A Nitsche extended finite element method for incompressible elasticity with discontinuous modulus of elasticity

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Abstract

In this note we propose a finite element method for incompressible (or compressible) elasticity problems with discontinuous modulus of elasticity (or, if compressible, Poisson's ratio). The problem is written on mixed form using P^1–continuous displacements and the space of piecewise P^0 pressures, leading to the possibility of eliminating the pressure beforehand in the compressible case. In the incompressible case, the method is augmented by a stabilization term, penalizing the pressure jumps. We show a priori error estimates under certain regularity hypothesis. In particular we prove that if the exact solution is sufficiently smooth in each subdomain then the convergence order is optimal.

1 Introduction

Hansbo and Hansbo [5,6] proposed an unfitted finite element method for elliptic interface methods using Nitsche's method to allow for discontinuous or weakly discontinuous solutions. In particular, [6] deals with the elasticity problem, however avoiding the incompressible case. In this paper we aim to fill this gap.

We note that the equations of incompressible elasticity written on mixed form with the pressure as an auxiliary variable is form–identical to the Stokes equations of creeping viscous flow, and thus this paper serves as a basis for computations involving also two– (or multi–) fluid problems, which has been an

important field of research during the last ten years. Techniques such as front tracking and front capturing have been introduced in order to keep track of the front. Both these techniques have qualities and drawbacks. Typically a front tracking algorithm will give a sharp resolution of the front and since the mesh follows the interface it is also well suited to solve the fluid mechanics equations with discontinuous coefficients that the two-fluid problem gives rise to; on the other hand it breaks down if topological changes occur in the exact solution. This is essentially the reason why front capturing method like volume of fluid or level set methods were introduced: they handle topological change automatically. However, since the position of the front is known only through intermediate of the interface capturing function the mesh is not fitted to the discontinuities of the coefficients in the exact solution. It is well known that for elliptic problems this leads to loss of convergence and pollution effects perturbing the solution close to the front. This paper thus addresses also the latter question in the setting of fluid mechanics.

We consider the space of piecewise affine, continuous, functions for the approximation of the velocities and the space of piecewise constant functions for the pressure. The pressure jumps between adjacent elements are penalized to enhance stability. Across the interface we enrich the finite element space so that both the velocities and the pressures are allowed to be discontinuous. The solutions in the two subdomains are then coupled using Nitsche's method.

We show that our formulation satisfies the inf-sup condition for stabilized methods related to the non-mixed constant strain method. This stability estimate is then used in combination with approximation estimates using suitable extension operators to prove prove optimal a priori estimates in energy and L_2–norms.

2   The elasticity problem on mixed form

Let Ω be a bounded domain in $\mathbb{R}^2$, with convex polygonal boundary $\partial\Omega$ and an internal smooth boundary Γ, with bounded measure, dividing Ω into two open sets Ω_1 and Ω_2. The mixed form of the elasticity equations that we propose as a model problem is given by seeking the displacements $\boldsymbol{u}$ and the pressure p such that

$$-\nabla \cdot (2\mu_i \boldsymbol{\varepsilon}(\boldsymbol{u})) + \nabla p = f \quad \text{in } \Omega_i,\ i = 1, 2,$$

$$\nabla \cdot \boldsymbol{u} - \frac{1}{\lambda_i} p = 0 \quad \text{in } \Omega_i,\ i = 1, 2, \tag{1}$$

$$\boldsymbol{u} = 0 \quad \text{on } \partial\Omega \setminus \Gamma.$$

where, in terms of Young's modulus E_i and Poisson's ratio ν_i, we have $\mu_i = E_i/(2(1+\nu_i))$ and $\lambda_i = E_i\nu_i/((1+\nu_i)(1-2\nu_i))$. (Strictly speaking, p can only be identified as the pressure in the incompressible case $\nu_i \to 1/2$, but for simplicity we shall omit this distinction.) Here $\boldsymbol{\varepsilon}(\boldsymbol{u})$ is the symmetric part of the displacement gradient and Γ denotes the interface between Ω_1 and Ω_2, $\Omega = \Omega_1 \cup \Omega_2$ and the Lamé parameters μ_i and λ_i are constant in each subdomain Ω_i. For any sufficiently regular function x in $\Omega_1 \cup \Omega_2$ we define the jump of x across Γ by $[\![x]\!] = x_1 - x_2$, where $x_i = x|_{\Omega_i}$. We also define the limits in domain Ω_1 or Ω_2 on Γ by $p(\mathbf{x})|_{\Gamma\pm} = \lim_{\epsilon\to\pm 0} p(\mathbf{x}+\epsilon\boldsymbol{n})$ where $\mathbf{x} \in \Gamma$ and where $\boldsymbol{n}$ is the outward pointing normal to Ω_1 on Γ. Let $(u,v)_X$ denote the L^2-scalar product on X,

$$(u,v)_X := \int_X uv \, \mathrm{d}x,$$

with associated norm $\|u\|_{0,X} := (u,u)_X^{\frac{1}{2}}$. The Sobolev-norms associated with the spaces $H^k(X)$ will be denoted by $\|u\|_{k,X}$. Define the the max and min of the shear moduli by $\mu_{\max} := \max_\Omega \mu$ and $\mu_{\min} := \min_\Omega \mu$. Formally we obtain the weak formulation by multiplying (1) by a function $(\boldsymbol{v},q) \in [H_0^1(\Omega)]^2 \times L^2(\Omega) = V \times Q$ and integrating by parts, leading to the problem of finding $(\boldsymbol{u},p) \in V \times Q$ such that

$$\sum_{i=1}^2 \Big((2\mu_i\boldsymbol{\varepsilon}(\boldsymbol{u}),\boldsymbol{\varepsilon}(\boldsymbol{v}))_{\Omega_i} - (p,\nabla\cdot\boldsymbol{v})_{\Omega_i} + (q,\nabla\cdot\boldsymbol{u})_{\Omega_i} - (\lambda_i^{-1}p,q)_{\Omega_i} \Big)$$
$$+ \int_\Gamma [\![p\,\boldsymbol{n} - 2\mu\boldsymbol{\varepsilon}(\boldsymbol{u})\cdot\boldsymbol{n}]\!]\cdot\boldsymbol{v}\,\mathrm{d}s = (\boldsymbol{f},\boldsymbol{v})_\Omega, \text{ for all } (\boldsymbol{v},q) \in V \times Q, \tag{2}$$

where

$$(\boldsymbol{\varepsilon}(\boldsymbol{u}),\boldsymbol{\varepsilon}(\boldsymbol{v}))_{\Omega_i} := \int_{\Omega_i} \boldsymbol{\varepsilon}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{v})\,dx.$$

Taking into account the following interface conditions

$$[\![p\,\boldsymbol{n} - 2\mu\boldsymbol{\varepsilon}(\boldsymbol{u})\cdot\boldsymbol{n}]\!]\cdot\boldsymbol{n} = -\sigma\kappa \tag{3}$$

and

$$[\![2\mu\boldsymbol{\varepsilon}(\boldsymbol{u})\cdot\boldsymbol{n}]\!]\cdot\boldsymbol{t} = 0, \tag{4}$$

where $\boldsymbol{t} = (-n_y, n_x)$ and $\sigma\kappa$ denotes the surface tension force term, with κ the interface curvature, we arrive at the following well posed formulation: Find $(u,p) \in V \times Q$ such that

$$B[(\boldsymbol{u},p),(\boldsymbol{v},q)] := (\boldsymbol{f},\boldsymbol{v})_\Omega + \int_\Gamma \sigma\kappa\boldsymbol{v}\cdot\boldsymbol{n}\,\mathrm{d}s, \text{ for all } (\boldsymbol{v},q) \in V \times Q, \tag{5}$$

where

$$B[(\boldsymbol{u},p),(\boldsymbol{v},q)] = \sum_{i=1}^2 \Big((2\mu_i\boldsymbol{\varepsilon}(\boldsymbol{u}),\boldsymbol{\varepsilon}(\boldsymbol{v}))_{\Omega_i} - (p,\nabla\cdot\boldsymbol{v})_{\Omega_i}$$
$$+ (q,\nabla\cdot\boldsymbol{u})_{\Omega_i} - (\lambda_i^{-1}p,q)_{\Omega_i} \Big).$$

3

For the analysis we will only consider the incompressible case to reduce the technical detail. When both materials are incompressible the pressure space must be chosen as $L_0^2(\Omega)$. The extension to the compressible case is straightforward combining the results below with those of [6] Moreover for the convergence analysis we will assume that $\boldsymbol{u}$ is in H^2 on each subdomain and p in H^1 in each subdomain. For a regularity analysis in the compressible case we refer to [8] and in the incompressible case to [9].

3 The finite element formulation

In a standard finite element method the jump in normal derivative resulting from the continuity of the traction, when $\mu_1 \neq \mu_2$ or $\lambda_1 \neq \lambda_2$, can be taken into account by letting Γ coincide with meshlines. However, for optimization or inverse identification purposes (or in general for two–fluid problems) where the interface will have to move, this implies remeshing during the simulation, at least close to the interface. Instead we propose to solve (1) approximately using piecewise linear displacements and piecewise constant pressures on a family of conforming triangulations $\mathcal{T}_h$ of Ω which are *independent* of the location of the interface. To avoid loss in convergence order we relax the continuity requirement over Γ and allow the approximation to be discontinuous inside elements which intersect the interface.

We will use the following notation for mesh related quantities. Let h_K be the diameter of K and $h = \max_{K \in \mathcal{T}_h} h_K$. For any element K, let $K_i = K \cap \Omega_i$ denote the part of K in Ω_i. By $G_h := \{K \in \mathcal{T}_h : K \cap \Gamma \neq \emptyset\}$ we denote the set of elements that are intersected by the interface. For an element $K \in G_h$, let $\Gamma_K := \Gamma \cap K$ be the part of Γ intersecting K.

We make the following assumptions regarding the mesh and the interface (from [5]).

