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A note on conservation laws for the singularly $\chi^{(2)}$ - model and the corresponding nonlocal $\chi^{(2)}$ - approximation.

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Abstract

We prove that both the singularly $\chi^{(2)}$ - model and the corresponding nonlocal $\chi^{(2)}$ - approximation can derived from variational principles. The conservation of power, linear momentum and hamiltonian for the $\chi^{(2)}$ - model follows as a consequence of Noethers theorem [E. Noether, Invariante Variationsprobleme, Nachr. d. König. Gesellsch. d. Wiss. zu Göttingen, Math-phys. Klasse, (1918), 235257]. Direct computation reveals that the power density, the linear momentum density and the hamiltonian density of the corresponding nonlocal $\chi^{(2)}$ - approximation do not satisfy conservation laws on standard differential form, i.e., these laws cannot be expressed in terms of a divergence - free vector field. In spite of that, it is shown by direct computation that the integrals of these densities, i.e., the power, linear momentum and the hamiltonian are constants of motion. It is conjectured that a generalized version of Noethers theorem, appropriate for dealing with nonlocal Euler- Lagrange equations, can be used to predict the existence of these conservation laws.

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1 Introduction

Spatial nonlocality of the nonlinear response is a generic property of a wide range of physical systems, which manifests itself in new and exciting properties of nonlinear waves [1,2,3,4,5,6]. The nonlocality implies that the response of the medium at a given point depends not only on the wave function at that point (as in local media), but also on the wave function in its vicinity. The nonlocal nature often results from transport processes, such as atom diffusion [7], heat transfer [8,9] or drift of electric charges [10]. It can also be induced by a long-range molecular interaction as in nematic liquid crystals, which exhibit orientational nonlocal nonlinearity [11,12]. Matter waves and Bose - Einstein condensates (BEC) inherently have a spatially nonlocal nonlinear response due to the finite range of the inter - particle interaction potential [13].

It turns out that the concept of nonlocality can be extended to nonlinear systems which are not nonlocal in the traditional sense as described above. This is the case of the type I wave interaction process in quadratic media, i.e., when we have interaction between a fundamental wave (FW) and its second harmonic (SH) propagating along the t - direction in a lossless quadratic nonlinear medium under the conditions for type I phase matching [14]. The normalized dynamical equations for the slowly varying envelopes $W(x,t)$ and $V(x,t)$ are then given as [15,17]

$$iW_t + dW_{xx} + \overline{W}V\exp(-i\beta t) = 0, \tag{1}$$

$$iV_t + cV_{xx} + W^2\exp(i\beta t) = 0, \tag{2}$$

In the spatial domain $d \approx 2c$, $d, c > 0$, and the coordinate x represents a transverse spatial direction. The terms W_{xx} and V_{xx} then model the beam diffraction. In the temporal domain d and c model the group velocity dispersion (GVD-) coefficient, which may take arbitrary values and x represents time. In that case W_{xx} and V_{xx} account for the pulse dispersion. The parameter β is the normalized phase - mismatch.

The modulational stability of plane/continuous waves described by the equations (1) - (2) was addressed in [16], where different regimes of modulational instability were found. A thorough survey of the nonlinear propagation of waves in $\chi^{(2)}$ media with respect to modulational instability/continuous waves, existence and stability of solitary waves is given in [17].

We conveniently rewrite the system (1) - (2) on the form

$$iw_t + dw_{xx} + \overline{w}v = 0 \tag{3}$$

$$iv_t - \beta v + cv_{xx} + w^2 = 0 \tag{4}$$

by means of the transformation

$$W = w, \quad V = v\exp(i\beta t) \tag{5}$$

In Nikolov et al [14] the problem of slow variation of the SH field with respect to the propagation direction is investigated, i.e., the assumption

$$|v_t| \ll |cv_{xx}| \sim |\beta v| \tag{6}$$

is imposed. Thus, neglecting $i\partial_t v$ in (4), we solve (4) exactly using Fourier transformation and the convolution theorem, treating w^2 as a source term. In this case we can express V as [14]

$$V = \beta^{-1}N\exp[i\beta t], \tag{7a}$$

$$N(w^2)(x) = (R * w^2)(x) \equiv \int_{-\infty}^{\infty} R(x-y)w^2(y)dy, \tag{7b}$$

with the response kernel $R(x)$ defining the nonlocality given as

$$R(x) = \frac{1}{\sigma}\Phi_{s_2}(\xi), \quad \xi = \frac{x}{\sigma} \tag{8}$$

where

$$\Phi_{s_2}(\xi) = \begin{cases} \frac{1}{2}\exp(-|\xi|) \text{ if } s_2 = +1 \\ \frac{1}{2}\sin(|\xi|) \quad \text{if } s_2 = -1 \end{cases} \tag{9}$$

Here

$$s_2 = sign(c\beta), \quad \sigma = |\frac{c}{\beta}|^{1/2} \tag{10}$$

Hence, when assuming slow variation of the second harmonic amplitude v with respect to the propagation coordinate, i.e., (6), the system (1) - (2) can formally be approximated by the nonlocal equation [14]

$$iw_t + dw_{xx} + \beta^{-1}N(w^2)\overline{w} = 0, \tag{11}$$

with $N(w^2)$ given as (7b).

