariance, we may assume that the patch starts at level zero. Let $\mathbf{e}_1 = (1, 0, \dots, 0)$. By a simple coupling argument; consider the process started at the origin, restarted at $j\mathbf{e}_1$, each time the infection reaches $j\mathbf{e}_1$, for each $j \geq 1$, we have

$$\tau_{S_k} \leq \sum_{j=1}^{|S_k|} T((j-1)\mathbf{e}_1, j\mathbf{e}_1).$$

The vertex $j\mathbf{e}_1$ can be reached from $(j-1)\mathbf{e}_1$, either via the horizontal edge e_0 , or via an alternative path using three other edges e_1, e_2, e_3 . Thus, $T((j-1)\mathbf{e}_1, j\mathbf{e}_1) \leq \min(\tau_{e_0}, \tau_{e_1} + \tau_{e_2} + \tau_{e_3})$. As in the proof of Lemma 2.3, let $\Lambda_n = \{|S_k| = (2N+1)n\}$. Arguing as in (2.9) gives, for some finite constant C ,

$$E[T((j-1)\mathbf{e}_1, j\mathbf{e}_1)^\alpha | \Lambda_n] \leq C E[\min(\tau_{e_0}, \tau_{e_1} + \tau_{e_2} + \tau_{e_3})^\alpha],$$

for any $j \geq 1$ and $\alpha > 0$. Together with (2.10), this allows us to estimate the moments of τ_{S_k} via the inequality

$$\begin{aligned} E[\tau_{S_k}^\alpha] &\leq E\left[|S_k|^\alpha \sum_{j=1}^{|S_k|} T((j-1)\mathbf{e}_1, j\mathbf{e}_1)^\alpha\right] \\ &= (2N+1)^\alpha \sum_{n=1}^\infty n^\alpha \sum_{j=1}^{(2N+1)n} E[T((j-1)\mathbf{e}_1, j\mathbf{e}_1)^\alpha | \Lambda_n] P(\Lambda_n) \\ &\leq (2N+1)^{\alpha+1} \sum_{n=1}^\infty n^{\alpha+1} C E[\min(\tau_{e_0}, \tau_{e_1} + \tau_{e_2} + \tau_{e_3})^\alpha] P(\Lambda_n) \\ &= C E[|S_k|^{\alpha+1}] E[\min(\tau_{e_0}, \tau_{e_1} + \tau_{e_2} + \tau_{e_3})^\alpha]. \end{aligned}$$

We know from Lemma 2.3 that $|S_k|$ has finite moments. It remains to see that $\mathbb{E}[\min(\tau_{e_0}, \tau_{e_1} + \tau_{e_2} + \tau_{e_3})^\alpha]$ is finite. Since for $x \geq 1$

$$\begin{aligned} P((\tau_{e_1} + \tau_{e_2} + \tau_{e_3})^\alpha \leq 3^\alpha x) &\geq P(\max(\tau_{e_1}, \tau_{e_2}, \tau_{e_3})^\alpha \leq x) \\ &= P(\tau_{e_1}^\alpha \leq x) P(\tau_{e_2}^\alpha \leq x) P(\tau_{e_3}^\alpha \leq x) \\ &= \left(1 - \frac{1}{x}\right)^3, \end{aligned}$$

we have for $x > 3^\alpha$

$$\begin{aligned} P((\tau_{e_1} + \tau_{e_2} + \tau_{e_3})^\alpha > x) &\leq 1 - \left(1 - \frac{3^\alpha}{x}\right)^3 \\ &= \frac{3^{\alpha+1}}{x} - \frac{3^{2\alpha+1}}{x^2} + \frac{3^{3\alpha}}{x^3}. \end{aligned}$$

We use this to see that

$$\begin{aligned} \mathbb{E}[\min(\tau_{e_0}, \tau_{e_1} + \tau_{e_2} + \tau_{e_3})^\alpha] &= \int_0^\infty P(\min(\tau_{e_0}, \tau_{e_1} + \tau_{e_2} + \tau_{e_3})^\alpha > x) dx \\ &= \int_0^\infty P(\tau_{e_0}^\alpha > x) P((\tau_{e_1} + \tau_{e_2} + \tau_{e_3})^\alpha > x) dx \\ &\leq 1 + \int_1^{3^\alpha} \frac{1}{x} dx + \int_{3^\alpha}^\infty \frac{1}{x} \left(\frac{3^{\alpha+1}}{x} - \frac{3^{2\alpha+1}}{x^2} + \frac{3^{3\alpha}}{x^3} \right) dx \\ &< \infty. \end{aligned}$$

We conclude that $\mathbb{E}[\tau_{S_k}^\alpha] < \infty$ for a passage times distribution as in (3.20). \square

4 Convergence of time constant

In this section we will compare the speed of a first-passage percolation infection on the \mathbb{Z}^d lattice with the speed on the $\mathbb{Z} \times \{-K, -(K-1), \dots, K-1, K\}^{d-1}$ nearest neighbour lattice. The latter graph is nothing else than a $(2K+1, d)$ -tube, and we introduce the notation Ω_K^d for that. Let $\partial\Omega_K^d := \{x \in \mathbb{Z}^d : |x_i| = K \text{ for some } i \geq 2\}$ denote the boundary of Ω_K^d . Consider first-passage percolation on the \mathbb{Z}^d lattice with an initial infection at the origin, and use for this section the convention

$$T(\mathbf{n}) = \inf\{T(\Gamma) : \Gamma \text{ is a path in } \mathbb{Z}^d \text{ from the origin to } \mathbf{n}\},$$

where $\mathbf{n} = (n, 0, \dots, 0)$. For the subgraph Ω_K^d of the \mathbb{Z}^d lattice, define

$$T_n^K := \inf\{T(\Gamma) : \Gamma \text{ is a path in } \Omega_K^d \text{ from the origin to } \mathbf{n}\}.$$

We consider T_n^K and $T(\mathbf{n})$ as defined on the same probability space, e.g., via the natural coupling where we identify the passage times for the set of edges shared by Ω_K^d and the \mathbb{Z}^d lattice.

Note that T_n^K denotes the time it takes for an infection started at the origin to infect the vertex \mathbf{n} of the $(2K+1, d)$ -tube. Recall that we had μ_K to denote the time constant associated with the (K, d) -tube. Then Corollary 3.7 of Theorem 1.2 states that

$$\lim_{n \rightarrow \infty} \frac{T_n^K}{n} = \mu_{2K+1}, \quad \text{almost surely and in } L^1.$$

Proof of Proposition 1.6. Since the sequence μ_1, μ_2, \dots is decreasing and bounded below by $\mu(\mathbf{e}_1)$, it suffices to show that $\lim_{K \rightarrow \infty} \mu_{2K+1} \leq \mu(\mathbf{e}_1)$. We first set out to show that for any $\delta > 0$ there exists an $N = N(\delta)$ such that for $n \geq N$ there is an $L = L(n, \delta)$ such that

$$\mathbb{E}[T_n^K] \leq (\mu(\mathbf{e}_1) + \delta)n, \quad (4.1)$$

for $K \geq L$. Note first that by (1.3) we have for some $N = N(\delta)$

$$\mathbb{E}[T(\mathbf{n})] \leq \left(\mu(\mathbf{e}_1) + \frac{\delta}{2}\right)n, \quad (4.2)$$

if $n \geq N$. Fix n and $\epsilon > 0$ and define the event $A_K = \{T_n^K \leq T(\mathbf{n}) + \epsilon\}$. Since $A_K \subseteq A_{K+1}$ for each K , then as $K \rightarrow \infty$

$$P(A_K) \rightarrow P(\cup_{K=1}^{\infty} A_K) = 1.$$

We may estimate the expected difference between T_n^K and $T(\mathbf{n})$ as follows.

