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# Safety Index of Fatigue Failure for Ship Structure Details

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## Abstract

One way to assess the uncertainty in fatigue damage analysis is to use a so-called safety index. In the computation of such an index the variation coefficient for the accumulated damage is required. In this paper the expected fatigue damage and its coefficient of variation is firstly estimated from measured stress. Secondly, when suitable stress measurements are not available these are computed from models for damage accumulation and sea states variability. Stresses during ship sailing period are known as the non-stationary, slowly changing, Gaussian processes and hence damage accumulation, during encountered sea state, can be approximated by an algebraic function of significant wave height, ship speed and heading angle; Further the space time variability of significant wave height is modeled as a lognormal field with parameters estimated from the satellite measurements.

The proposed methods to estimate uncertainties in the damage accumulation process are validated using full scale measurements carried out for a container vessel operating in the North Atlantic.

**Keywords:** Rainflow damage, fatigue risk of ship structure detail, safety index, damage variability.

# 1 Introduction

Material fatigue is one of the most important safety issues for structures subject to cyclic loads and the cause of failure in a majority of cases. Fatigue life of a structural detail is greatly influenced by a number of components and material dependent factors, such as geometry, size of the structure, surface smoothness, surface coating, residual stress, material grain size and defects. Further, the nature of the load process is important. The complex dependence between these factors and fatigue life makes predictions uncertain and even for controlled laboratory experiments the results from fatigue life tests exhibit a considerable scatter.

In this paper we present a simplified safety analysis showing how the different sources of uncertainties can be combined into a safety index using a Bayesian approach with material and structure detail dependent parameters modeled as random variables. We will particularly focuse on the variability of the loads a ship may encountered in a specified period of time.

When studying a variable environment, the average damage growth rate may not be sufficient to properly estimate the risk for fatigue failure. For example, the fatigue crack risk of a ship structure detail during one year depends on the age of such detail, and can be high during a year if a vessel encounters an extreme storm. The probability of meeting such a storm can be very small but may still influence the value of the estimated risk. Consequently the uncertainties in long term variability of load properties should be included in the risk analysis.

In this paper methods to estimate the fatigue risk of ship structure details will be presented. Data from an extensive measurement campaign will be used to validate the proposed methodology. The paper is organized as follows. In Section 2 some basic definitions of rainflow damage are given and in Section 2.1 variable amplitude tests are discussed. Safety index is introduced in Section 2.2. The computation of such index is illustrated in Section 3 where the measured stresses during half a year are used to compute the safety index of trade in different numbers of years. An important case of computation of the safety index when no stress measurements are available is discussed in Section 4. In this section the safety index will be estimated, by means of a model for the sea state variability estimated using the satellite data. The model is presented in the appendix. Some further mathematical details about computating the coefficient of variation of the accumulated damage are moved to the appendix. Finally, a numerical example is given in Section 5.

## 2 Fatigue review

Fatigue testing of structural details has traditionally been carried out using constant amplitude stress cycles. In these experiments the stress oscillates between the minimum and maximum value until fatigue failure occurs. Repeating the experiments for different amplitudes, keeping the ratio, $R$, between minimum and maximum load constant, result in what is known as a Wöhler curve, also called S-N curve, with a log-linear dependence between the number of cycles to failure, $N$, and the stress cycle range, $h$,

$$\log(N) = a - k\log(h) + e, \tag{1}$$

where parameters $a > 0$ and $k \geq 1$ depend on material and structural detail properties and the stress ratio $R$. When studying fatigue of welded ship structure, the parameters $a$, $k$ are usually categorized into different types based on the properties of structural details. In this paper, we will use the simple one slope S-N curve with $k = 3$ and $a = 12.76$, where the unit of stress cycle range $h$ should be "MPa", see DNV Fatigue Note (2005).

For random stresses the stress cycles and cycle ranges need to be defined using some cycle count procedure. In fatigue analysis the "rainflow" method, see Appendix I, has been shown to give the most accurate results. The method was originally introduced by Endo: The first paper in English is Matsuishi and Endo (1968). Here we shall use the alternative definition given in Rychlik (1987), which is more suitable for statistical analysis.

Fatigue damage from variable amplitude (random) stresses is commonly regarded as a cumulative process. Let $h_i$ be the ranges of the rainflow cycles, see Figure 5, found in the stress then using the linear Palmgren-Miner damage accumulation rule (Palmgren, 1924, Miner, 1945) one defines the pseudo rainflow damage $D^{rfc}(t)$ at time $t$ as

$$D^{rfc}(t) = \sum_i h_i^k \tag{2}$$

Finally it is assumed that fatigue failure occurs when $\log(D^{rfc}(t)) > a$. (Note that $h_i$ should have the same unit as the stress cycle range $h$ in (1)). In practice one is observing failures when $\log(D^{rfc}(t)) > a - 0.5$ due to variability of material properties and other factors, relevant for fatigue accumulation, see Johannesson et al. (2005a) for detailed discussion. A possible solution to incorporate these factors in the model is to estimate the parameters $a$ and $k$ of the S-N curve using tests with variable amplitude loads similar to the real load processes. We will discuss this issue further in the next section.

### 2.1 Variable amplitude S-N curve

Let us introduce the equivalent cycle range defined as

$$h^{eq}(n) = \left(\frac{1}{n}\sum_{i=1}^{n} h_i^k\right)^{1/k}, \tag{3}$$

where $\{h_i\}_1^n$ are stress ranges of rainflow cycles. Usually the stress signal is rainflow filtered, i.e. small cycles, with range smaller than some chosen threshold relative to the fatigue limit, are removed before computation of the equivalent range. Consequently $n$ in (3) is the number of remaining rainflow cycles used in the blocked test load. For stationary (ergodic) Gaussian loads $h^{eq}(n)$ fast approaches a limit $h^{eq}$, say,

$$h^{eq} = \lim_{n\to\infty} h^{eq}(n), \tag{4}$$

which can be computed from a single long measurement of the load.

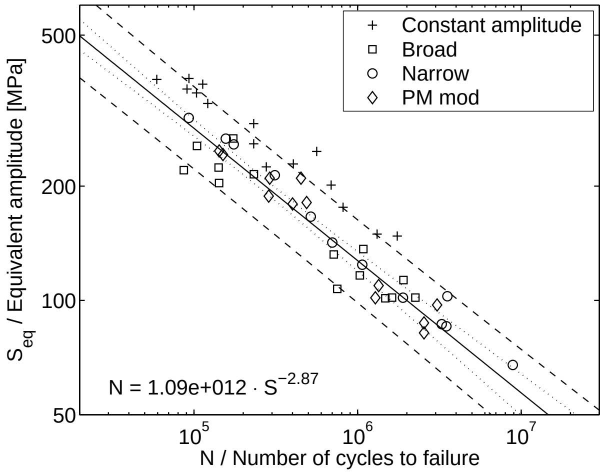



Figure 1: S-N curve estimated from variable amplitude tests using broad-banded, narrow-banded and Pierson-Moskowitz spectra compared with constant amplitude tests (Agerskov, 2000).