Notice the difference between the nonlocal model (7) - (11) for the $\chi^{(2)}$ system and the generic Kerr type nonlocal model treated in [1,2,3,4,5,6]: In the $\chi^{(2)}$ model the convolution integral involves w^2, while in the Kerr type it involves the intensity $|w|^2$, thus some phase information is retained in (7) - (11).

Due to the form of the response function $R(x)$, the case $s_2 = +1$ is referred to as the *non-resonant case* or the *exponentially decaying case*, while the $s_2 = -1$ is called the *resonant case* or *oscillatory case*. The Fourier transform of the response kernel $R(x)$ is given as

$$\widetilde{R}(k) = \frac{1}{1 + s_2\sigma^2 k^2} \tag{12}$$

from which it follows that $\widetilde{R}$ has real singularities given as

$$k = \pm\frac{1}{\sigma} = \pm|\frac{\beta}{c}|^{1/2} \tag{13}$$

when $s_2 = -1$. In this case the response function (8) - (9) is derived by means of the inversion theorem for Fourier - transforms in the principal value sense and that this function cannot be normalized. The role of the real singularities (13) becomes even more transparent when studying the Fourier - transformed version of the SH - equation (4):

$$i\widetilde{v}_t - \beta\widetilde{v} - ck^2\widetilde{v} + \widetilde{w^2} = 0 \tag{14}$$

Here $\widetilde{v}$ and $\widetilde{w^2}$ denote the Fourier transforms of v and w^2, respectively. The evolution equation (14) clearly displays that $-\beta\widetilde{v} - ck^2\widetilde{v}$ will be comparable with the small term $i\widetilde{v}_t$ in the vicinity of the singularities (13) in the case $s_2 = -1$. This naturally leads to the conjecture that the solutions of the initial value problem of (7) - (11) does not mimic the dynamical evolution of the full system (1) - (2) for this case. A similar type of behavior occurs in the multidimensional analogue of (1) - (2), which recently has been used to model light bullets and X - waves in quadratic media [18]. When neglecting the slowly varying term iv_t by assuming (6), one ends up with a multidimensional analogue to (7) - (11). By using the same type of arguments as in the 1D case, one might expect that the nonlocal approximation does not provide an appropriate description in the normal SH case as a dynamical model, since there is a wave number regime in the vicinity of the singularities of the spectrum of the response function for which the omission of the term iv_t is not justified [18].

These results are supported by the conclusions in Wyller et al [19]. Here the relationship between the $\chi^{(2)}$ - model (3) - (4) under the approximation (6) and the nonlocal approximation (7) - (11) is discussed with respect to modulational instability of plane waves. The key point in that analysis is to view (3) - (4) as a singularly perturbed system [20], [21], i.e., to incorporate a dimensionless parameter ε satisfying the condition $0 < \varepsilon \ll 1$ in the description by making the modification

$$
\begin{aligned}
&iw_t + dw_{xx} + \overline{w}v = 0 \\
&i\varepsilon v_t - \beta v + cv_{xx} + w^2 = 0
\end{aligned}
\tag{15}
$$

of (3) - (4). The perturbed problem (15) permits two branches of plane waves. One of these branches deforms to the plane wave of the nonlocal problem as $\varepsilon \to 0$. Moreover, it is shown that the stability results of the full model continuously deform to the MI results derived for the nonlocal model as the perturbational parameter tends to zero. The other plane wave branch which is singular in ε and which is absent in the nonlocal $\chi^{(2)}$ - model is always unstable as $\varepsilon \to 0$. Hence from a physical point of view, the nonlocal model yields an inadequate description in situations where that model predicts modulational stability.

This serves as a background for the present paper. We aim at investigating the relationship between the conservation laws of the two systems (7) - (11) and (15). It turns out that the conservation of power, linear momentum and hamiltonian for the singularly perturbed $\chi^{(2)}$ can be obtained by a straightforward application of the standard form of Noethers theorem [22], [23], i.e., the conservation equations on differential form can be expressed in terms of a divergence - free vector field. On the contrary, for the nonlocal approximation (7) - (11) this is not possible. In spite of that, one can prove by direct computation that the hamiltonian, the linear momentum and the power are constant of motions, thus leading to the conjecture that the existence of these conservation laws can be predicted by means of a symmetry group analysis of the variational formulation of the nonlocal action integral.