$$\begin{aligned} \mathbb{E}[T_n^K - T(\mathbf{n})] &= \mathbb{E}[(T_n^K - T(\mathbf{n}))\mathbf{1}_{A_K}] + \mathbb{E}[(T_n^K - T(\mathbf{n}))\mathbf{1}_{A_K^c}] \\ &\leq \epsilon + \mathbb{E}[T_n^1 - T(\mathbf{n})] - \mathbb{E}[(T_n^1 - T(\mathbf{n}))\mathbf{1}_{A_K}], \end{aligned}$$

which tends to ϵ as $K \rightarrow \infty$, according to the monotone convergence theorem, and $\mathbb{E}[T_n^1 - T(\mathbf{n})] \leq \mathbb{E}[T_n^1] = n\mathbb{E}[\tau_e] < \infty$. In particular, for any fixed n , we have

$$\mathbb{E}[T_n^K - T(\mathbf{n})] \leq \delta n/2 \quad (4.3)$$

for K larger than some $L = L(n, \delta)$. Combining (4.2) and (4.3), we conclude that for $n \geq N$

$$\begin{aligned} \mathbb{E} [T_n^K] &= \mathbb{E} [T(\mathbf{n})] + \mathbb{E} [T_n^K - T(\mathbf{n})] \\ &\leq \left(\mu(\mathbf{e}_1) + \frac{\delta}{2} \right) n + \delta n/2 \\ &= (\mu(\mathbf{e}_1) + \delta) n, \end{aligned}$$

for all $K \geq L$, and (4.1) is established.

If we let $T^K((j-1)\mathbf{n}, j\mathbf{n})$ denotes the time it takes for an infection started at $(j-1)\mathbf{n}$ to infect the vertex $j\mathbf{n}$, then $T^K((j-1)\mathbf{n}, j\mathbf{n})$ are not independent, but distributed as T_n^K for each $j \geq 1$. By a simple coupling argument; consider the process started at the origin, restarted at $j\mathbf{n}$, each time the infection reaches $j\mathbf{n}$, for each $j \geq 1$, we have

$$T_{mn}^K \leq \sum_{j=1}^m T^K((j-1)\mathbf{n}, j\mathbf{n}).$$

Taking expectation we use (4.1) to see that

$$\mathbb{E} [T_{mn}^K] \leq m \mathbb{E} [T_n^K] \leq (\mu(\mathbf{e}_1) + \delta) mn,$$

if n and K are large. As $\mathbb{E} [T_{mn}^K] / mn \rightarrow \mu_{2K+1}$, and $\delta > 0$ is arbitrary, the proof is complete. \square

5 Exact coupling and a 0–1 law

The aim for this section is to couple first-passage percolation infections with different initial configurations, i.e., different initially infected components, in such a way that both infections will eventually coincide. As an application of this, we shall prove a 0–1 law. The method of proof will once again make use of the regenerative behaviour explored in Section 2.

First we must state what we mean by a coupling. A *coupling* of two random variables $X \sim P$ and $Y \sim P'$ on a measurable space (E, \mathcal{E}) , is a joint distribution \hat{P} of (X, Y) , i.e., a measure on (E^2, \mathcal{E}^2) , such that its marginal distributions coincide with P and P' , respectively. When we couple two time-dependent random elements $\{X_t\}_{t \geq 0}$ and $\{Y_t\}_{t \geq 0}$, we say that the coupling is *exact* if with probability one there exists a $T_c < \infty$ such that $X_t = Y_t$, for all $t \geq T_c$.

We will present an exact coupling of the sets of infected vertices B_t and B'_t of two first-passage percolation processes with different initial configurations. Recall that we let $P_\tau(\cdot)$ denote the distribution of τ_e , and let \mathcal{R}_+ denote the Borel σ -algebra on $[0, \infty)$. Then $\{\tau_e\}_{e \in \mathbb{E}}$ and $\{\tau'_e\}_{e \in \mathbb{E}}$ are random elements on the product space $([0, \infty)^\mathbb{E}, \mathcal{R}_+^\mathbb{E})$, each with distribution given by the product measure $P_\tau^\mathbb{E}$. Let \mathbb{E}_n denote the set of edges between level $-n$ and n , but not including edges between two vertices at level $-n$ and n . In the same manner \mathbb{E}_n^c denotes the set of edges at and before level $-n$, as well as at level n and beyond.

We shall prove the following result which is slightly stronger than Proposition 1.8.

Proposition 5.1 (Coupling, continuous times). *Let I and I' be finite subsets of the set of vertices \mathbb{V} of an essentially 1-dimensional periodic graph \mathcal{G} . Assume that the passage time distribution P_τ has an absolutely continuous component (with respect to Lebesgue measure). For any $m \geq 0$, there exists a coupling of $\{\tau_e\}_{e \in \mathbb{E}_m^c}$ and $\{\tau'_e\}_{e \in \mathbb{E}_m^c}$ such that if $\{\tau_e\}_{e \in \mathbb{E}_m}$ and $\{\tau'_e\}_{e \in \mathbb{E}_m}$ each have distribution $P_\tau^{\mathbb{E}_m}$, then the marginal distributions of $\{\tau_e\}_{e \in \mathbb{E}}$ and $\{\tau'_e\}_{e \in \mathbb{E}}$ are given by the product measure $P_\tau^\mathbb{E}$, and such that if first-passage percolation is performed with $(I, \{\tau_e\}_{e \in \mathbb{E}})$ and $(I', \{\tau'_e\}_{e \in \mathbb{E}})$, respectively, then with probability one there exists an $N_c < \infty$ and a $T_c < \infty$, such that*

$$T(v_n) = T'(v_n) \quad \text{and} \quad B_t = B'_t, \quad (5.1)$$

for all $v_n \in \mathbb{V}_{\mathcal{G}_n}$ for $n \geq N_c$, and for all $t \geq T_c$.

When the passage time distribution P_τ is discrete, i.e., $P_\tau(\Lambda) = 1$ for the set of point masses

$$\Lambda := \{t_j \in [0, \infty) : P_\tau(t_j) > 0\},$$

the statement of Proposition 5.1 is not true in general. More precisely, there are essentially 1-dimensional periodic graphs on which no exact coupling is possible (cf. Remark 5.6). In the discrete case, we will therefore restrict our attention to the case of (K, d) -tubes.

Proposition 5.2 (Coupling, discrete times). *Let I and I' be finite subsets of the set of vertices \mathbb{V} of the (K, d) -tube, for $K, d \geq 2$. Assume that the passage time distribution P_τ is such that $P_\tau(\Lambda) = 1$ for the set of point masses Λ and that either of the following hold:*

a) there are $t_j \in \Lambda$ and integers n_j for j in some finite set of indices J^* , such that $\sum_{j \in J^*} n_j$ is odd, and

$$\sum_{j \in J^*} n_j t_j = 0.$$

b) $\text{dist}(\mathbf{x}, \mathbf{y})$ is even, for all $\mathbf{x} \in I$, $\mathbf{y} \in I'$.

For any $m \geq 0$, there exists a coupling of $\{\tau_e\}_{e \in \mathbb{E}_m^c}$ and $\{\tau'_e\}_{e \in \mathbb{E}_m^c}$ such that if $\{\tau_e\}_{e \in \mathbb{E}_m}$ and $\{\tau'_e\}_{e \in \mathbb{E}_m}$ each have distribution $P_\tau^{\mathbb{E}_m}$, then the marginal distributions of $\{\tau_e\}_{e \in \mathbb{E}}$ and $\{\tau'_e\}_{e \in \mathbb{E}}$ are given by the product measure $P_\tau^{\mathbb{E}}$, and such that if first-passage percolation is performed with $(I, \{\tau_e\}_{e \in \mathbb{E}})$ and $(I', \{\tau'_e\}_{e \in \mathbb{E}})$, respectively, then with probability one there exists an $N_c < \infty$ and a $T_c < \infty$, such that

$$T(v_n) = T'(v_n) \quad \text{and} \quad B_t = B'_t, \quad (5.2)$$

for all $v_n \in \mathbb{V}_{\mathcal{G}_n}$ for $n \geq N_c$, and for all $t \geq T_c$.