- A1: We assume that the triangulation is non-degenerate, i.e.,

$$h_K/\rho_K \leq C \quad \forall K \in \mathcal{T}_h$$

  where h_K is the diameter of K and ρ_K is the diameter of the largest ball contained in K.
- A2: We assume that Γ intersects each element boundary ∂K exactly twice and each (open) edge at most once.
- A3: Let $\Gamma_{K,h}$ be the straight line segment connecting the points of intersection between Γ and ∂K. We assume that Γ_K is a function of length on $\Gamma_{K,h}$; in local coordinates

$$\Gamma_{K,h} = \{(\xi, \eta) : 0 < \xi < |\Gamma_{K,h}|, \eta = 0\}$$

and
$$\Gamma_K = \{(\xi, \eta) : 0 < \xi < |\Gamma_{K,h}|, \eta = \delta(\xi)\}.$$

Since the curvature of Γ is bounded the assumptions A2 and A3 are always fulfilled on sufficiently fine meshes. These assumptions essentially demand that *the interface is well resolved by the mesh.*

To obtain the finite element formulation corresponding to (5) we replace the test function $\boldsymbol{v}$ by $\tilde{\boldsymbol{v}}$ such that $\tilde{\boldsymbol{v}}|_{\Omega_i} = \boldsymbol{v}_i$ with $\boldsymbol{v}_i \in [H^1(\Omega_i)]^2$ and $\boldsymbol{v}_i = 0$ on the boundary $\partial\Omega$ thus allowing for a discontinuity over the interface Γ.

For the average across Γ of any sufficiently smooth function x discontinuous across Γ we will use the notation $\{x\} = \kappa_1 x_1 + \kappa_2 x_2$ and $\langle x \rangle = \kappa_2 x_1 + \kappa_1 x_2$ with $\kappa_i|_{\Gamma_K} = |K_i|/|K|$. Note that $\kappa_1 + \kappa_2 = 1$. We have

$$x_1 = \{x\} + \kappa_2 \llbracket x \rrbracket \text{ and } x_2 = \{x\} - \kappa_1 \llbracket x \rrbracket, \tag{6}$$

and therefore
$$\llbracket ab \rrbracket = \{a\} \llbracket b \rrbracket + \llbracket a \rrbracket \langle b \rangle. \tag{7}$$
We note that since $\nabla \cdot (p\mathcal{I} - 2\mu\boldsymbol{\varepsilon}(\boldsymbol{u})) \in L^2(\Omega)$ we have $(p\boldsymbol{n} - 2\mu\boldsymbol{\varepsilon}(\boldsymbol{u}) \cdot \boldsymbol{n}) \in H^{-\frac{1}{2}}(\Gamma)$. Integrating by parts and using (7) we now obtain for the interface term

$$\int_\Gamma \llbracket (p\boldsymbol{n} - 2\mu\boldsymbol{\varepsilon}(\boldsymbol{u}) \cdot \boldsymbol{n})\tilde{\boldsymbol{v}} \rrbracket \, \mathrm{d}s =$$
$$\int_\Gamma \{p\boldsymbol{n} - 2\mu\boldsymbol{\varepsilon}(\boldsymbol{u}) \cdot \boldsymbol{n}\} \llbracket \tilde{\boldsymbol{v}} \rrbracket \, \mathrm{d}s + \int_\Gamma \llbracket p\boldsymbol{n} - 2\mu\boldsymbol{\varepsilon}(\boldsymbol{u}) \cdot \boldsymbol{n} \rrbracket \langle \tilde{\boldsymbol{v}} \rangle \, \mathrm{d}s. \tag{8}$$

Since $\tilde{\boldsymbol{v}} \in [H^1(\Omega)]^2$ the quantities in (8) are well defined if the interface integrals are interpreted as duality pairings $H^{-\frac{1}{2}}(\Gamma)/H^{\frac{1}{2}}(\Gamma)$.

Using now the interface conditions (3) and (4), with $\kappa = 0$, in the second term we have
$$\int_\Gamma \llbracket p\boldsymbol{n} - 2\mu\boldsymbol{\varepsilon}(\boldsymbol{u}) \cdot \boldsymbol{n} \rrbracket \langle \tilde{\boldsymbol{v}} \rangle \, \mathrm{d}s = - \int_\Gamma \sigma\kappa \langle \tilde{\boldsymbol{v}} \cdot \boldsymbol{n} \rangle \, \mathrm{d}s. \tag{9}$$
After these preliminary considerations we are now ready to propose a finite element discretization of the problem, with this aim we introduce two conforming triangulations $\mathcal{T}_1$ and $\mathcal{T}_2$ such that the union of $\mathcal{T}_1$ and $\mathcal{T}_2$ gives a conforming triangulation of all Ω and for every triangle $K \in \mathcal{T}_1 \cap \mathcal{T}_2$ we have $K \cap \Gamma \neq \emptyset$. We also denote the set of *interior* faces of $\mathcal{T}_i$ by $\mathcal{F}_i$. Associated with $\mathcal{T}_1$ and $\mathcal{T}_2$ we have the finite element spaces

$$V_i^h = \{\boldsymbol{v} \in [C^0(\Omega_i)]^2 : \boldsymbol{v}|_K \in [P^1(K)]^2, \forall K \in \mathcal{T}_i, \boldsymbol{v}|_{\partial\Omega} = 0\},$$

$$Q_i^h = \{v \in L_0^2(\Omega_i) : v|_K \in P^0(K), \forall K \in \mathcal{T}_i\}.$$

Moreover we denote $V_h = V_1^h \times V_2^h$ and $Q_h = Q_1^h \times Q_2^h$. The finite element discretisation now takes the form: find $(\boldsymbol{u}_{i,h}, p_{i,h}) \in V_i^h \times Q_i^h$, $i = 1, 2$ such

that $\boldsymbol{u}_h = (\boldsymbol{u}_{1,h}, \boldsymbol{u}_{2,h})$, $p_h = (p_{1,h}, p_{2,h})$ satisfy

$$B_h[(\boldsymbol{u}_h, p_h), (\boldsymbol{v}_h, q_h)] + J(p_h, q_h) = (\boldsymbol{f}, \boldsymbol{v}_h) + \int_\Gamma \sigma\kappa\langle \boldsymbol{v}_h \cdot \boldsymbol{n}\rangle \, \mathrm{d}s, \qquad (10)$$

for all $(\boldsymbol{v}_h, q_h) \in V_h \times Q_h$, where

$$B_h[(\boldsymbol{u}_h, p_h), (\boldsymbol{v}_h, q_h)] = a_h(\boldsymbol{u}_h, \boldsymbol{v}_h) + b_h(p_h, \boldsymbol{v}_h) - b_h(q_h, \boldsymbol{u}_h)$$

$$a_h(\boldsymbol{u}_h, \boldsymbol{v}_h) = \int_\Omega 2\mu\boldsymbol{\varepsilon}(\boldsymbol{u}_h) : \boldsymbol{\varepsilon}(\boldsymbol{v}_h)\mathrm{d}x - \int_\Gamma \{2\mu\boldsymbol{\varepsilon}(\boldsymbol{u}_h) \cdot \boldsymbol{n}\} [\![\boldsymbol{v}_h]\!] \, \mathrm{d}s$$
$$- \int_\Gamma \{2\mu\boldsymbol{\varepsilon}(\boldsymbol{v}_h) \cdot \boldsymbol{n}\} [\![\boldsymbol{u}_h]\!] \, \mathrm{d}s + \int_\Gamma \gamma_u \mu_{\max} h^{-1} [\![\boldsymbol{u}_h]\!] [\![\boldsymbol{v}_h]\!] \, \mathrm{d}s,$$

$$b_h(p_h, v_h) = -\int_\Omega p_h \nabla \cdot \boldsymbol{v}_h \, \mathrm{d}x + \int_\Gamma \{p_h\} [\![\boldsymbol{v}_h \cdot \boldsymbol{n}]\!] \, \mathrm{d}s,$$

and

$$J(p_h, q_h) = \sum_{F \in \mathcal{F}_1} \int_F \frac{\gamma_1}{\mu_1} h_K[p_{1,h}][q_{1,h}]\mathrm{d}s + \sum_{F \in \mathcal{F}_2} \int_F \frac{\gamma_2}{\mu_2} h_K[p_{2,h}][q_{2,h}]\mathrm{d}s.$$

where $[p]$ denotes the jump of p across an element face F and $(\gamma_u, \gamma_1, \gamma_2)$ are constants, the size of which will be discussed in the following.

Remark 1 *Note that the velocities and the pressures are double valued in the overlap region. Moreover the pressure from both sides are stabilized over the whole interior face in the overlapping domain.*

The formulation (10) satisfies the following consistency relation

Lemma 2 *(Galerkin orthogonality) Let $(\boldsymbol{u}_h, p_h)$ be the solution of the finite element formulation (10) and $(\boldsymbol{u}, p) \in [H^2(\Omega_1) \cup H^2(\Omega_2)]^2 \times H^1(\Omega_1) \cup H^1(\Omega_2)$ be the solution of (5) then*

$$B_h[(\boldsymbol{u}_h - \boldsymbol{u}, p_h - p), (\boldsymbol{v}_h, q_h)] = -J(p_h, q_h) \quad \forall (\boldsymbol{v}_h, q_h) \in V_h \times Q_h. \qquad (11)$$

PROOF. Using the formulation (10) we may write

$$a_h(\boldsymbol{u}, \boldsymbol{v}_h) + b_h(p, \boldsymbol{v}_h) - b_h(q_h, \boldsymbol{u}) = (\boldsymbol{f}, \boldsymbol{v}_h) + \int_\Gamma \sigma\kappa\langle v_h \cdot \boldsymbol{n}\rangle \, \mathrm{d}s. \qquad (12)$$

The claim follows by noting that $b_h(q_h, \boldsymbol{u}) = 0$ and by integrating by parts in the remaining two terms on the left hand side and using equality (9) and equation (1).

A FE basis for V_h is easily obtained from a standard FE basis on the mesh by the introduction of new basis functions for the elements that intersect Γ.