Empirical tests, see Agerskov (2000) and Figure 1, have shown that the S-N curve (1) is valid also for Gaussian random loads if the constant stress range $s$ is replaced by $h^{eq}$,

$$\log(N) = a - k\log\left(h^{eq}\right) + e. \tag{5}$$

The S-N curve (5) tells us that, if an undamaged structure details loaded by a stationary Gaussian stress under time $t$, then the load is safe for this structure detail if

$$a - \log\left(D^{rfc}(t)\right) + e > 0. \tag{6}$$

In Figure 1 the results of constant amplitude experiments are marked by pluses and one can see that the S-N relation for the constant amplitude load would give the same $k$ but also a higher value of the parameter $a$. It indicates that using $a$ in (6) from the constant amplitude experiments will give some (nonconservative) bias.

## 2.2 Fatigue in variable environment

Measurements show that the Gaussian processes are often good models for variability of the wave induced stresses to ship structure details under stationary sea conditions, from about 30 minutes to several hours. However the sea-states vary along the route and hence the stress is in fact a non-stationary Gaussian process. Since the fatigue tests leading to S-N curve (5) were performed under stationary conditions and hence it is not obvious that one can again use the S-N based criterion (6) to estimate the risk of fatigue failure. In fact some additional assumptions are needed to extend applicability of the criterion from stationary to non-stationary loading. For example one needs to neglect the possible sequential effects and then use the S-N curve obtained for stationary Gaussian loads.

More precisely, suppose that during period $T$, $M$ voyages were undertaken. Further, assume that the damage accumulated in harbors during loading and unloading operations can be neglected. If the stresses are known during the voyages then the pseudo rainflow damage $D_j^{rfc}$, during $j$th voyage and defined by (2), can be evaluated and the total pseudo damage defined by

$$D^{rfc}(T) = \sum_{j=1}^{M} D_j^{rfc} \tag{7}$$

Now the Palmgren-Miner hypothesis is equivalent to the criterion that the stress history is safe for fatigue if

$$a - \log\left(D^{rfc}(T)\right) + e + \tilde{e} > 0 \tag{8}$$

where $a$ and $e$ are taken from variable amplitude fatigue tests. Further, the additional error term $\tilde{e}$ represents the uncertainties caused by possible modeling errors, e.g. using Palmgren-Miner rule, neglecting sequential effects between voyages, using stress concentration factor and other simplifications. (The mean of $\tilde{e}$ is often assumed to be zero while the variance of $\tilde{e}$ needs to be determined by means of experience.)

The total accumulated pseudo damage $D^{rfc}(T)$, defined in (7), is a function of the magnitudes of stresses experienced by structure details during the period $T$. However most often the stresses are unknown. In such situation one can model the uncertain value of the damage $D^{rfc}(T)$ by a distribution of the possible values it can take, in other words $D^{rfc}(T)$ is a random variable. And then one is interested in the failure probability

$$P_f = P(a - \log(D^{rfc}(T)) + e + \tilde{e} \leq 0).$$

Here, using the Bayesian ideas $a$, $D^{rfc}(T)$, $e$ and $\tilde{e}$ are random variables. If

$$G = a - \log(D^{rfc}(T)) + e + \tilde{e}$$

is normally distributed then the probability of cracking occurrences for the structural detail $P_f = 1 - \Phi(I_C)$, where $\Phi$ is the cumulative distribution function (cdf) of a standard normal variable, while $I_C$ is the so called Corell's safety index defined as follows

$$I_C = \frac{\mathrm{E}\left[G\right]}{\sqrt{\mathrm{Var}(G)}} = \frac{\mathrm{E}[a - \log(D^{rfc}(T)) + e + \tilde{e}]}{\sqrt{\mathrm{Var}(a - \log(D^{rfc}(T)) + e + \tilde{e})}} \tag{9}$$

Most often $G$ is not normally distributed and hence $P_f \neq 1 - \Phi(I_C)$ but the index is still a useful measure for the risk of cracking for the structure detail.

In the case when the distributions of $a$, $D^{rfc}(T)$, $e$ and $\tilde{e}$ are not well known one is further simplifying the safety index (assuming independence and employing Gauss formulas) by

$$I_C \approx \frac{\mathrm{E}\left[a\right] - \log(\mathrm{E}[D^{rfc}(T)])}{\sqrt{\mathrm{Var}(a) + \mathrm{Var}(\log(D^{rfc}(T))) + \mathrm{Var}(e) + \mathrm{Var}(\tilde{e})}} \tag{10}$$

In the following examples we shall use $k = 3$ and $\mathrm{E}[a] = 12.76$ as mentioned before. The value of variance of $a$ (and the two other variances as well) is not available and we shall use typical values taken from literature. The variability of $a$, and $e$ were studied in Johannesson et al. (2005a), typical values are $\mathrm{Var}(a) = 0.005$, $\mathrm{Var}(e) = 0.14$ while $\mathrm{Var}(\tilde{e}) = 0.1$. Further, $\mathrm{Var}(\log(D^{rfc}(T)))$ can be approximated by $\mathrm{CoV}(D^{rfc}(T))^2 = \frac{\mathrm{Var}(D^{rfc}(T))}{\mathrm{E}[D^{rfc}(T)]^2}$, coefficient of variation of pseudo damage. Then the safety index is approximated as

$$I_C \approx \frac{\mathrm{E}\left[a\right] - \log(\mathrm{E}[D^{rfc}(T)])}{\sqrt{\mathrm{Var}(a) + \mathrm{CoV}(D^{rfc}(T))^2 + \mathrm{Var}(e) + \mathrm{Var}(\tilde{e})}}. \tag{11}$$

Hence only the orders of $\mathrm{E}[D^{rfc}(T)]$ and $\mathrm{CoV}(D^{rfc}(T))^2$, have to be estimated.

In what follows two simplifying assumptions, both realistic, are employed to estimate the order of $\mathrm{CoV}(D^{rfc}(T))^2$: firstly, if routes $i$, $j$ and their starting dates are known then $D_i^{rfc}$, $D_j^{rfc}$ are independent; and secondly, the errors of time series of stress measurement can be neglected. Suppose that one wishes to compute variation coefficient after $M$ voyages for which routes are known then

$$\mathrm{CoV}(D^{rfc}(T))^2 = \mathrm{CoV}(\sum_{j=1}^{M} D_j^{rfc})^2 = \frac{\sum_{j=1}^{M} \mathrm{Var}(D_j^{rfc})}{(\sum_{j=1}^{M} \mathrm{E}[D_j^{rfc}])^2}. \tag{12}$$

We shall use (12) in the following situations. Suppose that a ship will operate in similar conditions for $T$ years, and that we found a way to estimate the variation coefficient for $T_0$ period, such as 0.5 or 1 year, then

$$\mathrm{CoV}(D^{rfc}(T))^2 = \frac{T_0}{T}\,\mathrm{CoV}(D^{rfc}(T_0))^2. \tag{13}$$

There is a vast literature proposing different means for estimating of $\mathrm{E}[D^{rfc}(T)]$ and one is often assuming that the uncertainty in the damage, i.e. $\mathrm{CoV}(D^{rfc}(T))^2$, is negligible relatively to other uncertainties. Formula (13) could be used to motivate this practice. However sometimes the shipping for an old vessel can be drastically changed and then the $\mathrm{CoV}(D^{rfc}(T))^2$ for short time period $T$ is not negligible and should be included in evaluation of the safety index.