The present note is organized as follows: In Section 2 the system singularly perturbed system (15) is derived by standard variational calculus, and the conservation laws for power, linear momentum and hamiltonian are derived by means of symmetry group analysis of the variational principle (Noethers theorem). In Section 3 it is proved that the nonlocal approximation (7) - (11) represents Euler - Lagrange equation of a nonlocal action integral. In Section 4 it is proved by direct computation that variational formulation of the nonlocal action integral of (7) - (11) permits invariance under temporal translation -, spatial translation - and rotational group, and that the power, the linear momentum and the hamiltonian are constant of motion. For the sake of completeness we include the standard proof of Noethers theorem in Appendix A.

2 The variational formulation of the singularly perturbed $\chi^{(2)}$ - model. Conservation laws.

The system (15) can be derived from a field theoretical variational principle where the Lagrangian density L_ε is given as

$$
\begin{aligned}
L_\varepsilon &= i\frac{1}{2}(\overline{w}w_t - w\overline{w}_t) + \frac{1}{4}i\varepsilon(\overline{v}v_t - v\overline{v}_t) \\
&-d|w_x|^2 - \frac{1}{2}c|v_x|^2 - \frac{1}{2}\beta|v|^2 + \frac{1}{2}\overline{w}^2 v + \frac{1}{2}w^2\overline{v}
\end{aligned}
\tag{16}
$$

and the action integral reads

$$
S_\varepsilon[v,w] = \int\int L_\varepsilon dx dt \tag{17}
$$

The version of Noethers theorem elaborated in Appendix A is applicable. It implies that we get conservation laws for the total power, linear momentum and hamiltonian.

(1) *Conservation of total power.* The rotational group

$$
\begin{gathered}
w \to we^{-i\alpha}, \quad \overline{w} \to \overline{w}e^{i\alpha}, \quad v \to ve^{-i2\alpha}, \quad \overline{v} \to \overline{v}e^{i2\alpha} \\
x \to x, \quad t \to t
\end{gathered}
\tag{18}
$$

is a symmetry group for the variational principle. The corresponding infinitesimal generators of this group is given by

$$
\begin{gathered}
w \to w - i\alpha w, \quad \overline{w} \to \overline{w} + i\alpha\overline{w} \\
v \to v - i2\alpha v, \quad \overline{v} \to \overline{v} + i2\alpha\overline{v} \\
x \to x, \quad t \to t
\end{gathered}
\tag{19}
$$

The corresponding conserved density is given by

$$
\begin{aligned}
\mathcal{P}_\varepsilon &= \frac{\partial L_\varepsilon}{\partial w_t} iw + \frac{\partial L_\varepsilon}{\partial \overline{w}_t}(-i\overline{w}) + \frac{\partial L_\varepsilon}{\partial v_t} 2iv + \frac{\partial L_\varepsilon}{\partial \overline{v}_t}(-2i\overline{v}) \\
&= |w|^2 + \varepsilon|v|^2
\end{aligned}
$$

leading to the conservation of the total power

$$
P_\varepsilon = \|w\|_2^2 + \varepsilon\|v\|_2^2 \tag{20}
$$

Here in the sequel we employ the standard notation

$$\|\phi\|_p^p = \int |\phi|^p dx$$

for norms in L^p - spaces.

(2) *Conservation of linear momentum.* The spatial translation

$$x \to x + \alpha, \quad t \to t, \quad w \to w, \quad \overline{w} \to \overline{w}, \quad v \to v, \quad \overline{v} \to \overline{v} \tag{21}$$

is a symmetry group, which leads to linear momentum density

$$\begin{aligned}\mathcal{Q}_\varepsilon &= \frac{\partial L_\varepsilon}{\partial w_t} w_x + \frac{\partial L_\varepsilon}{\partial \overline{w}_t} \overline{w}_x + \frac{\partial L_\varepsilon}{\partial v_t} v_x + \frac{\partial L_\varepsilon}{\partial \overline{v}_t} \overline{v}_x \\ &= i[\frac{1}{2}(\overline{w} w_x - w \overline{w}_x) + \frac{1}{4}\varepsilon(\overline{v} v_x - v \overline{v}_x)]\end{aligned}$$

from which we get conservation of linear momentum

$$Q_\varepsilon = \int i[\frac{1}{2}(\overline{w} w_x - w \overline{w}_x) + \frac{1}{4}\varepsilon(\overline{v} v_x - v \overline{v}_x)]dx \tag{22}$$

