Chebyshev transforms on Okounkov bodies

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Abstract

Let L be a big holomorphic line bundle on a compact complex manifold X. We show how to associate a convex function on the Okounkov body of L to any continuous metric $e^{-\psi}$ on L. We will call this the Chebyshev transform of ψ, denoted by $c[\psi]$. Our main theorem states that the integral of the difference of the Chebyshev transforms of two weights is equal to the relative energy of the weights, which is a well-known functional in Kähler-Einstein geometry and Arakelov geometry. We show that this can be seen as a generalization of classical results on Chebyshev constants and the Legendre transform of invariant metrics on toric manifolds. As an application we prove the differentiability of the relative energy in the ample cone.

Contents

1 Introduction

In [8] and [9] Khovanskii-Kaveh and Lazarsfeld-Mustaţă initiated a systematic study of Okounkov bodies of divisors and more generally of linear series. Our goal is to contribute with an analytic viewpoint.

It was Okounkov who in his papers [10] and [11] introduced a way of associating a convex body in $\mathbb{R}^n$ to any ample divisor on a n-dimensional projective variety. This convex body, called the Okounkov body of the divisor and denoted by $\Delta(L)$, can then be studied using convex geometry. It was recognized in [9] that the construction works for arbitrary big divisors.

We will restrict ourselves to a complex projective manifold X, and instead of divisors we will for the most part use the language of holomorphic line bundles. Because of this, in the construction of the Okounkov body, we prefer choosing local holomorphic coordinates instead of the equivalent use of a flag of subvarieties (see [9]). We use additive notation for line bundles, i.e. we will write kL instead of $L^{\otimes k}$ for the k:th tensor power of L. We will also use the additive notation for metrics. If h is a hermitian metric on a line bundle, we may write it as $h = e^{-\psi}$, and call ψ a weight. Thus if ψ is a weight on L, $k\psi$ is a weight on kL.

The main motivation for studying Okounkov bodies has been their connection to the volume function on divisors. Recall that the volume of a line bundle L is defined as

$$\mathrm{vol}(L) := \limsup_{k\to\infty} \frac{n!}{k^n} \dim(H^0(kL)).$$

A line bundle is said to be big if it has positive volume. From here on, all line bundles L we consider will be assumed to be big. By Theorem A in [9], for any big line bundle L it holds that

$$\mathrm{vol}_{\mathbb{R}^n}(\Delta(L)) = \frac{1}{n!}\mathrm{vol}(L).$$

We are interested in studying certain functionals on the space of weights on L that refine $\mathrm{vol}(L)$ (see below).

A weight ψ is said to be *psh* if

$$dd^c\psi \geq 0$$

as a current. Given two locally bounded psh weights ψ and φ we define $\mathcal{E}(\psi, \varphi)$ as

$$\frac{1}{n+1} \sum_{j=0}^{n} \int_X (\psi - \varphi)(dd^c\psi)^j \wedge (dd^c\varphi)^{n-j},$$

which we will refer to as the relative energy of ψ and φ. This bifunctional first appeared in the works of Mabuchi and Aubin in Kähler-Einstein geometry (see [1] and references therein).

If ψ and φ are continuous but not necessarily psh, we may still define a relative energy, by first projecting them down to the space of psh weights,

$$P(\psi) := \sup\{\psi' : \psi' \leq \psi, \psi' \text{ psh}\}.$$

We are therefore led to consider the functional

$$\mathcal{E}(\psi,\varphi) := \frac{1}{n+1}\sum_{j=0}^{n}\int_{\Omega}(P(\psi)-P(\varphi))(dd^cP(\psi))^j \wedge (dd^cP(\varphi))^{n-j}, \quad (1)$$

where Ω denotes the Zariski open set where both $P(\psi)$ and $P(\varphi)$ are locally bounded. For psh weights ψ, trivially $P(\psi) = \psi$, therefore there is no ambiguity in the notation. The relative energy can be seen as a generalization of the volume since if we let ψ be equal to $\varphi + 1$, from e.g. [1] we have that

$$\mathcal{E}(\psi,\varphi) = \int_{\Omega}(dd^cP(\varphi))^n = \text{vol}(L).$$

Given a continuous weight ψ, we will show how to construct an associated convex function on the interior of the Okounkov body of L which we will call the Chebyshev transform of ψ, denoted by $c[\psi]$. The construction can be seen to generalize both the Chebyshev constants in classical analysis and the Legendre transform of convex functions (see subsections 9.2 and 9.3 respectively).

First we construct $\Delta(L)$. Choose a point $p \in X$ and local holomorphic coordinates $z_1, ..., z_n$ centered at p. Choose also a trivialization of L around p. With respect to this trivialization any holomorphic section $s \in H^0(L)$ can be written as a convergent power series in the coordinates z_i,

$$s = \sum_{\alpha} a_\alpha z^\alpha.$$

Consider the lexicographic order on $\mathbb{N}^n$, and let $v(s)$ denote the smallest index α (i.e. with respect to the lexicographic order) such that

$$a_\alpha \neq 0.$$

We let $v(H^0(L))$ denote the set $\{v(s) : s \in H^0(L)\}$, and finally let the Okounkov body of L, denoted by $\Delta(L)$, be defined as closed convex hull in $\mathbb{R}^n$ of the union

$$\bigcup_{k\geq 1}\frac{1}{k}v(H^0(kL)).$$

Observe that the construction depends on the choice of p and the holomorphic coordinates. For other choices, the Okounkov bodies will in general differ.

Now let ψ be a continuous weight on L. There are associated supremum norms on the spaces of sections $H^0(kL)$,

$$||s||^2_{k\psi} := \sup_{x\in X}\{|s(x)|^2 e^{-k\psi(x)}\}.$$

If $v(s) = k\alpha$ for some section $s \in H^0(kL)$, we let $A_{\alpha,k}$ denote the affine space of sections in $H^0(kL)$ of the form

$$z^{k\alpha} + \text{ higher order terms.}$$

We define the discrete Chebyshev transform $F[\psi]$ on $\bigcup_{k\geq 1} v(H^0(kL)) \times \{k\}$ as

$$F[\psi](k\alpha, k) := \inf\{\ln ||s||^2_{k\psi} : s \in A_{\alpha,k}\}.$$

Theorem 1.1. *For any point $p \in \Delta(L)^\circ$ and any sequence $\alpha(k) \in \frac{1}{k}v(H^0(kL))$ converging to p, the limit*

$$\lim_{k\to\infty} \frac{1}{k} F[\psi](k\alpha(k), k)$$

exists and only depends on p. We may therefore define the Chebyshev transform of ψ by letting

$$c[\psi](p) := \lim_{k\to\infty} \frac{1}{k} F[\psi](k\alpha(k), k),$$

for any sequence $\alpha(k)$ converging to p.

The main observation underlying the proof is the fact that the discrete Chebyshev transforms are subadditive. Our proof is thus very much inspired by the work of Zaharjuta, who in [14] used subadditive functions on $\mathbb{N}^n$ when studying the classical Chebyshev constants, and also by the article [4] where Bloom-Levenberg recognize the importance of subadditivity, extending Zaharjutas results to a more general weighted setting, but still in $\mathbb{C}^n$ (we show in section 7 how to recover the formula of Bloom-Levenberg from Theorem 1.1).

We prove a general statement concerning subadditive functions on subsemigroups of $\mathbb{N}^d$ that generalizes a result of Zaharjuta.

Theorem 1.2. *Let $\Gamma \subseteq \mathbb{N}^d$ be a semigroup which generates $\mathbb{Z}^d$ as a group, and let F be a subadditive function on Γ which is locally bounded from below by some linear function. Then for any sequence $\alpha(k) \in \Gamma$ such that $|\alpha(k)| \to \infty$ and $\frac{\alpha(k)}{|\alpha(k)|} \to p \in \Sigma(\Gamma)^\circ$ ($\Sigma(\Gamma)$ denotes the convex cone generated by Γ) for some point p in the interior of $\Sigma(\Gamma)$, the limit*

$$\lim_{k\to\infty} \frac{F(\alpha(k))}{|\alpha(k)|}$$

exists and only depends on F and p. Furthermore the function

$$c[F](p) := \lim_{k\to\infty} \frac{F(\alpha(k))}{|\alpha(k)|}$$

thus defined on $\Sigma(\Gamma)^\circ \cap \Sigma^\circ$ is convex.