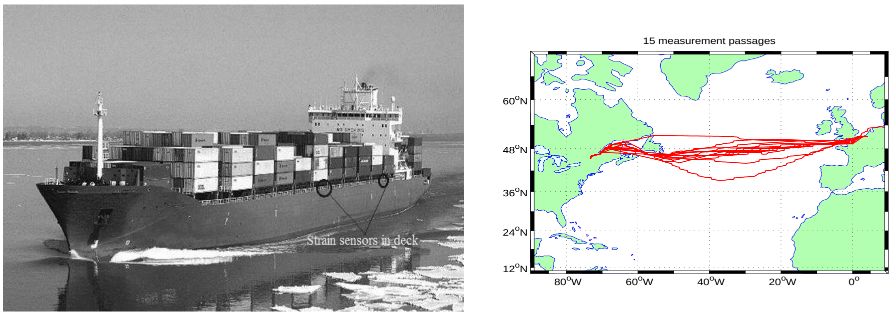
Two principally different approaches to estimate $\mathrm{E}[D^{rfc}(T)]$ and $\mathrm{Var}(D^{rfc}(T))$ will be presented in the following sections. The first one is the statistical (nonparametric) approach when the information from historical data (measured stresses) will be used and the second, parametric one, is when a model for the stress variability will be employed to compute $\mathrm{E}[D^{rfc}(T)]$ and $\mathrm{Var}(D^{rfc}(T))$. (Obviously one needs data to estimate the parameters in the model.)

# 3 Safety index, extrapolation of measurement to longer periods

Often in practice when long time series of stresses have been measured one may assume that the future damage increase is stationary, i.e. varies in the same way as during the measured period, e.g. when a vessel is operated in the similar routes. In what follows we shall denote the observed rainflow pseudo damages during $j$th voyage by $d_j^{rfc}$. We assume that measurement errors are negligible and denote as $d^{rfc} = \sum d_j^{rfc}$ the rainflow pseudo damage computed from the measured stresses during a period $T_0$, e.g. a year. If one is planning to use a vessel for $T$-years on similar transports (routes, cargo) as during the measured period $T_0$ ($T \geq T_0$) then, as will be shown in Appendix II, the safety index can be evaluated according to the following formula

$$I_C \approx \frac{\mathrm{E}\,[a] - \log(T/T_0) - \log(d^{rfc})}{\sqrt{\mathrm{Var}(a) + K\ \mathrm{Var}(D^{rfc}(T_0))/(d^{rfc})^2 + \mathrm{Var}(e) + \mathrm{Var}(\tilde{e})}}, \quad K = \frac{T - T_0}{T}. \tag{14}$$

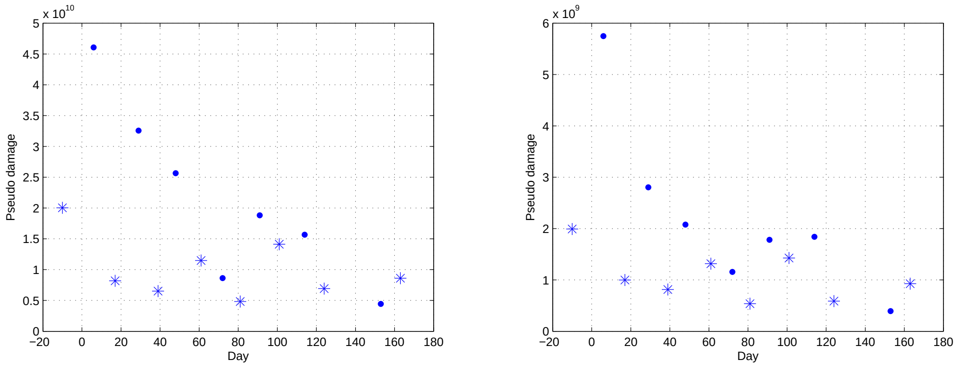
(Obviously $K$ grows from zero to one as extrapolation period increases.) In order to evaluate $I_C$ one still needs to estimate $\mathrm{Var}(D^{rfc}(T_0))$. Since damages accumulated during individual voyages are independent the variance can be estimated by means of standard statistical method if there are voyages that have the same expectation. For example voyages on similar routes undertaken at the same month should have the same mean and can be used to estimate both mean and variance, see the following example.




(a) Measured locations at the 2800 TEU vessel (b) Measured courses in the North Atlantic

Figure 2: The locations and detailed routes of 15 measured voyages for the 2800TEU container ship operating in the North Atlantic during the first half of year 2008.

**Example 1:** One container vessel is now operating in the North Atlantic between EU and Canada. The time series of stress were measured at 2 critical locations at this vessel, shown in Figure 2(a), during the first half of year 2008 (detailed description about the measurement see Storhaug and Heggelund (2008a)). There are 15 voyages measured during this period, and the detailed courses are shown in Figure 2(b). The rainflow estimated pseudo damages during different voyages are provided in Figure 3, where (a) presents the observed pseudo damage $d_j^{rfc}$ of the structure detail in the midship, while (b)




(a) $d_j^{rfc}$ of midship position

(b) $d_j^{rfc}$ of aftership position

Figure 3: The observed pseudo damages $d_j^{rfc}$ for a container ship: stars are the passages from Canada to EU while dots are passages from EU to Canada. The $x$ axis gives the day of the year when trips are finished.

shows $d_j^{rfc}$ of the other structure detail in the aftership. In Figure 3 the stars represent the pseudo damages $d_j^{rfc}$ measured on voyages from Canada to EU while dots are pseudo damages $d_j^{rfc}$ when sailing from EU to Canada. For the structure detail in the midship, the accumulated damage $d_{mid}^{rfc} = 23.3 \cdot 10^{10}$, for the one in the aftership, the corresponding damage $d_{aft}^{rfc} = 2.44 \cdot 10^{10}$. In both cases the measuring period $T_0 = 0.5$ year. The variances of $\mathrm{Var}(D_{mid}^{rfc}(T_0))$ and $\mathrm{Var}(D_{aft}^{rfc}(T_0))$ will be estimated next.

The data consists of 15 passages over North Atlantic and the measured $d_j^{rfc}$ are presented in Figure 3. Inspired by the figure, we split the voyages into two groups: three most damaging winter passages from EU to Canada and the remaining 12 less damaging passages. For structure detail in the midship, variance of a winter passage from EU to Canada is estimated to be $1.08 \cdot 10^{20}$, while the less damaging type passage has variance $2.82 \cdot 10^{19}$. Consequently the variance $\mathrm{Var}(D_{mid}^{rfc}(T_0)) \approx 6.62 \cdot 10^{20}$ and hence $\mathrm{Var}(D_{mid}^{rfc}(T_0))/(d_{mid}^{rfc})^2 \approx 0.012$. Meanwhile for the aftership detail, taking the same approach, variance of a winter passage from EU to Canada is estimated to be $3.78 \cdot 10^{18}$ while the less damaging type passage has variance $2.85 \cdot 10^{17}$, thus $\mathrm{Var}(D_{aft}^{rfc}(T_0)) \approx 1.48 \cdot 10^{19}$ and $\mathrm{Var}(D_{aft}^{rfc}(T_0))/(d_{aft}^{rfc})^2 \approx 0.025$.