Before we construct the couplings, we focus on the promised 0–1 law that follows from Proposition 5.1 and 5.2. For this we will use Lévy's 0–1 law. It states that for σ -algebras \mathcal{F}_t such that $\mathcal{F}_t \uparrow \mathcal{F}_\infty$ as $t \rightarrow \infty$, if $A \in \mathcal{F}_\infty$, then $P(A|\mathcal{F}_t) \rightarrow 1_A$, almost surely. A proof for the discrete case can be found in e.g. Durrett (2005, Theorem 4.5.8). The following version follows from the martingale convergence theorem.

Recall that we defined the σ -algebra $\mathcal{T}_t = \sigma(\{B_s\}_{s \geq t})$, and define $\mathcal{F}_t := \sigma(\{B_s\}_{0 \leq s \leq t})$, where as before B_s is the set of infected vertices at time s . We may think of \mathcal{T}_t as the σ -algebra of events $A \in \cup_{t \geq 0} \mathcal{F}_t$ that do not depend on the times at which vertices were infected before time t . The 0–1 law we shall prove deals with the tail σ -algebra $\mathcal{T} = \cap_{t \geq 0} \mathcal{T}_t$.

Theorem 5.3 (0–1 law). *Consider first-passage percolation performed under the assumptions of either Proposition 5.1 or 5.2. Then $P(A) \in \{0, 1\}$, for any event $A \in \mathcal{T}$.*

Proof of Theorem 5.3 from Propositions 5.1 and 5.2. Consider two infections with the respective sets of passage times $\{\tau_e\}_{e \in \mathbb{E}}$ and $\{\tau'_e\}_{e \in \mathbb{E}}$. For $t \geq 0$, let \mathcal{F}_t and \mathcal{F}'_t be σ -fields generated by their respective realizations up to time t . Let

$$\nu_t = \max\{n \geq 0 : (B_t \cup B'_t) \cap (\mathbb{V}_{\mathcal{G}_n} \cup \mathbb{V}_{\mathcal{G}_{-n}}) \neq \emptyset\}$$

denote the furthest level (in positive or negative direction) infected at time t . For any fixed $t \geq 0$, by Propositions 5.1 and 5.2, there is a coupling of $\{\tau_e\}_{e \in \mathbb{E}_{v_t+1}^c}$ and $\{\tau'_e\}_{e \in \mathbb{E}_{v_t+1}}$, such that there exists an almost surely finite time T_c , such that $B_s = B'_s$ for all $s \geq T_c$. Since $A \in \mathcal{T}_{T_c}$, the outcome of A only depends on B_s for $s \geq T_c$. In particular it has to hold that

$$P(A|\mathcal{F}_t) = P(A|\mathcal{F}'_t).$$

Thus, $P(A|\mathcal{F}_t)$ is nonrandom and equals $P(A)$, for all $t \geq 0$. But, according to Lévy's 0–1 law, $P(A|\mathcal{F}_t) \rightarrow 1_A$ as $t \rightarrow \infty$, almost surely. Hence, $P(A) = 1_A$ almost surely, and therefore $P(A)$ equals either 0 or 1. \square

It remains only to prove Propositions 5.1 and 5.2.

5.1 Exact coupling of time-delayed infections on \mathbb{Z}

Before proving Proposition 5.1 and 5.2, we shall first prove a lemma where we consider two infections on \mathbb{Z} . For first-passage percolation on \mathbb{Z} , T_n simply takes the form $T_n = \sum_{k=1}^n \tau_k$. If we let the latter infection be delayed for some time T_{delay} , i.e., started at time T_{delay} instead of time zero, then $T'_n = T_{\text{delay}} + \sum_{k=1}^n \tau'_k$. We will construct a coupling of the passage times such that $T_n = T'_n$ for large n . The precise statement is as follows.

Lemma 5.4. *Let T_{delay} be any nonnegative constant, and assume that either of the following hold:*

- a) P_τ has an absolutely continuous component (with respect to Lebesgue measure).
- b) P_τ is such that for some finite index set J , there are nonnegative integers n_j and n'_j , such that $\sum_{j \in J} n_j = \sum_{j \in J} n'_j$, and for atoms $t_j \in \Lambda$ of P_τ

$$\sum_{j \in J} n_j t_j = \sum_{j \in J} n'_j t_j + T_{\text{delay}}. \quad (5.3)$$

There exists a coupling of $\{\tau_k\}_{k \geq 1}$ and $\{\tau'_k\}_{k \geq 1}$ such that their marginal distributions are that of i.i.d. random variables with distribution P_τ , and such that

$$\sum_{k=1}^n \tau_k = T_{\text{delay}} + \sum_{k=1}^n \tau'_k, \quad (5.4)$$

for large n .

The proof of this lemma heavily exploits ideas similar to those found in Lindvall (1992, Chapter II.12–17) for the discrete case, and Lindvall (1992, Chapter III.5) for the continuous case.

Proof of case a). Let $[a, b]$ be an interval on which P_τ has density $\geq c$, for some $c > 0$. Choose $\delta \leq \frac{b-a}{2}$ such that $m\delta = T_{\text{delay}}$, for some $m \in \mathbb{N}$. To be specific, choose

$$\delta = \max \left\{ d \geq 0 : d \leq \frac{b-a}{2}, d = \frac{T_{\text{delay}}}{m} \text{ for some } m \in \mathbb{N} \right\}.$$

Couple $\{\tau_k\}_{k \geq 1}$ and $\{\tau'_k\}_{k \geq 1}$ in the following way. With probability $1 - c2\delta$ we choose $\tau_k = \tau'_k$, drawn from the distribution

$$\tilde{P}_\tau(\cdot) := (P_\tau(\cdot) - c\mu(\cdot \cap [a, a + 2\delta])) / (1 - c2\delta),$$

where μ denotes Lebesgue measure. With the remaining probability $c2\delta$, draw τ_k uniformly on the interval $[a, a + 2\delta]$, and choose τ'_k as

$$\tau'_k = \begin{cases} \tau_k + \delta, & \text{if } \tau_k \leq a + \delta \\ \tau_k - \delta, & \text{if } \tau_k > a + \delta \end{cases}.$$

That τ'_k also is uniformly distributed on $[a, a + 2\delta]$ is immediate. Thus, it is easy to see that the marginal distribution of both τ_k and τ'_k is P_τ , and this is indeed a coupling of the two infections.

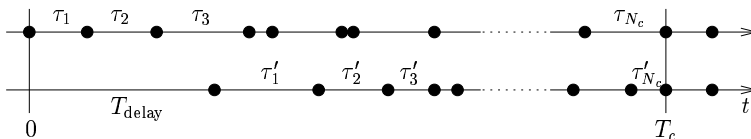


Figure 5: The dots represent the times at which the respective infection spreads. In this realization $\tau_1 = \tau'_1 - \delta$, $\tau_2 = \tau'_2$ and $\tau_3 = \tau'_3 + \delta$. The coupling is constructed such that after some time T_c , both infections reach some level N_c simultaneously.

The coupling is such that each time τ_k and τ'_k are chosen differently, the difference $\{D_n\}_{n \geq 1}$, where $D_n := T_{\text{delay}} + \sum_{k=1}^n (\tau'_k - \tau_k)$ will jump $\pm \delta$. Since $T_{\text{delay}} = m\delta$, for some integer m , D_n constitutes a simple

random walk on $\delta\mathbb{Z}$. Let N_c denote the first n for which D_n hits zero. From this moment on, τ_k and τ'_k are chosen identically, and (5.4) holds for $n \geq N_c$. That the coupling is successful is easily seen, since

$$\begin{aligned} P(N_c < \infty) &= P(\exists n : D_n = 0) \\ &\geq P(\exists n : D_n = 0 | \tau_k \neq \tau'_k \text{ i.o.}) P(\tau_k \neq \tau'_k \text{ i.o.}) \\ &= 1, \end{aligned}$$

where i.o. abbreviates 'infinitely often'. The last equality follows from the recurrence of a 1-dimensional simple random walk, and Borel-Cantelli's second lemma.