For piecewise linears, the standard interior nodal basis functions in Ω may be partitioned into the two sets $\{\psi_i^j\}_{j=1}^{N_i}$ of basis functions with support entirely in Ω_i, $i = 1, 2$, and the set $\{\psi_\Gamma^k\}_{k=1}^M$ of basis functions which are non-zero on Γ. For each of the latter, let $\psi_{\Gamma,i}^j := \psi_\Gamma^j|_{\Omega_i}$. Then $\{\psi_i^j\}_{j=1}^{N_i} \cup \{\psi_{\Gamma,i}^k\}_{k=1}^M$ is a basis for V_i^h. As a consequence, the number of basis functions on each element that intersects Γ is doubled.

4 Approximation properties

The solution of (1) looses regularity at the interface. Indeed the pressure and normal component of the deformation tensor may jump. We introduce the broken Sobolev spaces

$$H^k(\Omega_1 \cup \Omega_2) := \{v : v|_{\Omega_i} \in H^k(\Omega_i), i = 1, 2\}$$

with the corresponding norms

$$\|u\|_{k,\Omega_1 \cup \Omega_2}^2 := \sum_{i=1}^2 \|u\|_{k,\Omega_i}^2$$

We need to show that our approximating spaces V_h and Q_h have optimal approximation properties on $H^2(\Omega_1 \cup \Omega_2)$ and $H^1(\Omega_1 \cup \Omega_2)$ respectively. This follows from some minor modifications of the analysis in [5]. Let G_h denote the set of elements that are intersected by the interface. We will use the following mesh dependent norms defined for functions in $[H^2(\Omega_1 \cup \Omega_2)]^2$ or $[H^2(\Omega_1 \cup \Omega_2)]^2 \times H^1(\Omega)$:

$$\|\boldsymbol{v}\|_{1/2,h,\Gamma}^2 := \sum_{K \in G_h} h_K^{-1} \|\boldsymbol{v}\|_{0,\Gamma_K}^2 ,$$

$$\|\boldsymbol{v}\|_{-1/2,h,\Gamma}^2 := \sum_{K \in G_h} h_K^1 \|\boldsymbol{v}\|_{0,\Gamma_K}^2 ,$$

and

$$\|\|\boldsymbol{v}\|\|^2 := \|\boldsymbol{v}\|_{0,\Omega}^2 + \left\|\mu^{1/2}\nabla\boldsymbol{v}\right\|_{0,\Omega}^2 + \|\{2\mu\boldsymbol{\varepsilon}(\boldsymbol{v}) \cdot \boldsymbol{n}\}\|_{-1/2,h,\Gamma}^2 + \|[\boldsymbol{v}]\|_{1/2,h,\Gamma}^2 .$$

On the product space $V \times Q$ we introduce the norm

$$\|\|(\boldsymbol{u}_h, p_h)\|\|^2 := \|\|\boldsymbol{u}_h\|\|^2 + \frac{1}{\mu_{\max}}\|p_h\|_{0,\Omega}^2$$

and on $V_h \times Q_h$ the norm

$$\|\|(\boldsymbol{u}_h, p_h)\|\|_h^2 := \|\|\boldsymbol{u}_h\|\|^2 + \frac{1}{\mu_{\max}}\|p_h\|_{0,\Omega}^2 + J(p_h, p_h).$$

We note for future reference that

$$(\boldsymbol{u}, \boldsymbol{v})_\Gamma \leq \|\boldsymbol{u}\|_{1/2,h,\Gamma} \|\boldsymbol{v}\|_{-1/2,h,\Gamma}. \tag{13}$$

To show that functions in the space (V_h, Q_h) approximate functions $\boldsymbol{v} \in [H_0^1(\Omega) \cap H^2(\Omega_1 \cup \Omega_2)]^2$ and $q \in L_0^2(\Omega) \cap H^1(\Omega_1 \cup \Omega_2)$ to the order h in the norm $\|\|\cdot\|\|$ and $\|\cdot\|_{0,\Omega}$, respectively, we construct an interpolant of $(\boldsymbol{v}, q)$ by standard interpolants of $[H^2]^2 \times H^1$–extensions of $(\boldsymbol{v}_1, q_1)$ and $(\boldsymbol{v}_2, q_2)$ as follows. Choose extension operators $\mathsf{E}_i^k : H^k(\Omega_i) \to H^k(\Omega)$ such that $(\mathsf{E}_i^2 \boldsymbol{v}, \mathsf{E}_i^1 q)|_{\Omega_i} = (\boldsymbol{v}, q)$ and

$$\|\mathsf{E}_i^k w\|_{s,\Omega} \leq C \|w\|_{s,\Omega_i}, \quad \forall w \in H^s(\Omega_i), \ s = 0, \ldots, k. \tag{14}$$

Let I_h be the standard interpolant and C_h be the local projection on the piecewise constants. We define

$$(I_h^* \boldsymbol{v}, C_h^* q) := ((I_{h,1}^* \boldsymbol{v}_1, I_{h,2}^* \boldsymbol{v}_2), (C_{h,1}^* q_1, C_{h,2}^* q_2)), \tag{15}$$

where $I_{h,i}^* \boldsymbol{v}_i := (I_h \mathsf{E}_i^2 \boldsymbol{v}_i)|_{\Omega_i}$ and $C_{h,i}^* q_i := (C_h \mathsf{E}_i^1 q_i)|_{\Omega_i}$. The following theorem is valid.

Theorem 3 *Let (I_h^*, C_h^*) be a pair of interpolation operators defined as in (15). Then*

$$\|\|\boldsymbol{v} - I_h^* \boldsymbol{v}\|\| \leq Ch \|\boldsymbol{v}\|_{2,\Omega_1 \cup \Omega_2}$$

and

$$\|q - C_h^* q\|_{0,\Omega} \leq Ch \|q\|_{1,\Omega_1 \cup \Omega_2}$$

for all $\boldsymbol{v} \in [H_0^1(\Omega) \cap H^2(\Omega_1 \cup \Omega_2)]^2$ and $p \in L_0^2(\Omega) \cap H^1(\Omega_1 \cup \Omega_2)$.

For the proof of this we need the following variant of a trace inequality on a reference element that we recall from [5] and state here without proof.

Lemma 4 *Map a triangle $K \in G_h$ onto the unit reference triangle $\tilde{K}$ by an affine map and denote by $\tilde{\Gamma}_{\tilde{K}}$ the corresponding image of Γ_K. Under the the assumptions A1-A3 of Section 3 there exists a constant C, depending on Γ but independent of the mesh, such that*

$$\|w\|_{0,\tilde{\Gamma}_{\tilde{K}}}^2 \leq C \|w\|_{0,\tilde{K}} \|w\|_{1,\tilde{K}}, \quad \forall w \in H^1(\tilde{K}). \tag{16}$$

PROOF. (Theorem 3)
Recall that $K_i = K \cap \Omega_i$ and let $\boldsymbol{v}_i^* = \mathsf{E}_i^2 \boldsymbol{v}_i$ denote the extension of $\boldsymbol{v}_i$ to Ω and similarly $q_i^* = \mathsf{E}_i^1 q_i$ denote the extension of q_i to Ω. By standard interpolation estimates for I_h and C_h respectively we now obtain

$$\|\nabla(\boldsymbol{v}_i - I_{h,i}^* \boldsymbol{v}_i)\|_{0,K_i}^2 = \|\nabla(\boldsymbol{v}_i^* - I_{h,i}^* \boldsymbol{v}_i^*)\|_{0,K_i}^2 \leq \|\nabla(\boldsymbol{v}_i^* - I_{h,i}^* \boldsymbol{v}_i^*)\|_{0,K}^2$$

$$\leq Ch_K^2 \|\boldsymbol{v}_i^*\|_{2,K}^2$$

and in the same fashion

$$\|q_i - C_{h,i}^* q_i\|_{0,K_i}^2 \leq C h_K^2 \|q_i^*\|_{1,K}^2$$

Summing over all triangles we obtain using (14)

$$\|\nabla(\boldsymbol{v}_i - I_{h,i}^* \boldsymbol{v}_i)\|_{0,\Omega_i}^2 \leq C h^2 \sum_{K \cap \Omega_i \neq \emptyset} \|\boldsymbol{v}_i^*\|_{2,K}^2 \leq C h^2 \|\boldsymbol{v}_i\|_{2,\Omega_i}^2 \qquad (17)$$

and

$$\|q_i - C_{h,i}^* q_i\|_{0,\Omega_i}^2 \leq C h^2 \|q_i\|_{1,\Omega_i}^2. \qquad (18)$$

We turn now to the jumps on the interface. Since the mesh is non-degenerate, it follows from Lemma 4, scaled by the map from the reference triangle, that for $s \in \mathbb{R}$

$$h_K^{-s} \|w\|_{0,\Gamma_K}^2 \leq C\Big(h_K^{-1-s}\|w\|_{0,K}^2 + h_K^{1-s}\|w\|_{1,K}^2\Big), \quad \forall w \in H^1(K)$$

for $s \in \mathbb{R}$. Hence it follows, using once again standard interpolation estimates, that

$$h_K^{-1}\|\, [\![\boldsymbol{v} - I_h^*\boldsymbol{v}]\!]\, \|_{0,\Gamma_K}^2 \leq C h_K^{-1} \sum_i \|\boldsymbol{v}_i - I_{h,i}^*\boldsymbol{v}_i\|_{0,\Gamma_K}^2$$

$$= C h_K^{-1} \sum_i \|\boldsymbol{v}_i^* - I_h\boldsymbol{v}_i^*\|_{0,\Gamma_K}^2$$

$$\leq C \sum_i \Big(h_K^{-2}\|\boldsymbol{v}_i^* - I_h\boldsymbol{v}_i^*\|_{0,K}^2 + \|\boldsymbol{v}_i^* - I_h\boldsymbol{v}_i^*\|_{1,K}^2\Big)$$

$$\leq C h_K^2 \sum_i \|\boldsymbol{v}_i^*\|_{2,K}^2$$

and Summing the contributions from $K \in G_h$, we get from (14) that

$$\|\, [\![\boldsymbol{v} - I_h^*\boldsymbol{v}]\!]\, \|_{1/2,h,\Gamma} \leq C h \sum_i \|\boldsymbol{v}_i^*\|_{2,\bigcup K \in G_h} \leq C h \|\boldsymbol{v}\|_{2,\Omega_1 \cup \Omega_2} \quad \text{and} \qquad (19)$$

$$\|\, [\![q - C_h^*q]\!]\, \|_{-1/2,h,\Gamma} \leq C h \sum_i \|q_i^*\|_{1,\bigcup K \in G_h} \leq C h \|q\|_{1,\Omega_1 \cup \Omega_2}. \qquad (20)$$