Now for any $T \geq 0.5$ the safety indexes of structure details in midship and aftership are respectively computed as follows:

$$I_{C\_mid} \approx \frac{12.76 - \log(d_{mid}^{rfc}) - \log(T/T_0)}{\sqrt{0.005 + 0.012\,K + 0.14 + 0.1}},$$

$$I_{C\_aft} \approx \frac{12.76 - \log(d_{aft}^{rfc}) - \log(T/T_0)}{\sqrt{0.005 + 0.025\,K + 0.14 + 0.1}},$$

where $K = (T - T_0)/T$. In following table 1, we give indexes for different periods $T$.

| T | $I_C(T)$ - MidSect | $1 - \Phi(I_C)$ | $I_C(T)$ - AftSect | $1 - \Phi(I_C)$ |
|---|---|---|---|---|
| 0.5 | 2.81 | 0.002 | 4.79 | $8{\cdot}10^{-7}$ |
| 1 | 2.18 | 0.015 | 4.08 | 0.00002 |
| 2 | 1.57 | 0.06 | 3.45 | 0.0002 |
| 3 | 1.22 | 0.11 | 3.09 | 0.001 |
| 5 | 0.78 | 0.22 | 2.65 | 0.004 |

Table 1: Column 1 - time period $T$; Column 2 - safety index for midship position; Column 3 - nominal probability of fatigue failure (crack) for structure detail in the midship; Column 4 - safety index for aftership position; Column 5 - nominal probability of fatigue failure for aftership structure detail.

Base on this simplified analysis, we conclude that the risk of fatigue cracking for the structure detail in midship is not negligible even for the time period of 1 year; and that the safety of aftership detail is also low for the time period exceeding 5 years, in the sense of possibility of crack development. However it has to be noted that the consequence of existence of a crack may not affect the hull integrity. (Cracks are often accepted after 20 years of age.)

# 4 Safety index, parametric approach

In the previous section we derived the safety index by means of the extrapolation of the measured damage during a period of time $T_0$. Here we will consider the case when one cannot use the extrapolation approach because either measured stresses are not representative for the future loads or there are no measurements of stresses at all. In such situation one needs to estimate $\mathrm{E}[D^{rfc}(T)]$ and $\mathrm{CoV}(D^{rfc}(T))^2$ by proposing a model for the distribution of $D^{rfc}(T)$. The following properties of the wave induced stresses are basis of our model:

(a) The waves are build up from rather long period, about 30 minutes, when the loading conditions can be assumed to be stationary.

(b) The mean stress remains almost constant over long time period, i.e. for a voyage between two harbors.

(c) Wave load has short memory, i.e. load process becomes independent after couple of minutes.

Properties (a-b) allow us to approximate the damage accumulated during a voyage by the sum of damages caused by loads during the stationarity periods, see Bogsjö and Rychlik (2007), Bengtsson et al. (2008) for more detailed discussion. In Mao et al. (2008) it is shown that the error of such an approximation was less than 1% for stresses measured during 15 voyages over North Atlantic.

Although cycles vary in unpredictable manner during the stationarity periods the variability of the pseudo damage $D^{rfc}(T)$ is still negligible, because of (c), in comparison with other sources of uncertainties, see Bengtsson and Rychlik (2008). Consequently, as it is often done in practice, one can approximate the damage increments during stationarity periods by their expected values. The expectations can then be bounded by means of the narrow-band approximation, reviewed next.

## 4.1 Narrow-band bound

Let $Y(t)$ be a Gaussian stress, and $h_s = 4\sqrt{\mathrm{Var}(Y(0))}$ be the significant stress range while $f_z = \frac{1}{2\pi}\sqrt{\mathrm{Var}(\dot{Y}(0))/\,\mathrm{Var}(Y(0))}$, the apparent frequency (the intensity of mean stress level upcrossings by $Y$), then the expected pseudo rainflow damage in the period $t$ is bounded by

$$\mathrm{E}\left[D^{rfc}(t)\right] \le 0.5\, t\, f_z\, h_s^k, \tag{15}$$

for $2 \le k \le 4$, see Rychlik (1993) for the proof. (This is the so called narrow-band approximation introduced in Bendat (1964).) Furthermore, as it was reported in Bengtsson and Rychlik (2008), the coefficient of variation of $D^{rfc}(t)$ converges fast to zero as $t$ increases and, for typical wave spectra, one can assume that even $D^{rfc}(t) \le 0.5\, t\, f_z\, h_s^k$. During a voyage if the stress properties change slowly (conditions (a-c) are valid) then, approximately, the accumulated pseudo damage:

$$D_j^{rfc} \le 0.5\,\Delta t \sum f_z(i)\, h_s^k(i) = D_j^{nb}, \tag{16}$$

Here $D_j^{rfc}$ is the increase of the rainflow pseudo damage during $j$th voyage, $\Delta t$ is the common length of stationary period, usually 1800 seconds, $h_s(i)$ and $f_z(i)$ are the significant stress range and apparent frequency estimated from $i$th stationarity period in the $j$th voyage. In the previous work Mao et al. (2008) it was demonstrated that $(d_j^{nb} - d_j^{rfc})/d_j^{rfc}$ was less than 0.3. As before we denote by $d_j^{nb}, d_j^{rfc}$

the measured damages, by narrow-band approximation and rainflow analysis, respectively. (In addition (16) is used in many dedicated softwares to estimate the damage accumulation during a voyage.) What remains is to find a model for variability of significant stress range $h_s$ and apparent frequencies $f_z$, which is done in the following subsections.

### 4.1.1 Model for $f_z$

Suppose that the sea contains only one cosine wave with period $T$. For a vessel sailing with heading angle $\beta$ and speed $v$, then the encountered frequency is

$$f^e = \left| \frac{1}{T} + \frac{2\pi\, v \cos(\beta)}{g} \frac{1}{T^2} \right| = f_z \tag{17}$$

by assumed linear relation between stresses and encountered waves. Since the sea is composed of many waves having different periods and since the heading angle, to these waves, may also vary hence we propose to replace $T$ and $\beta$ in (17) by the peak period $T_p$ and the average heading angle $\bar{\beta}$, respectively, giving the following approximation of $f_z$

$$f_z \approx \left| \frac{1}{T_p} + \frac{2\pi\, v_s \cos(\bar{\beta})}{g} \frac{1}{T_p^2} \right|. \tag{18}$$

Here it is assumed that main wave period does not deviate much from main response spectrum, but this can happen for "narrow" band transfer function $S_R(\omega) = |H(\omega)|^2\, S(\omega)$, where $S(\omega)$ is the encountered wave spectrum.

Both the average heading angle and the peak period have to be estimated onboard of the vessel. Finally we also propose to estimate as follows $T_p \approx 4.9\sqrt{H_s}$, approximately valid for fully developped sea, see DNV Fatigue Note (2005) now replaced by new recommendation in DNV Environment Note (2007), giving

$$f_z \approx \left| \frac{1}{4.9\sqrt{H_s}} + \frac{2\pi\, v_s \cos(\bar{\beta})}{g} \frac{1}{24\, H_s} \right|. \tag{19}$$

**Example 1 cntd.:** For a container ship the directional spectrum $S_i(\omega, \alpha)$ were measured by means of a radar and hence one can estimate the average heading angle $\bar{\beta}_i$, during stationary periods, by means of

$$\bar{\beta}_i = \frac{\int_0^{180} \int_0^{\infty} \alpha S_i(\omega, \alpha)\, d\omega\, d\alpha}{\int_0^{180} \int_0^{\infty} S_i(\omega, \alpha)\, d\omega\, d\alpha}. \tag{20}$$

(Note that we defined directional spectrum only for angles $[0, 180]$, instead for more commonly used $[0, 360]$.)