Proof of case b). By assumption, for some set $\{t_j\}_{j \in J} \subseteq \Lambda$ of atoms for the distribution P_τ , there are nonnegative integers n_j and n'_j such that $\sum_{j \in J} n_j = \sum_{j \in J} n'_j$ and (5.3) holds. Since T_{delay} satisfies (5.3), it suffices to make sure that for some level k , each infection will have passed n_j , respectively n'_j , edges with passage time t_j , for all $j \in J$. In addition, the sum of the passage times of the remaining edges will have to be equal. Then $T_k = T'_k$.

It is easily seen from (5.3) that we may assume that the set J is chosen such that for each $j \in J$, exactly one of the integers n_j and n'_j is positive. We introduce integer valued random variables

$$\begin{aligned} X_j^n &= \#\{k \leq n : \tau_k = t_j\} - n_j, \\ Y_j^n &= \#\{k \leq n : \tau'_k = t_j\} - n'_j. \end{aligned}$$

Define $Z_j^n = X_j^n - Y_j^n$. From (5.3) we conclude that (5.4) holds, if $Z_j^n = 0$ for all $j \in J$ and $\tau_k = \tau'_k$ for all $k \leq n$ such that $\tau_k \notin \{t_j\}_{j \in J}$ or $\tau'_k \notin \{t_j\}_{j \in J}$.

Let $J_n = \{j \in J : Z_j^n \neq 0\}$, and let p_j denote the probability $P_\tau(t_j)$, and $q_n = \sum_{j \in J_n} p_j$. In particular, $J_0 = J$. Couple $\{\tau_k\}_{k \geq 1}$ and $\{\tau'_k\}_{k \geq 1}$ by choosing τ_k and τ'_k identically from the distribution

$$\tilde{P}_\tau(\cdot) := \frac{1}{1 - q_{k-1}} \left(P_\tau(\cdot) - \sum_{j \in J_{k-1}} p_j \mathbf{1}_{\{t_j\}}(\cdot) \right)$$

with probability $1 - q_{k-1}$. With remaining probability q_{k-1} we choose τ_k and τ'_k independently to equal t_j with probability $\frac{p_j}{q_{k-1}}$, for each $j \in J_{k-1}$.

The marginal distribution of τ_k and τ'_k is seen to be P_τ , whence this is a coupling of $\{\tau_k\}_{k \geq 1}$ and $\{\tau'_k\}_{k \geq 1}$.

Note that $\tau_k = \tau'_k$ for all k such that $\tau_k \notin \{t_j\}_{j \in J}$ and $\tau'_k \notin \{t_j\}_{j \in J}$. For each fixed $j \in J$, $\{Z_j^n\}_{n \geq 0}$ will, as n increases, jump ± 1 with equal probability. Hence, for fixed j $\{Z_j^n\}_{n \geq 0}$ constitutes a simple random walk on \mathbb{Z} . Let n^* be the first n such that $Z_j^n = 0$. Then, j is contained in J_k for $k < n$, but will not be contained in J_k for $k \geq n$.

By assumption we have that

$$\sum_{j \in J} Z_j^0 = \sum_{j \in J} (n_j - n'_j) = 0.$$

Moreover, the sum of Z_j^n is constant for all n , i.e.,

$$\sum_{j \in J} Z_j^n = \sum_{j \in J} Z_j^0 = 0.$$

It follows that it is not possible for J_n to end up with only one index j . From this conclusion, Borel-Cantelli's second lemma and the recurrence of 1-dimensional simple random walks, we may further conclude that $P(\exists n : Z_j^n = 0) = 1$ for each $j \in J$. Let $N_c = \min\{n \geq 0 : J_n = \emptyset\}$. At this moment $Z_j^n = 0$ for all $j \in J$, and (5.4) holds for every $n \geq N_c$. The coupling is complete. The coupling is successful since

$$P(N_c < \infty) = P\left(\bigcap_{j \in J} \{\exists n : Z_j^n = 0\}\right) = 1.$$

□

5.2 Exact coupling of two infections

First, we need some notation. Recall from Section 2.3 that E_n denotes the set of edges between level n and $n + 2N$, including edges at level n and level $n + 2N$. In (2.11) we defined $\hat{E}_n = \gamma_n \cup \mathbb{V}_{\mathcal{G}_n} \cup \mathbb{V}_{\mathcal{G}_{n+2N}}$, where γ_n is a path of shortest distance between $\mathbb{V}_{\mathcal{G}_n}$ and $\mathbb{V}_{\mathcal{G}_{n+2N}}$. Denote by \hat{e}_{n+N} the edge in γ_n with endpoints \hat{v}_{n+N} and u , where \hat{v}_{n+N} is the vertex in $\mathbb{V}_{\mathcal{G}_{n+N}}$ first reached by γ_k , and u reached after. Let constants t' , t'' and N be chosen in accordance with Lemma 2.5, and define the event

$$A_n^* = \left\{ \tau_e \leq t', \forall e \in \hat{E}_n \setminus \{\hat{e}_{n+N}\} \right\} \cap \left\{ \tau_e \geq t'', \forall e \in E_n \setminus \hat{E}_n \right\}.$$

Note that $A_n = A_n^* \cap \{\tau_{\hat{e}_{n+N}} \leq t'\}$ for A_n as defined in (2.12).

We will next prove Proposition 5.1, which is a slightly stronger version of Proposition 1.8. The idea is to make sure that both infections will have to pass certain edges, and that they from each such edge will be “responsible” for the infection of all vertices at higher levels. We then alter the passage times for these particular edges in order for the difference $T(v_n) - T'(v_n)$ to vary. We will accomplish this by choosing the passage times for both infections equal for all but the edges \hat{e}_{l_k+N} , for a certain sequence $\{l_k\}_{k \geq 0}$. For k such that $A_{l_k}^*$ occurs, we either choose both passage times for the edge \hat{e}_{l_k+N} equal, or both passage times at most t' . This generates a sequence of edges with passage times for which we invoke Lemma 5.4. Thus, we makes sure that with probability one, from some level on, the difference $T(v_n) - T'(v_n)$ is zero.

Proof of Proposition 5.1. By assumption, P_τ has an absolutely continuous component, so suppose that $[a, b]$ is an interval on which P_τ has density $\geq c > 0$. Lemma 2.5 allows us to make sure that $t' > a$. We may further assume that $I \cup I'$ contains no vertex beyond level m . Let $l_k := m + k(2N + 1)$ for integers $k \geq 0$. Couple $\{\tau_e\}_{e \in \mathbb{E}_m^c}$ and $\{\tau'_e\}_{e \in \mathbb{E}_m^c}$ by choosing $\tau_e = \tau'_e$ with distribution P_τ , independently for all e at level m or beyond such that $e \neq \hat{e}_{l_k+N}$ for $k \geq 0$. Independently for $k \geq 0$, let

$$(\xi_k, \xi'_k) = \begin{cases} (\rho_k, \rho'_k), & \text{with probability } P_\tau([0, t']) \\ (\eta_k, \eta_k), & \text{with probability } 1 - P_\tau([0, t']), \end{cases}$$

where ρ_k and ρ'_k have marginal distribution $P_\tau(\cdot | \tau \leq t')$, and η_k has distribution $P_\tau(\cdot | \tau > t')$. For the set of edges $\{\hat{e}_{l_k+N}, \text{ for } k \geq 0\}$, we choose the pair

$$\left(\tau_{\hat{e}_{l_k+N}}, \tau'_{\hat{e}_{l_k+N}} \right) = \begin{cases} (\xi_k, \xi'_k), & \text{if } A_{l_k}^* \text{ occurs} \\ (\tau_k, \tau_k), & \text{otherwise,} \end{cases}$$

where τ_k is distributed according to P_τ , independently for all k . One realizes from the coupling that the marginal distributions of both τ_e and τ'_e is P_τ , for every edge e .