Next, we consider the mean viscous stress at the interface. Lemma 4 applied to $\boldsymbol{n} \cdot \nabla \boldsymbol{w}$ and scaling gives

$$h_K \|\boldsymbol{n} \cdot \nabla \boldsymbol{w}\|_{0,\Gamma_K}^2 \leq C(\|\boldsymbol{w}\|_{1,K}^2 + h_K^2 \|\boldsymbol{w}\|_{2,K}^2), \quad \forall \boldsymbol{w} \in [H^2(K)]^2.$$

Using this result applied to $\boldsymbol{w} = \boldsymbol{v}_i^* - I_h \boldsymbol{v}_i^*$ and again standard interpolation estimates, it follows that

$$
\begin{aligned}
h_K \|\{\boldsymbol{n} \cdot \nabla(\boldsymbol{v} - I_h^* \boldsymbol{v})\}\|_{0,\Gamma_K}^2 &\le C h_K \sum_i \|\nabla(\boldsymbol{v}_i - I_{h,i}^* \boldsymbol{v}_i) \cdot \boldsymbol{n}\|_{0,\Gamma_K}^2 \\
&= C h_K \sum_i \|\nabla(\boldsymbol{v}_i^* - I_h \boldsymbol{v}_i^*) \cdot \boldsymbol{n}\|_{0,\Gamma_K}^2 \\
&\le C \sum_i \left(h_K \|\boldsymbol{v}_i^* - I_h \boldsymbol{v}_i^*\|_{1,K}^2 + h_K^2 \|\boldsymbol{v}_i^* - I_h \boldsymbol{v}_i^*\|_{2,K}^2 \right) \\
&\le C h_K^2 \sum_i \|\boldsymbol{v}_i^*\|_{2,K}^2 .
\end{aligned}
$$

Summing again the contributions from $K \in G_h$, we from (14) that

$$
\|\{\boldsymbol{n} \cdot \nabla(\boldsymbol{v}_i - I_{h,i}^* \boldsymbol{v}_i)\}\|_{-1/2,h,\Gamma} \le C h \|\boldsymbol{v}_i\|_{2,\Omega_i}. \tag{21}
$$

The claim now follows from equations (17), (18), (19).

Lemma 5 *Let $q \in H^1(\Omega_1 \cup \Omega_2)$ then there holds*

$$
J(C_h^* q^*, C_h^* q^*) \le C h^2 \|q\|_{H^1(\Omega_1 \cup \Omega_2)}^2 .
$$

PROOF. This is easily proved using the standard trace inequality in the following fashion

$$
\begin{aligned}
h_K \|[q_i^* - C_h^* q_i^*]\|_{0,F}^2 &\le C \left(\|q_i^* - C_h^* q_i^*\|_{0,K}^2 + h_K^2 \|q_i^*\|_{1,K}^2 \right) \\
&\le C h_K^2 \sum \|q_i^*\|_{1,K}^2 .
\end{aligned}
$$

This leads to

$$
\begin{aligned}
J(C_h^* q^*, C_h^* q^*) &\le \sum_{K \in \mathcal{T}_1} C h_K^2 \sum \|q_1^*\|_{1,K}^2 + \sum_{K \in \mathcal{T}_2} C h_K^2 \sum \|q_2^*\|_{1,K}^2 \\
&\le C h^2 (\|q\|_{1,\Omega_1}^2 + \|q\|_{1,\Omega_2}^2)
\end{aligned}
$$

5   The inf-sup condition

We need to show that the formulation satisfies an *inf-sup* condition with regard to the norm $\|\|(\cdot,\cdot)\|\|_h$ in the case $\lambda_i \to \infty$. We remark that if λ_i remain bounded from above, the pressure does not need stabilization and can be

eliminated elementwise, recovering the constant–strain finite element method. In this case there is no stability problem in the pressure, and the results of [6] can be directly applied in the compressible subdomain. We shall therefore in the following concentrate on the case when both materials are assumed incompressible.

Observe that the pressure stabilization in $J(\cdot, \cdot)$ acts over the whole interior faces in G_h, i.e. not only the part of the faces in the computational domain Ω. This is of fundamental importance for the below analysis, indeed this "ghost penalty" is what gives us stability of the interface pressure.

Here we prove the *inf-sup* condition only in the particular case of the mixed stabilized $P_1 \times P_0$ formulation for convenience. However, the general framework holds also when using for instance the pressure stabilisations proposed in [4] or [1] and we expect the *inf-sup* condition to hold uniformly in these cases as well, provided the stabilization operator acts in all elements of $\mathcal{T}_1$ and $\mathcal{T}_2$.

We need the following inverse inequality, for a proof of which we refer to [5].

Lemma 6 *For $v \in V_h$, the following inverse inequality holds:*

$$\|\{\boldsymbol{n} \cdot \nabla \boldsymbol{v}\}\|^2_{-1/2,h,\Gamma} \leq C_I \|\nabla \boldsymbol{v}\|^2_{0,\Omega_1 \cup \Omega_2}$$

5.1 Fortin interpolant

The key result for the inf-sup condition is the construction of the Fortin interpolant. The upshot in the case of Nitsche XFEM for Stokes' problem is the need to construct a Fortin interpolant with some additional orthogonality on the interface. The interpolant defined on mesh $\mathcal{T}_i$ will be denoted by π^i_h and the global interpolant by π_h. Note that the global interpolant is not continuous across the interface. Let us consider the construction of the Fortin interpolant on the space V^h_1 the construction on V^h_2 then follows by symmetry. Let $\mathcal{T}_\Gamma$ by the subset of triangles of $\mathcal{T}_1$ such that

$$\mathcal{T}_\Gamma = \{T \in \mathcal{T}_1 : T \cap (\mathcal{T}_1 \cap \mathcal{T}_2) \neq \emptyset\}.$$

That is all the elements cut by the interface and the elements sharing a face or a vertex with them. This is a subset of the mesh of width approximately $2h$. Let us now regroup the elements in $\mathcal{T}_\Gamma$ in patches $\{\mathcal{P}_i\}$, cutting up the interface zone in macroelements, each containing a "sufficient" amount of basis functions to construct a patch function $\varphi_i > 0$ associated to each P_i that is zero on the interior patch boundary and takes the value 1 on at least one face cut by the interface. The $\mathcal{P}_i$ can be constructed so that

- $\exists c_1, c_2 > 0$ such that $c_1 h \leq \text{diam}(P_i) \leq c_2 h$.

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- $\exists c_1, c_2 > 0 \ c_1 h \leq \int_{\Gamma \cap P_i} \varphi_i \ \mathrm{d}s \leq c_2 h.$
- $\exists c_1, c_2 > 0 \ c_1 h^{-1} \leq |\nabla \varphi_i| \leq c_2 h^{-1}.$

We want to prove that we can construct an interpolant $\pi_h : [H^1(\Omega)]^2 \to V^h$ such that

$$\|\boldsymbol{v} - \pi_h \boldsymbol{v}\|_{0,\Omega} + h\|\nabla(\boldsymbol{v} - \pi_h \boldsymbol{v})\|_{0,\Omega_1 \cup \Omega_2} \leq ch\|\nabla \boldsymbol{v}\|_{0,\Omega}$$

and

$$\int_{\Gamma \cap P_i} (\boldsymbol{v} - \pi_h^k \boldsymbol{v}) \ \mathrm{d}s = 0, \ \text{ for } k = 1, 2 \tag{22}$$

for all $\mathcal{P}_i$.

The idea is to start from the global Clément interpolant of $\boldsymbol{v}$ and then construct a correction using the space $\Phi = \{\varphi_i\}$ so that the condition (22) is satisfied. Let $I_h \boldsymbol{v} \in [C^0(\bar{\Omega})]^2$ denote the global Clément interpolant of $\boldsymbol{v}$. In each patch $\mathcal{P}_i$ there is exactly one (vector valued) degree of freedom $\boldsymbol{y}_i$ associated to the basis function φ_i. We know fix the $\boldsymbol{y}_i$ by requiring

$$\int_{\Gamma \cap \mathcal{P}_i} \boldsymbol{\varphi}_i \boldsymbol{y}_i \ \mathrm{d}s = \int_{\Gamma \cap \mathcal{P}_i} (\boldsymbol{v} - I_h \boldsymbol{v}) \ \mathrm{d}s.$$

where

$$\boldsymbol{\varphi} = \begin{bmatrix} \varphi_i & 0 \\ 0 & \varphi_i \end{bmatrix}.$$

Finally we set $\pi_h^1 \boldsymbol{v} = I_h \boldsymbol{v}|_{\mathcal{T}_1} + \sum_i \boldsymbol{\varphi}_i \boldsymbol{y}_i$. Clearly, by construction (22) holds. The approximability is a consequence of the stability of the projection. Since $\pi_h^1 \boldsymbol{v} = I_h \boldsymbol{v}$, except in the interface zone, we only need to prove stability for the interpolant in $\mathcal{T}_\Gamma$. This is where the position of the interface could upset things. The fact that we have chosen the patches to include the neighbours seems to work around this problem. Consider, for some $K \in \mathcal{P}_i$

$$\|\nabla \pi_h^1 \boldsymbol{v}\|_{K,0} \leq \|\nabla I_h \boldsymbol{v}\|_{K \cap \Omega_1, 0} + \|\nabla(\boldsymbol{\varphi}_i \boldsymbol{y}_i)\|_{K \cap \Omega_1, 0}$$

$$\leq c\|\nabla \boldsymbol{v}\|_{K,0} + ch^{-1}|\boldsymbol{y}_i|\mathrm{meas}(K)^{\frac{1}{2}}$$

$$\leq c(\|\nabla \boldsymbol{v}\|_{K,0} + |\boldsymbol{y}_i|).$$

It remains to show that $|\boldsymbol{y}_i|$ is bounded by the norm of $\nabla \boldsymbol{v}$ over some patch of diameter h. This follows by using the definition, a Cauchy-Schwarz inequality,