(3) *Conservation of hamiltonian.* The temporal translation

$$t \to t + \alpha, \quad x \to x, \quad w \to w, \quad \overline{w} \to \overline{w}, \quad v \to v, \quad \overline{v} \to \overline{v} \tag{23}$$

is a symmetry group for the problem. The corresponding density is the hamiltonian density which is given as

$$\begin{aligned}\mathcal{H}_\varepsilon &= \frac{\partial L_\varepsilon}{\partial w_t} w_t + \frac{\partial L_\varepsilon}{\partial \overline{w}_t} \overline{w}_t + \frac{\partial L_\varepsilon}{\partial v_t} v_t + \frac{\partial L_\varepsilon}{\partial \overline{v}_t} \overline{v}_t - L_\varepsilon \\ &= d|w_x|^2 + \frac{1}{2}c|v_x|^2 + \frac{1}{2}\beta|v|^2 - \frac{1}{2}\overline{w}^2 v - \frac{1}{2}w^2\overline{v}\end{aligned}$$

from which conservation of total energy (hamiltonian)

$$H_\varepsilon = \int [d|w_x|^2 + \frac{1}{2}c|v_x|^2 + \frac{1}{2}\beta|v|^2 - \frac{1}{2}\overline{w}^2 v - \frac{1}{2}w^2\overline{v}]dx \tag{24}$$

follows.

3 Variational formulation of the nonlocal $\chi^{(2)}$ - model

In order to derive the action integral for the nonlocal approximation (11), we eliminate the field v from the action integral of the full $\chi^{(2)}$ - model. We proceed as follows: The $\varepsilon = 0$ version of the SH equation reads

$$cv_{xx} - \beta v + w^2 = 0, \quad c\overline{v}_{xx} - \beta\overline{v} + \overline{w}^2 = 0$$

from which the identity

$$\int [c|v_x|^2 + \beta|v|^2]dx = \frac{1}{2}\int [w^2\overline{v} + \overline{w}^2 v)]dx$$

follows. By plugging the latter identity into the action integral (16) - (17) and exploiting (7) we find that the action integral decomposes as

$$S[w] = S_{loc}[w] + S_{nonloc}[w] \tag{25}$$

where

$$S_{loc}[w] = \int\int[\frac{1}{2}i(\overline{w}w_t - w\overline{w}_t) - d|w_x|^2]dtdx$$

$$S_{nonloc}[w] = \frac{1}{4}\beta^{-1}\int\int\int R(y-x)[w^2(x)\overline{w}^2(y) + w^2(y)\overline{w}^2(x)]dtdxdy$$

Since variational calculus of action integrals involving nonlocal terms is not standard, we will now prove in detail that the equation (11) can be derived from a variational principle. [1]

The proof of this statement proceeds by evaluating the variation of the action integrals $S_{loc}[w]$ and $S_{nonloc}[w]$ separately. It is tacitly assumed that the complex valued function h is sufficiently smooth and vanishes on the boundary of the region of integration.

(1) *The contribution from S_{loc}*

We readily find that

$$\delta S_{loc}[w](h) = \int\int[iw_t + dw_{xx}](t,x)\overline{h}(t,x)dtdx + c.c. \tag{27}$$

by means of standard variational calculus. Here and in the sequel "c.c." means "complex conjugate".

(2) *The contribution from S_{nonloc}*

In the evaluation of the variation of the nonlocal part of the action functional special care must be taken. We first get

[1] Notice the similarity with the variational principle formulated in Abe and Ogura [24] for the conventional nonlinear, nonlocal Schrödinger equation, in which the nonlocal terms appear as a weighted mean.

$$\delta S_{nonloc}[w](h) = \frac{1}{2}\beta^{-1}[\int\int\int R(y-x)w^2(t,y)\overline{w}(t,x)\overline{h}(t,x)dtdxdy$$
$$+\int\int\int R(y-x)w^2(t,x)\overline{w}(t,y)\overline{h}(t,y)dtdxdy] + c.c. \qquad (28)$$

The next step consists of exploiting the fact that $R(x) = R(-x)$ in the second integral of the variation δS_{nonloc}:

$$\int\int\int R(y-x)w^2(t,x)\overline{w}(t,y)\overline{h}(t,y)dtdxdy + c.c.$$
$$= \int\int\int R(x-y)w^2(t,x)\overline{w}(t,y)\overline{h}(t,y)dtdxdy + c.c.$$

Finally, by viewing x and y as integration variables ("dummy variables") we interchange x and y in the latter integral, from which it follows that

$$\int\int\int R(x-y)w^2(t,x)\overline{w}(t,y)\overline{h}(t,y)dtdxdy + c.c.$$
$$= \int\int\int R(y-x)w^2(t,y)\overline{w}(t,x)\overline{h}(t,x)dtdxdy + c.c.$$

which is identical with the first integral in the variation $\delta S_{nonloc}[w](h)$. Hence we have

$$\delta S[w](h) = \int\int [iw_t + dw_{xx} + \beta^{-1}R * w^2\overline{w}](t,x)\overline{h}(t,x)dtdx + c.c.$$

and thus

$$\delta S[w](h) = 0 \Leftrightarrow iw_t + dw_{xx} + \beta^{-1}(R * w^2)\overline{w} = 0$$

for all sufficiently regular functions h vanishing on the boundary of the region of integration by exploiting the previous results.