Theorem 1.1 will follow from Theorem 1.2.

It should be pointed out that related Chebyshev transforms play an important role in [12] in the context of Arakelov geometry.

Our main result on the Chebyshev transform is the following.

Theorem 1.3. *Let ψ and φ be two continuous weights on L. Then it holds that*

$$\mathcal{E}(\psi,\varphi) = n! \int_{\Delta(L)^\circ} (c[\varphi] - c[\psi]) d\lambda, \tag{2}$$

where $d\lambda$ denotes the Lebesgue measure on $\Delta(L)$.

The proof of Theorem 1.3 relies on the fact that one can use certain L^2-norms related to the weight, called Bernstein-Markov norms, to compute the Chebyshev transform. With the help of these one can interpret the right-hand side in equation (2) as a limit of Donaldson bifunctionals $\mathcal{L}_k(\psi,\varphi)$. On the other hand, the main theorem in [1] says that the bifunctionals $\mathcal{L}_k(\psi,\varphi)$ converges to the relative energy when k tends to infinity, which gives us our theorem.

Because of the homogeneity of the Okounkov body, i.e.

$$\Delta(kL) = k\Delta(L),$$

one may define the Okounkov body of an arbitrary $\mathbb{Q}$-divisor D by letting

$$\Delta(D) := \frac{1}{p}\Delta(pD),$$

for any integer p clearing all denominators in D. Theorem B in [9] states that one may in fact associate an Okounkov body to an arbitrary big $\mathbb{R}$-divisor, such that the Okounkov bodies are fibers of a closed convex cone in $\mathbb{R}^n \times N^1(X)_{\mathbb{R}}$, where $N^1(X)_{\mathbb{R}}$ denotes the Neron-Severi space of $\mathbb{R}$-divisors. We show that this can be done also on the level of Chebyshev transforms, i.e. there is a continuous and indeed convex extension of the Chebyshev transforms to the space of continuous weights on big $\mathbb{R}$-divisors. We prove Theorem 1.3 for weights on ample $\mathbb{R}$-divisors.

As an application we prove that the relative energy is differentiable in the ample cone. In [1] Berman-Boucksom consider as a function of t the relative energy of weights ψ_t and φ, where ψ_t vary smoothly with t. Theorem B in [1] states that the function

$$F(t) := \mathcal{E}_L(\psi_t,\varphi)$$

then is differentiable in t, and that the derivative is given by

$$F'(0) = \int_\Omega \psi_t'(0)(dd^c P(\psi_0))^n,$$

where $\psi_t'(0)$ denotes the derivative of ψ_t in zero. In section 9 we prove a generalization of this in the ample setting where the underlying $\mathbb{R}$-divisor L_t varies with t within the ample cone.

Theorem 1.4. *Let A_i, $i = 1, ..., m$ be a finite collection of ample line bundles, and for each i let ψ_i and φ_i be two continuous weights on A_i. Let O denote the open cone in $\mathbb{R}^d$ such that $a \in O$ iff $\sum a_i A_i$ is an ample $\mathbb{R}$-divisor. Then the function*

$$F(a) := \mathcal{E}_{\sum a_i A_i}(\sum a_i\psi_i, \sum a_i\varphi_i)$$

is $\mathcal{C}^1$ on O.

We also calculate the differential. If we consider the special case where A is ample and Ψ is some positive continuous weight on A, and let

$$f(t) := \mathcal{E}_{L+tA}(\psi + t\Psi, \varphi + t\Psi)$$

for some continuous weights ψ and φ on an ample divisor L. Then our calculations show that

$$f'(0) = \sum_{j=0}^{n-1} \int_X (P(\psi) - P(\varphi)) dd^c\Psi \wedge (dd^c P(\psi))^j \wedge (dd^c P(\varphi))^{n-j-1}. \quad (3)$$

Another special case is the following. If A is an ample divisor and s_A is a defining section for A, by multiplying with $s_A^{\otimes tk}$ we get embeddings of the spaces $H^0(k(L - tA))$ into $H^0(kL)$. There is also an associated map between the spaces of weights, where ψ_L maps to

$$\psi_{L-tA} := \psi_L - t \ln |s_A|^2.$$

It follows from the proof of Theorem 1.4 that

$$\frac{d}{dt}_{|0} \mathcal{E}_X(\psi_{L-tA}, \varphi_{L-tA}) = -\mathcal{E}_A(\psi_L, \varphi_L).$$

Our proof uses the same approach as the proof of the differentiability of the volume in [9]. Since the relative energy is given by the integral of Chebyshev transforms over Okounkov bodies, when we differentiate we get one term coming from the variation of the Okounkov body, as studied in [9], and one term coming from the variation of the Chebyshev transforms. One can show that if one in formula (3) as Ψ chooses the positive weight $\ln |s|^2$, and let $\psi_0 = \varphi_0 + 1$, using the Lelong-Poincaré formula one recovers the formula for the derivative of the volume in the ample cone, i.e.

$$\frac{d}{dt}_{|0} \mathrm{vol}_X(L + tA) = n \mathrm{vol}_{[A]}(L_{|[A]}),$$

where $[A]$ denotes the divisor $\{s = 0\}$.

1.1 Organization

In section 2 we start by defining the Okounkov body of a semigroup, and we recall a result on semigroups by Khovanskii that will be of great use later on.

Section 3 deals with subadditive functions on subsemigroups of $\mathbb{N}^{n+1}$ and contains the proof of Theorem 1.2.

The definition of the Okounkov body of a line bundle follows in section 4.

In section 5 we define the discrete Chebyshev transform of a weight, and prove that this function has the properties needed for Thereom 1.2 to be applicable. We thus prove Theorem 1.1. We also show that the difference between two Chebyshev transforms is bounded on the interior of the Okounkov body.

The relative energy of weights is introduced in section 6. Here we also state our main theorem, Theorem 1.3.

In section 7 we show how one can use Bernstein-Markov norms instead of supremum norms in the construction of the Chebyshev transform.

The proof of Theorem 1.3 follows in section 8.

Section 9 discusses previuos results.

In subsection 9.1 we observe that if we in (2) let φ be equal to $\psi + 1$, then we recover Theorem A in [9], i.e. that

$$\mathrm{vol}_{\mathbb{R}^n}(\Delta(L)) = \frac{1}{n!}\mathrm{vol}(L).$$

In subsection 9.2 we move on to clarify the connection to the classical Chebyshev constants. We see that if we embed $\mathbb{C}$ into $\mathbb{P}^1$ and choose our weights wisely then formula (2) gives us the classical result in potential theory that the Chebyshev constant and transfinite diamter of a regular compact set in $\mathbb{C}$ coincides. See subsection 9.2 for definitions.

Subsection 9.3 studies the case of a toric manifold, with a torus invariant line bundle and invariant weights. We calculate the Chebyshev transforms, and observe that for invariant weights, the Chebyshev transform equals the Legendre transform of the weight seen as a function on $\mathbb{R}^n$.

We show in section 10 that if the line bundle is ample, the Chebyshev transform is defined on the zero-fiber of the Okounkov body, not only in the interior. Using the Ohsawa-Takegoshi extension theorem we prove that

$$\mathcal{E}_Y(P(\varphi)_{|Y}, P(\psi)_{|Y}) = (n-1)! \int_{\Delta(L)_0} (c[\psi] - c[\varphi])(0, \alpha) d\alpha, \tag{4}$$

where $\Delta(L)_0$ denotes the zero-fiber of $\Delta(L)$, and Y is a submanifold locally given by the equation $z_1 = 0$.

In section 11 we show how to translate the results of Bloom-Levenberg to our language of Chebyshev transforms. We reprove Theorem 2.7 in [4] using our Theorem 1.3, equation (4) and a recursion formula from [1].

We show in section 12 how to construct a convex and therefore continuous extension of the Chebyshev transform to arbitrary big $\mathbb{R}$-divisors.

In section 13 we move on to prove Theorem 1.4 concerning the differentiability of the relative energy in the ample cone.