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a trace inequality and the interpolation properties of the Clement interpolant:

$$
\begin{aligned}
|\boldsymbol{y}_i| &\leq \left| \frac{\displaystyle\int_{\Gamma \cap \mathcal{P}_i} (\boldsymbol{v} - I_h \boldsymbol{v}) \, \mathrm{d}s}{\displaystyle\int_{\Gamma \cap \mathcal{P}_i} (\boldsymbol{\varphi}_i : \boldsymbol{\varphi}_i)^{1/2} \, \mathrm{d}s} \right| \leq ch^{-1} \left| \int_{\Gamma \cap \mathcal{P}_i} (\boldsymbol{v} - I_h \boldsymbol{v}) \, \mathrm{d}s \right| \\[2ex]
&\leq ch^{-\frac{1}{2}} \|\boldsymbol{v} - I_h \boldsymbol{v}\|_{0,\Gamma \cap \mathcal{P}_i} \\[2ex]
&\leq c(h^{-1} \|\boldsymbol{v} - I_h \boldsymbol{v}\|_{0,\mathcal{P}_i} + \|\nabla(\boldsymbol{v} - I_h \boldsymbol{v})\|_{0,\mathcal{P}_i}) \leq c \|\nabla \boldsymbol{v}\|_{0,\Delta_{\mathcal{P}_i}},
\end{aligned}
$$

where $\Delta_{\mathcal{P}_i}$ denotes the patch consisting of the elements in $\mathcal{P}_i$ and those sharing a node with an element in $\mathcal{P}_i$. In all interior elements the H^1-stability is a consequence of the properties of the Clément-interpolant.

The approximability in L^2 is now easily obtained, noting that for $K \in \mathcal{P}_i$

$$
\|\boldsymbol{v} - \pi_h^1 \boldsymbol{v}\|_{0,K} \leq \|\boldsymbol{v} - I_h \boldsymbol{v}\|_{0,K} + \|\boldsymbol{\varphi}_i \boldsymbol{y}_i\|_{0,K} \leq \|\boldsymbol{v} - I_h \boldsymbol{v}\|_{0,K} + ch_K |\boldsymbol{y}_i|
$$

and applying the previous result. Using arguments similar to that of Section 4 one may readily prove the stability estimate

$$
\|\|\pi_h \boldsymbol{v}\|\| \leq c \|\boldsymbol{v}\|_{1,\Omega}, \quad \forall \boldsymbol{v} \in [H^1(\Omega)]^2. \tag{23}
$$

We will now use the proposed interpolant to prove the inf-sup condition for our formulation. Let us start with a preliminary lemma

Lemma 7 *Let* $v \in [H^1(\Omega)]^2$; *then there holds, for* $i = 1, 2$,

$$
|(\llbracket p_h \rrbracket, \boldsymbol{n} \cdot (\boldsymbol{v} - \pi_h^i \boldsymbol{v}))_\Gamma| \leq c J(p_h, p_h)^{\frac{1}{2}} \|\mu_{\max}^{\frac{1}{2}} \nabla \boldsymbol{v}\|_{0,\Omega}
$$

PROOF. Using the orthogonality on the patches $\mathcal{P}_k$, we may write, with $\xi_k \in \mathbb{R}$,

$$
\begin{aligned}
|(\llbracket p_h \rrbracket, \boldsymbol{n} \cdot (\boldsymbol{v} - \pi_h^i \boldsymbol{v}))_\Gamma| &\leq \sum_k |(\llbracket p_h \rrbracket, \boldsymbol{n} \cdot (\boldsymbol{v} - \pi_h^i \boldsymbol{v}))_{\mathcal{P}_k \cap \Gamma}| \\[2ex]
&\leq \sum_k \| \llbracket p_h \rrbracket - \xi_k \|_{0,\mathcal{P}_k \cap \Gamma} \|\boldsymbol{v} - \pi_h^i \boldsymbol{v}\|_{0,\mathcal{P}_k \cap \Gamma} \\[2ex]
&\leq C \sum_k \| \llbracket p_h \rrbracket - \xi_k \|_{0,\mathcal{P}_k \cap \Gamma} h^{\frac{1}{2}} \|\nabla \boldsymbol{v}\|_{0,\mathcal{P}_i} \\[2ex]
&\leq C \left(h \sum_k \| \llbracket p_h \rrbracket - \xi_k \|_{0,\mathcal{P}_k \cap \Gamma}^2 \right)^{\frac{1}{2}} \|\nabla \boldsymbol{v}\|_{0,\Omega}
\end{aligned}
$$

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We now choose ξ_k as the arithmetic mean of all the jumps in the patch $\mathcal{P}_k$,

$$\xi_k = \frac{1}{N_k} \sum_{K \in \mathcal{P}_k : K \cap \Gamma \neq \emptyset} [\![p_h]\!] \, |_{K \cap \Gamma}$$

where N_k denotes the number of triangles in $\mathcal{P}_k \cap \mathcal{T}_\Gamma$. It then follows, since N_k is bounded from above by a moderate constant, uniformly in h and k and using the relation $(a-b)^2 \leq 2a^2 + 2b^2$, that for each $K \in \mathcal{P}_k$ such that $K \cap \Gamma \neq \emptyset$

$$\| [\![p_h]\!] - \xi_k \|_{K \cap \Gamma}^2 \leq 2 \| N_k^{-1} \textstyle\sum_{K' \in \mathcal{K}'} (p_{1,h}|_K - p_{1,h}|_{K'}) \|_{0,K \cap \Gamma}^2$$

$$+ 2 \| N_k^{-1} \textstyle\sum_{K' \in \mathcal{K}'} (p_{2,h}|_K - p_{2,h}|_{K'}) \|_{0,K \cap \Gamma}^2$$

$$\leq C \left(\sum_{F \in \mathcal{E}_k} \int_F [p_{1,h}]^2 \, \mathrm{d}s + \int_F [p_{2,h}]^2 \, \mathrm{d}s \right)$$

where $\mathcal{K}' := \{K' \in \mathcal{P}_i : K' \cap \Gamma \neq \emptyset; K' \neq K\}$, $\mathcal{E}_k := \{F \in \mathcal{P}_k : F \cap \Gamma \neq \emptyset\}$ and where the constant C depends only on the N_k. We conclude by summing over all the patches $\mathcal{P}_i$.

We will now prove the main result of this section:

Theorem 8 *Let* $(\boldsymbol{u}_h, p_h) \in V_h \times Q_h$. *Then*

$$c_s \, |\!|\!| \, (\boldsymbol{u}_h, p_h) |\!|\!|_h \leq \sup_{(\boldsymbol{w}_h, q_h) \in V_h \times Q_h} \frac{B_h[(\boldsymbol{u}_h, p_h), (\boldsymbol{v}_h, q_h)] + J(p_h, q_h)}{|\!|\!|(\boldsymbol{v}_h, q_h)|\!|\!|_h}. \tag{24}$$

PROOF. We start by choosing $(\boldsymbol{v}_h, q_h) = (\boldsymbol{u}_h, p_h)$ to obtain

$$B_h[(\boldsymbol{u}_h, p_h), (\boldsymbol{u}_h, p_h)] + J(p_h, p_h) = a_h(\boldsymbol{u}_h, \boldsymbol{u}_h) + J(p_h, p_h).$$

We will need a version of Korn's inequality for functions in V_h. From [3] (eq. (1.19)), using $V_h \subset [H^1(\Omega_1 \cup \Omega_2)]^d$ and $\boldsymbol{v}|_{\partial \Omega} = 0$ for $\boldsymbol{v} \in V_h$, we have:

$$\| \boldsymbol{\varepsilon}(\boldsymbol{v}) \|_{0,\Omega_1 \cup \Omega_2}^2 + \| [\boldsymbol{v}] \|_{1/2,h,\Gamma}^2 \geq C |\boldsymbol{v}|_{1,\Omega_1 \cup \Omega_2}^2 \quad \forall v \in V_h,$$

while from [2], using $\boldsymbol{v}|_{\partial \Omega} = 0$, $\boldsymbol{v}|_{\Gamma_K} \in L^2(\Gamma_K)$ for $\boldsymbol{v} \in V_h$ and a standard error estimate, we have

$$|\boldsymbol{v}|_{1,\Omega_1 \cup \Omega_2}^2 + \| [\boldsymbol{v}] \|_{1/2,h,\Gamma}^2 \geq C \|\boldsymbol{v}\|_{0,\Omega}^2 \quad \forall \boldsymbol{v} \in V_h$$

and thus

$$\| \boldsymbol{\varepsilon}(\boldsymbol{v}) \|_{0,\Omega_1 \cup \Omega_2}^2 + \| [\boldsymbol{v}] \|_{1/2,h,\Gamma}^2 \geq C_K \|\boldsymbol{v}\|_{1,\Omega_1 \cup \Omega_2}^2 \quad \forall \boldsymbol{v} \in V_h. \tag{25}$$