Notice that it is not necessary to require the response function to be given by means of the formulas (8) - (9) in order to show that the nonlocal model (7) - (11). The only thing which matters is that the function R is real and symmetric.

4 Conservation laws of the nonlocal $\chi^{(2)}$ - model

As the variational principle for the nonlocal model (7) - (11) is on non - standard form due to the presence of the nonlocal term, the version of Noethers theorem presented in Appendix A is not applicable. However, as expected, it turns out that the power, linear momentum and the hamiltonian of the nonlocal model (11) are constants of motion. We show these results by direct computation. Thus we may suspect that a generalized version of Noethers theorem apply to the present situation.

In the process of deriving these results we make use of the following results: Introduce the Fourier representations

$$w^2(x) = \int_{-\infty}^{\infty} \widetilde{f}(k)\, \mathrm{e}^{ikx}\, dk, \quad R(x) = \int_{-\infty}^{\infty} \widetilde{R}(k)\, \mathrm{e}^{ikx}\, dk$$

where R is assumed to be a real and even function. Then, by using the definition (7), we find that

$$\int_{-\infty}^{\infty} (R * w^2)(x)\overline{w}^2(x)dx = 4\pi^2 \int_{-\infty}^{\infty} \widetilde{R}(k)|\widetilde{f}(k)|^2 dk \tag{29}$$

and

$$\int_{-\infty}^{\infty} (R * w^2)(x)(\overline{w}^2)_x(x)dx = -i4\pi^2 \int_{-\infty}^{\infty} k\widetilde{R}(k)|\widetilde{f}(k)|^2 dk \tag{30}$$

Moreover, we make use of the facts that

$$\begin{gathered} \int (R * w^2)\overline{w}\overline{w}_{xx} dx = \int (R * \overline{w}\overline{w}_{xx}) w^2 dx \\ \int R * (|w|^2 R * \overline{w}^2) w^2 dx = \int R * (|w|^2 R * w^2)\overline{w}^2 dx \end{gathered} \tag{31}$$

which can be derived by considering the integration variables as "dummy variables" and exploiting the assumption that the integral kernel R is a real and even function of its argument.

(1) *Conservation of power.* We readily find that

$$\begin{aligned} &(|w|^2)_t + (id(w\overline{w}_x - \overline{w}w_x))_x \\ &= i\beta^{-1}[(R * w^2)\overline{w}^2 - (R * \overline{w}^2)w^2] \end{aligned} \tag{32}$$

By integrating this equation and exploiting the result (29), we readily find that

$$\partial_t(\| w \|_2^2) = \partial_t(\int |w|^2 dx) = 0 \tag{33}$$

We observe that the variational principle is invariant under the rotational group

$$w \to we^{i\alpha}, \quad \overline{w} \to \overline{w}e^{-i\alpha}, \quad x \to x, \quad t \to t \tag{34}$$

whose infinitesimal generator is given by means of

$$w \to w + \alpha i w, \quad \overline{w} \to \overline{w} - \alpha i \overline{w}, \quad x \to x, \quad t \to t \tag{35}$$

We conjecture that the present conservation law can be obtained directly from a version of Noethers theorem generalizing the one elaborated in Appendix A.

(2) *Conservation of linear momentum.* Direct computation shows that the evolution equation for the linear momentum density $\mathcal{Q} = i\frac{1}{2}(\overline{w}w_x - w\overline{w}_x)$ is given as

$$\mathcal{Q}_t + \mathcal{M}_x = \Delta$$

where

$$\mathcal{M} = \frac{1}{2}d(\overline{w}w_{xx} + w\overline{w}_{xx} - 2d|w_x|^2) + \frac{1}{2}\beta^{-1}(N(w^2)\overline{w}^2 + N(\overline{w}^2)w^2)$$

$$\Delta = \frac{1}{2}\beta^{-1}[(R * w^2)(\overline{w}^2)_x) + (R * \overline{w}^2)(w^2)_x]$$

By assuming the flux Q to vanish at the boundary in x - space and exploiting the result (30), integration yields

$$\partial_t[\int_{-\infty}^{-\infty} \mathcal{Q}dx] = \int_{-\infty}^{\infty} \Delta dx$$

Due to the property (30) we find that the RHS of this equation is identically equal to zero, from which conservation of linear momentum follows. We notice that the variational problem is invariant under the translation group

$$t \to t, \quad x \to x + \alpha, \quad y \to y + \alpha, \quad w \to w, \quad \overline{w} \to \overline{w}$$

which gives rise to the conjecture that the present result also can be derived from a Lie symmetry group analysis applied to a generalized version of Noethers theorem.