### 4.1.2 Model for $h_s$

Suppose that we use a linear wave model then sea state, under stationary condition, is defined by a directional spectrum $S(\omega, \alpha)$. A typical model for $S(\omega, \alpha)$, is obtained by combining Piearson-Moskowitz spectrum $S(\omega)$ and $\cos^2 \alpha$ spreading function. Such a directional spectrum is characterized by significant wave height $H_s$ and $T_z$ only. The linear transfer function, estimated by means of dedicated software, give a relation

$$h_s = C(T_z, \beta, v)\, H_s$$

where as before, $\beta$ is the heading angle, while $v$ is the ship speed. Next, for fixed $v$, the constant $C(\beta)$ is defined as the average $C(\beta) = \mathrm{E}[C(T_z, \beta)]$ where $T_z$ has a long term distribution that ship would encountered in the particular route. Here a simplification is done by choosing the $T_z$-distribution used by DNV for the North Atlantic scatter diagram which does not reflect the seasonal variability, see DNV Environment Note (2007).

For the particular details, respectively located in the midship and aftership of investigated container vessel, the constant $C(\beta)$ is computed using the linear strip software Waveship (see Waveship User's Manual (1993)) and given in the following Table 2, where $C_{mid}(\beta)$ is for midship detail and $C_{aft}(\beta)$ is for aftership detail. (We have assumed that the ship is sailing with the constant service speed 10 [m/s]).

Combining the proposed model the following approximation, say the new narrow band approximation, for the increase of the pseudo damage during the $j$th voyage is proposed

$$D_j^{nb} \approx \Delta t \sum_i C(\bar{\beta}_i)^3 \left| H_s(i)^{2.5} \frac{1}{9.8} + H_s(i)^2 \frac{\pi\, v_s\, \cos(\bar{\beta}_i)}{24\, g} \right|, \qquad D^{nb}(T) = \sum_{j=1}^{M} D_j^{nb}, \tag{21}$$

where $\Delta t$ is the common length of the stationary period taken to be 1800 seconds here. Obviously the values of significant wave height encountered during a voyage, as well as heading angles are not known in advance and hence $D_j^{nb}$ is a random variable. For a specific voyage, i.e. when starting date, ship speed and the route is defined, then one could bound $D^{nb}$ by taking heading angle $\beta = 0$. Then what remains is to model the variability of encountered significant wave height $H_s$ along the route. Using model for $H_s$ variability, presented in Appendix III, one can simulate the sequence of $H_s(i)$ and then compute values of $D_j^{nb}$. Repeating independently the simulations one can obtain the distribution $D^{nb}$ by a standard statistical method.

| $\beta$ | 0 | 10 | 20 | 30 | 40 | 50 | 60 | 70 | 80 |
|---|---|---|---|---|---|---|---|---|---|
| $C_{mid}(\beta)$ | 25.66 | 25.77 | 25.58 | 25.10 | 24.37 | 23.47 | 22.47 | 21.48 | 20.65 |
| $C_{aft}(\beta)$ | 12.73 | 12.76 | 12.62 | 12.35 | 11.96 | 11.46 | 10.99 | 10.51 | 10.13 |
| $\beta$ | 90 | 100 | 110 | 120 | 130 | 140 | 150 | 160 | 170 |
| $C_{mid}(\beta)$ | 20.10 | 19.92 | 20.16 | 20.77 | 21.65 | 22.66 | 23.67 | 24.56 | 25.24 |
| $C_{aft}(\beta)$ | 9.89 | 9.84 | 9.99 | 10.32 | 10.78 | 11.29 | 11.79 | 12.23 | 12.55 |

Table 2: The constant $C(\beta)$ computed using linear strip software Waveship and to be used in (21) to approximate the increament of pseudo damage during a voyage.

### 4.2 Estimation of safety index $I_C$

Let $T$ be the computed period, usually measured in years, the safety index $I_C(T)$, given by (11), can be now estimated by replacing $D^{rfc}(T)$ by $D^{nb}(T)$. (Note that this is an conservative approximation and hence we do not add any additional uncertainty into denominator of the index). Now the safety index based on the proposed model becomes

$$I_C \approx \frac{\mathrm{E}\,[a] - \log(\mathrm{E}[D^{nb}(T)])}{\sqrt{\mathrm{Var}(a) + \mathrm{CoV}(D^{nb}(T))^2 + \mathrm{Var}(e) + \mathrm{Var}(\tilde{e})}}. \tag{22}$$

Hence only the orders of $\mathrm{E}[D^{nb}(T)]$ and $\mathrm{CoV}(D^{nb}(T))^2$, have to be estimated. Consequently by (21) one needs to have a model for variability of encountered significant wave height $H_s$ and heading angles $\beta$.

Finally in order to easy comparison between non-parametric and parametric approaches to estimate the index we will now give a parametric version of formula (14). Suppose that there is a period $T_0$, for example one year, and that the similar shipping is planned for the whole period of $T$ years then

$$I_C \approx \frac{\mathrm{E}\,[a] - \log(T/T_0) - \log(\mathrm{E}[D^{nb}(T_0)])}{\sqrt{\mathrm{Var}(a) + K\ \mathrm{CoV}(D^{nb}(T_0)) + \mathrm{Var}(e) + \mathrm{Var}(\tilde{e})}}, \quad K = \frac{T_0}{T}. \tag{23}$$

## 5 Validation of the proposed approach

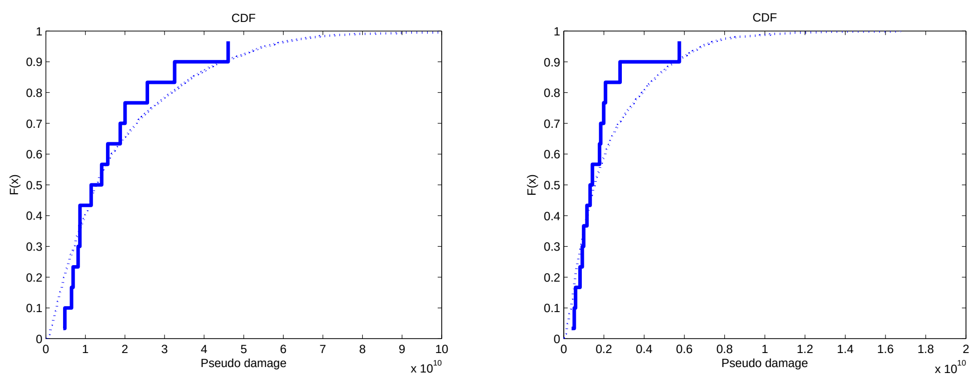
We say that operation schedule of a vessel is specified if: a number of voyages are given together with planned time of the year when voyage starts; positions in latitude, longitude and ship velocity for the