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See also the proof on discrete spaces of [7]. Applying now (25), (13) and Lemma 6, we get

$$
\begin{aligned}
a_h(\boldsymbol{u}_h, \boldsymbol{u}_h) &+ J(p_h, p_h) \\
&\geq \|\mu^{1/2}\boldsymbol{\varepsilon}(\boldsymbol{u}_h)\|^2_{0,\Omega_1 \cup \Omega_2} - 2\|\{2\mu\boldsymbol{n} \cdot \boldsymbol{\varepsilon}(\boldsymbol{u}_h)\}\|_{-1/2,h,\Gamma}\|[\boldsymbol{u}_h]\|_{1/2,h,\Gamma} \\
&\quad + \gamma_u\|\mu^{1/2}_{\max}[\boldsymbol{u}_h]\|^2_{1/2,h,\Gamma} + \sum_{i=1}^{2}\sum_{F \in \mathcal{F}_i}\frac{\gamma_i}{\mu_i}h_K\|[p_{i,h}]\|^2_{0,F} \\
&\geq \frac{C_K}{2}\mu_{\min}\|\boldsymbol{u}_h\|^2_{1,\Omega_1 \cup \Omega_2} + \left(\frac{\mu_{\min}}{2} - \frac{8C_I\mu_{\max}}{\epsilon}\right)\|\boldsymbol{\varepsilon}(\boldsymbol{u}_h)\|^2_{0,\Omega_1 \cup \Omega_2} \\
&\quad + \frac{1}{\epsilon}\|\{2\mu\boldsymbol{n} \cdot \boldsymbol{\varepsilon}(\boldsymbol{u}_h)\}\|^2_{-1/2,h,\Gamma} + (\gamma_u\mu_{\max} - \epsilon\mu_{\max})\|[\boldsymbol{u}_h]\|^2_{1/2,h,\Gamma} \\
&\qquad\qquad\qquad\qquad\qquad + \sum_{i=1}^{2}\sum_{F \in \mathcal{F}_i}\frac{\gamma_i}{\mu_i}h_K\|[p_{i,h}]\|^2_{0,F}. \quad (26)
\end{aligned}
$$

Taking $\epsilon = 16C_I\frac{\mu_{\max}}{\mu_{\min}}$,

$$
c\,\|\|\,\boldsymbol{u}_h\,\|\|^2 \leq B_h[(\boldsymbol{u}_h, p_h), (\boldsymbol{u}_h, p_h)] + J(p_h, p_h)
$$

for $\gamma_u > \epsilon$.

By the surjectivity of the divergence operator from $[H_0^1(\Omega)]^2$ to $L_0^2(\Omega)$ there exists $\boldsymbol{v}_p \in [H_0^1(\Omega)]^2$ such that $\mu_{\max}^{-1}p_h = -\nabla \cdot \boldsymbol{v}_p$ and $\|\boldsymbol{v}_p\|_{1,\Omega} \leq c\mu_{\max}^{-1}\|p_h\|_{0,\Omega}$. Taking now $(\boldsymbol{w}_h, q_h) = (\pi_h\boldsymbol{v}_p, 0)$ where π_h is the interpolation operator introduced in the previous section we obtain

$$
B_h[(\boldsymbol{u}_h, p_h), (\pi_h\boldsymbol{v}_p, 0)] = a_h(u_h, \pi_h\boldsymbol{v}_p) + b_h(p_h, \pi_h\boldsymbol{v}_p - \boldsymbol{v}_p) + \mu_{\max}^{-1}\|p_h\|^2_{0,\Omega}.
$$

Using the Cauchy-Schwarz inequality we have

$$
a_h(u_h, \pi_h\boldsymbol{v}_p) \geq - \|\|\,\boldsymbol{u}_h\,\|\| \; \|\|\pi_h\boldsymbol{v}_p\|\|
$$

$$
\geq -C\,\|\|\,\boldsymbol{u}_h\,\|\|\; \mu_{\max}^{\frac{1}{2}}\|\nabla\boldsymbol{v}_p\|_{0,\Omega} \geq -C\,\|\|\,\boldsymbol{u}_h\,\|\|\; \mu_{\max}^{-\frac{1}{2}}\|p_h\|_{\Omega}.
$$

Using integration by parts in the term $b_h(p_h, \pi_h\boldsymbol{v}_p - \boldsymbol{v}_p)$ we have

$$
\begin{aligned}
b_h(p_h, \pi_h\boldsymbol{v}_p - \boldsymbol{v}_p) &= \sum_{K \in \mathcal{T}_h}\int_{\partial K}[p_h]\,\boldsymbol{n} \cdot (\boldsymbol{v}_p - \pi_h\boldsymbol{v}_p)\,\mathrm{d}s \\
&\quad + \sum_i\int_{\mathcal{P}_i \cap \Gamma}\left([\![p_h\,\boldsymbol{n} \cdot (\boldsymbol{v}_p - \pi_h\boldsymbol{v}_p)]\!] - \{p_h\}\,[\![\boldsymbol{n} \cdot (\boldsymbol{v}_p - \pi_h\boldsymbol{v}_p)]\!]\right)\,\mathrm{d}s \\
&= \sum_{K \in \mathcal{T}_h}\int_{\partial K}[p_h]\,\boldsymbol{n} \cdot (\boldsymbol{v}_p - \pi_h\boldsymbol{v}_p)\,\mathrm{d}s + \sum_i\int_{\mathcal{P}_i \cap \Gamma}[\![p_h]\!]\,\boldsymbol{n} \cdot \langle\boldsymbol{v}_p - \pi_h\boldsymbol{v}_p\rangle\,\mathrm{d}s \\
&= I + II.
\end{aligned}
$$

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Treating these term separately we first note that by using the approximation properties of π_h we have

$$I \geq -C \, J(p_h, p_h)^{\frac{1}{2}} \|\mu_{\max}^{-\frac{1}{2}} p_h\|_{0,\Omega}.$$

For the second term we use Lemma 7 which gives

$$II \geq -\left| \sum_i \int_{\mathcal{P}_i \cap \Gamma} [\![p_h]\!] \, \boldsymbol{n} \cdot \langle \boldsymbol{v}_p - \pi_h \boldsymbol{v}_p \rangle \, \mathrm{d}s \right| \geq -C \, J(p_h, p_h)^{\frac{1}{2}} \|\mu_{\max}^{-\frac{1}{2}} p_h\|_{0,\Omega}.$$

Note that the different weights here are of no importance since the orthogonality property is satisfied on each side separately and since the weights are upper bounded by 1. We conclude that

$$B_h[(\boldsymbol{u}_h, p_h), (\pi_h \boldsymbol{v}_p, 0)] \geq \|\mu_{\max}^{-\frac{1}{2}} p_h\|_{0,\Omega}^2 - C \left(J(p_h, p_h)^{\frac{1}{2}} + \|\!|\boldsymbol{u}_h|\!\|\right) \|\mu_{\max}^{-\frac{1}{2}} p_h\|_{0,\Omega} \tag{27}$$

Using now an arithmethic-geometric inequality we get

$$B_h[(\boldsymbol{u}_h, p_h), (\pi_h \boldsymbol{v}_p, 0)] \geq \tfrac{1}{2}\|\mu_{\max}^{-\frac{1}{2}} p_h\|_{0,\Omega}^2 - C J(p_h, p_h) + C \, \|\!|\, \boldsymbol{u}_h \,|\!\|^2 \tag{28}$$

for $\epsilon < C^{-1}$ there holds

$$c \, \|\!|\!| \, (\boldsymbol{u}_h, p_h) \|\!|\!|_h^2 \leq B[(\boldsymbol{u}_h, p_h), (\boldsymbol{u}_h + \epsilon \pi_h \boldsymbol{v}_p, p_h)].$$

The claim now follows noting that there exists c_s such that

$$2c_s \, \|\!|\!| \, (\boldsymbol{u}_h + \epsilon \pi_h \boldsymbol{v}_p, p_h) \|\!|\!|_h \leq \|\!|\!|(\boldsymbol{u}_h, p_h)\|\!|\!|_h,$$

which is a consequence of the stability of $\pi_h \boldsymbol{v}_p$.

5.2 A sketch of the case of one compressible and one incompressible material

Let assume that λ_1 is bounded but not λ_2. We will then work in the norm

$$\|\!|\!|(\boldsymbol{u}_h, p_h)\|\!|\!|_h^2 = \|\!|\boldsymbol{u}_h \,|\!\|^2 + \frac{1}{\mu_{\max}}\|p_h\|_{0,\Omega_2}^2 + \lambda_1^{-1}\|p_h\|_{0,\Omega_1}^2 + J(p_h, p_h).$$

Note that coercivity now gives us

$$\|\!|\boldsymbol{u}_h \,|\!\|^2 + \lambda_1^{-1}\|p_h\|_{0,\Omega_1}^2 + J(p_h, p_h) \leq B_h[(u_h, p_h), (v_h, q_h)] + J(p_h, q_h).$$

However the pressure is no longer in $L_0^2(\Omega)$. On the other hand we can repeat the above analysis using v_p such that $\nabla \cdot v_p = \mu_{\max}^{-1}(p_h - \bar{p}_h)$, where $\bar{p}_h$ denotes the average of p_h over Ω. Clearly

$$\mu_{\max}^{-1}\|p_h - \bar{p}_h\|_{0,\Omega}^2 = (p_h, \nabla \cdot v_p)_\Omega$$

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and we proceed as before in order to obtain a bound of the quantity:

$$\lambda_1^{-1}\|p_1\|_{0,\Omega_1}^2 + \mu_{\max}^{-1}\|p_h - \bar{p}_h\|_{0,\Omega}^2. \tag{29}$$

Now we stress that by the decomposition $p_{i,h} = (p_{i,h} - \bar{p}_{i,h}) + \bar{p}_{i,h}$ and the orthogonality $((p_{i,h} - \bar{p}_{i,h}), \bar{p}_{i,h} - \bar{p}_h)_{\Omega_i} = 0$ there holds

$$\|p_h - \bar{p}_h\|_{0,\Omega}^2 = \|p_{2,h} - \bar{p}_{2,h}\|_{0,\Omega_2}^2 + \|p_{1,h} - \bar{p}_{1,h}\|_{0,\Omega_1}^2 + \|\bar{p}_h - \bar{p}_{1,h}\|_{0,\Omega_1}^2 + \|\bar{p}_h - \bar{p}_{2,h}\|_{0,\Omega_2}^2.$$

On the other hand observing that for $i = 1, 2$

$$\|p_{i,h}\|_{0,\Omega_i}^2 = \|\bar{p}_{i,h}\|_{0,\Omega_i}^2 + \|p_{i,h} - \bar{p}_{i,h}\|_{0,\Omega_i}^2$$

it follows that we only need to bound $\|\bar{p}_{2,h}\|_{0,\Omega_2}^2$. Observing that

$$\begin{aligned}
\|\bar{p}_{2,h}\|_{0,\Omega_2} &\le \|\bar{p}_h\|_{0,\Omega_2} + \|\bar{p}_h - \bar{p}_{2,h}\|_{0,\Omega_2} \\
&\le \|\bar{p}_{1,h}\|_{0,\Omega_2} + \|\bar{p}_h - \bar{p}_{1,h}\|_{0,\Omega_2} + \|\bar{p}_h - \bar{p}_{2,h}\|_{0,\Omega_2} \\
&\le (\operatorname{meas}(\Omega_2)/\operatorname{meas}(\Omega_1))^{\frac{1}{2}}(\|\bar{p}_{1,h}\|_{0,\Omega_1} + \|\bar{p}_h - \bar{p}_{1,h}\|_{0,\Omega_1}) + \|\bar{p}_h - \bar{p}_{2,h}\|_{0,\Omega_2},
\end{aligned}$$

the required bound now follows from (29), since we have *inf-sup* control of the right hand.

