CHALMERS | UNIVERSITY OF GOTHENBURG

PREPRINT 2008:48

Wythoff Nim Extensions and Certain Beatty Sequences

URBAN LARSSON

Department of Mathematical Sciences
Division of Mathematics
CHALMERS UNIVERSITY OF TECHNOLOGY
UNIVERSITY OF GOTHENBURG
Göteborg Sweden 2008

Preprint 2008:48

Wythoff Nim Extensions and Certain Beatty Sequences

Urban Larsson

Department of Mathematical Sciences
Division of Mathematics
Chalmers University of Technology and University of Gothenburg
SE-412 96 Göteborg, Sweden

Göteborg, December 2008

Preprint 2008:48
ISSN 1652-9715

Matematiska vetenskaper
Göteborg 2008

WYTHOFF NIM EXTENSIONS AND CERTAIN BEATTY SEQUENCES

URBAN LARSSON

Abstract. The solution of Wythoff's game—the set of P-positions—may be represented as a pair of sequences of non-negative integers, satisfying Beatty's well-known theorem.

We generalize the solution of Wythoff's game to a pair of so-called m-complementary Beatty sequences. Our main result is that these sequences give the solution to three new extensions of Wythoff's game—of which one has a certain blocking manoeuvre on the rook-type options.

1. Introduction and some notation

The game of Wythoff Nim is an impartial game played on two piles of tokens, see [Wy]. As an addition to the rules of the game of Nim, where a player may remove tokens from precisely one of the piles, Wythoff's game also allows removal of the same number of tokens from both piles. The game is more commonly known as "Corner the queen", invented by Rufus P. Isaacs (1960), because the game can be played on a chessboard with one single queen. The queen may be moved as in a game of chess but with the restriction that for each move, the 'distance' to the lower left corner must decrease. The two players move alternately until one of them cannot do so any longer. The last player to move wins.

The solution of Wythoff's game is given by $\{\{a_i, b_i\}^1 \mid i \in \mathbb{N}_0\}$, where, for all n,

$$a_n = \left\lfloor \frac{n(1+\sqrt{5})}{2} \right\rfloor$$

and

$$b_n = \left\lfloor \frac{n(3+\sqrt{5})}{2} \right\rfloor .$$

Then $(a_i)_{i\in\mathbb{N}}$ and $(b_i)_{i\in\mathbb{N}}$ are so-called complementary sequences ([Kim07, Kim08]), which is a consequence of Beatty's theorem ([Ray, Bea, OsHy]):

Theorem 1.1. Suppose α, β are positive irrational numbers such that

$$\frac{1}{\alpha} + \frac{1}{\beta} = 1.$$

Date: September 30, 2008.

Key words and phrases. Beatty sequence, Blocking manoeuvre, Complementary sequences, Congruence, Impartial game, Integer sequence, Muller Twist, Wythoff Nim.

[1]As usual, $\{x, y\}$ denotes an unordered pair of integers, that is (x, y) and (y, x) are considered the same.

Then for all $x \in \mathbb{N}$ there is an $n = n(x) \in \mathbb{N}$, such that precisely one of αx and βx lies in the interval $[n-1, n)$.

This result is a special case of Theorem 3.1. In Section 2 we give some background material for our extensions of Wythoff's game, including one of the main ideas in this paper - a variation of Wythoff's game where a move consists of two parts, the move in itself and a blocking manoeuvre, also known as a Muller Twist (see for example [GaSt, HeLa, HoRe, Lar, SmSt]). In Section 3 we discuss the sequences that constitute the solutions of our three families of games. We rely on a generalization of Beatty's theorem to m-complementary sequences.

Definition 1. Let $m \in \mathbb{N}$. Two sequences (a_i) and (b_i) of (positive) integers are *m-complementary*, if for any (positive) integer, say n,

$$\#\{i \mid a_i = n\} + \#\{i \mid b_i = n\} = m.$$

A 1-complementary pair of sequences is simply denoted *complementary.*

In Section 4 we define three (families of) games—here we give a rough outline: Fix positive integers k and m as game constants (for our third game there is also another integer constant $0 \le l < m$).

For each game the players may move along a widened bishop-type diagonal, namely for the first two games a player may 'deviate' at most $k-1$ squares and for the third game at most $km-1$ squares. In addition, for the game

(I) $k\mathrm{WN}^m$: The previous player may block off at most $m-1$ non-diagonal positions;

(II) $k\mathrm{WN}^{(m)}$: The rook-type moves are restricted to moving a multiple of m squares;

(III) $k{\times}m\mathrm{WN}_l$: A rectangle with circumference $2m$ and one side l, is removed from the lower left corner of the game board. A rook-type move is as in Nim.

Then we illustrate our games by some examples. In Section 5, by making use of the results in Section 3, we prove that for *each* game—given the constants k and m—a winning strategy is (if possible) to move to a position of the form

$$\left\{ \left\lfloor \frac{n\Phi_{km}}{m} \right\rfloor, \left\lfloor \frac{n(\Phi_{km}+km)}{m} \right\rfloor \right\},$$

where

$$\Phi_x = \frac{2 - x + \sqrt{x^2+4}}{2}. \tag{1}$$

Finally, in Section 6 we consider some other ways to define the solutions of our games, namely via the notion of minimal exclusive algorithms.