Note that the only edges for which τ_e and τ'_e may differ, are the edges \hat{e}_{l_k+N} for $k \geq 0$ such that A_{l_k} occurs. Let κ_j denote the index k for which A_{l_k} occurs for the j th time. That

$$\left(\tau_{\hat{e}_{l_{\kappa_j}+N}}, \tau'_{\hat{e}_{l_{\kappa_j}+N}} \right) = (\rho_{\kappa_j}, \rho'_{\kappa_j}) \quad (5.5)$$

is equivalent to that A_{l_k} occurs. Since $P(A_{l_k}) > 0$, we will have an infinite sequence $\{\kappa_j\}_{j \geq 1}$ such that (5.5) holds. We now claim that the proposition will follow if we apply Lemma 5.4 to the sequences $\{\rho_{\kappa_j}\}_{j \geq 1}$ and $\{\rho'_{\kappa_j}\}_{j \geq 1}$, with distribution $P_\tau(\cdot | \tau \leq t')$, and

$$T_{\text{delay}} = |T(\hat{v}_{l_{\kappa_1}+N}) - T'(\hat{v}_{l_{\kappa_1}+N})|.$$

To see this, we use Lemma 2.5. Given A_{l_k} , the path along which any vertex at level $l_k + 2N$ or beyond is infected has to pass the edge \hat{e}_{l_k+N} . By the coupling $\tau_e = \tau'_e$ for all e at level l_{κ_1} or beyond such that $e \neq \hat{e}_{l_{\kappa_j}+N}$ for $j \geq 1$. Moreover, $\tau_e = \rho \leq t'$ and $\tau'_e = \rho' \leq t'$ for $e \in \{\hat{e}_{l_{\kappa_j}+N}, \text{ for } j \geq 1\}$. It follows that each vertex at level $l_{\kappa_1} + 2N + 1$ and beyond, will be reached in the same order. Since P_τ is absolutely continuous on $[a, b]$ and $t' > a$, $P_\tau(\cdot | \tau \leq t')$ is absolutely continuous on $[a, t']$. Case a) of Lemma 5.4 is therefore fulfilled. Coupling $\{\rho_{\kappa_j}\}_{j \geq 1}$ and $\{\rho'_{\kappa_j}\}_{j \geq 1}$ according to the lemma we will have with probability one that, from some level on, both infections will reach each vertex at the same time, i.e.,

$$T(v_n) = T'(v_n) \tag{5.6}$$

for any $v_n \in \mathbb{V}_{\mathcal{G}_n}$, for n sufficiently large.

The infections may in the same manner be coupled along the negative coordinate axis. Doing this, then there is $N_c \in \mathbb{N}$ such that (5.6) holds for $|n| \geq N_c$. In almost surely finite time, each vertex at level n , for $|n| \leq N_c$, will be infected. Hence, we conclude that for some almost surely finite time T_c ,

$$B_t = B'_t$$

for each $t \geq T_c$. □

In preparation for the proof of Proposition 5.2, we restrict our attention to (K, d) -tubes. Let F_n denote the set of edges between level n and $n + 2N + 4\nu$, for integers

$$N \geq \frac{(d-1)(K-1)t'}{t''-t'} \quad \text{and} \quad \nu \geq \frac{t'}{t''-t'}.$$

Denote by $e_{u,n}$ the edge between $(n + N + \nu, 0, 0, \dots, 0)$ and $(n + N + \nu, 1, 0, \dots, 0)$, and by $e_{d,n}$ the edge between $(n + N + 3\nu, 1, 0, \dots, 0)$ and $(n + N + 3\nu, 0, 0, \dots, 0)$. Let \hat{F}_n be defined as

$$\begin{aligned} \hat{F}_n := & \{e \in F_n : e \text{ is at height zero, level } n, \text{ or at level } n + 2N + 4\nu\} \\ & \cup \{e_{u,n}, e_{d,n}\}. \end{aligned}$$

Let \hat{H}_n denote the set of edges at level n , level $n + 2N + 4\nu$, and at height zero between level n and $n + N + \nu$, the edge $e_{u,n}$, the edges along the path of shortest length between the vertices $(n + N + \nu, 1, 0, \dots, 0)$ and $(n + N + 3\nu, 1, 0, \dots, 0)$, the edge $e_{d,n}$, and the edges at height zero between level $n + N + 3\nu$ and $n + 2N + 4\nu$ (see Figure 6). For constants t' and t'' such that $m_\tau < t' < t'' < M_\tau$, define the events

$$C_n := \left\{ \tau_e \leq t', \forall e \in \hat{F}_n \right\} \cap \left\{ \tau_e \geq t'', \forall e \in F_n \setminus \hat{F}_n \right\},$$

$$D_n := \left\{ \tau_e \leq t', \forall e \in \hat{H}_n \right\} \cap \left\{ \tau_e \geq t'', \forall e \in F_n \setminus \hat{H}_n \right\}.$$

Trivially $P(C_n) = P(D_n) > 0$, since \hat{F}_n and \hat{H}_n contain equally many edges.

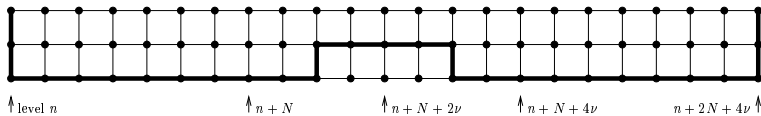


Figure 6: The $(3, 2)$ -tube between level n and $n + 2N + 4\nu$. If D_n occurs, the infection will advance along the thick edges.

Let $\rho_0 = \max\{n \geq 0 : \exists v \in I \text{ at level } n\}$. The following lemma says that given the event C_n (or D_n) occurs, the infection will in order to reach level $n + 2N + 4\nu$ from level n use $2N + 4\nu$ (or $2N + 4\nu + 2$) edges.

Lemma 5.5. *Let t' and t'' be constants such that $m_\tau < t' < t'' < M_\tau$, and assume that $n \geq \rho_0$. Given C_n (respectively D_n), then $T(\hat{v}_{n+N}) = T_{n+N}$ and*

$$T(v) = T(\hat{v}_{n+N}) + T(\hat{v}_{n+N}, \hat{v}_{n+N+4\nu}) + T(\hat{v}_{n+N+4\nu}, v),$$

for each v at level $n + 2N + 4\nu$. Moreover $T(\hat{v}_{n+N}, \hat{v}_{n+N+4\nu})$ is minimized by the path using edges of \hat{F}_n (respectively \hat{H}_n) only.

Proof. Arguing as in the proof of Lemma 2.1, we see that \hat{v}_{n+N} is the first vertex reached at level $n + N$. Moreover, if $\hat{v}_{n+N+4\nu}$ is the first vertex reached at level $n + N + 4\nu$, then each vertex v at level $n + 2N + 4\nu$ is infected via this vertex.

If C_n occurs, the path of shortest length between level $n + N$ and $n + N + 4\nu$ is the one using edges of \hat{F}_n only. It follows that it is also the path of shortest passage time.

Given that D_n occurs, since by the choice of ν ,

$$(\nu + 1)t' \leq \nu t'',$$

then $(n + N + \nu, 1, 0, \dots, 0)$ and $\hat{v}_{n+N+3\nu}$ are both infected via \hat{v}_{n+N} , using edges of \hat{H}_n only. It follows that the path of shortest passage time between level $n + N$ and $n + N + 4\nu$ is the one between \hat{v}_{n+N} and $\hat{v}_{n+N+4\nu}$ using edges of \hat{H}_n only. \square

The events C_n and D_n were defined with respect to passage times from the sequence $\{\tau_e\}_{e \in \mathbb{E}}$. Let C'_n and D'_n denote the analogous events with respect to the sequence $\{\tau'_e\}_{e \in \mathbb{E}}$.