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2 Semigroups and Okounkov bodies

Let $\Gamma \subseteq \mathbb{N}^{n+1}$ be a semigroup. We denote by $\Sigma(\Gamma) \subseteq \mathbb{R}^{n+1}$ the closed convex cone spanned by Γ. By $\Delta_k(\Gamma)$ we will denote the set

$$\Delta_k(\Gamma) := \{\alpha : (k\alpha, k) \in \Gamma\} \subseteq \mathbb{R}^n.$$

Definition 2.1. *The Okounkov body $\Delta(\Gamma)$ of the semigroup Γ is defined as*

$$\Delta(\Gamma) := \{\alpha : (\alpha, 1) \in \Sigma(\Gamma)\} \subseteq \mathbb{R}^n.$$

It is clear that for all non-negative k,

$$\Delta_k(\Gamma) \subseteq \Delta(\Gamma).$$

The next theorem is a result of Khovanskii from [7].

Theorem 2.2. *Assume that $\Gamma \subseteq \mathbb{N}^{n+1}$ is a finitely generated semigroup which generates $\mathbb{Z}^{n+1}$ as a group. Then there exists an element $z \in \Sigma(\Gamma)$, such that*

$$(z + \Sigma(\Gamma)) \cap \mathbb{Z}^{n+1} \subseteq \Gamma.$$

When working with Okounkov bodies of semigroups it is sometimes useful to reformulate Theorem 2.2 into the following lemma.

Lemma 2.3. *Suppose that Γ is finitely generated, generates $\mathbb{Z}^{n+1}$ as a group, and also that $\Delta(\Gamma)$ is bounded. Then there exists a constant C such that for all k, if*

$$\alpha \in \Delta(\Gamma) \cap \left(\frac{1}{k}\mathbb{Z}\right)^n$$

and if the distance between α and the boundary of $\Delta(\Gamma)$ is greater than C/k, then in fact we have that

$$\alpha \in \Delta_k(\Gamma).$$

Proof. By definition we that

$$\alpha \in \Delta(\Gamma) \cap \left(\frac{1}{k}\mathbb{Z}\right)^n \qquad \text{iff} \qquad (k\alpha, k) \in \Sigma(\Gamma) \cap \mathbb{Z}^{n+1}.$$

Also by definition

$$\alpha \in \Delta_k(\Gamma) \qquad \text{iff} \qquad (k\alpha, k) \in \Gamma.$$

By Theorem 2.2 we have that

$$(k\alpha, k) \in \Gamma \qquad \text{if} \qquad (k\alpha, k) - z \in \Sigma(\Gamma),$$

and since $\Sigma(\Gamma)$ is a cone, $(k\alpha, k) - z \in \Sigma(\Gamma)$ iff $(\alpha, 1) - z/k \in \Sigma(\Gamma)$. If $(\alpha, 1)$ lies further than $|z|/k$ from the boundary of $\Sigma(\Gamma)$, then trivially $(\alpha, 1) - z/k \in \Sigma(\Gamma)$. Since by assumtion the Okounkov body is bounded, the distance between $(\alpha, 1)$ and the boundary of $\Sigma(\Gamma)$ is greater than some constant times the distance between α and the boundary of $\Delta(\Gamma)$. The lemma follows. ☐

Corollary 2.4. *Suppose that Γ generates $\mathbb{Z}^{n+1}$ as a group, and also that $\Delta(\Gamma)$ is bounded. Then $\Delta(\Gamma)$ is equal to the closure of the union $\cup_{k\geq 0}\Delta_k(\Gamma)$.*

Proof. That

$$\overline{\cup_{k\geq 0}\Delta_k(\Gamma)} \subseteq \Delta(\Gamma)$$

is clear. For the opposite direction, we exhaust $\Delta(\Gamma)$ by Okounkov bodies of finitely generated subsemigroups of Γ. Therefore, without loss of generality we may assume that Γ is finitely generated. We apply Lemma 2.3 which says that all the $(\frac{1}{k}\mathbb{Z})^n$ lattice points in $\Delta(\Gamma)$ whose distance to the boundary of $\Delta(\Gamma)$ is greater that some constant depending on the element z in (2.2), divided by k, actually lie in $\Delta_k(\Gamma)$. The corollary follows. □

3 Subadditive functions on semigroups

Let Γ be a semigroup. A real-valued function F on Γ is said to be *subadditive* if for all $\alpha, \beta \in \Gamma$ it holds that

$$F(\alpha + \beta) \leq F(\alpha) + F(\beta).$$

If $\alpha \in \mathbb{R}^{n+1}$, we denote the sum of its coordinates $\sum \alpha_i$ by $|\alpha|$. We also let $\Sigma^0 \subseteq \mathbb{R}^{n+1}$ denote the set

$$\Sigma^0 := \{(\alpha_1, ..., \alpha_{n+1}) : |\alpha| = 1, \theta_i > 0\}.$$

In [4] Bloom-Levenberg observe that one can extract from [14] the following theorem on subadditive functions on $\mathbb{N}^{n+1}$.

Theorem 3.1. *Let F be a subadditive function on $\mathbb{N}^{n+1}$ which is bounded from below by some linear function. Then for any sequence $\alpha(k) \in \mathbb{N}^{n+1}$ such that $|\alpha(k)| \to \infty$ when k tends to infinity and such that*

$$\alpha(k)/|\alpha(k)| \to \theta \in \Sigma^0,$$

it holds that the limit

$$c[F](\theta) := \lim_{k\to\infty} \frac{F(\alpha(k))}{|\alpha(k)|}$$

exists and does only depend on θ. Furthermore, the function $c[F]$ thus defined is convex on Σ^0.

We will give a proof of this theorem which also shows that it holds locally, i.e. that F does not need to be subadditive on the whole of $\mathbb{N}^{n+1}$ but only on some open convex cone and only for large $|\alpha|$. Then Zaharjutas theorem still holds for the part of Σ^0 lying in the open cone. We will divide the proof into a couple of lemmas.

Lemma 3.2. *Let O be an open convex cone in $\mathbb{R}^{n+1}_+$ and let F be a subadditive function on $(O \setminus B(0, M)) \cap \mathbb{N}^{n+1}$, where $B(0, M)$ denotes the ball of radius M centered at the origin, and M is any positive number. Then for any closed convex cone $K \subseteq O$ there exists a constant C_K such that*

$$F(\alpha) \leq C_K|\alpha|$$

on $(K \setminus B(0, M)) \cap \mathbb{N}^{n+1}$.

Proof. Pick points in $(O \setminus B(0, M)) \cap \mathbb{N}^{n+1}$ such that if we denote by Γ the semigroup generated by the points, the convex cone $\Sigma(\Gamma)$ should contain $(K \setminus B(0, M))$ and the distance between the boundaries should be positive. The points should also generate $\mathbb{Z}^{n+1}$ as a group. Then from Theorem 2.2 it follows that there exists an M' such that

$$(K \setminus B(0, M')) \cap \mathbb{N}^{n+1} \subseteq \Gamma. \tag{5}$$

Let α_i denote the generators of Γ we picked. The inclusion (5) means that for all $\alpha \in (K \setminus B(0, M')) \cap \mathbb{N}^{n+1}$ there exist non-negative integers a_i such that

$$\alpha = \sum a_i \alpha_i.$$

By the subadditivity we therefore get that

$$F(\alpha) \leq \sum a_i F(\alpha_i) \leq C \sum a_i \leq C|\alpha|.$$

Since only finitely many points in $(K \setminus B(0, M)) \cap \mathbb{N}^{n+1}$ do not lie in $(K \setminus B(0, M')) \cap \mathbb{N}^{n+1}$ the lemma follows. □

Lemma 3.3. *Let O, K and F be as in the statement of Lemma 3.2. Let α be a point in $(K^\circ \setminus B(0, M)) \cap \mathbb{N}^{n+1}$, and let $\gamma(k)$ be a sequence in $(K \setminus B(0, M)) \cap \mathbb{N}^{n+1}$ such that*

$$|\gamma(k)| \to \infty$$

when k tends to infinity and that

$$\frac{\gamma(k)}{|\gamma(k)|} \to p \in K^\circ$$

for some point p in the interior of K. Let l be the ray starting in $\alpha/|\alpha|$, going through p, and let q denote the first intersection of l with the boundary of K. Denote by t the number such that

$$p = t\frac{\alpha}{|\alpha|} + (1-t)q.$$

Then there exists a constant C_K depending only of F and K such that

$$\limsup_{k\to\infty} \frac{F(\gamma(k))}{|\gamma(k)|} \leq t\frac{F(\alpha)}{|\alpha|} + (1-t)C_K.$$