(3) *Conservation of the hamiltonian.* The hamiltonian density of the nonlocal $\chi^{(2)}$ - model (11) is given as

$$\mathcal{H} = d|w_x|^2 - \frac{1}{4}\beta^{-1}[(R * w^2)\overline{w}^2 + (R * \overline{w}^2)w^2] \tag{36}$$

Just as in the previous cases we first derive an evolution equation for this density:

$$\partial_t\mathcal{H} + \partial_x\mathcal{W} = \Omega \tag{37}$$

where the hamiltonian flux $\mathcal{W}$ is given as

$$\mathcal{W} = id^2(\overline{w}_{xx}w_x - w_{xx}\overline{w}_x)$$

$$+i\beta^{-1}d[(R * \overline{w}^2)ww_x - (R * w^2)\overline{w}\overline{w}_x]$$

while

$$\Omega = i\frac{1}{2}\beta^{-1}d[-(R*w^2)\overline{w}\overline{w}_{xx} + (R*\overline{w}\overline{w}_{xx})w^2$$

$$-(R*ww_{xx})\overline{w}^2 + (R*\overline{w}^2)ww_{xx}] \tag{38}$$

$$+\frac{1}{2}i\beta^{-2}[-R*(|w|^2(R*\overline{w}^2))w^2 + R*(|w|^2(R*w^2))\overline{w}^2]$$

Integration of (37) yields conservation of the hamiltonian:

$$\partial_t H = 0$$

$$H = \int \mathcal{H} dx$$

In the process of deriving this result we have exploited the identities (31). We conjecture that even this result can be derived using the generalized version of Noethers theorem: The variational principle permits invariance with respect to the t - coordinate

$$t \to t+\alpha, \quad x \to x, \quad w \to w, \quad \overline{w} \to \overline{w}$$

Notice that the conservation laws for power and linear momentum of the singularly perturbed $\chi^{(2)}$ - model (15) deform to the corresponding laws of the nonlocal approximation (11). In order to show that the same property holds true for the hamiltonian we eliminate the field v from the hamiltonian of (15) in the case $\varepsilon = 0$ in the same way as we did when we deduced the action integral for the nonlocal approximation (11).

One can prove by means of the conservation laws that the gradient norm of the solution w of the nonlocal $\chi^{(2)}$ - model is bounded from above by a constant. Then it is concluded that the Cauchy problem is globally wellposed. However, this argument presupposes that one has proved local existence. It is conjectured that a proof of the latter property is done either by means of the *contraction principle* or by using *compactness arguments.*

A Variational formulation and Noethers theorem

Let $S : A \to \mathbf{R}$ be a functional defined on an appropriate space A of complex valued functions u_1, u_2,..., u_n and their respective complex conjugates $\overline{u}_1$, $\overline{u}_2$,..., $\overline{u}_n$ and possibly also explicitly on the coordinates $x_\nu; \nu = 0, 1, 2, ..., D$. This functional is given by the integral

$$S[u_a, \overline{u}_a] = \int_\Omega L(u_a, \overline{u}_a, \partial_\nu u_a, \partial_\nu \overline{u}_a, x_\mu) dx_\mu \tag{A.1}$$

where the integrand (Lagrangian density) is supposed to be real sufficient regular real valued function and Ω is some subset of $\mathbf{R}^{D+1}$. We refer to the above functional as the *action integral.* Variational calculus leads then to n Euler - Lagrange equations for the extremal of this functional, i.e.,

$$\begin{aligned} &\partial_\mu[\frac{\partial L}{\partial(\partial_\mu \overline{u}_a)}] - \frac{\partial L}{\partial \overline{u}_a} = 0 \\ &\partial_\mu[\frac{\partial L}{\partial(\partial_\mu u_a)}] - \frac{\partial L}{\partial u_a} = 0 \end{aligned} \tag{A.2}$$

for $a = 1, 2, 3, ..., n$. Here and in the sequel we employ the Einstein summation convention.

Noethers theorem says that to every Lie group which keeps the action integral invariant there exists a conservation law [22]. This conservation law is expressed in terms of the infinitesimal generators of this group. We will refer to these groups as *symmetry groups.*

Let us specify what type of symmetry groups we are considering. Symmetry under coordinate transformation refers to the translation

$$x_\mu \rightarrow \widetilde{x}_\mu = x_\mu + \delta x_\mu \tag{A.3}$$

where the infinitesimal change δx_μ may be a function of the other coordinates x_ν. Noethers theorem also considers the effect of the transformation of in the field quantities themselves, which may be described by means of the infinitesimal generators as

$$u_a(x_\mu) \rightarrow \widetilde{u}_a(\widetilde{x}_\mu) = u_a(x_\mu) + \delta u_a(x_\mu) \tag{A.4}$$