2. Background and some discussion of rules

Denote the positive integers by $\mathbb{N}$ and the non-negative integers by $\mathbb{N}_0$. Let G be a 2-player impartial game; for an introduction to impartial games see [BeCoGu]. We follow the convention to denote our players as the *next player* (the one whose turn it is) and the *previous player.* A P-position is

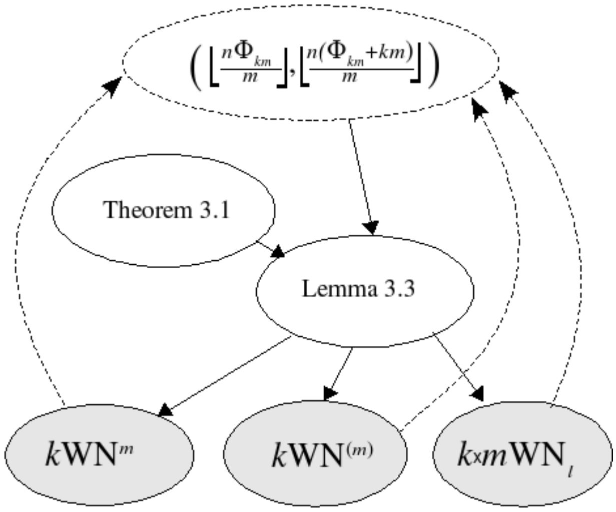

FIGURE 1. A rough sketch: The Beatty-pair at the top, together with the theorem and the lemma motivate the rules for our games (in grey)—namely, the formula represents a winning strategy (the dotted lines).

a position from which the previous player can win (given perfect play). An N-position is a position from which the next player can win. A position is either a P-position or an N-position. The set of all P-positions is denoted by $\mathcal{P} = \mathcal{P}(G)$ and the set of all N-positions by $\mathcal{N} = \mathcal{N}(G)$.

2.1. **Connell's game.** In [Con] I.G. Connell studies the solution of a variation of Wythoff's game, which we call Connell's game and denote by $\mathrm{WN}^{(m)}$: The queen's *bishop-type* options are as in Wythoff's original game, but the *rook-type* options consist of all integer multiples of a fixed positive integer.

Hence, for a positive integer m, a rook-type moves is $(x, y) \to (x, y - mi)$ or $(x, y) \to (x - mj, y)$, where $i, j \in \mathbb{N}$ and $mi \le y$, $mj \le x$.

Example 1. The first few P-positions $\{c_n, d_n\}$ of $\mathrm{WN}^{(3)}$ are:

d_n	0	1	2	4	5	7	8	10	11	12	14	15	17	18	20	21	22
c_n	0	0	0	1	1	2	2	3	3	3	4	4	5	5	6	6	6
n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16

TABLE 1. Some values of $c_n = \lfloor \frac{n\Phi_3}{3} \rfloor$ and $d_n = c_n + n$.

The values in Table 1 are actually given by:

$$c_n = \left\lfloor \frac{n(\sqrt{13} - 1)}{6} \right\rfloor,$$

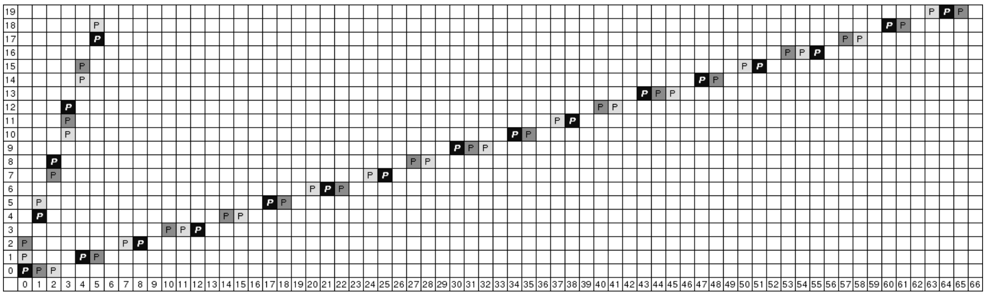

FIGURE 2. P-positions of Connell's game, $\mathrm{WN}^{(3)}$; the positions nearest the origin such that there are precisely three positions in each row and column and one position in each NE-SW-diagonal.

and

$$d_n = \left\lfloor \frac{n(\sqrt{13}+5)}{6} \right\rfloor .$$

In fact, the general solution, which can be derived from [Con], is given by

$$c_n = \left\lfloor \frac{n\Phi_m}{m} \right\rfloor ,$$

and

$$d_n = c_n + n.$$

Remark 1. In Connell's presentation, for the proof of the above formulas, he rather uses m pairs of complementary sequences of integers (by analogy with the discovery of a new formulation of Beatty's theorem in [Sko]). We have indicated this pattern of P-positions by different shades in Figure 1. We will see that, in fact, the squares of darkest shade, starting by $(0,0)$, are P-positions of the game in the next paragraph (for k=3) and the lighter shades are P-positions of special cases of games that we study in Sections 4 and 5.

2.2. **Fraenkel's k-Wythoff Nim.** We define an extension of Wythoff's game which is studied in [Fra82] by A. Fraenkel. Our definition of Fraenkel's game differs somewhat from the original definition. However, it does not change the rules of the game; namely, we have introduced an overlap of the type (I) and (II) options below. The reason for this minor technical change will become clear in Section 3.

Definition 2. Let $k \in \mathbb{N}$. For the game of *k-Wythoff Nim*, or just kWN, the queen's moves (options) are of two types:

(I) A rook-type move as in Nim;
(II) A k-bishop-type move is a composition of a (restricted) rook-type move and a bishop-type move, all in one and the same move. Namely, first move $0 \le i < k$ rook-type positions and then, from the new position, move $0 \le j$ bishop-type positions as long as $i + j > 0$.

The first few P-positions of 3WN are $\{a_n, b_n\}$, where:

b_n	0	4	8	12	17	21	25	30	34	38	43	47	51	55	60	64	68
a_n	0	1	2	3	5	6	7	9	10	11	13	14	15	16	18	19	20
n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16

TABLE 2. Some values of $a_n = \lfloor n\Phi_3 \rfloor$ and $b_n = a_n + 3n$.