Proof. We can pick points β_i in $(K \setminus B(0, M)) \cap \mathbb{N}^{n+1}$ with $\beta_i/|\beta_i|$ lying arbitrarily close to q, such that if Γ denotes the semigroup generated by the points β_i and α, Γ generates $\mathbb{Z}^{n+1}$ as a group and

$$p \in \Sigma(\Gamma)^\circ.$$

Therefore from Theorem 2.2 it follows that for large k $\gamma(k)$ can be written

$$\gamma(k) = a\alpha + \sum a_i \beta_i$$

for non-negative integers a_i and a. The subadditivity of F gives us that

$$F(\gamma(k)) \leq aF(\alpha) + \sum a_i F(\beta_i) \leq aF(\alpha) + C_K \sum a_i |\beta_i|,$$

where we in the last inequality used Lemma 3.2. Dividing by $|\gamma(k)|$ we get

$$\frac{F(\gamma(k))}{|\gamma(k)|} \leq \frac{a|\alpha|}{|\gamma(k)|} \frac{F(\alpha)}{|\alpha|} + C_K \sum \frac{a_i|\beta_i|}{|\gamma(k)|}.$$

Our claim is that $\frac{a|\alpha|}{|\gamma(k)|}$ will tend to t and that $\sum \frac{a_i|\beta_i|}{|\gamma(k)|}$ will tend to $(1-t)$. Consider the equations

$$\frac{\gamma(k)}{|\gamma(k)|} = \frac{a|\alpha|}{|\gamma(k)|} \frac{\alpha}{|\alpha|} + \sum \frac{a_i|\beta_i|}{|\gamma(k)|} \frac{\beta_i}{|\beta_i|}$$

and

$$p = t\frac{\alpha}{|\alpha|} + (1-t)q.$$

Observe that

$$t = \frac{|p - \frac{\alpha}{|\alpha|}|}{|q - \alpha|}.$$

If $|\frac{\gamma(k)}{|\gamma(k)|} - p| < \delta$ and $|\frac{\beta_i}{|\beta_i|} - q| < \delta$ for all i, then we see that

$$\frac{a|\alpha|}{|\gamma(k)|} \leq \frac{|p - \frac{\alpha}{|\alpha|}| + \delta}{|q - \frac{\alpha}{|\alpha|}| - \delta} \leq t + \varepsilon(\delta),$$

where $\varepsilon(\delta)$ goes to zero as δ goes to zero. Similarly we have that

$$\frac{a|\alpha|}{|\gamma(k)|} \geq \frac{|p - \frac{\alpha}{|\alpha|}| - \delta}{|q - \frac{\alpha}{|\alpha|}| + \delta} \geq t - \varepsilon'(\delta), \tag{6}$$

where $\varepsilon'(\delta)$ goes to zero as δ goes to zero. Since

$$\frac{a|\alpha|}{|\gamma(k)|} + \sum \frac{a_i|\beta_i|}{|\gamma(k)|} = 1,$$

inequality (6) implies that

$$\sum \frac{a_i|\beta_i|}{|\gamma(k)|} \leq 1 - t + \varepsilon'(\delta).$$

The lemma follows. □

Corollary 3.4. *Let O and F be as in the statement of Lemma 3.2. Then for any sequence $\alpha(k)$ in $O \cap \mathbb{Z}^{n+1}$ such that $|\alpha(k)| \to \infty$ when k tends to infinity and such that $\alpha(k)/|\alpha(k)|$ converges to some point p in O the limit*

$$\lim_{k\to\infty} \frac{F(\alpha)}{|\alpha(k)|}$$

exists and only depends on F and p.

Proof. Let $\alpha(k)$ and $\beta(k)$ be two such sequences converging to p. Let $K \subseteq O$ be some closed cone such that $p \in K^\circ$. Let us as in Lemma 3.3 write

$$p = t_k \frac{\beta(k)}{|\beta(k)|} + (1 - t_k) q_k.$$

For any $\varepsilon > 0$, t_k is greater than $1 - \varepsilon$ when k is large enough. By Lemma 3.3 we have that for such k

$$\limsup_{m\to\infty} \frac{F(\alpha(m))}{|\alpha(m)|} \leq (1 - t_k) \frac{F(\beta(k))}{|\beta(k)|} + \varepsilon C_K \leq \frac{F(\beta(k))}{|\beta(k)|} + \varepsilon C_K + \varepsilon C,$$

where C comes from the lower bound

$$\frac{F(\beta)}{|\beta|} \geq C$$

which holds for all β by assumption. Since ε tends to zero when k gets large we have that

$$\limsup_{k\to\infty} \frac{F(\alpha(k))}{|\alpha(k)|} \leq \liminf_{k\to\infty} \frac{F(\beta(k))}{|\beta(k)|}.$$

By letting $\alpha(k) = \beta(k)$ we get existence of the limit, and by symmetry the limit is unique. □

Proposition 3.5. *The function $c[F]$ on $O \cap \Sigma^\circ$ defined by*

$$c[F](p) := \lim_{k\to\infty} \frac{F(\alpha(k))}{|\alpha(k)|}$$

for any sequence $\alpha(k)$ such that $|\alpha(k)| \to \infty$ and $\frac{\alpha(k)}{|\alpha(k)|} \to p$, which is well-defined according to Corollary 3.4, is convex, and therefore continuous.

Proof. First we wish to show that $c[F]$ is lower semicontinuous. Let p be a point in $O \cap \Sigma^\circ$ and q_n a sequence converging to p. From Lemma 3.3 it follows that

$$c[F](p) \leq \liminf_{q_n\to p} c[F](q_n),$$

which is equivalent to lower semicontinuity.

Using this the lemma will follow if we show that for any two points p and q in $O \cap \Sigma^\circ$ it holds that

$$2c[F](\frac{p+q}{2}) \leq c[F](p) + c[F](q). \tag{7}$$

Choose sequences $\alpha(k), \beta(k) \in O \cap \mathbb{N}^{n+1}$ such that

$$\frac{\alpha(k)}{|\alpha(k)|} \to p, \qquad \frac{\beta(k)}{|\beta(k)|} \to q,$$

and for simplicity assume that $|\alpha(k)| = |\beta(k)|$. Then

$$\frac{\alpha(k)+\beta(k)}{|\alpha(k)+\beta(k)|} \to \frac{p+q}{2}.$$

Hence

$$2c[F](\frac{p+q}{2}) = \lim_{k\to\infty} \frac{F(\alpha(k)+\beta(k))}{|\alpha(k)|} \leq \lim_{k\to\infty} \frac{F(\alpha(k))}{|\alpha(k)|} + \lim_{k\to\infty} \frac{F(\beta(k))}{|\beta(k)|} = \\ = c[F](p) + c[F](q).$$

□

Together with Theorem 2.2 these lemmas yield a general result for subadditive funcitons on subsemigroups of $\mathbb{N}^{n+1}$.

A function F defined on a cone O is said to be *locally linearly bounded from below* if for each point $p \in O$ there exists an open subcone $O' \subseteq O$ containing p and a linear function λ on O' such that $F \geq \lambda$ on O'.

Theorem 3.6. *Let $\Gamma \subseteq \mathbb{N}^{n+1}$ be a semigroup which generates $\mathbb{Z}^{n+1}$ as a group, and let F be a subadditive function on Γ which is locally linearly bounded from below. Then for any sequence $\alpha(k) \in \Gamma$ such that $|\alpha(k)| \to \infty$ and $\frac{\alpha(k)}{|\alpha(k)|} \to p \in \Sigma(\Gamma)^\circ$ for some point p in the interior of $\Sigma(\Gamma)$, the limit*

$$\lim_{k\to\infty} \frac{F(\alpha(k))}{|\alpha(k)|}$$

exists and only depends on F and p. Furthermore the function

$$c[F](p) := \lim_{k\to\infty} \frac{F(\alpha(k))}{|\alpha(k)|}$$

thus defined on $\Sigma(\Gamma)^\circ \cap \Sigma^\circ$ is convex.

Proof. By Theorem 2.2 it follows that for any point $p \in \Sigma(\Gamma)^\circ$ there exists an open convex cone O and a number M such that

$$(O \setminus B(0,M)) \cap \mathbb{N}^{n+1} \subseteq \Gamma.$$

We can also choose O such that F is bounded from below by a linear function on O. Therefore the theorem follows immediately from Corollary 3.4 and Proposition 3.5. □

We will show how this theorem can be seen as the counterpart to Theorem 2.2 for subadditive functions.