Note that the constant still degenerates for large λ_1. This reflects the instability of the problem due to the average value of p_h that has to be fixed by the small L^2-coercivity. The same holds for the case where λ_1 is moderate on a very small subdomain Ω_1. The quantities $\mu_{\max}^{-1}\|p_{1,h} - \bar{p}_{1,h}\|_{0,\Omega_1}^2$, $\mu_{\max}^{-1}\|p_{2,h} - \bar{p}_{2,h}\|_{0,\Omega_2}^2$ and $\mu_{\max}^{-1}\|p_h - \bar{p}_h\|_{0,\Omega}^2$ however can be controlled independently of λ_1.

6 A priori error estimates

Proposition 9 *Assume that the solution $(\boldsymbol{u}, p)$ to problem (2) resides in $[H^2(\Omega_1 \cup \Omega_2) \cap H^1(\Omega)]^2 \times H^1(\Omega_1 \cup \Omega_2) \cap L_0^2(\Omega)$; then the finite element solution (10) satisfies the error estimate*

$$\||(\boldsymbol{u} - \boldsymbol{u}_h, p - p_h)\|| \le c_\mu h(\|\boldsymbol{u}\|_{2,\Omega_1\cup\Omega_2} + \|p\|_{1,\Omega_1\cup\Omega_2})$$

where c_μ is independent of h but not of the ratio $\left(\frac{\mu_{\max}}{\mu_{\min}}\right)$.

PROOF. In view of Theorem 3 we only need to show the inequality for $\||(I_h^*\boldsymbol{u} - \boldsymbol{u}_h, C_h^*p - p_h)\||$. First we set $\boldsymbol{\eta}_h = \boldsymbol{u}_h - I_h^*\boldsymbol{u}$ and $\zeta_h = p_h - C_h^*p$ and note that

$$\||(\boldsymbol{u} - \boldsymbol{u}_h, p - p_h)\|| \le \||(\boldsymbol{u} - I_h^*\boldsymbol{u}, p - C_h^*p)\|| + \||(\boldsymbol{\eta}_h, \zeta_h)\||_h.$$

By Theorem 8 and using Galerkin orthogonality we obtain,

$$\||(\boldsymbol{\eta}_h, \zeta_h)\||_h \leq \frac{1}{c_s} \sup_{(\boldsymbol{v}_h, q_h) \in V_h \times Q_h} \frac{B_h[(\boldsymbol{\eta}_h, \zeta_h), (\boldsymbol{v}_h, q_h)] + J(\zeta_h, q_h)}{\||(\boldsymbol{v}_h, q_h)\||_h}$$

$$\leq \frac{1}{c_s} \sup_{(\boldsymbol{v}_h, q_h) \in V_h \times Q_h} \frac{B_h[(\boldsymbol{u} - I_h^* \boldsymbol{u}, p - C_h^* p), (\boldsymbol{v}_h, q_h)] - J(C_h^* p, q_h)}{\||(\boldsymbol{v}_h, q_h)\||_h}.$$

It remains to use interpolation estimates to bound the terms on the right hand side. We will make repeated use of inequality (13), Theorem 3 and Lemma 4 below. Treating $B_h[(\boldsymbol{u} - I_h^* \boldsymbol{u}, p - C_h^* p), (\boldsymbol{v}_h, q_h)]$ termwise we obtain

$$a(\boldsymbol{u} - I_h^* \boldsymbol{u}, \boldsymbol{v}_h) = (2\mu\boldsymbol{\varepsilon}(\boldsymbol{u} - I_h^* \boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}_h))_{0,\Omega_1 \cup \Omega_2} - (\{2\mu\boldsymbol{\varepsilon}(\boldsymbol{u} - I_h^* \boldsymbol{u})\}, [\![\boldsymbol{v}_h]\!])_\Gamma$$

$$- (\{2\mu\boldsymbol{\varepsilon}(\boldsymbol{v}_h)\}, [\![\boldsymbol{u} - I_h^* \boldsymbol{u}]\!])_\Gamma + (\gamma_u h^{-1} [\![\boldsymbol{u} - I_h^* \boldsymbol{u}]\!], [\![\boldsymbol{v}_h]\!])_\Gamma$$

$$\leq \||\boldsymbol{u} - I_h^* \boldsymbol{u}\|| \quad \||\boldsymbol{v}_h\||$$

$$\leq ch\|\boldsymbol{u}\|_{2,\Omega_1 \cup \Omega_2} \||(\boldsymbol{v}_h, q_h)\||_h,$$

$$b(p - C_h^* p, \boldsymbol{v}_h) = -(p - C_h^* p, \nabla \cdot \boldsymbol{v}_h) + (\{p - C_h^* p\}, [\boldsymbol{v}_h])_\Gamma$$

$$\leq (\|\mu_{\max}(p - C_h^* p)\|_{0,\Omega_1 \cup \Omega_2} + \|h(p - C_h^* p)|_{\Gamma^+}\|_{0,\Gamma} + \|h(p - C_h^* p)|_{\Gamma^-}\|_{0,\Gamma})$$

$$\times \left(\frac{\mu_{\max}}{\mu_{\min}}\right) \||(\boldsymbol{v}_h, q_h)\||_h$$

$$\leq c\left(\frac{\mu_{\max}}{\mu_{\min}}\right) h\|p\|_{1,\Omega_1 \cup \Omega_2} \||(\boldsymbol{v}_h, q_h)\||_h,$$

$$b(q_h, \boldsymbol{u} - I_h^* \boldsymbol{u}) = (q_h, \nabla \cdot (\boldsymbol{u} - I_h^* \boldsymbol{u})) + (\{q_h\}, [\boldsymbol{u} - I_h^* \boldsymbol{u}])_\Gamma$$

$$\leq \left(\frac{\mu_{\max}}{\mu_{\min}}\right) \||(\boldsymbol{u} - I_h^* \boldsymbol{u}, 0)\|| \quad \||(\boldsymbol{v}_h, q_h)\||_h$$

$$+ \frac{1}{\gamma_u} \||(\boldsymbol{u} - I_h^* \boldsymbol{u}, 0)\|| \quad \||(\boldsymbol{v}_h, q_h)\||_h$$

$$\leq c\left(\frac{\mu_{\max}}{\mu_{\min}}\right) h\|\boldsymbol{u}\|_{2,\Omega_1 \cup \Omega_2} \||(\boldsymbol{v}_h, q_h)\||_h,$$

and for the weakly consistent term we have

$$J(C_h^* p, q_h) \leq J_2(p^* - C_h^* p^*, p^* - C_h^* p^*)^{1/2} J_2(q_h, q_h)^{1/2}$$

$$\leq c\left(\sum_{K \in \mathcal{T}} \|h(p^* - C_h^* p^*)\|_{\partial K}\right)^{1/2} J(q_h, q_h)$$

$$\leq ch\|\mu^{-\frac{1}{2}} p\|_{1,\Omega_1 \cup \Omega_2} \||(\boldsymbol{v}_h, q_h)\||_h.$$

Using the Aubin-Nitsche duality argument we prove the following $L_2(\Omega)$–estimate for the velocities.

Proposition 10 *Under the same assumptions as Proposition 9 there holds*

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{0,\Omega} \leq c_\mu h^2 (\|\boldsymbol{u}\|_{2,\Omega_1 \cup \Omega_2} + \|p\|_{1,\Omega_1 \cup \Omega_2})$$

where c_μ is independent of h but not of the ratio $\left(\frac{\mu_{\max}}{\mu_{\min}}\right)$.