| Voyage date | MidSect | | | | AftSect | | | |
|---|---|---|---|---|---|---|---|---|
| 2007-12-20 | 2.00 | (0.40) | 2.53 | 2.03 | 0.20 | (0.34) | 0.30 | 0.24 |
| 2008-01-06 | 4.61 | (0.44) | 4.85 | 1.76 | 0.57 | (0.48) | 0.59 | 0.21 |
| 2008-01-17 | 0.82 | (0.27) | 1.86 | 1.73 | 0.10 | (0.28) | 0.22 | 0.21 |
| 2008-01-29 | 3.26 | (0.28) | 4.46 | 2.06 | 0.28 | (0.14) | 0.54 | 0.24 |
| 2008-02-09 | 0.65 | (0.30) | 1.42 | 1.42 | 0.08 | (0.30) | 0.17 | 0.17 |
| 2008-02-18 | 2.56 | (0.22) | 3.83 | 1.64 | 0.21 | (0.10) | 0.46 | 0.20 |
| 2008-03-01 | 1.15 | (0.76) | 0.63 | 0.72 | 0.13 | (0.75) | 0.07 | 0.08 |
| 2008-03-12 | 0.86 | (0.12) | 2.04 | 1.02 | 0.12 | (0.15) | 0.25 | 0.13 |
| 2008-03-21 | 0.48 | (0.23) | 1.40 | 1.24 | 0.05 | (0.23) | 0.17 | 0.15 |
| 2008-04-01 | 1.88 | (0.58) | 1.74 | 0.75 | 0.18 | (0.36) | 0.21 | 0.09 |
| 2008-04-11 | 1.41 | (0.50) | 1.42 | 1.05 | 0.14 | (0.41) | 0.17 | 0.12 |
| 2008-04-24 | 1.57 | (0.72) | 1.24 | 0.56 | 0.18 | (0.70) | 0.15 | 0.07 |
| 2008-05-04 | 0.69 | (0.53) | 0.66 | 0.42 | 0.06 | (0.34) | 0.08 | 0.05 |
| 2008-06-03 | 0.44 | (0.44) | 0.47 | 0.20 | 0.04 | (0.23) | 0.06 | 0.03 |
| 2008-06-13 | 0.86 | (1.00) | 0.24 | 0.20 | 0.09 | (1.00) | 0.03 | 0.02 |

Table 3: Column 1 - day the voyage ends; Column 2 to 5 list the results of structure detail located in the midship: Column 2 - the observed pseudo damage $10^{-10} \cdot d_j^{rfc}$ computed using the measured stresses (stress concentration factor 2), Column 3 - a Monte Carlo estimation of the probability $P(D_j^{nb} \leq d_j^{rfc})$, Column 4,5 - the expected accumulated damage $10^{-10} \cdot \mathrm{E}[D_j^{nb}]$ and the standard deviation $10^{-10} \cdot \sqrt{\mathrm{Var}(D_j^{nb})}$, where $D_j^{nb}$ defined as in (21), and the model for $H_s$ variability estimated using satellite measurements of significant wave height presented in Appendix III. Columns 6 to 9 are the results for aftership structure detail with the same meaning as column 2 to 5.

routes are chosen. In such situation uncertainties in values of accumulated damages are results of "lack of knowledge" of the significant wave heights and heading angles which will be encountered during the planned voyages. The heading angles can be taken to zero giving the conservative estimates of damages and what remains is finding a statistical model for $H_s$ variability. Such a model has been proposed in Baxevani et al. (2005) and Baxevani et al. (2008b). The parameters of the model, estimated from the satellite measurements of $H_s$, are presented in Baxevani et al. (2008a) and hence one can find the distribution of $D_j^{nb}$ for almost any route.

Since in this section we are primarily interested in checking the accuracy of the proposed approach by validating it against the measured data the distribution of $D_j^{nb}$ will be found only for the 15 routes for




(a) Result of location in Midship (b) Result of location in Aftership

Figure 4: Comparison between empirical cumulative distributions of the observed rainflow pseudo damages $d_j^{rfc}$ for the 15 voyages and the cumulative distribution of $D_j^{nb}$ (dotted line) defined by means of (21), for structure details respectively in the midship and aftership. (The distributions describe variability of pseudo damages on a route taken at random from the 15 passages.)

which measured values of $d_j^{rfc}$ are available. In order to increase precision we also assume that heading angle $\beta_i$ on that 15 routes are known, i.e. the same as measured (each half hour) on that voyages (the speed is kept constant 9 [m/s] for the whole voyage). Two type of checks will be performed. The first one is to compute probability that, for a voyage indexed by $j = 1, ..., 15$, $D_j^{nb}$ is smaller than the observed damage $d_j^{rfc}$. Values below 0.01 and above 0.99 would indicate a significant difference between the observed damages and the variability of $D_j^{nb}$. The results are presented in Table 3, third and seventh columns respectively for locations in midship and aftership. (In second and sixth columns of this table we have $10^{-10} \cdot d_j^{rfc}$.) Results presented in the table show that observed variability of rainflow damages is well modeled by $D_j^{nb}$ in (21).

The second comparison is presented in Figure 4, where (a) is for midship location and (b) is for the aftership location. The solid line is the cumulative distribution function (cdf) of the observed values of $d_j^{rfc}$. Such cdf describes variability of rainflow damages that are selected at random from the second column (for midship) or seventh column (for aftership) in Table 3. The dotted cdf describes variability of the corresponding random experiment for the damages $D_j^{nb}$, i.e. drawing at random one of the 15 routes and simulating the value of $D_j^{nb}$. Two distributions agree surprisingly well for both of the structure details in the midship and aftership. Hence we conclude that $D_j^{nb}$ seems to be a very good approximation for $d_j^{rfc}$ and one can compute the safety index $I_C$ for the 15 voyages by replacing $D^{rfc}(T)$ with $D^{nb}(T) = \sum_{j=1}^{15} D_j^{nb}$.

In order to compute safety index $I_C$, for extension in sailing for additional $T = 0.5$ year, one needs $\mathrm{E}[D_j^{nb}]$ and $\mathrm{Var}(D_j^{nb})$. Those are given in columns 4, 5 (for midship detail) and column 8, 9 (for aftership detail) in Table 3, respectively. (The details of the computations of $\mathrm{E}[D_j^{nb}]$, $\mathrm{Var}(D_j^{nb})$ are given in Appendix III). Finally the safety indexes of locations in midship and aftership are respectively given by

$$I_{C\_mid} = \frac{12.76 - \log(2.88) - \log(T/T_0) - 11}{\sqrt{0.005 + 0.0295 \cdot (T_0/T) + 0.14 + 0.1}},$$

$$I_{C\_aft} = \frac{12.76 - \log(3.46) - \log(T/T_0) - 10}{\sqrt{0.005 + 0.0291 \cdot (T_0/T) + 0.14 + 0.1}}.$$

In the following table, we give indexes for different periods $T$.

| T | $I_C(T)$ - MidSect | $1 - \Phi(I_C)$ | $I_C(T)$ - AftSect | $1 - \Phi(I_C)$ |
|---|---|---|---|---|
| 0.5 | 2.48 | 0.006 | 4.24 | $2 \cdot 10^{-5}$ |
| 1 | 1.96 | 0.025 | 3.77 | 0.00008 |
| 2 | 1.39 | 0.08 | 3.22 | 0.0007 |
| 3 | 1.05 | 0.15 | 2.89 | 0.002 |
| 5 | 0.60 | 0.27 | 2.45 | 0.007 |

Table 4: Column 1 - time period $T$; Column 2 - safety index for midship location; Column 3 - nominal probability of failure for midship location; Column 4 - safety index for aftership location; Column 5 - nominal probability of failure for aftership location.