Definition 3.7. *Let Γ be a subsemigroup of $\mathbb{N}^{n+1}$ and let F be a subadditive function of Γ which is locally linearly bounded from below. One defines the convex envelope of F, denoted by $P(F)$, as the supremum of all linear functions on $\Sigma(\Gamma)^\circ$ dominated by F, or which ammounts to the same thing, the supremum of all convex one-homogeneous functions on $\Sigma(\Gamma)^\circ$ dominated by F.*

Theorem 3.8. *If Γ generates $\mathbb{Z}^{n+1}$ as a group, then for any subadditive function F on Γ which is locally linearly bounded from below it holds that*

$$F(\alpha) = P(F)(\alpha) + o(|\alpha|)$$

for $\alpha \in \Gamma \cap \Sigma(\Gamma)^\circ$.

Proof. That

$$F(\alpha) \geq P(F)(\alpha)$$

follows from the definition. If we let $c[F]$ be defined on the whole of $\Sigma(\Gamma)^\circ$ by letting

$$c[F](\alpha) := |\alpha| c[F](\frac{\alpha}{|\alpha|}),$$

it follows from Theorem 3.6 that $c[F]$ will be convex and one-homogeneous. It will also be dominated by F since by the subadditivity

$$\frac{F(\alpha)}{|\alpha|} \geq \frac{F(k\alpha)}{|k\alpha|}$$

for all positive integers and therefore

$$\frac{F(\alpha)}{|\alpha|} \geq \lim_{k\to\infty} \frac{F(k\alpha)}{|k\alpha|} = c[F](\frac{\alpha}{|\alpha|}).$$

It follows that

$$P(F) \geq c[F].$$

For $\alpha \in \Gamma$ by definition we have that

$$P(F)(\alpha) \leq \frac{F(k\alpha)}{k}$$

for all positive integers k. At the same time

$$c[F](\alpha) = \lim_{k\to\infty} \frac{F(k\alpha)}{k},$$

hence we get that

$$P(F)(\alpha) \leq c[F](\alpha)$$

for $\alpha \in \Gamma$ Since both $P(F)$ and $c[F]$ are convex they are continuous, so by the homogeneity we get that

$$P(F) \leq c[F]$$

on $\Sigma(\Gamma)^\circ$, and therefore $P(F) = c[F]$. The theorem now follows from Theorem 3.6. □

4 Okounkov body of a line bundle

In this section we will show how to associate a semigroup to a line bundle.

Definition 4.1. *An order $<$ on $\mathbb{N}^n$ is additive if $\alpha < \beta$ and $\alpha' < \beta'$ implies that*

$$\alpha + \alpha' < \beta + \beta'.$$

One example of an additive order is the lexicographic order where

$$(\alpha_1, ..., \alpha_n) <_{\text{lex}} (\beta_1, ..., \beta_n)$$

iff there exists an index j such that $\alpha_j < \beta_j$ and $\alpha_i = \beta_i$ for $i < j$.

Let X be a compact projective complex manifold of dimension n, and L a holomorphic line bundle, which we will assume to be big. Suppose we have chosen a point p in X, and local holomorphic coordinates $z_1, ..., z_n$ around that point, and let $e_p \in H^0(U, L)$ be a local trivialization of L around p. Any holomorphic section $s \in H^0(X, kL)$ has an unique represention as a convergent power series in the variables z_i,

$$\frac{s}{e_p^k} = \sum a_\alpha z^\alpha,$$

which for convenience we will simply write as

$$s = \sum a_\alpha z^\alpha.$$

We consider the lexicographic order on the multiindices α, and let $v(s)$ denote the smallest index α such that $a_\alpha \neq 0$.

Definition 4.2. *Let $\Gamma(L)$ denote the set*

$$\bigcup_{k \geq 0} \big(v(H^0(kL)) \times \{k\}\big) \subseteq \mathbb{N}^{n+1}.$$

It is a semigroup, since for $s \in H^0(kL)$ and $t \in H^0(mL)$

$$v(st) = v(s) + v(t). \tag{8}$$

The Okounkov body of L, denoted by $\Delta(L)$, is defined as the Okounkov body of the associated semigroup $\Gamma(L)$.

We write $\Delta_k(\Gamma(L))$ simply as $\Delta_k(L)$.

From the article [9] by Lazarsfeld-Mustaţă we recall some results on Okounkov bodies of line bundles.

Lemma 4.3. *The number of points in $\Delta_k(L)$ is equal to the dimension of the vector space $H^0(kL)$.*

This is part of Lemma 1.3 in [9].

Lemma 4.4. *The Okounkov body of a big line bundle is bounded, hence compact.*

This is Lemma 1.10 in [9].

Lemma 4.5. *If L is a big line bundle, $\Gamma(L)$ generates $\mathbb{Z}^{n+1}$ as a group. In fact $\Gamma(L)$ contains a translated unit simplex.*

It is proved as part of Lemma 2.2 in [9].

Remark 4.6. *Note that the additivity of v as seen in equation (8) only depends on the fact that the lexicographic order is additive. Therefore we could have used any total additive order on $\mathbb{N}^n$ to define a semigroup $\tilde{\Gamma}(L)$, and the associated Okounkov body $\tilde{\Delta}(L)$. We will only consider the case where the Okounkov body $\tilde{\Delta}(L)$ is bounded, and the semigroup $\tilde{\Gamma}(L)$ generates $\mathbb{N}^n$ as a group.*

Lemma 4.7. *For any closed set K contained in the convex hull of $\Delta_M(L)$ for some M, there exists a constant C_K such that if*

$$\alpha \in K \cap (\frac{1}{k}\mathbb{Z})^n$$

and the distance between α and the boundary of K is greater than $\frac{C_K}{k}$, then $\alpha \in \Delta_k(L)$.

Proof. Let Γ be the semigroup generated by the elements $(M\beta, M)$ where $\beta \in \Delta_M(L)$, and some unit simplex in $\Gamma(L)$. Applying Lemma 2.3 gives the lemma. □

Lemma 4.8. *If K is relatively compact in the interior of $\Delta(L)$, there exists a number M such that for $k > M$,*

$$\alpha \in K \cap (\frac{1}{k}\mathbb{Z})^n$$

implies that $\alpha \in \Delta_k(L)$.

Proof. This is a consequence of Lemma 4.7 by choosing M such that the distance between K and the convex hull of $\Delta_M(L)$ is strictly positive, therefore greater than $\frac{C_K}{k}$ for large k. □

5 The Chebyshev transform

Definition 5.1. *A continuous hermitian metric $h = e^{-\psi}$ on a line bundle L is a continuous choice of scalar product on the complex line L_p at each point p on the manifold. If f is a local frame for L on U_f, then one writes*

$$|f|^2 = h_f = e^{-\psi_f},$$

where ψ_f is a continuous function on U_f. If $h = e^{-\psi}$ is a metric, ψ is called a weight.

We will show how one to a given continuous weight associates a subadditive function on the semigroup $\Gamma(L)$.

For all $(k\alpha, k) \in \Gamma(L)$, let us denote by $A_{\alpha,k}$ the affine space of sections in $H^0(kL)$ of the form

$$z^{k\alpha} + \text{higher order terms}$$

. Consider the supremum norm $||.||_{k\psi}$ on $H^0(kL)$ given by

$$||s||^2_{k\psi} := \sup_{x \in X}\{|s(x)|^2 e^{-k\psi(x)}\}.$$

Definition 5.2. *We define the discrete Chebyshev transform $F[\psi]$ on $\Gamma(L)$ by*

$$F[\psi](k\alpha, k) := \inf\{\ln ||s||^2_{k\psi} : s \in A_{\alpha,k}\}.$$

Lemma 5.3. *The function $F[\psi]$ is subadditive.*

Proof. Let $(k\alpha, k)$ and $(l\beta, l)$ be two points in $\Gamma(L)$, and denote by γ

$$\gamma := \frac{k\alpha + l\beta}{k + l}.$$

Thus we have that

$$(k\alpha, k) + (l\beta, l) = ((k + l)\gamma, k + l).$$

Let s be some section in $A_{\alpha,k}$ and s' some section in $A_{\beta,l}$. Since

$$\begin{aligned} ss' = (z^{k\alpha} + \text{higher order terms})(z^{l\beta} + \text{higher order terms}) = \\ = z^{(k+l)\gamma} + \text{higher order terms}, \end{aligned}$$

we see that $ss' \in A_{\gamma,k+l}$. We also note that the supremum of the product of two functions is less or equal to the product of the supremums, i.e.

$$||ss'||^2_{(k+l)\psi} \leq ||s||^2_{k\psi}||s'||^2_{l\psi}.$$

It follows that

$$\inf\{||s||^2_{k\psi} : s \in A_{\alpha,k}\} \inf\{||s'||^2_{l\psi} : s' \in A_{\beta,l}\} \leq \inf\{||t||^2_{\gamma,k+l} : t \in A_{\gamma,k+l}\},$$

which gives the lemma by taking the logarithm. ☐

Lemma 5.4. *There exists a constant C such that for all $(k\alpha, k) \in \Gamma(L)$,*

$$F[\psi](k\alpha, k) \geq C|(k\alpha, k)|$$

.