As we have remarked at the end of the last paragraph, and according to Table 2, at least for the first few n we get that $(a_n, b_n) = (c_{3n}, d_{3n})$. That this holds for all n is evident by the following result (where the minimal exclusive algorithm is defined as usual: for a strict subset X of the non-negative integers $\operatorname{mex} X := \min(\mathbb{N}_0 \setminus X)$):

Theorem 2.1. Let k be a positive integer. The P-positions $\{a_n, b_n\}$ of k-Wythoff Nim are characterized by either of the two equivalent formulas:

(I) For $n \geq 0$ put $a_n = \operatorname{mex}\{a_i, b_i \mid i \in [0, n-1]\}$ and $b_n = a_n + kn$;
(II) For $n \in \mathbb{N}_0$ put $a_n = \lfloor n\Phi_k \rfloor$, $b_n = a_n + kn$, where Φ_k is as above.

The minimal exclusive algorithm in (I) gives an exponential time solution to kWN, in succinct input size, but the Beatty-type solution in (II) gives a polynomial time solution. For a discussion on complexity issues for combinatorial games, see for example [Fra04]. For the main part of this paper, we will focus on the Beatty-representation of a solution. It is not only faster, but also more explicit in its formulation.

We note that k-Wythoff Nim is included as a special case of a variant of Wythoff's game in [FrBo], where Fraenkel and I. Borosh study an extension of Connell's game (different from ours in Section 4), which includes a Beatty-type characterization of the P-positions.

Before we study the solution to our generalizations of Connell's and Fraenkel's games, let us give some background to the so-called blocking manoeuvre, in the context of Wythoff Nim.

2.3. **A bishop-type blocking variation of k-Wythoff Nim.** Let $p, k \in \mathbb{N}$. In [HeLa] the authors give an exponential time solution to a variation of k-Wythoff Nim with a blocking manoeuvre, denoted by p-Blocking k-Wythoff Nim (and by (p, k)-Wythoff Nim in [Lar]). For this game there is a certain "twist" included to each move. The rules are as in k-Wythoff Nim, except that before the next player moves, the previous player is allowed to select (at most) $p - 1$ bishop-type options and declare that the next player may not make any of these moves.

The solution of this game is in a certain sense "very close" to pairs of Beatty sequences (see also the Appendix of [Lar]) of the form

$$\left(\left\lfloor \frac{n(\sqrt{k^2 + 4p^2} - k)}{2p} \right\rfloor\right) \text{and} \left(\left\lfloor \frac{n(\sqrt{k^2 + 4p^2} + k)}{2p} \right\rfloor\right).$$

But the authors explain why there can be no Beatty-type solution to this game for $p > 1$ and $k \nmid p$ (For $k \mid p$, the 'Beatty-type solution' is given in [Lar]). For these type of questions, see also [BoFr]. However, a recent discovery, in [Had], provides a polynomial time algorithm for the solution of (p,k)-Wythoff Nim (for any combination of k and p).

An interesting connection to 4-Blocking 2-Wythoff Nim is presented in [DuGr], where the authors give an explicit bijection of solutions to a variation of Wythoff's original game, where a player's bishop-type move is restricted to jumps by an even number of squares.

For another variation, [Lar] defines the rules of a so-called move-size dynamic variation of two-pile Nim, (p,k)-Imitation Nim, for which the P-positions, treated as starting positions are identical to the P-positions of (p,k)-Wythoff Nim.

We will not discuss (p,k)-Wythoff Nim any more in this paper.

3. Solutions represented as pairs of integer sequences

As we have seen, it is customary to represent the solution of a "Wythoff-type" game as a sequence of pairs of non-negative integers; or more precisely, as pairs of increasing sequences of non-negative integers. This leads us to a certain extension of Beatty's original theorem.

3.1. Beatty sequences.

We generalize Beatty's original theorem to the notion of (a pair of) m-complementary sequences. In the literature we have found a proof of this theorem in [Bry02], where K. O'Bryant uses generating functions (a method adapted from [BoBo]). Here, we have chosen to include an elementary proof, in analogy to ideas presented in [OsHy] and [Fra82].

Theorem 3.1. Let $0 < \alpha < \beta$ be irrational numbers such that

$$\frac{1}{\alpha} + \frac{1}{\beta} = 1$$

and let $m \in \mathbb{N}$. Then $n \in \mathbb{N}$ implies

$$m = \#\left\{ i \in \mathbb{N} \mid n = \left\lfloor \frac{i\alpha}{m} \right\rfloor \right\} + \#\left\{ i \in \mathbb{N} \mid n = \left\lfloor \frac{i\beta}{m} \right\rfloor \right\}.$$

In other words, $(A_i) = (\lfloor \frac{i\alpha}{m} \rfloor)$ and $(B_i) = (\lfloor \frac{i\beta}{m} \rfloor)$ are m-complementary.

Proof. It suffices to establish that exactly m members of the set

$$S = \{0, \alpha, \beta, 2\alpha, 2\beta, \ldots\}$$

are in the interval $[n, n+1)$ for each $n \in \mathbb{N}$, or to say slightly more, that $N \in \mathbb{N}_0$ implies $\#S \cap [0, N) = mN$. But

$$\begin{aligned}\#(S \cap [0, N] &= \#(\{0, \alpha, 2\alpha, \ldots\} \cap [0, N]) + \#(\{\beta, 2\beta, \ldots\} \cap [1, N]) \\ &= \lfloor mN/\alpha \rfloor + 1 + \lfloor mN/\beta \rfloor,\end{aligned}$$

and since

$$\begin{aligned} mN/\alpha + mN/\beta - 1 &< \lfloor mN/\alpha \rfloor + 1 + \lfloor mN/\beta \rfloor \\ &< mN/\alpha + mN/\beta + 1, \end{aligned}$$

we are done.

For one of our games (see Figure 4) we will need somewhat more precise information on the sequences (A_i) and (B_i). In fact, the following corollary contains the essence of the ideas in Connell's proof in [Con]. It is a special case of the generalization of Beatty's theorem in [Sko, Fra69, Bry03]. We have included a proof to make the paper self-contained.