Comparing the results presented in table 1, we conclude that derived indexes obtained by the methods are equivalent. However applying both methods we neglected some uncertainties, statistical errors when estimating $\mathrm{Var}(D^{rfc}(T_0))$ (this error can be large due to crudeness of our estimation method) while for the parametric method we neglected the possibility of modeling errors.

# 6 Conclusions

In this paper nonparametric (extrapolation of measured damages) and parametric (based on a model for significant wave height variable along a route) were presented and validated. The application of the nonparametric approach is limited to the case of stationary shipping.

The second approach, defined in (21), seems to provide with a very accurate approximation of the damage accumulation process. It has a clear advantage that no measurements of stresses or significant wave height are explicitly needed and could be applied to any route and ship. However the deficiency of this approach is possibility of "modeling errors", i.e. that the linear transfer function is too simple model to describe relation between waves and stresses. Further the transfer function itself may be not estimated accurately enough. There could be similar uncertainty in the modeled wave environment (e.g. by assuming Pierson-Moskowitz spectrum and $\cos^2\alpha$ spreading function). Consequently measurements of stresses could still be needed to validate the results of numerical computations.

The safety index indicate that the current ship has relatively good fatigue strength, but that fatigue cracks may be anticipated before ending of the ships life. The method does not yet reflect the possibility to reduce the fatigue damage risk and corresponding safety index, but this is subject of future work.

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## Appendix I

In the rainflow cycle count each local maximum of the load process is paired with one particular local minimum, determined as follows:

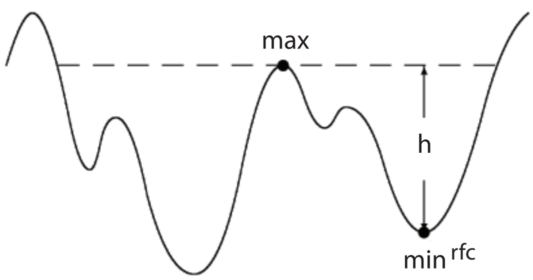


Figure 5: A rainflow pair

- From the local maximum one determines the lowest values in forward and backward directions between the time point of the local maximum and the nearest points at which the load exceeds the value of the local maximum.

- The larger of those two values is the rainflow minimum paired with that specific local maximum, i.e. the rainflow minimum is the least drop before reaching the value of the local maximum again on either side.

- The cycle range, $h$, is the difference between the local maximum and the paired rainflow minimum.

Note that for some local maxima, the corresponding rainflow minimum could lie outside the measured load sequence. In such situations, the incomplete rainflow cycle constitutes the so called residual and has to be handled separately. In this approach, we assume that, in the residual, the maxima form cycles with the preceding minima.

## Appendix II

In this appendix we shall motivate the approximation (14). Suppose that one has measured stresses during a period $T_0$, and let denote the accumulated pseudo damage by $d^{rfc}$. (We assume that measurements errors are negligible.) Obviously $d^{rfc} \neq \mathrm{E}[D^{rfc}(T_0)]$ and let $e_{T_0}$ be the error

$$e_{T_0} = \mathrm{E}[D^{rfc}(T_0)] - d^{rfc}, \qquad \mathrm{E}[e_{T_0}] = 0, \qquad \mathrm{Var}(e_{T_0}) = \mathrm{Var}(D^{rfc}(T_0)).$$

The safety index for $T$ years of trade, given by (11), is equal to

$$I_C \approx \frac{\mathrm{E}\,[a] - \log(\mathrm{E}[D^{rfc}(T)])}{\sqrt{\mathrm{Var}(a) + \mathrm{CoV}(D^{rfc}(T))^2 + \mathrm{Var}(e) + \mathrm{Var}(\tilde{e})}}.$$

If one is planning similar trade (routes, cargo) for $T$ years as during the measured period $T_0$ then $I_C$ can be computed as follows. From stationarity of damage accumulation process and independence of $D_i, D_j$ it follows that

$$\begin{aligned} \mathrm{E}[D^{rfc}(T)] &= \frac{T}{T_0} d^{rfc} + \frac{T - T_0}{T_0}\, e_{T_0}, \\ \mathrm{Var}[D^{rfc}(T)] &= \frac{T - T_0}{T_0}\, \mathrm{Var}[D^{rfc}(T_0)]. \end{aligned}$$

Next, using $\log(a + x) \approx \log(a) + x/a$,

$$\log(\mathrm{E}[D^{rfc}(T)]) \approx \log(d^{rfc}) + \log(T/T_0) + \frac{T - T_0}{T}\, \frac{e_{T_0}}{d^{rfc}}$$

Since $\mathrm{E}[e_{T_0}/d^{rfc}] = 0$ we further approximate the index as follows

$$I_C \approx \frac{\mathrm{E}\,[a] - \log(d^{rfc}) - \log(T/T_0)}{\sqrt{\mathrm{Var}(a) + \mathrm{CoV}(D^{rfc}(T))^2 + \mathrm{Var}(e) + \mathrm{Var}(\tilde{e}) + \mathrm{Var}(X)}}.$$

where $X = \frac{T - T_0}{T}\, \frac{e_{T_0}}{d^{rfc}}$. Since $\mathrm{E}[D^{rfc}(T_0)] \approx d^{rfc}$ we obtain

$$\mathrm{CoV}(D^{rfc}(T))^2 = \frac{(T - T_0)\, T_0}{T^2} \frac{\mathrm{Var}(D^{rfc}(T_0))}{(d^{rfc})^2}$$

and hence

$$\mathrm{CoV}(D^{rfc}(T))^2 + \mathrm{Var}(X) = \frac{T - T_0}{T}\, \frac{\mathrm{Var}(D^{rfc}(T_0))}{(d^{rfc})^2}$$

giving (14).

## Appendix III

As reported in Baxevani et al. (2005) the significant wave height at position $\mathbf{p}$ and time $t$ is accurately model by means of lognormal cdf. Let $X(\mathbf{p}, t) = \ln(H_s(\mathbf{p}, t))$ denote a field of logarithms of significant wave height that evolves in time. Suppose $t_0$ be the starting date of a voyage, $\mathbf{p}(t) = (x(t), y(t))$, $[t_0, t_1]$, the planned route, while $\mathbf{v}(t) = (v_x(t), v_y(t))$ a velocity a ship will move with. For a route let $z(t) = X(\mathbf{p}(t), t)$ be the encountered logarithms of the significant wave heights. (The encountered significant wave heights are $H_s(t) = \exp(z(t))$.) The $z(t)$ is a non stationary Gaussian process and in this appendix we give a model for the covariance function $r_z(t_1, t_2) = \mathrm{C}\,(z(t_1), z(t_2))$.

*Locally stationary field:* Suppose that for a fixed geographical region and season (e.g. January) $X$ is a stationary Gaussian field with mean $m$, variance $\sigma^2$ and separable correlation structure. We also assume that the field is drifting (moving) with a constant velocity $\mathbf{V} = (V_x, V_y)$, say. By this we mean that there are two autocorrelation functions $\rho_S$ correlation between $\log H_s$ at two positions at the same time and $\rho_T$ the correlation of $\log H_s$ at the same location but different time instances that defines the covariance between log $H_s$ at different locations and time instances, viz.

$$\mathrm{C}\,(X(\mathbf{p}_1, t_1), X(\mathbf{p}_2, t_2)) = \sigma^2\, \rho_S(x_2 - x_1 - V_x(t_2 - t_1), y_2 - y_1 - V_y(t_2 - t_1)) \cdot \rho_T(t_2 - t_1).$$

(The correlation $\rho_S$ could be estimated from a map of $H_s$ derived by means of Hindcast data (ERA40) or satellite measurements while $\rho_T$ comes from the buoy measurements.)