Proof. Let $r > 0$ be such that the polydisc D of radius r centered at p is fully contained in the coordinate chart of $z_1, ..., z_n$. We can also assume that our trivialization $e_p \in H^0(U, L)$ of L is defined on D, i.e. $D \subseteq U$. Let s be a section in $A_{\alpha,k}$ and let

$$\tilde{s} := \frac{s}{e_p^k}.$$

Denote by ψ_p the trivialization of ψ. Hence

$$|s|^2 e^{-k\psi} = |\tilde{s}|^2 e^{-k\psi_p}.$$

Since ψ_p is continuous,

$$e^{-\psi_p} > A$$

on D for some constant A. This yields that

$$||s||^2 \geq \sup_{x \in D}\{|\tilde{s}(x)|^2 e^{-k\psi_p(x)}\} \geq A^k \sup_{x \in D}\{|\tilde{s}(x)|^2\}.$$

We claim that

$$\sup_{x \in D}\{|\tilde{s}(x)|^2\} \geq r^{k|\alpha|}.$$

Observe that

$$\sup_{z \in D}\{|z^{k\alpha}|^2\} = r^{k|\alpha|}.$$

One now shows that

$$\sup_{z \in D}\{|z^{k\alpha}|^2\} \leq \sup_{z \in D}\{|z^{k\alpha} + \text{higher order terms}|^2\}$$

by simply reducing it to the case of one variable where it is immediate. We get that

$$||s||^2 \geq A^k r^{k|\alpha|}$$

and hence

$$F[\psi](k\alpha, k) \geq k \ln A + k|\alpha| \ln r \geq C(k + k|\alpha|),$$

if we choose C to be less than both $\ln A$ and $\ln r$. □

Definition 5.5. *We define the Chebyshev transform of ψ, denoted by $c[\psi]$ as the convex envelope of $F[\psi]$ on $\Sigma(\Gamma)^\circ$. It is convex and one-homogeneous. We will also identify it with its restriction to $\Delta(L)^\circ$, the interior of the Okounkov body of L. Recall that by definition*

$$\Delta(L) := \Sigma(L) \cap (\mathbb{R}^n \times \{1\}).$$

Proposition 5.6. *For any sequence $(k\alpha(k), k)$ in $\Gamma(L)$, $k \to \infty$, such that*

$$\lim_{k \to \infty} \alpha(k) = p \in \Delta(L)^\circ,$$

it holds that

$$c[\psi](p) = \lim_{k \to \infty} \frac{1}{k} \ln ||t_{\alpha(k),k}||^2.$$

Proof. By Lemma 5.3 and Lemma 5.4 we can apply Theorem 3.8 to the function $F[\psi]$ and get that

$$c[\psi](p) = |(p,1)| c[\psi](\frac{(p,1)}{|(p,1)|}) = |(p,1)| \lim_{k \to \infty} \frac{F[\psi](k\alpha, k)}{k|(\alpha(k),1)|} =$$
$$= \lim_{k \to \infty} \frac{F[\psi](k\alpha, k)}{k} = \lim_{k \to \infty} \frac{1}{k} \ln ||t_{\alpha(k),k}||^2.$$

□

Lemma 5.7. *Let ψ be a continuous weight on L and consider the continuous weight on L given by $\psi + C$ for some constant C. Then it holds that*

$$F[\psi + C](k\alpha, k) = F[\psi](k\alpha, k) - kC, \tag{9}$$

and that

$$c[\psi + C] = c[\psi] - C$$

on $\Delta(L)^\circ$.

Proof. For any section $s \in H^0(kL)$ we have that

$$||s||^2_{k(\psi+C)} = e^{-kC}||s||^2_{k\psi},$$

therefore

$$\ln ||s||^2_{k(\psi+C)} = \ln ||s||^2_{k\psi} - kC.$$

The same holds true when taking the infimum, which gives equation (9). The second part then follows from Proposition 5.6. □

Proposition 5.8. *If ψ and φ are two continuous weights such that*

$$\psi \leq \varphi,$$

then

$$F[\psi] \geq F[\varphi],$$

and also

$$c[\psi] \geq c[\varphi].$$

Proof. For any $s \in H^0(kL)$ we get that

$$\sup_{x\in X}\{|s(x)|^2 e^{-k\varphi(x)}\} \leq \sup_{x\in X}\{|s(x)|^2 e^{-k\psi(x)}\}.$$

The inequality still holds when taking the logarithm and the infimum over $A_{\alpha,k}$. □

Proposition 5.9. *For any two continuous weights on L, ψ and φ, the difference of the Chebyshev transforms, $c[\psi] - c[\varphi]$, is continuous and bounded on $\Delta(L)^\circ$.*

Proof. It is the difference of two convex hence continuous functions, and is therefore continuous. Since $\psi - \varphi$ is a continuous function on the compact space X, there exists a constant C such that

$$\psi \leq \varphi + C.$$

Thus by Lemma 5.8 and Lemma 5.7 we have that

$$c[\psi] \leq c[\varphi + C] = c[\varphi] - C.$$

By symmetry we see that $|c[\psi] - c[\varphi]|$ is bounded on $\Delta(L)^\circ$. □

For Okounkov bodies we have that

$$\Delta(mL) = m\Delta(L),$$

see e.g. [9]. The Chebyshev transforms also exhibit a homogeneity property.

Proposition 5.10. *Let ψ be a continuous weight on L. Consider the weight $m\psi$ on mL. For any $p \in \Delta(L)^\circ$ it holds that*

$$c[m\psi](mp) = mc[\psi](p).$$

Proof. We observe that trivially $A_{m\alpha,k} = A_{\alpha,km}$, as affine subspaces of $H^0(kmL)$, and hence

$$F[m\psi](km\alpha, k) = F[\psi](km\alpha, km).$$

Let $\alpha(k) \to p \in \Delta(L)^\circ$.We get that

$$c[m\psi](mp) = |(mp,1)|c[m\psi](\frac{(mp,1)}{|(mp,1)|}) = |(mp,1)| \lim_{k\to\infty} \frac{F[m\psi](km\alpha(k),k)}{k|(m\alpha(k),1)|} =$$

$$= \lim_{k\to\infty} \frac{F[\psi](km\alpha(k),km)}{k} = mc[\psi](p).$$

□

6 Relative energy of weights

One may define a partial order on the space of weights to a given line bundle. Let $\psi <_w \varphi$ if

$$\psi \leq \varphi + O(1)$$

on X. If a weight is maximal with respect to the order $<_w$, it is said to have minimal singularities. It is a fact that a weight with minimal singularities on a big line bundle is locally bounded on a dense Zariski-open subset of X (see e.g. [1]). On an ample line bundle, the weights with minimal singularities are exactly those who are locally bounded.

Let ψ and φ be two locally bounded psh-weights. By $\mathrm{MA}_m(\psi, \varphi)$ we will denote the positive current

$$\sum_{j=0}^{m} (dd^c\psi)^j \wedge (dd^c\varphi)^{m-j},$$

and by $\mathrm{MA}(\psi)$ we will mean the positive measure $(dd^c\psi)^n$.

Definition 6.1. *If ψ and φ are two psh weights with minimal singularities, then we define the relative energy of ψ with respect to φ as*

$$\mathcal{E}(\psi,\varphi) := \frac{1}{n+1}\int_\Omega (\psi-\varphi) MA_n(\psi,\varphi),$$

where Ω is a Zariski open subset of X on which ψ and φ are locally bounded.