Corollary 3.2. With notation as in Theorem 3.1 and provided $m > 1$, for any integer $0 < l < m$, the sequences $(A_{mi+l})_{i\in\mathbb{N}_0}$ and $(B_{m(i+1)-l})_{i\in\mathbb{N}_0}$ are complementary. If $l = 0$ then $(A_{mi})_{i\in\mathbb{N}}$ and $(B_{mi})_{i\in\mathbb{N}_0}$ are complementary.

Proof. Clearly $1 < \alpha < 2 < \beta$. Then for all i, $0 < B_{i+1} - B_i$, so that by Theorem 3.1,

$$\#\left\{i \in \mathbb{N} \mid n = \left\lfloor \frac{i\alpha}{m} \right\rfloor\right\} = m - 1 \text{ and } \#\left\{i \in \mathbb{N} \mid n = \left\lfloor \frac{i\beta}{m} \right\rfloor\right\} = 1 \tag{2}$$

or

$$\#\left\{i \in \mathbb{N} \mid n = \left\lfloor \frac{i\alpha}{m} \right\rfloor\right\} = m \text{ and } \#\left\{i \in \mathbb{N} \mid n = \left\lfloor \frac{i\beta}{m} \right\rfloor\right\} = 0. \tag{3}$$

Then, for any n, the number of elements of (A_i) in the interval $[B_n, B_{n+m})$ is

$$cm + m(m-1) \equiv 0 \pmod{m}, \tag{4}$$

where c is some non-negative integer constant.
The result follows by an inductive argument, where the base cases are

$$\begin{aligned}
0 = B_0 &= A_1, A_2, \ldots, A_{m-1};\\
1 = B_1 &= A_m, A_{m+1}, \ldots, A_{2m-2};\\
2 = B_2 &= A_{2m-1}, A_{2m}, \ldots, A_{3m-3};\\
&\ \vdots\\
m - 1 = B_{m-1} &= A_{m^2-2m+2}, A_{m^2-2m+3}, \ldots, A_{m^2-m}.
\end{aligned}$$

Namely, let n be as in (2). Then, since (A_i) is non-decreasing, (4) together with the base cases give that there is precisely one x in each congruence class modulo m, such that $n = A_x$ (and these indexes are consecutive), except for, say the congruence class z, where $n = B_{m'm-z}$ for some integer m'. If n is as in (3), then, trivially, there is precisely one x in each congruence class modulo m, such that $n = A_x$.

3.2. **Some special sequences.** Let Φ_x be as in (1). Then Φ_x is irrational and so, since

$$\frac{1}{\Phi_x} + \frac{1}{\Phi_x + x} = 1,$$

the next result follows from Theorem 3.1 together with some elementary arithmetics (we omit the proof).

Lemma 3.3. For $k, m \in \mathbb{N}$ with $m > 1$, for each $n \in \mathbb{N}_0$, let

$$a_n = a_n(k, m) = \left\lfloor \frac{n\Phi_{km}}{m} \right\rfloor$$

and

$$b_n = b_n(k, m) = \left\lfloor \frac{n(\Phi_{km} + km)}{m} \right\rfloor.$$

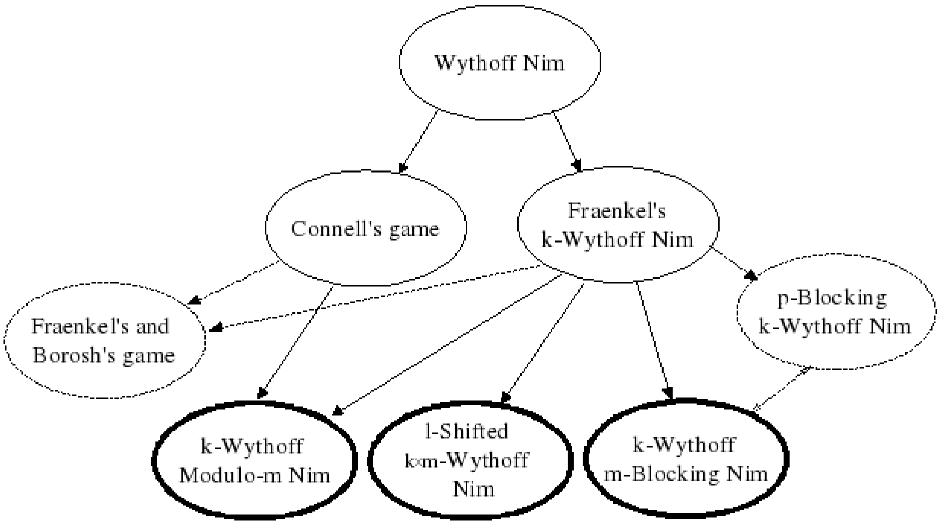

FIGURE 3. A family tree of some extensions of Wythoff's original game.

Then

(I) $\#\{a_i, b_i \mid i \in \mathbb{N}\} = m$;
(II) $b_n - a_n = kn$;
(III) if $m = 1$, then
  (a) $a_{n+1} - a_n = 1$ and $b_{n+1} - b_n = k + 1$, or
  (b) $a_{n+1} - a_n = 2$ and $b_{n+1} - b_n = k + 2$.
(IV) if $m > 1$, then
  (a) $a_{n+1} - a_n = 0$ and $b_{n+1} - b_n = k$, or
  (b) $a_{n+1} - a_n = 1$ and $b_{n+1} - b_n = k + 1$.

4. Three Extensions of Wythoff's, Connell's and Fraenkel's games

As we have remarked in Section 2.2, the rook-type options intersect the k-bishop-type options precisely when $k > 1$. For example, for 2-Wythoff Nim $(2, 3) \to (1, 3)$ is both a "diagonal" and a rook-type move. We will make use of this fact when defining the blocking manoeuvre. Therefore, let us introduce the following non-standard notation.