Now suppose that a vessel is sailing with constant velocity $(v_x, v_y)$ and let $z(t)$ be encountered $\log(H_s)$ at time $t$. If variability in time and space of log $H_s$ is modeled by the stationary Gaussian field $X$ then $z$ is also stationary Gaussian process with mean $m$ and the covariance function

$$\mathrm{C}\,(z(t_1), z(t_2)) = \sigma^2\, \rho_S(v_1\,(t_2 - t_1), v_2\,(t_2 - t_1))\rho_T(t_2 - t_1) = r_z(t_2 - t_1), \tag{24}$$

where $v_1 = v_x - V_x$ and $v_2 = v_y - V_y$. In Baxevani et al. (2008b) one used, in (24),

$$\rho_S(x, y) = \exp(-(x^2 + y^2)/2L^2), \qquad \rho_T(t) = \exp(-\lambda|t|), \tag{25}$$

$t$ in hours, where parameters $L$ and $\lambda$ are slowly varying over oceans and seasons.

Since $z$ is a stationary process it has power spectral density (psd) $S(\omega)$, say. Here the psd depends on parameters $\sigma^2$, $L$, $\lambda$ and the relative ship velocity $\mathbf{v} = (v_1, v_2)$. (The parameters $\sigma^2, L$ were estimated by means of satellite observation while $\lambda$ is estimated using $H_s$ measured by buoys, see Baxevani et al. (2008a) where the variability of the parameters in season and geographical location over the globe is presented.)

We have assumed that the process $z$ is stationary however in practice the assumption may be valid for short period of time because the statistical properties of sea changes with the geographical locations. Consequently, as has been observed in data, parameters $m, \sigma^2$, $L, \lambda$ and velocity $\mathbf{v}$ varies between different geographical locations on the oceans. Hence the encountered log $H_s$ process, i.e. $z(t)$, cannot be stationary for the whole voyage. Since the properties of $z$ changes slowly we shall model it by means of locally stationary processes defined next.

Let $S_t(\omega)$ be the spectrum of a stationary process $z$ with covariance function defined by formulas (24-25) where the parameters $\sigma^2(t), L(t), \lambda(t)$ and $\mathbf{v}(t)$ are functions of position of a ship $\mathbf{p}(t)$. If $S_t$ is known for all $t \in [t_0, t_1]$ then a "locally stationary" process $z$ can be defined, by means of spectral representation and moving averages construction, as folllows

$$z(t) = \int \exp(-it\,\omega)\sqrt{S_t(\omega)}\,dB(\omega), \tag{26}$$

where $B(\omega)$ is a Brownian motion. This is somewhat technical construction which results in a nonstationary Gaussian model for $z$, with $\mathrm{E}\,[z(t)] = m(t)$ and

$$\mathrm{C}\,(z(t_1), z(t_2)) = \int \exp(-i(t_2 - t_1)\,\omega)\sqrt{S_{t_1}(\omega)S_{t_2}(\omega)}\,d\omega = r_z(t_1, t_2), \tag{27}$$

say. Since the Gaussian process $z$ is uniquely defined by its mean $m(t))$ and covariance function $r_z(t_1, t_2)$ hence also $H_s(t) = \exp(z(t))$ is uniquely defined when the encountered local spectra $S_t(\omega)$ and means $m(t)$ are known. Here it means that one have to estimate parameters defining spectra for geographical locations and time of the year of interest for shipping, see Baxevani et al. (2008a).

Having $r_z$ and $m$ then, by means of methods presented in Baxevani and Rychlik (2007), one can compute $\mathrm{E}[D_j]$ and variance $\mathrm{Var}(D_j)$ if the heading angles $\beta(t)$ and speed of the vessel are known. However in order to estimate the distribution of damage $D_j$ a Monte Carlo approach is the most convenient. Simply one can generate sequences of $H_s(i)$ of possible values of significant wave heights along routes and then compute the damage $D_j$.

More precisely, let times $t_i$, $i = 0, \dots, n$, with $t_{i+1} - t_i = \Delta t$ equal 30 minutes, be the times a vessel is passing positions $(x_i, y_i) = (x(t_i), y(t_i))$ and the values of significant wave height at the position $H_s(i) = \exp(z_i)$, where $z_i = z(t_i)$ are correlated normal variables. It is a simple task to generate a sequence of $z_i$ when the vector of means $\mathbf{m} = [m_i]$, $m_i = m(t_i)$, and the covariance matrix $\Sigma = [r_{ij}]$, where $r_{ij} = r_z(t_i, t_j)$ are known.

However in order to make computation fast one would like to have explicit formula for covariance $r_z$ instead of the integral (27) that has to be evaluated numerically. In addition for the particular choice of the autocorrelations $\rho_S$ and $\rho_T$, given in (25), even spectrum $S_t(\omega)$ has to be computed by means of numerical procedure. e.g. FFT transform, for all $t$ values. In the following subsection we shall modify the autocorrelation function $\rho_T$ in such a way that covariance $r_z$ will be given by an explicit algebraic expression depending only on easily interpretable parameters.

## Approximation of $r_z(t_1, t_2)$

In previous work we have used (24) with $\rho_T(t) = \exp(-\lambda|t|)$ to define time correlation structure of the significant wave field at a fixed position. A typical value for parameter $\lambda$ estimated from buoys is 0.0125, which means that correlation length $\tau_T$, say, is about 40 hours. (Here we define correlation length as a time lag the correlation drops to 0.6.) In order to simplify computation we propose to approximate the covariance $\rho_T(t) = \exp(-0.0125\,|t|)$, where $t$ is defined in hours, by the Gaussian covariance with the same correlation length, viz. $\rho(t) = \exp(-0.5(t/\tau_T)^2)$, $\tau_T = 2/\lambda$.

Using (24), some simple algebra gives

$$r_z(t) = \sigma^2 \exp(-0.5t^2/C^2), \quad C = \frac{\tau_T\,\tau_S}{\sqrt{\tau_T^2 + \tau_S^2}}, \quad \tau_S = \sqrt{v_1^2 + v_2^2}/L. \tag{28}$$

Note that $\tau_S$ is the space related correlation length and has interpretation as the time it takes for a vessel to move between two positions $\mathbf{p}_1$ and $\mathbf{p}_2$ for which the log of significant wave heights spatial correlation drops to 0.6. Parameters $\tau_T$ and $\tau_S$ characterize the spatial and time sizes of storms, respectively. The covariance (28) is particularly convenient since the power spectrum $S_t$, used in (26), can be given in an explicit way

$$S_t(\omega) = \sigma^2 \frac{C}{\sqrt{2\pi}} \exp(-\omega^2\, C^2/2).$$

The spectrum depends on $t$ because the values of parameters $\sigma^2$ and $C$ are changing along the route $\mathbf{p}(t)$. Knowing $\sigma(t)$ and $C(t)$ the integral in (27) can be computed giving

$$r_z(t, s) = 2\sigma(t)\sigma(s) \frac{C(t)C(s)}{C(t)^2 + C(s)^2}\, e^{-(t-s)^2/(C(s)^2 + C(t)^2)}.$$