Remark 6.2. *In [1] Berman-Boucksom use the notation $\mathcal{E}(\psi) - \mathcal{E}(\varphi)$ for what we denote by $\mathcal{E}(\psi, \varphi)$. Thus they consider $\mathcal{E}(\psi)$ as a functional defined only up to a constant.*

An important aspect of the relative energy (and a motivation for calling it an energy) is its cocycle property, i.e. that

$$\mathcal{E}(\psi, \varphi) + \mathcal{E}(\varphi, \psi') + \mathcal{E}(\psi', \psi) = 0$$

for all weights ψ, φ and ψ', (see e.g. [1]).

Definition 6.3. *If ψ is a continuous weight and K a compact subset of X, the psh envelope of ψ with respect to K, $P_K(\psi)$, is given by*

$$P_K(\psi) := \sup\{\varphi : \varphi \textit{ psh weight on } L, \varphi \leq \psi \textit{ on } K\}.$$

For any ψ and K, as one may check, $P_K(\psi)$ will be psh and have minimal singularities. When $K = X$, we will simply write $P(\psi)$ for $P_X(\psi)$.

If ψ and φ are continuous weights, we will call

$$\mathcal{E}(P(\psi), P(\varphi))$$

the relative energy of ψ with respect to φ, and we will denote it by $\mathcal{E}(\psi, \varphi)$. Since for psh weights ψ, trivially $P(\psi) = \psi$, therefore the notation is unambiguous.

Remark 6.4. *In [1] Berman-Boucksom use the notation $\mathcal{E}_{eq}(X, \psi)$ for the expression*

$$\frac{1}{vol(L)} \mathcal{E}(P(\psi)),$$

hence it is the same as our $\mathcal{E}(\psi)$ except with a different normalization.

We refer the reader to [1] for a more thourough exposition on Monge-Ampère measures and psh envelopes.

We now state our main result.

Theorem 6.5. *Let ψ and φ be continuous weights on L. Then it holds that*

$$\mathcal{E}(\varphi, \psi) = n! \int_{\Delta(L)^\circ} (c[\psi] - c[\varphi]) d\lambda, \tag{10}$$

where $d\lambda$ denotes the Lebesgue measure on $\Delta(L)^\circ$.

The proof of Theorem 6.5 will depend on the fact that one can also use L^2-norms to compute the Chebyshev transform of a continuous weight. This will be explained in the next section.

7 Bernstein-Markov norms

Definition 7.1. *Let μ be a positive measure on X, and ψ a continuous weight on a line bundle L. One says that μ satisfies the Bernstein-Markov property with respect to ψ if for each $\varepsilon > 0$ there exists $C = C(\varepsilon)$ such that for all non-negative k and all holomorphic sections $s \in H^0(kL)$ we have that*

$$\sup_{x\in X}\{|s(x)|^2 e^{-k\psi(x)}\} \le Ce^{\varepsilon k}\int_X |s|^2 e^{-k\psi} d\mu. \tag{11}$$

If ψ is a continuous weight on L and μ a Bernstein-Markov measure on X with respect to ψ, we will call the L^2-norm on $H^0(kL)$ defined by

$$||s||^2_{k\psi,\mu} := \int_X |s|^2 e^{-k\psi} d\mu$$

a Bernstein-Markov norm. We will also call the pair (ψ, μ) a Bernstein-Markov pair on (X, L).

For any continuous weight ψ on L there exist measures μ such that (ψ, μ) is a Bernstein-Markov pair. In fact it is easy to show that any smooth volume form dV on X satisfies the Bernstein-Markov property with respect to any continuous weight, see e.g. [1].

A pair (E, ψ) where E is a subset of X and ψ is a continuous weight on L is called a weighted subset. The equilibrium weight ψ_E of (E, ψ) is defined as

$$\psi_E := \sup\{\varphi : \varphi \text{ is psh}, \varphi \le \psi \text{ on } E\}.$$

A weighted set (E, ψ) is said to be regular if the equilibrium weight ψ_E is upper semicontinuous.

Definition 7.2. *If a compact $K \subseteq X$ is the support of a positive measure μ, one says that μ satisfies the Bernstein-Markov property with respect to the weighted set (K, ψ) if for all k and $s \in H^0(kL)$ inequality (11) holds when X is replaced with K.*

Lemma 7.3. *If μ is a smooth volume form and (K, ψ) is a compact regular weighted subset, then the restriction of μ to K satisfies the Bernstein-Markov property with respect to (K, ψ).*

Proof. This follows e.g. from Theorem 2.4 in [1]. □

We want to be able to use a Bernstein-Markov norm instead of the supremum norm to calculate the Chebyshev transform of a continuous weight ψ.

We pick a positive measure μ with the Bernstein-Markov property with respect to ψ. For all $(k\alpha, k) \in \Gamma(L)$, let $t_{\alpha,k}$ be the section in $H^0(kL)$ of the form

$$z^{k\alpha} + \text{higher order terms}$$

that minimizes the L^2-norm

$$||t_{\alpha,k}||^2_{k\psi,\mu} := \int_X |t_{\alpha,k}|^2 e^{-k\psi} d\mu.$$

It follows that

$$< t_{\alpha,k}, t_{\beta,k} >_{k\psi} = 0$$

for $\alpha \neq \beta$, since otherwise the sections $t_{\alpha,k}$ would not be minimizing. Hence

$$\{t_{\alpha,k} : \alpha \in \Delta_k(L)\}$$

is an orthogonal basis for $H^0(kL)$ with respect to $||.||_{k\psi,\mu}$. Indeed they are orthogonal, and by Lemma 4.3 we have that

$$\#\{t_{\alpha,k} : \alpha \in \Delta_k(L)\} = \#\Delta_k(L) = \dim(H^0(kL)),$$

therefore it must be a basis.

Definition 7.4. *We define the discrete Chebyshev transform $F[\psi,\mu]$ of (ψ,μ) on Γ by*

$$F[\psi,\mu](k\alpha,k) := \ln ||t_{\alpha,k}||^2_{k\psi,\mu}.$$

We also denote $\frac{1}{k}F[\psi,\mu](k\alpha,k)$ by $c_k[\psi,\mu](\alpha)$.

We will sometimes write $c_k[\psi]$ when we mean $c_k[\psi,\mu]$, considering μ as fixed.

Proposition 7.5. *For any sequence $(k\alpha(k),k)$ in $\Gamma(L)$, $k \to \infty$, such that*

$$\lim_{k\to\infty} \alpha(k) = p \in \Delta(L)^\circ,$$

it holds that

$$c[\psi](p) = \lim_{k\to\infty} c_k[\psi,\mu](\alpha(k)).$$

Proof. For a point $(k\alpha,k) \in \Gamma$, let $t_{\alpha,k}$ be the minimizer with respect to the Bernstein-Markov norm. By the Bernstein-Markov property we get that

$$||t_{\alpha,k}||^2_{\sup} \leq Ce^{\varepsilon k}||t^{\mu}_{\alpha,k}||^2_{\mu},$$

and hence

$$F[\psi](k\alpha,k) \leq F[\psi,\mu](k\alpha,k) + \ln C + \varepsilon k. \qquad (12)$$

Let s be any section in $A_{\alpha,k}$. We have that by definition

$$||t_{\alpha,k}||^2_{\mu} \leq ||s||^2_{\mu} \leq \mu(X)||s||^2_{\sup},$$

so

$$F[\psi,\mu](k\alpha,k) \leq F[\psi](k\alpha,k) + \ln \mu(X). \qquad (13)$$

Equations (12) and (13) put together gives that

$$F[\psi](k\alpha,k) - \ln C - \varepsilon k \leq F[\psi,\mu](k\alpha,k) \leq F[\psi](k\alpha,k) + \ln \mu(X). \qquad (14)$$

It follows that

$$\lim_{k\to\infty} \frac{F[\psi,\mu](k\alpha(k),k)}{k} = \lim_{k\to\infty} \frac{F[\psi](k\alpha(k),k)}{k} = c[\psi](p),$$

which gives the proposition. □

Lemma 7.6. *Let ψ be a continuous weight on L and consider the continuous weight on L given by $\psi + C$ for some constant C. Then it holds that*

$$F[\psi + C, \mu](k\alpha, k) = F[\psi, \mu](k\alpha, k) - kC.$$

Proof. This follows exactly as in the case of the suprumum norm, see proof of Lemma 5.7. □

Proposition 7.7. *Let (ψ, μ) and (φ, ν) be two Bernstein-Markov pairs, and assume that*

$$\psi \leq \varphi$$

Then for every $varepsilon > 0$ there exists a constant C' such that

$$F[\psi, \mu](k\alpha, k) \geq F[\varphi, \nu](k\alpha, k) - C' - \varepsilon k.$$

Proof. Let $t^{\psi}_{\alpha,k}$ and $t^{\varphi}_{\alpha,k}$ be the minimizing sections with respect to the Bernstein-Markov norms $||.||_{k\psi,\mu}$ and $||.||_{k\varphi}$ respectively. From equation (14) and Proposition 7.7 we get that

$$F[\psi, \mu](k\alpha, k) \geq F[\psi](k\alpha, k) - \ln C - \varepsilon k \geq F[\varphi](k\alpha, k) - \ln C - \varepsilon k \geq \\ \geq F[\varphi, \nu] - \ln \nu(X) - \ln C - \varepsilon k.$$

□

Proposition 7.8. *For any two Bernstein-Markov pairs on (X, L), (ψ, μ) and (φ, ν) the difference of the discrete Chebyshev transforms*

$$c_k[\psi, \mu] - c_k[\varphi, \nu]$$

is uniformly bounded on $\Delta(L)^\circ$.