Fix positive integers k and m. If an option of k-Wythoff Nim is not of the form of the k-bishop as in Definition 2 (II), it is a *roob*(-type)[2] option. Then for 2WN $(2, 3) \to (2, 1)$ is a roob option, but $(2, 3) \to (2, 2)$ is not a roob option (both options are rook options).

Let us define our games:

$k\mathrm{WN}^m$: *k-Wythoff m-Blocking Nim* is a variation of Wythoff's game with a roob-type blocking manoeuvre. The players move as in k-Wythoff Nim, but with one exception: before the next player moves, the previous player may *block off* (at most) $m - 1$ of the next player's

[2]Think of 'roob' as 'ROOk minus k-Bishop', or maybe 'ROOk Blocking'

roob options. The blocked off options are excluded from the next player's set of options. Each blocking manoeuvre is particular to a specific move; that is, when the next player has moved, the previous player's blocking manoeuvre has no further impact of the game.

$k\mathrm{WN}^{(m)}$: *k-Wythoff Modulo-m Nim* is a variation of Wythoff's game, where the players move as in k-Wythoff Nim (where the k-bishop-type options are as in Definition 2) with one exception. The length of a rook-type move must be some positive multiple of m. Note: By this definition there is no congruence restriction on a k-bishop-type option (even when it intersects the rook-type options).

$k{\times}m\mathrm{WN}_l$: *l-Shifted (final positions of) k×m-Wythoff Nim.* For an integer

$$0 < l < m,$$

a player moves as in (km)-Wythoff Nim, except that a player cannot move to any of the positions in the rectangle

$$\{(i,j) \mid 0 \le i < kl, 0 \le j < k(m-l)\}^3.$$

Hence the new terminal positions are $(kl, 0)$ and $(0, k(m-l))$. On the other hand, if $l = 0$, $k \times m\mathrm{WN}_0$ reduces to the game (km)-Wythoff Nim.

We let the two players *Alice* and *Bob* illustrate our games with some easy examples; Alice makes the first move (and Bob makes the first blocking manoeuvre in case the game has a Muller twist).

Example 2. Suppose the starting position is $(0, 2)$ and the game is $2\mathrm{WN}^2$. Then the only bishop-type move is $(0, 2) \to (0, 1)$. There is precisely one roob option, namely $(0, 0)$. Since this is a terminal position it will be blocked off from the next player's options, so that Alice has to move to $(0, 1)$. Then the move $(0, 1) \to (0, 0)$ cannot be blocked off for the same reason, so Bob wins. Hence $(0, 2)$ is a P-position.

Example 3. Suppose the starting position is $(0, 2)$ and the game is $2\mathrm{WN}^{(2)}$. Alice can move to $(0, 0)$, since $0 \equiv 2 \pmod 2$, and wins. Hence $(0, 2)$ is an N-position.

Example 4. Suppose the starting position is $(0, 4)$ and the game is $2\mathrm{WN}^3$. Then the only bishop-type move is $(0, 4) \to (0, 3)$, so that the roob options are $(0, 0), (0, 1), (0, 2)$. Bob may block off 2 of these positions, say $(0, 0), (0, 2)$. Then if Alice moves to $(0, 1)$ she will loose (since she may not block off $(0, 0)$), so suppose rather that she moves to $(0, 3)$. Than she may not block off $(0, 2)$ so Bob moves $(0, 3) \to (0, 2)$ and blocks off $(0, 0)$. Hence $(0, 4)$ is a P-position.

Example 5. Suppose the starting position is $(0, 4)$ and the game is $2\mathrm{WN}^{(3)}$. Alice can only move to $(0, 1)$. Then Bob may move $(0, 1) \to (0, 0)$, a 2-bishop-type option. This shows that $(0, 4)$ is a P-position.

Notice in Examples 2 and 3 that the P-positions are distinct inspite the identical constants ($k = m = 2$). But in the Examples 4 and 5 the P-positions are the same.

[3] One may think of the game as if this lower left rectangle is cut out from the gameboard.

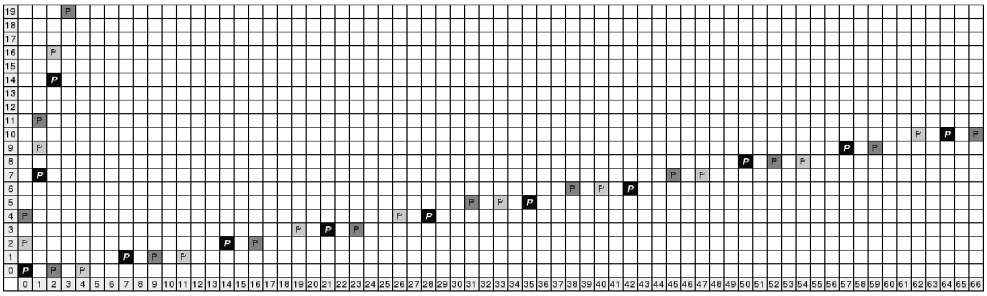

FIGURE 4. P-positions of $2\mathrm{WN}^{(3)}$, $2\mathrm{WN}^3$ and $2{\times}3\mathrm{WN}_l$ for $0 \le l \le 2$; the positions nearest the origin such that there are precisely three positions in each row and column and one position in every second NE-SW-diagonal.

Example 6. If the starting position is $(0,4)$ and the game is $2{\times}3\mathrm{WN}_1$, then Alice cannot move so that Bob wins. If, on the other hand, the game is $2{\times}3\mathrm{WN}_2$, the position $(0,2)$ is terminal and so Alice wins (by moving $(0,4) \to (0,2)$).

Suppose now that the starting position of $2{\times}3\mathrm{WN}_2$ is $(1,8)$. Then, Alice may move to $(0,2)$ to win. But if the starting position of $2{\times}3\mathrm{WN}_0$ is $(1,7)$ Alice may not move to $(0,0)$ and hence Bob wins.