Here $\delta u_a(x_\mu)$ measures the effect of both the changes in x_μ and in u_a, and may be a function of the other field quantities. Note that the change of one of the field quantities at a fixed, particular point in x_μ - space is a different quantity:

$$\widetilde{u}_a(x_\mu) = u_a(x_\mu) + \overline{\delta} u_a(x_\mu) \tag{A.5}$$

As a consequence of the transformations of both the coordinates and the field quantities the Lagrangian appears in general as a different function of both the field variables and the coordinates x_μ

$$\begin{gathered} L(u_a(x_\mu), \overline{u}_a(x_\mu), \partial_{x_\nu} u_a(x_\mu), \partial_{x_\nu} \overline{u}_a(x_\mu), x_\mu) \\ \downarrow \\ \widetilde{L}(\widetilde{u}_a(\widetilde{x}_\mu), \overline{\widetilde{u}}_a(\widetilde{x}_\mu), \partial_\nu \widetilde{u}_a(\widetilde{x}_\mu), \partial_\nu \overline{\widetilde{u}}_a(\widetilde{x}_\mu), \widetilde{x}_\mu) \end{gathered} \tag{A.6}$$

The version of Noethers theorem that will be presented is not the most general form possible, but it is such as to facilitate the derivation without significantly restricting the scope of the theorem or the usefulness of the conclusions.

The following three conditions will be assumed to hold:

(1) The x_μ - space is Euclidean.

(2) The Lagrangian is *form-invariant*, i.e.,

$$\begin{aligned} &L(u_a(x_\mu), \overline{u}_a(x_\mu), \partial_{x_\nu} u_a(x_\mu), \partial_{x_\nu} \overline{u}_a(x_\mu), x_\mu) \\ &= L(\widetilde{u}_a(\widetilde{x}_\mu), \overline{\widetilde{u}}_a(\widetilde{x}_\mu), \partial_\nu \widetilde{u}_a(\widetilde{x}_\mu), \partial_\nu \overline{\widetilde{u}}_a(\widetilde{x}_\mu), \widetilde{x}_\mu) \end{aligned} \tag{A.7}$$

Notice that this condition ensures that the Euler - Lagrange equations have the same form whether expressed in terms of the old and the new coordinates.

(3) The action integral is invariant under the transformation, i.e.,

$$\begin{aligned} &\int_\Omega L(u_a(x_\mu), \overline{u}_a(x_\mu), \partial_\nu u_a(x_\mu), \partial_\nu \overline{u}_a(x_\mu), x_\mu) dx_\mu = \\ &\int_{\widetilde{\Omega}} \widetilde{L}(\widetilde{u}_a(\widetilde{x}_\mu), \overline{\widetilde{u}}_a(\widetilde{x}_\mu), \partial_\nu \widetilde{u}_a(\widetilde{x}_\mu), \partial_\nu \overline{\widetilde{u}}_a(\widetilde{x}_\mu), \widetilde{x}_\mu) d\widetilde{x}_\mu \end{aligned} \tag{A.8}$$

The latter condition is referred to as the condition of *scaling-invariance*.

Now, by combining (A.7) and (A.8) we get

$$\begin{aligned} &\int_{\widetilde{\Omega}} L(\widetilde{u}_a(\widetilde{x}_\mu), \overline{\widetilde{u}}_a(\widetilde{x}_\mu), \partial_\nu \widetilde{u}_a(\widetilde{x}_\mu), \partial_\nu \overline{\widetilde{u}}_a(\widetilde{x}_\mu), \widetilde{x}_\mu) d\widetilde{x}_\mu \\ &- \int_\Omega L(u_a(x_\mu), \overline{u}_a(x_\mu), \partial_\nu u_a(x_\mu), \partial_\nu \overline{u}_a(x_\mu), x_\mu) dx_\mu = 0 \end{aligned} \tag{A.9}$$

In the first integral $\widetilde{x}_\mu$ represents a dummy variable and can therefore be relabeled x_μ:

$$\begin{aligned} &\int_{\widetilde{\Omega}} L(\widetilde{u}_a(x_\mu), \overline{\widetilde{u}}_a(x_\mu), \partial_\nu \widetilde{u}_a((x_\mu)), \partial_\nu \overline{\widetilde{u}}_a(x_\mu), x_\mu) dx_\mu \\ &- \int_\Omega L(u_a(x_\mu), \overline{u}_a(x_\mu), \partial_\nu u_a(x_\mu), \partial_\nu \overline{u}_a(x_\mu), x_\mu) dx_\mu = 0 \end{aligned}$$

It is now possible to express the difference on the left hand side of this equation as

$$\int_{\widetilde{\Omega}} L(\widetilde{u}, x_\mu) dx_\mu - \int_{\Omega} L(u, x_\mu) dx_\mu$$

$$= \int_{\Omega} [L(\widetilde{u}, x_\mu) - L(u, x_\mu) + \partial_\nu (L(u, x_\mu)\delta x_\nu)] dx_\mu \tag{A.10}$$

in the linear approximation. Here we for convenience have introduced the abbreviated notation

$$L(\widetilde{u}, x_\mu) \equiv L(\widetilde{u}_a(x_\mu), \overline{\widetilde{u}}_a(x_\mu), \partial_\nu \widetilde{u}_a(x_\mu), \partial_\nu \overline{\widetilde{u}}_a(x_\mu), x_\mu)$$

$$L(u, x_\mu) \equiv L(u_a(x_\mu), \overline{u}_a(x_\mu), \partial_\nu u_a(x_\mu), \partial_\nu \overline{u}_a(x_\mu), x_\mu)$$