The coupling of Proposition 5.2 will be constructed in two steps. The second part is similar to the coupling in the proof of Proposition 5.1. The first part is needed to make sure that case b) of Lemma 5.4 will be satisfied. Before we give the somewhat technical proof, we will give the idea of the first step. Assign identical passage times for both infections, except for some edges in F_n , for certain n , which we flip. Doing so we make the events C_n occur simultaneously as D'_n , and D_n occur simultaneously as C'_n , for these n . When they happen, the difference in length of the minimizing paths in $T_{n+2N+4\nu}$ and $T'_{n+2N+4\nu}$ will either increase or decrease by 2. Thus, the difference constitutes a random walk, and end the first step when it hits either 0 or the odd number $\alpha = \sum_{j \in J^*} n_j$, for $\{n_j\}_{j \in J^*}$ as in case a) of Proposition 5.2. We will see that case b) of Lemma 5.4 is then satisfied for $T_{\text{delay}} = |T'(v) - T(v)|$.

Proof of Proposition 5.2. We may assume that $I \cup I'$ contains no vertex beyond level m . Set $l_k := m + k(2N + 4\nu + 1)$ for $k \geq 0$. For $j = 1, 2, \dots, 2\nu$, let $f_{k,j}$ (respectively $h_{k,j}$) denote the edge in \hat{F}_{l_k} (respectively \hat{H}_{l_k}) between level $l_k + N + \nu + j - 1$ and $l_k + N + \nu + j$.

Couple $\{\tau_e\}_{e \in \mathbb{E}_m^c}$ and $\{\tau'_e\}_{e \in \mathbb{E}_m^c}$ in the following way. For one k at the time, choose $\tau_e = \tau'_e$ with distribution P_τ , independently for every edges e between level l_k and l_{k+1} , not at level l_{k+1} or among $\{f_{k,j}, h_{k,j} : j = 1, 2, \dots, 2\nu\}$. For $j = 1, 2, \dots, 2\nu$, choose $\tau_{f_{k,j}}$ and $\tau_{h_{k,j}}$ independently with distribution P_τ , and set

$$\left(\tau'_{f_{k,j}}, \tau'_{h_{k,j}} \right) = \begin{cases} (\tau_{h_{k,j}}, \tau_{f_{k,j}}), & \text{if } C_{l_k} \cup D_{l_k} \\ (\tau_{f_{k,j}}, \tau_{h_{k,j}}), & \text{otherwise.} \end{cases}$$

Trivially τ_e has distribution P_τ , and it is easy to see that the marginal distribution of τ'_e for each e also is P_τ . Note that the coupling is such that C'_{l_k} occurs if and only if D_{l_k} occurs. As well, D'_{l_k} occurs if and only if C_{l_k} does.

Let z_k , for $k \geq 1$, denote the length of the path of shortest passage time from I to level $l_k + 2N + 4\nu$, with respect to $\{\tau_e\}_{e \in \mathbb{E}}$. When several paths are possible, choose one. Similarly, let z'_k denote the length of the path of shortest passage time from I' to level $l_k + 2N + 4\nu$, with respect to $\{\tau'_e\}_{e \in \mathbb{E}}$. Set $\zeta_k := z_k - z'_k$.

With help from Lemma 5.5, we draw the following conclusions. Given $C_{l_k} \cup D_{l_k}$, then \hat{v}_{l_k+N} is “responsible” for the infection of each vertex at level l_{k+1} and beyond. For each k such that C_{l_k} (and therefore also D'_{l_k}) occurs, ζ_k takes two steps down. When D_{l_k} (and therefore also C'_{l_k}) occurs, ζ_k takes two steps up. In every other case, $\zeta_k = \zeta_{k-1}$. Thus, $\{\zeta_k\}_{k \geq 1}$ constitutes a simple random walk on either $2\mathbb{Z}$ or $2\mathbb{Z} + 1$, depending on the value of ζ_1 . Such walk is recurrent and will with probability one, reach either zero or the odd number $\alpha := \sum_{j \in J^*} n_j$, respectively. Let κ denote the first k for which this happens. In the same way the infections may be coupled along the negative coordinate axis.

The first part of the coupling is done, and before we continue with the second part, we shall verify that assumption b) of Lemma 5.4 is satisfied. We may assume that $T_{l_\kappa+2N+4\nu} \leq T'_{l_\kappa+2N+4\nu}$. Set

$$T_{\text{delay}} = T'_{l_\kappa+2N+4\nu} - T_{l_\kappa+2N+4\nu}.$$

Given $\{(\tau_e, \tau'_e)\}_{e \in \mathbb{E}_{\kappa+2N+4\nu}}$, we may represent the passage time for each infection as

$$T_{l_\kappa+2N+4\nu} = \sum_{j \in J} m_j t_j \quad \text{and} \quad T'_{l_\kappa+2N+4\nu} = \sum_{j \in J'} m'_j t_j,$$

for index sets J and J' , $t_j \in \Lambda$, and positive integers m_j and m'_j that indicate the number of edges e in the minimizing path to level $l_\kappa + 2N + 4\nu$ such that $\tau_e = t_j$ and $\tau'_e = t_j$, respectively.

If $\zeta_\kappa = 0$, then $\sum_{j \in J} m_j = \sum_{j \in J'} m'_j$, and assumption b) of Lemma 5.4 is directly satisfied, since

$$T_{\text{delay}} + T_{l_\kappa+2N+4\nu} = T'_{l_\kappa+2N+4\nu}.$$

Note that this will be the case if $\text{dist}(\mathbf{x}, \mathbf{y})$ is even, for all $\mathbf{x} \in I$, $\mathbf{y} \in I'$, since then $\zeta_k \in 2\mathbb{Z}$. If rather $\zeta_\kappa = \alpha$ is odd, we need the additional

assumption that $\sum_{j \in J^*} n_j t_j = 0$ for some index set J^* , point masses t_j , and integers n_j such that $\sum_{j \in J^*} n_j = \alpha$. Then, assumption b) of Lemma 5.4 is again satisfied, since

$$T_{\text{delay}} + T_{l_\kappa + 2N + 4\nu} = T'_{l_\kappa + 2N + 4\nu} + \sum_{j \in J^*} n_j t_j.$$

We will now get on with the second part of the coupling. Let $t^* := \max\{t_j \in \Lambda : j \in J \cup J' \cup J^*\}$. It may be the case that $t^* = M_\tau$ as defined in (2.3). This makes it necessary to introduce some extra notation. Let $\lambda_k := l_{\kappa+1} + k(2N + 2)$ for $k \geq 0$. Write $E_{\lambda_k}^*$ for the set of edges between level λ_k and $\lambda_k + 2N + 1$, and let

$$\hat{E}_{\lambda_k}^* = \{e \in E_{\lambda_k}^* : e \text{ is at height zero, or at level } \lambda_k \text{ or } \lambda_k + 2N + 1\}.$$

Denote by $\hat{e}_{\lambda_k + N}$ the edge at height zero connecting level $\lambda_k + N$ and $\lambda_k + N + 1$. Let X_{λ_k} denote the set of edges connecting a vertex at level $\lambda_k + N$ with one at level $\lambda_k + N + 1$, excluding the edge $\hat{e}_{\lambda_k + N}$. Define the event

$$\begin{aligned} A_{\lambda_k}^{**} &= \{\tau_e \geq t^*, \forall e \in X_{\lambda_k}\} \cap \left\{ \tau_e \leq t', \forall e \in \hat{E}_{\lambda_k}^* \setminus \{\hat{e}_{\lambda_k + N}\} \right\} \\ &\quad \cap \left\{ \tau_e \geq t'', \forall e \in E_{\lambda_k}^* \setminus \left(\hat{E}_{\lambda_k}^* \cup X_{\lambda_k} \right) \right\}. \end{aligned}$$

Continue the coupling of $\{\tau_e\}_{e \in \mathbb{E}_m}$ and $\{\tau'_e\}_{e \in \mathbb{E}_m}$ by choosing $\tau_e = \tau'_e$ with distribution P_τ , independently for all e at level λ_0 or beyond such that $e \neq \hat{e}_{\lambda_k + N}$ for $k \geq 0$. Independently for $k \geq 0$, let

$$(\xi_k, \xi'_k) = \begin{cases} (\rho_k, \rho'_k), & \text{with probability } P_\tau([0, t^*]) \\ (\eta_k, \eta_k), & \text{with probability } 1 - P_\tau([0, t^*]), \end{cases}$$

where ρ_k and ρ'_k have marginal distribution $P_\tau(\cdot | \tau \leq t^*)$, and η_k has distribution $P_\tau(\cdot | \tau > t^*)$. For the set of edges $\{\hat{e}_{\lambda_k + N}, \text{ for } k \geq 0\}$ we couple their passage times as

$$\left(\tau_{\hat{e}_{\lambda_k + N}}, \tau'_{\hat{e}_{\lambda_k + N}} \right) = \begin{cases} (\xi_k, \xi'_k), & \text{if } A_{\lambda_k}^{**} \text{ occurs} \\ (\tau_k, \tau_k), & \text{otherwise,} \end{cases}$$

where τ_k is distributed according to P_τ , independently for all k . One realizes from the coupling that the marginal distributions of both τ_e and τ'_e is P_τ .