In Table 3 we present the values of $a_n(2,3)$ and $b_n(2,3)$ as defined in Lemma 3.3. In Figure 3 we see that $\{a_n, b_n\}$ are the first few P-positions of $2\mathrm{WN}^{(3)}$ and $2\mathrm{WN}^3$ respectively and that the positions (a_{3n+l}, b_{3n+l}) and $(b_{3(n+1)-l}, a_{3(n+1)-l})$ correspond to the P-positions of $2{\times}3\mathrm{WN}_l$ for $0 \le l \le 2$ and $n \ge 0$. For example, the dark positions in Figure 3 are the first few P-positions of $2{\times}3\mathrm{WN}_0$.

b_n	0	2	4	7	9	11	14	16	19	21	23	26	28	31	33	35	38
a_n	0	0	0	1	1	1	2	2	3	3	3	4	4	5	5	5	6
n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16

TABLE 3. Some values of $a_n = \lfloor \frac{n\Phi_6}{3} \rfloor$ and $b_n = a_n + 2n$

5. THREE GAMES—ONE SOLUTION

The solution of our congruence variation, $k\mathrm{WN}^{(m)}$ (see also Figure 4), depends on a certain structure of our sequences. It suffices to use Lemma 3.3 (I),(II) and (IV) to establish the next result. Namely, let $n \in \mathbb{N}_0$, $i' = \min\{i \mid a_i = b_n\}$ and $j' = \min\{i \mid a_i = b_{n+1}\}$. Then

$$\begin{aligned} b_{j'} - b_{i'} &= a_{j'} - a_{i'} + k(j' - i') \\ &= b_{n+1} - b_n + k(j' - i') \\ &= a_{n+1} - a_n + k(j' - i' + 1) \\ &\equiv a_{n+1} - a_n \pmod{km}, \end{aligned}$$

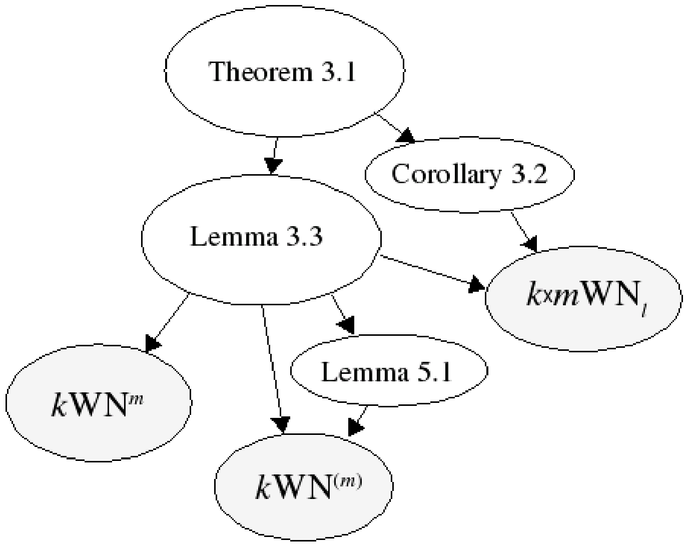

FIGURE 5. The structure of the proof of a winning strategy for our respective games. The solution of the blocking variation follows directly from Lemma 3.3. For a Beatty-type (as in Lemma 3.3) solution of $k\mathrm{WN}^{(m)}$ we demand $\gcd(k, m) = 1$.

since $j' - i' = km - 1$ or $(k+1)m - 1$. We may as a base case use $a_0 = 0, b_1 = k, \ldots, b_{m-1} = k(m-1)$. Then, via an inductive argument, we obtain:

Lemma 5.1. Let $x \in \mathbb{N}_0$. With notation as in Lemma 3.3, either

(I) there is an i, such that $a_i = a_{i+1} = \ldots = a_{i+m-1} = x$, where

$$\frac{b_i - x}{k}, \frac{b_{i+1} - x}{k}, \ldots, \frac{b_{i+m-1} - x}{k}$$

are consecutive integers; or

(II) there is an n such that $b_n = x$, and if $m > 1$, there is an $i > n$ such that $x = a_i = a_{i+1} = \ldots = a_{i+m-2}$, where

$$a_n \equiv b_i - k \pmod{km},$$

and such that

$$\frac{b_i - x}{k}, \frac{b_{i+1} - x}{k}, \ldots, \frac{b_{i+m-2} - x}{k}$$

are consecutive integers.

Let X, Y and Z be sets. Then we denote $Y \sqcup X = Z$ if $Y \cup X = Z$ and $Y \cap X = \emptyset$. Let us state the main theorem.

Theorem 5.2. Let $k, m \in \mathbb{N}$ and let (a_i) and (b_i) be the sequences of integers defined in Lemma 3.3. Then

(I) $\mathcal{P}(k\mathrm{WN}^m) = \{\{a_i, b_i\} \mid i \in \mathbb{N}_0\}$;

(II) $\mathcal{P}(k\mathrm{WN}^{(m)}) = \{\{a_i, b_i\} \mid i \in \mathbb{N}_0\}$ if and only if $\gcd(k, m) = 1$;

(III) $\bigsqcup_{0 \le l < m} \mathcal{P}(k{\times}m\mathrm{WN}_l) = \{\{a_i, b_i\} \mid i \in \mathbb{N}_0\}$.

Proof. For $m = 1$, the games have identical rules. This case has been established in [Fra82]. The case $k = 1$ has been studied in [Con] for games of form (II) (and implicitly for $1{\times}m\mathrm{WN}_l$).

For the rest of the proof assume that $m > 1$. Let us first explain the 'only if' direction of (II). Denote by

$$\gamma = \gcd(k, m),\ m' = m/\gamma,\ k' = k/\gamma.$$

Then the positions of the form $(0, ki)$, where $0 \le i < m'$, are P-positions of $k\mathrm{WN}^{(m)}$. Now, $(0, km')$ is an N-position because $k'm = km'$ implies $(0, m'k) \to (0, 0)$. But, by definition, $b_{m'} = km'$ if $m' < m$, which holds if and only if $\gamma > 1$.