The difference term in the square brackets can be written to the first order as

$$L(\widetilde{u}, x_\mu) - L(u, x_\mu) =$$

$$\frac{\partial L}{\partial \overline{u}_a} \overline{\delta} \overline{u}_a + \frac{\partial L}{\partial u_a} \overline{\delta} u_a + \frac{\partial L}{\partial(\partial_\nu \overline{u}_a)} \overline{\delta}(\partial_\nu \overline{u}_a) + \frac{\partial L}{\partial(\partial_\nu u_a)} \overline{\delta}(\partial_\nu u_a)$$

According to the definition of the operator $\overline{\delta}$, partial differentiation and $\overline{\delta}$ commute and we get

$$L(\overline{u}, x_\mu) - L(u, x_\mu) =$$

$$\frac{\partial L}{\partial \overline{u}_a} \overline{\delta} \overline{u}_a + \frac{\partial L}{\partial u_a} \overline{\delta} u_a + \frac{\partial L}{\partial(\partial_\nu \overline{u}_a)} \partial_\nu(\overline{\delta} \overline{u}_a) + \frac{\partial L}{\partial(\partial_\nu u_a)} \partial_\nu(\overline{\delta} u_a)$$

The Euler - Lagrange equations (A.2) now enable us convert the RHS of this equation to

$$L(\widetilde{u}, x_\mu) - L(u, x_\mu) =$$

$$\partial_\nu [\frac{\partial L}{\partial(\partial_\nu \overline{u}_a)} \overline{\delta} \overline{u}_a + \frac{\partial L}{\partial(\partial_\nu u_a)} \overline{\delta} u_a]$$

Hence the invariance condition (A.9) appears as the continuity equation

$$\int_{\Omega} \partial_\nu [\frac{\partial L}{\partial(\partial_\nu \overline{u}_a)} \overline{\delta} \overline{u}_a + \frac{\partial L}{\partial(\partial_\nu u_a)} \overline{\delta} u_a + L\delta x_\nu] dx_\mu = 0$$

It is helpful, however, to develop this condition further by specifying the form of infinitesimal generators of the Lie group in terms of parameters $\varepsilon_r, r =$

$1, 2, ..., R$ such that the change in x_μ, u_a and $\overline{u}_a$ is linear in ε_r:

$$\delta x_\nu = \varepsilon_r X_{r\nu}, \quad \delta u_a = \varepsilon_r \Psi_{ra}, \quad \delta \overline{u}_a = \varepsilon_r \overline{\Psi}_{ra} \tag{A.11}$$

The functions $X_{r\nu}$ and Ψ_{ra} may depend on the other coordinates and field variables. By taking into account that

$$\delta u_a = \overline{\delta} u_a + \partial_\sigma u_a \delta x_\sigma$$

we readily find that

$$\overline{\delta} u_a = \varepsilon_r (\Psi_{ra} - \partial_\sigma u_a X_{r\sigma})$$

The invariance condition (A.9) now reads

$$\int_\Omega \partial_\nu [(\frac{\partial L}{\partial(\partial_\nu \overline{u}_a)} \partial_\sigma \overline{u}_a + \frac{\partial L}{\partial(\partial_\nu u_a)} \partial_\sigma u_a - L\delta_{\nu\sigma}) X_{r\sigma} - (\frac{\partial L}{\partial(\partial_\nu \overline{u}_a)} \overline{\Psi}_{ra} + \frac{\partial L}{\partial(\partial_\nu \overline{u}_a)} \Psi_{ra})] dx_\mu = 0$$

Since the ε_r - parameters are arbitrary, there exist R conserved currents with conservation equations on differential form given by

$$\partial_\nu [(\frac{\partial L}{\partial(\partial_\nu \overline{u}_a)} \partial_\sigma \overline{u}_a + \frac{\partial L}{\partial(\partial_\nu u_a)} \partial_\sigma u_a - L\delta_{\nu\sigma}) X_{r\sigma} - (\frac{\partial L}{\partial(\partial_\nu \overline{u}_a)} \overline{\Psi}_{ra} + \frac{\partial L}{\partial(\partial_\nu u_a)} \Psi_{ra})] = 0 \tag{A.12}$$

for $r = 1, 2, ..., R$. The equations (A.12) form the main conclusion of Noethers theorem: A variational principle permitting R symmetry groups such that form- and scale invariance hold for the Lie groups acting on the space of dependent and independent variables, possesses R conserved currents.

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