Note that the only edges for which τ_e and τ'_e may differ, are the edges \hat{e}_{λ_k+N} for $k \geq 0$ such that $A_{\lambda_k}^{**} \cap \{\tau_{\hat{e}_{\lambda_k+N}} \leq t^*\}$ occurs. Let κ_j denote the index k for which $A_{\lambda_k}^{**} \cap \{\tau_{\hat{e}_{\lambda_k+N}} \leq t^*\}$ occurs for the j th time. That

$$\left(\tau_{\hat{e}_{\lambda_k+N}}, \tau'_{\hat{e}_{\lambda_k+N}}\right) = (\rho_k, \rho'_k) \quad (5.7)$$

is equivalent to that $A_{\lambda_k}^{**} \cap \{\tau_{\hat{e}_{\lambda_k+N}} \leq t^*\}$ occurs. Since $P(A_{\lambda_k}^{**} \cap \{\tau_{\hat{e}_{\lambda_k+N}} \leq t^*\}) > 0$, we will have an infinite sequence $\{\kappa_j\}_{j \geq 1}$ such that (5.7) holds. We now claim that the proposition will follow if we apply Lemma 5.4 to the sequences $\{\rho_{\kappa_j}\}_{j \geq 1}$ and $\{\rho'_{\kappa_j}\}_{j \geq 1}$, with distribution $P_\tau(\cdot | \tau \leq t^*)$ and T_{delay} as defined above.

To see this, we use Lemma 2.1. Given $A_{\lambda_k}^{**} \cap \{\tau_{\hat{e}_{\lambda_k+N}} \leq t^*\}$, the path along which any vertex at level $\lambda_k + 2N + 1$ or beyond is infected inevitably has to pass the edge \hat{e}_{λ_k+N} . By the coupling, $\tau_e = \tau'_e$ for all e at level λ_{κ_1} or beyond such that $e \neq \hat{e}_{\lambda_{\kappa_j}+N}$ for $j \geq 1$. Moreover, $\tau_e = \rho \leq t'$ and $\tau'_e = \rho' \leq t'$ for $e \in \{\hat{e}_{\lambda_{\kappa_j}+N} \text{ for } j \geq 1\}$. Therefore, each vertex at level $\lambda_{\kappa_1} + 2N + 1$ and beyond, will be reached in the same order for both infections. Coupling $\{\rho_{\kappa_j}\}_{j \geq 1}$ and $\{\rho'_{\kappa_j}\}_{j \geq 1}$ according to Lemma 5.4 we will have with probability one that, from some level on, both infections will reach each vertex at the same time, i.e.,

$$T(v_n) = T'(v_n) \quad (5.8)$$

for any $v_n \in \mathbb{V}_{\mathcal{G}_n}$ for n sufficiently large. Since we chosen t^* as large as we did, we made sure that $P_\tau(\cdot | \tau \leq t^*)$ meets assumption b) of Lemma 5.4.

The infections may in the same manner be coupled along the negative coordinate axis. Doing this, then there is $N_c \in \mathbb{N}$ such that (5.6) holds for $|n| \geq N_c$. In almost surely finite time, each vertex at level n , for $|n| \leq N_c$, will be infected. Hence, we conclude that for some almost surely finite time T_c ,

$$B_t = B'_t$$

for each $t \geq T_c$. □

Remark 5.6. There exists in general no exact coupling of two infections with discrete passage time distribution on arbitrary 1-dimensional periodic graphs. Consider the distribution $P_\tau(1) = P_\tau(1 + 3/5) = 1/2$. P_τ satisfies the assumption of Proposition 5.2, whence there is an exact coupling of two infections on the (K, d) -tube, for $K, d \geq 2$.

Consider instead the graph with set of vertices $\mathbb{Z} \times \{0, 1\}$ and where two vertices are connected by an edge if their euclidean distance is $\leq \sqrt{2}$. Note that with the above passage time distribution, in order to reach any vertex at level n , an infection will always pass exactly n edges. This is easily seen by realizing that no vertical edge will ever be used in order to reach an uninfected vertex. Thus, for two infections started with $I = \{(0, 0)\}$ and $I' = \{(m, 0)\}$, we will have

$$\begin{aligned} |T_n - T'_n| &\geq \inf_{\substack{a+b=n \\ a'+b'=n-m}} \left| a - a' + (b - b') \left(1 + \frac{3}{5} \right) \right| \\ &= \inf_{b-b' \in \mathbb{Z}} \left| m - \frac{3(b - b')}{5} \right| \geq \frac{1}{5}, \end{aligned}$$

for any m that is not a multiple of 3. As we can see, an exact coupling is not possible. \square

Remark 5.7. Condition a) of Proposition 5.2 is due to the fact that the (K, d) -tube is bipartite, i.e., that every circuit has even length. As seen in Remark 5.6, it is not for just any non-bipartite graph possible to have an exact coupling without condition a). But, condition a) of Proposition 5.2 could be dropped for e.g. the class of triangular graphs with vertex set $\mathbb{Z} \times \{0, 1, \dots, K - 1\}$ and where two vertices at euclidean distance is 1 and every two vertices (n, m) and $(n + 1, m + 1)$ for any $n \in \mathbb{Z}$ and $m = 0, 1, \dots, K - 2$, are connected by an edge. The necessary modifications of the first part of the proof, and of the event D_n in particular, are easily made. \square

Remark 5.8. If $\text{dist}(\mathbf{x}, \mathbf{y})$ is odd, for all $\mathbf{x} \in I$, $\mathbf{y} \in I'$, then condition a) of Proposition 5.2 is necessary. To see this, assume that an exact coupling is possible. In particular, $T(v) = T'(v)$ for some vertex v . But, if one infection has an even number of edges to pass in order to reach v , the other has an odd number of edges to pass. Thus,

$$0 = T(v) - T'(v) = \sum_{j \in J} n_j t_j - \sum_{j \in J} n'_j t_j,$$

for integers n_j and n'_j such that $\sum_{j \in J} (n_j - n'_j)$ is odd. Hence, condition a) holds. \square

Remark 5.9. Condition a) of Lemma 5.4 can be weakened to distributions P_τ whose convolution with itself has an absolutely continuous component.

In fact, it is sufficient if P_τ convoluted with itself n times, for some $n \geq 0$, has an absolutely continuous component. Since the distribution of a sum of independent random variables is the convolution of the individual distributions, we may instead of specifying how to choose (τ_j, τ'_j) for $j \geq 1$, choose $(\sum_{k=(j-1)n+1}^{jn} \tau_k, \sum_{k=(j-1)n+1}^{jn} \tau'_k)$ according to the same specification. Consequently, the assumption on P_τ of Proposition 5.1 can be weakened to involve distributions whose convolution with itself n times has an absolutely continuous component. The modifications are left to the reader.