For convenience, we may think of the games in (III) as one game, denoted by $k{\times}m\mathrm{WN}$, where, before the next player makes her first move the previous player decides the parameter l which then stays fixed for the rest of the game. Otherwise the rules are as in (III). Then, if the proposition holds, the P-positions of $k{\times}m\mathrm{WN}$, *treated as starting positions*, are identical to (I) (and (II) for relative prime k and m).

Fix k and m and let G be one of our games. Suppose there exists a least index $\xi > 0$ such that, either

(i) $(a_\xi, b_\xi) \in \mathcal{N}$, or
(ii) there is a P-position $(a_\xi, y) \notin \{(a_i, b_i)\}$.

Suppose first that (i) holds and if (ii) holds then $y > b_\xi$.

Case $k\mathrm{WN}^m$: By Lemma 3.3 (IV) and our assumption, there is no k-bishop-type move to a P-position. Again, by Lemma 3.3 (I) and (IV) the previous player can block off all followers of (a_ξ, b_ξ) in the set $\{(a_i, b_i)\}$. But then, by assumption, there is no rook-type move to a P-position.

Case $k\mathrm{WN}^{(m)}$: As before, there is no k-bishop-type move to a P-position. By Lemma 5.1

$$b_\xi \equiv b_i \pmod m$$

implies $a_\xi \neq a_i$, so there is no rook-type move

$$(a_\xi, b_\xi) \to (a_i, b_i)$$

for any $i < \xi$.

Case $k{\times}m\mathrm{WN}$: There is an $0 \le l < m$ and an index i such that

$$(a_\xi, b_\xi) = (a_{mi+l}, b_{mi+l}).$$

By Corollary 3.2 there can be no rook-type move to a position

$$(a_{mj+l}, b_{mj+l})$$

or

$$(b_{m(j+1)-l}, a_{m(j+1)-l})$$

for any $0 \le j < i$. By Lemma 3.3 (IV)

$$b_{mi+l} - b_{m(i-1)+l} - (a_{mi+l} - a_{m(i-1)+l}) = mk$$

for any i. But, by Definition 2, a (km)-bishop-type option is distanced less than $mk - 1$ rook-type positions from the queens main diagonal. Since (b_i) is strictly increasing, there is no (km)-bishop-type move to a position of the form

$$(a_{mj+l}, b_{mj+l})$$

(or trivially of the form $(b_{m(j+1)-l}, a_{m(j+1)-l})$). By assumption, this implies that the previous player can choose l as the parameter for this game and be certain that the next player must move to an N-position of $k{\times}m\mathrm{WN}_l$.

Then, if (i) holds for some game, we must have $y \le b_\xi$, which leads us to consider (ii). Suppose that (ii) holds with $y > b_j$ for each j such that $x := a_j = a_\xi$.

Case $k\mathrm{WN}^m$: Suppose the next player is about to move from (x, y). By the blocking rule the previous player may block off at most $m - 1$ positions in a rook-type direction. Then, by Lemma 3.3 (I) together with our assumption, (x, y) must be an N-position.

Case $k\mathrm{WN}^{(m)}$: Lemma 5.1, gives a contradiction for this case. Namely, by $\gcd(k, m) = 1$ there has to be an index j such that $a_j = x$ and $y > b_j$ where

$$y \equiv b_j \pmod m$$

or a $j' < j$ such that $b_{j'} = x$ and

$$y \equiv a_{j'} \pmod m.$$

Hence there is a move

$$(x, y) \to (x, b_j)$$

or

$$(x, y) \to (x, a_{j'})$$

and so $(x, y) \in \mathcal{N}$.

Case $k{\times}m\mathrm{WN}$: Suppose (x, y) is the starting position of G. There are precisely m possible choices of the parameter l for the previous player, hence by Corollary 3.2 and the assumption, the pigeonhole principle gives a contradiction.

Suppose that (ii) holds with $b_j > y$ for some j such that $a_j = a_\xi$.

Case $k\mathrm{WN}^m$: We may assume that there is a position (a_i, b_i) with $0 < i < \xi$ such that

$$b_i - a_i \le y - x < b_{i+1} - a_{i+1}.$$

Then, by Lemma 3.3,

$$0 \le y - x - (b_i - a_i) < k(i + 1) - ki = k,$$

so that there is a k-bishop-type move

$$(x, y) \to (a_i, b_i);$$

which may not be blocked off by the previous player even if $a_i = x$, since in this case $y - b_i < k$ and so (a_i, b_i) is not a roob option.

Case $k\mathrm{WN}^{(m)}$: This case is a consequence of the previous argument (omit the discussion of the blocking manoeuvre).

Case $k{\times}m$WN: For any choice of l, by assumption, there is a position (a_{mi+l}, b_{mi+l}) with $mi + l < \xi$ such that

$$b_{mi+l} - a_{mi+l} \le y - x < b_{m(i+1)+l} - a_{m(i+1)+l}$$

and so, by Lemma 3.3,

$$0 \le y - x - (b_{mi+l} - a_{mi+l}) < km(i+1) - kmi = km.$$

Hence, for each choice of l, (x, y) must be an N-position.

6. And finally, the minimal exclusive algorithms

As a round off, we will give a minimal exclusive algorithm for each one of our games—where each algorithm 'mimics' the idea of a particular game.

To this purpose, we need another notation. A *multiset* is a sequence of non-negative integers, say $(\xi^i)_{i\in\mathbb{N}_0}$, where for each $i \in \mathbb{N}_0$, ξ^i represents the unique non-negative integer that counts the number of occurences of i in (ξ^i). For a positive integer m, let $\text{mex}^m(\xi^i)$ denote the least non-negative integer $i \in (\xi^i)$ such that $\xi^i < m$.