PROOF. Let $(\boldsymbol{w}, r) \in V \times Q$ be the solution of the dual equation

$$B[(\boldsymbol{v}, q), (\boldsymbol{w}, r)] = (\boldsymbol{\psi}, v)_{0,\Omega} \quad \forall (\boldsymbol{v}, q) \in V \times Q, \tag{30}$$

and we assume that this dual solution enjoys the additional regularity

$$\|\boldsymbol{w}\|_{2,\Omega_1 \cup \Omega_2} + \|r\|_{1,\Omega_1 \cup \Omega_2} \leq c\|\boldsymbol{\psi}\|_{0,\Omega}. \tag{31}$$

Choosing $\boldsymbol{v} = \boldsymbol{u} - \boldsymbol{u}_h$, $q = 0$ and $\boldsymbol{\psi} = \boldsymbol{u} - \boldsymbol{u}_h$, we may write

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{0,\Omega}^2 = a_h(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{w}) + (\nabla \cdot (\boldsymbol{u} - \boldsymbol{u}_h), r)_{0,\Omega}$$

and proceed using Galerkin orthogonality and the Cauchy-Schwartz inequality to obtain

$$\begin{aligned}
\|\boldsymbol{u} - \boldsymbol{u}_h\|_{0,\Omega_1 \cup \Omega_2}^2 = {} & a_h(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{w} - I_h^* \boldsymbol{w}) + (\nabla \cdot (\boldsymbol{u} - \boldsymbol{u}_h), (r - C_h^* r))_{0,\Omega_1 \cup \Omega_2} \\
& + (\nabla \cdot I_h^* \boldsymbol{w}, p - p_h)_{0,\Omega_1 \cup \Omega_2} + J(-p_h, C_h^* r) \\
\leq {} & \||(\boldsymbol{u} - \boldsymbol{u}_h, 0)\||_h \; \||(\boldsymbol{w} - I_h^* \boldsymbol{w}, 0)\||_h \\
& + \|\nabla \cdot (\boldsymbol{u} - \boldsymbol{u}_h)\|_{0,\Omega_1 \cup \Omega_2} \|r - C_h^* r\|_{0,\Omega_1 \cup \Omega_2} \\
& + |I_h^* \boldsymbol{w} - \boldsymbol{w}|_{1,\Omega_1 \cup \Omega_2} \|p - p_h\|_{0,\Omega_1 \cup \Omega_2} \\
& + J(-p_h, C_h^* r^*) \\
\leq {} & c(\||(\boldsymbol{u} - \boldsymbol{u}_h, p - p_h)\|| + J(p_h, p_h)^{\frac{1}{2}}) \\
& \times (\||(\boldsymbol{w} - I_h^* \boldsymbol{w}, r - C_h^* r)\|| + J(C_h^* r^*, C_h^* r^*)^{\frac{1}{2}}).
\end{aligned}$$

As a consequence of Proposition 9, Theorem 3, Lemma 5 and the regularity hypothesis (31) we may conclude that

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{0,\Omega_1 \cup \Omega_2}^2 \leq Ch \left(\|\boldsymbol{w}\|_{2,\Omega_1 \cup \Omega_2} + \|r\|_{1,\Omega_1 \cup \Omega_2}\right) \||(\boldsymbol{u} - \boldsymbol{u}_h, p - p_h)\||_h$$

$$\leq Ch^2 \|\boldsymbol{u} - \boldsymbol{u}_h\|_{0,\Omega_1 \cup \Omega_2} \left(\|\boldsymbol{u}\|_{2,\Omega_1 \cup \Omega_2} + \|p\|_{1,\Omega_1 \cup \Omega_2}\right),$$

and the result follows.

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7 Numerical results

7.1 An artificial interface in an incompressible medium

Our first example shows that we have optimal convergence in the case of an artificial interface in an incompressible solid with constant material parameters. The problem, often used in the context of Stokes problem of creeping flow, is posed on the unit square with exact solution given by $\boldsymbol{u} = (20xy^3, 5x^4 - 5y^4)$ and $p = 60x^2y - 20y3 - 5$. Choosing $\gamma_{1,2} = 1$ and $\gamma_\mu = 10$, we obtain the optimal convergence shown in Figure 1, first order for the pressure and second order for the displacements. The position of the artificial interface is shown in Fig. 2 on one of the computational meshes in the sequence.

7.2 An elastic interface problem with $\nu_1 \to 1/2$

We have considered a radially symmetric problem with different material properties in concentric discs around the origin. The inner disc has material parameters E_1, ν_1, and the outer E_2, ν_2. At any point, the displacement vector can be written $\boldsymbol{u} = (u_r, u_\theta)$, where u_r is the radial component of the displacement and u_θ is the circumferential component. The material is subjected to a boundary displacement $\boldsymbol{u} = \mathbf{x}$ (in Cartesian coordinates), and the exact solution to the problem is given by (cf. [10])

$$u_r(r) = \begin{cases} \left(\left(1 - \dfrac{b^2}{a^2}\right) c + \dfrac{b^2}{a^2} \right) r, & 0 \le r \le a, \\ \left(r - \dfrac{b^2}{r} \right) c + \dfrac{b^2}{r}, & a \le r \le b, \end{cases}$$

$$u_\theta = 0,$$

with

$$c = \frac{(\lambda_1 + \mu_1 + \mu_2)\, b^2}{(\lambda_2 + \mu_2)\, a^2 + (\lambda_1 + \mu_1)(b^2 - a^2) + \mu_2 b^2}.$$

When $\nu_1 = 1/2$, the last relation degenerates to $c = 1/(b^2 - a^2)$. We have set $E_1 = E_2 = 1$, $\nu_2 = 0.25$ and set $\nu_1 \in \{0.3, 0.49, 0.5\}$. The problem was solved on a quarter of a disc with symmetry boundary conditions on the vertical and horizontal boundaries and with the given boundary condition on the circumference. The stability parameters were chosen as in the previous example.

In Figure 3 we show the convergence in displacements for the different ν_1. We obtain second order convergence in all cases, though the error on a given mesh

increases with ν_1. We also show, in Figure 4, an elevation of the length of the displacement vectors and the p variable for $\nu_1 = 0.49$.

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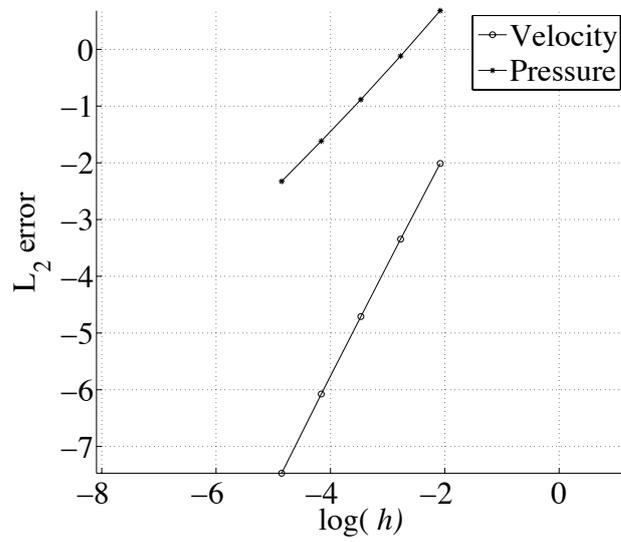
Fig. 1. Convergence for an artificial interface.
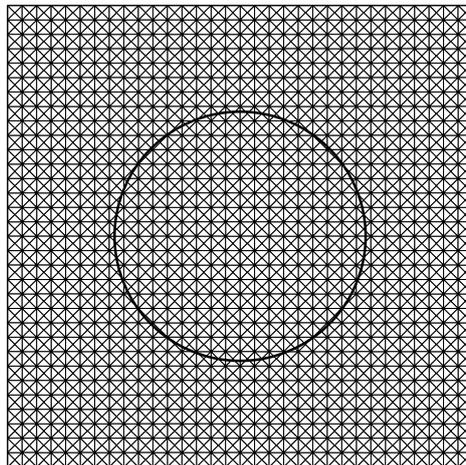
Fig. 2. Position of the artificial interface.

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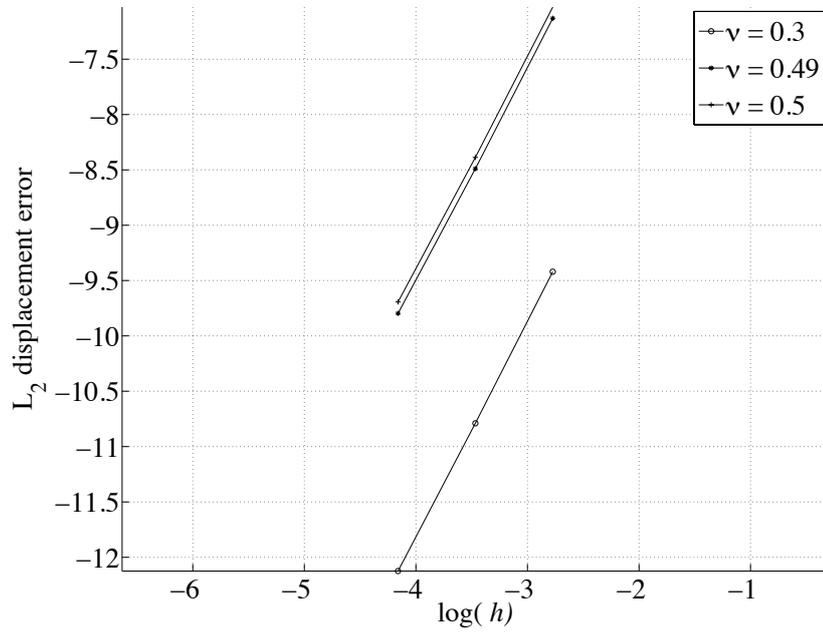
Fig. 3. Convergence for the elasticity problem.
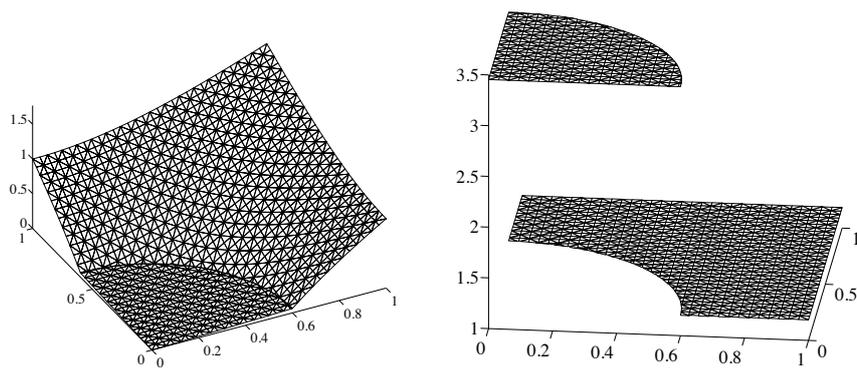
Fig. 4. Elevation of $|u|$ and p for $\nu_1 = 0.49$.

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