As an example of a distribution that does not have an absolutely continuous component, but whose convolution does, consider the following. Let ξ_0, ξ_1, \dots be i.i.d. Bernoulli(1/2)-distributed random variables. Define τ to have binary expansion

$$\tau := \begin{cases} (0, \xi_1, 0, \xi_3, 0, \dots), & \text{with probability } \frac{1}{2} \\ (\xi_0, 0, \xi_2, 0, \xi_4, \dots), & \text{otherwise.} \end{cases}$$

Let τ_1 and τ_2 be two independent random variables distributed as τ , and let A denote the event that one of τ_1 and τ_2 has all even coordinates equal to zero and the other has all odd coordinates equal to zero. Neither τ_1 nor τ_2 is absolutely continuous, but the conditional distribution of $\tau_1 + \tau_2$ given A is uniformly distributed on $[0, 1]$. Hence the distribution of $\tau_1 + \tau_2$ has an absolutely continuous component. \square

5.3 No exact coupling possible on trees

We have seen that there is an exact coupling of two first-passage percolation infections on any essentially 1-dimensional periodic graph when the passage time distribution has an absolutely continuous component. We also saw how this sort of coupling gave rise to a 0–1 law. One may ask whether a continuous component is sufficient for an analogous coupling, and corresponding 0–1 law, on any graph? We will answer this question no, by showing that the binary tree \mathbb{T}^2 constitutes a counterexample. \mathbb{T}^2 is the infinite graph that do not contain any circuit, and where each vertex has three neighbours. The graph is completely homogeneous and one vertex, called the *root*, is chosen for reference. Let $\{\tau_e\}_{e \in \mathbb{E}}$ be a set of independent and exponentially distributed passage times associated with

the edge set \mathbb{E} of \mathbb{T}^2 , and analogous to before, let

$$B_t = \{v \in \mathbb{V} : T(\text{root}, v) \leq t\}.$$

For the reader familiar with the theory of continuous branching processes, an argument can be given as follows (a somewhat longer but more elementary argument will be presented afterwards). Define the front line of the infection at time t as

$$F_t := \#\{v \notin B_t : v \text{ shares an edge with some } u \in B_t\}.$$

Note that $F_0 = 3$ and that F_t increases by one, when B_t does. Hence, F_t can be seen as a continuous time branching process with F_t individuals at time t . Each individual gives with probability one birth to two children (and dies) after an exponentially distributed time, independent of one another. It is well-known (see e.g. Athreya and Ney (1972, Theorems III.7.1–2)) that, for some Malthusian parameter $\lambda > 0$,

$$\exists W := \lim_{t \rightarrow \infty} F_t e^{-\lambda t}, \quad \text{almost surely,} \quad (5.9)$$

and that $\mathbb{E}[W] = 1$. Let τ_{e_1} , τ_{e_2} and τ_{e_3} denote the passage time of the edges connected to the root, and let \tilde{F}_t denote F_t conditioned on $\{\tau_{e_1}, \tau_{e_2}, \tau_{e_3} \geq 1\}$. Then, by the lack-of-memory property of the exponential distribution, we have that $\tilde{F}_{t+1} \stackrel{d}{=} F_t$ for any $t \geq 0$. Thus, by (5.9) we have almost surely

$$\lim_{t \rightarrow \infty} \tilde{F}_t e^{-\lambda t} \stackrel{d}{=} e^{-\lambda} \lim_{t \rightarrow \infty} F_t e^{-\lambda t} = e^{-\lambda} W,$$

and we conclude that W is almost surely nonconstant. Note that the event $\{W = \lim_{t \rightarrow \infty} F_t e^{-\lambda t} \leq x\} \in \mathcal{T}$, for every x . Then, a 0–1 law analogous to Theorem 5.3 cannot hold for first-passage percolation on \mathbb{T}^2 , since this would imply that $P(W \leq x) \in \{0, 1\}$, i.e., that W is almost surely constant.

The reader who is not familiar with the theory of branching processes may appreciate the following more elementary argument. To the root of the tree, there are three identical (connected) components \mathbb{T}_1^2 , \mathbb{T}_2^2 and \mathbb{T}_3^2 connected. Denote the respective vertex sets of these components by \mathbb{V}_1 , \mathbb{V}_2 and \mathbb{V}_3 . Define for $k = 1$ and 2

$$B_k(t) = \{v \in \mathbb{V}_k : T(\text{root}, v) \leq t\}.$$

Since \mathbb{T}_1^2 , \mathbb{T}_2^2 and \mathbb{T}_3^2 are identical, it is clear that $B_1(t)$ and $B_2(t)$ are i.i.d. In particular,

$$P(|B_1(t)| > |B_2(t)|) \leq \frac{1}{2}, \quad (5.10)$$

for any $t \geq 0$. To the vertex $v_1 \in \mathbb{V}_1$ joined to the root by an edge, there are two identical (connected) subcomponents of \mathbb{T}_1^2 connected. Denote their vertex sets by $\mathbb{V}_{1,1}$ and $\mathbb{V}_{1,2}$ respectively, and define for $k = 1$ and 2

$$B_{1,k}(t) = \{v \in \mathbb{V}_{1,k} : T(v_1, v) \leq t\}.$$

Set $B'_1(t) = B_{1,1}(t) \cup B_{1,2}(t) \cup \{v_1\}$. Since

$$|B'_1(t)| \leq |B_2(t)| \quad \Rightarrow \quad |B_2(t)| > \max(|B_{1,1}(t)|, |B_{1,2}(t)|),$$

we conclude that

$$\begin{aligned} P(|B'_1(t)| > |B_2(t)|) &= 1 - P(|B'_1(t)| \leq |B_2(t)|) \\ &\geq 1 - P\left(|B_2(t)| > \max(|B_{1,1}(t)|, |B_{1,2}(t)|)\right) \\ &\geq 1 - \frac{1}{3} = \frac{2}{3}, \end{aligned} \quad (5.11)$$

for any $t \geq 0$, since $B_{1,1}(t)$, $B_{1,2}(t)$ and $B_2(t)$ are i.i.d.

It is clear from their definition that we may interpret $B'_1(t)$ and $B'_2(t) := B_2(t)$ as the sets of infected vertices in \mathbb{V}_1 respective \mathbb{V}_2 for an infection started in $\{\text{root}, v_1\}$. Doing so, we can combine (5.10) and (5.11) to conclude that it is not possible to couple two infections with initial conditions $I = \{\text{root}\}$ and $I' = \{\text{root}, v_1\}$ such that $B_t = B'_t$ for large t , with probability one. If such a coupling would exist, the probabilities in (5.10) and (5.11) would have to be equal for sufficiently large t .

We may use (5.10) and (5.11) as well to conclude that the 0–1 law analogue to Theorem 5.3 cannot hold in our present setting. Consider the events

$$\begin{aligned} A &= \{\exists T > 0 : \forall t \geq T : |B_1(t)| > |B_2(t)|\}, \\ B &= \{\exists T > 0 : \forall t \geq T : |B_1(t)| < |B_2(t)|\}. \end{aligned}$$

It is obvious that $P(A) = P(B) \leq \frac{1}{2}$, and that $A, B \in \mathcal{T}$. To see that these events have positive probability, we shall condition on the event

$\{\tau_{e_1} < \tau_{e_2}\}$.

$$P(A) \geq P(A \mid \tau_{e_1} < \tau_{e_2})P(\tau_{e_1} < \tau_{e_2}) = \frac{1}{2}P(A \mid \tau_{e_1} < \tau_{e_2}).$$

We may from the lack-of-memory property of the exponential distribution, and (5.11) conclude that

$$P(|B_1(t)| > |B_2(t)| \mid \tau_{e_1} < \tau_{e_2}) = P(|B_1'(t)| > |B_2(t)|) \geq \frac{2}{3}, \quad (5.12)$$

for $t \geq \tau_{e_1}$. Assume that $P(A \mid \tau_{e_1} < \tau_{e_2}) = 0$. Then, with probability one, there is an almost surely finite constant $T \geq \tau_{e_1}$ such that $|B_1(T)| = |B_2(T)|$. This in turn, together with (5.10), implies that

$$P(|B_1(t)| > |B_2(t)| \mid \tau_{e_1} < \tau_{e_2}) = P(|B_1(t)| > |B_2(t)|) \leq \frac{1}{2},$$

for $t \geq T$. This contradicts (5.12).

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