One may verify that the properties in Lemma 3.3 are satisfied for each pair of sequences in the following proposition. Then the result follows by induction (we omit the proof).

Proposition 6.1. Let $k > 0$ and $m > 1$ be integers. Then the algorithms I, II and III are equivalent, and, for each $i \in \mathbb{N}_0$, the pair (a_i, b_i) is as in Lemma 3.3. For $n \ge 0$,

Algorithm I:

$$\begin{aligned}
&a_n = \text{mex}^m(\zeta_n^i), \text{where } \zeta_n \text{ is the multiset, where for each } i \in \mathbb{N}_0,\\
&\zeta_n^i = \#\{j \mid i = a_j \text{ or } i = b_j \mid 0 \le j < n\},\\
&b_n = a_n + kn;
\end{aligned}$$

Algorithm II:

$$\begin{aligned}
&a_n = \text{mex}\{a_i, b_i \mid a_i + kn \equiv b_i \pmod{km},\ b_i + kn \equiv a_i \pmod{km},\ 0 \le i < n\},\\
&b_n = a_n + kn;
\end{aligned}$$

Algorithm III: for each $0 < l < m$,

$$\begin{aligned}
a_{mn} &= \text{mex}\{a_{mi}, b_{mi} \mid 0 \le i < n\},\\
b_{mn} &= a_{mn} + kmn,\\
a_{mn+l} &= \text{mex}\{a_{mi+l}, b_{m(i+1)-l} \mid 0 \le i < n\},\\
b_{mn+l} &= a_{mn+l} + k(mn + l).
\end{aligned}$$

We are left with the following question: What is the solution of $k\text{WN}^{(m)}$ whenever $\gcd(k, m) \ne 1$?

Acknowledgements: The author would like to thank Niklas Eriksen, Aviezri Fraenkel, Peter Hegarty and Johan Wästlund for their informed comments and inspiring suggestions which helped to improve this paper.

References

[Bea] S. Beatty, Problem 3173, *Amer. Math. Monthly*, **33** (1926) 159.

[BeCoGu] E.R. Berlekamp, J.H. Conway and R.K. Guy, *Winning Ways for Your Mathematical Plays* Volume **1** and **2** (1982).

[BoBo] J. M. Borwein and P. B. Borwein, On the generating function of the integer part: $[\alpha n + \gamma]$, *J. Number Theory* 43 (1993), pp. 293-318.

[BoFr] M. Boshernitzan and A. S. Fraenkel, A linear algorithm for nonhomogeneous spectra of numbers, *J. Algorithms*, **5**, no. 2, pp. 187-198, 1984.

[Bry02] K. O'Bryant, A Generating Function Technique for Beatty Sequences and Other Step Sequences, *J. Number Theory* 94, 299–319 (2002).

[Bry03] K. O'Bryant, Fraenkel's Partition and Brown's Decomposition *Integers*, **3** (2003), A11, 17 pp.

[Con] I.G. Connell, A Generalization of Wythoff's game *Can. Math. Bull.* **2** no. 3 (1959), 181-190.

[DuGr] E.Duchêne, S. Gravier, Geometrical Extensions of Wythoff's Game, to appear in *Discrete Math* (2008).

[Fra69] A.S. Fraenkel, The bracket function and complementary sets of integers, Canad. J. Math. **21** (1969), 6-27.

[FrBo] A.S. Fraenkel, I. Borosh, A Generalization of Wythoff's Game, *Jour. of Comb. Theory* (A) **15** (1973) 175-191.

[Fra82] A.S. Fraenkel, How to beat your Wythoff games' opponent on three fronts, *Amer. Math. Monthly* **89** (1982) 353-361.

[Fra04] A.S. Fraenkel, Complexity, appeal and challanges of combinatorial games. *Theoret. Comp. Sci.*, **313** (2004) 393-415.

[GaSt] H. Gavel and P. Strimling, Nim with a Modular Muller Twist, *Integers: Electr. Jour. Comb. Numb. Theo.* **4** (2004).

[Had] U. Hadad, Msc Thesis, Polynomializing Seemingly Hard Sequences Using Surrogate Sequences, *Fac. of Math. Weiz. In. of Sci.*, (2008).

[HeLa] P. Hegarty and U. Larsson, Permutations of the natural numbers with prescribed difference multisets, *Integers* **6** (2006), Paper A3, 25pp.

[HoRe] A. Holshouser and H. Reiter, Three Pile Nim with Move Blocking, http://citeseer.ist.psu.edu/470020.html.

[Kim07] C. Kimberling, Complementary equations, *J. Integer Sequences* **10** (2007), Article 07.1.4.

[Kim08] C. Kimberling, Complementary equations and Wythoff sequences, *J. Integer Sequences* **11** (2008), Article 08.3.3.

[Lar] U. Larsson, 2-pile Nim with a Restricted Number of Move-size Imitations, submitted to *Integers*, available at http://arxiv.org/abs/0710.3632

[OsHy] A. Ostrowski and J. Hyslop, Solution to Problem 3177, *Amer. Math. Monthly*, **34** (1927), 159-160.

[Ray] J. W. Rayleigh. The Theory of Sound, *Macmillan, London*, (1894) p. 122-123.

[Sko] Th. Skolem, Über einige Eigenschaften der Zahlenmengen $[\alpha n + \beta]$ bei irrationalem α mit einleitenden Bemerkungen über eine kombinatorishe Probleme, *Norske Vid. Selsk. Forh., Trondheim* **30** (1959), 42-49.

[SmSt] F. Smith and P. Stănică, Comply/Constrain Games or Games with a Muller Twist, *Integers*, **2**, (2002).

[Wy] W.A. Wythoff, A modification of the game of Nim, *Nieuw Arch. Wisk.* **7** (1907) 199-202.

E-mail address: `urban.larsson@chalmers.se`

Mathematical Sciences, Chalmers University of Technology and University of Gothenburg, Göteborg, Sweden