Proof. By symmetry it suffices to find an upper bound. Let $\tilde{C}$ be a constant such that $\psi \leq \varphi + \tilde{C}$. By Lemma 7.6 and Proposition 7.7 we get that

$$c_k[\psi, \mu](\alpha) = \frac{1}{k} F[\psi, \mu](k\alpha, k) \geq \frac{1}{k} F[\varphi + C, \nu](k\alpha, k) - \frac{C'}{k} - \varepsilon = \\ = \frac{1}{k} F[\varphi, \nu](k\alpha, k) - C - \frac{C'}{k} - \varepsilon = c_k[\varphi, \nu](\alpha) - C - \frac{C'}{k} - \varepsilon.$$

The proposition follows. □

8 Proof of main theorem

8.1 Preliminary results

Let $\mathcal{B}^2(\mu, k\varphi)$ denote the unit ball in $H^0(kL)$ with respect to the norm $\int_X |.|^2 e^{-k\varphi} d\mu$, i.e.

$$\mathcal{B}^2(\mu, k\varphi) := \{s \in H^0(kL) : \int_X |s|^2 e^{-k\varphi} d\mu \leq 1\}.$$

Consider the quotient of the volume of two unit balls

$$\frac{\mathrm{vol}\mathcal{B}^2(\mu, k\varphi)}{\mathrm{vol}\mathcal{B}^2(\nu, k\psi)}$$

with respect to the Lebesgue measure on $H^0(kL)$, where we by some linear isomorphism identify $H^0(kL)$ with $\mathbb{C}^N$, $N = h^0(kL)$. In fact the quotient of the volumes does not depend on how we choose to represent $H^0(kL)$.

Lemma 8.1.

$$\frac{\mathit{vol}\mathcal{B}^2(\mu, k\varphi)}{\mathit{vol}\mathcal{B}^2(\nu, k\psi)} = \frac{\mathit{det}(\int s_i\bar{s}_j e^{-k\psi}d\nu)_{ij}}{\mathit{det}(\int s_i\bar{s}_j e^{-k\varphi}d\mu)_{ij}}, \tag{15}$$

where $\{s_i\}$ *is any basis for* $H^0(kL)$.

Proof. First we show that the right hand side does not depend on the basis. Let $\{t_i\}$ be some orthonormal basis with respect to $\int |.|^2 e^{-k\psi}d\nu$, and let $A = (a_{ij})$ be the matrix such that

$$s_i = \sum a_{ij}t_j.$$

Then we see that

$$\int s_i\bar{s}_j e^{-k\psi}d\nu = \int (\sum a_{ik}t_k)(\overline{\sum a_{jl}t_l})e^{-k\psi}d\nu = \sum a_{ik}\bar{a}_{jk}. \tag{16}$$

Therefore by linear algebra we get that

$$\det\left(\int s_i\bar{s}_j e^{-k\psi}d\nu\right)_{ij} = \det(AA^*) = |\det A|^2. \tag{17}$$

If we let $\{s_i'\}$ be a new basis,

$$s_i' = \sum b_{ij}s_j, \qquad B = (b_{ij}),$$

then

$$\det\left(\int s_i'\bar{s}_j' e^{-k\psi}d\nu\right)_{ij} = |\det B|^2 \det\left(\int s_i\bar{s}_j e^{-k\psi}d\nu\right)_{ij}.$$

Since $|\det B|^2$ also will show up in the denominator, we see that the quotient does not depend on the choice of basis.

Let as above $\{t_i\}$ be an orthonormal basis with respect to $\int |.|^2 e^{-k\psi}d\nu$ and let $\{s_i\}$ be an orthonormal basis with respect to $\int |.|^2 e^{-k\varphi}d\mu$ and let

$$s_i = \sum a_{ij}t_j, \qquad A = (a_{ij}).$$

It is clear that

$$\frac{\mathrm{vol}\mathcal{B}^2(\mu, k\varphi)}{\mathrm{vol}\mathcal{B}^2(\nu, k\psi)} = |\det A|^2.$$

Note that the square in the right-hand side comes from the fact that we take the determinant of A as a complex matrix. By equations (16) and (17) we also have that

$$\det\left(\int s_i \bar{s}_j e^{-k\psi} d\nu\right)_{ij} = |\det A|^2,$$

and since $\{s_i\}$ were chosen to be orthonormal

$$\det\left(\int s_i \bar{s}_j e^{-k\varphi} d\mu\right)_{ij} = 1.$$

The lemma follows. □

Definition 8.2. *Let (φ, μ) and (ψ, ν) be two Bernstein-Markov pairs on (X, L). The Donaldson $\mathcal{L}_k$ bifunctional on (φ, ψ) is defined as*

$$\mathcal{L}_k(\varphi, \psi) := \frac{n!}{2k^{n+1}} \ln\left(\frac{vol\mathcal{B}^2(\mu, k\varphi)}{vol\mathcal{B}^2(\nu, k\psi)}\right).$$

Theorem A in [1] states that for Bernstein-Markov pairs the Donaldson $\mathcal{L}_k$ bifunctional converges to the relative energy.

Theorem 8.3. *Let (φ, μ) and (ψ, ν) be two Bernstein-Markov pairs on (X, L). Then it holds that*

$$\lim_{k\to\infty} \mathcal{L}_k(\varphi, \psi) = \mathcal{E}(\varphi, \psi).$$

We will use this result to prove our main result, Theorem 6.5, stating that the relative energy of two continuous weights is equal to the integral of the difference of the respective Chebyshev transforms over the Okounkov body.

8.2 Proof of Theorem 6.5

Proof. We let $\{s_i\}$ be a basis for $H^0(kL)$ such that

$$s_i = z^{k\alpha_i} + \text{higher order terms},$$

where $\alpha_i \in \Delta_k(L)$ is some ordering of $\Delta_k(L)$. Let

$$s_i = \sum a_{ij} t^{\psi}_{\alpha_j,k}, \qquad A = (a_{ij}).$$

From the proof of Lemma 8.1 we see that

$$\det\left(\int_X s_i \bar{s}_j e^{-k\psi} d\nu\right)_{ij} = |\det A|^2 \det\left(\int_X t^{\psi}_{\alpha_i,k} \bar{t}^{\psi}_{\alpha_j,k} e^{-k\psi} d\nu\right)_{ij} =$$
$$= |\det A|^2 \prod_{\alpha\in\Delta_k(L)} ||t^{\psi}_{\alpha,k}||^2,$$

since $t^{\psi}_{\alpha,k}$ constitute an orthogonal basis. Also since the lowest term of s_i is $z^{k\alpha_i}$ we must have that $a_{ij} = 0$ for $j < i$ and $a_{ii} = 1$. Hence det$A = 1$, and consequently

$$\det\left(\int_X s_i \bar{s}_j e^{-k\psi} d\nu\right)_{ij} = \prod_{\alpha \in \Delta_k(L)} ||t^{\psi}_{\alpha,k}||^2.$$

From equation (15) we get that

$$\mathcal{L}_k(\varphi, \psi) = \frac{n!}{k^n} \sum_{\alpha \in \Delta_k(L)} (c_k[\psi](\alpha) - c_k[\varphi](\alpha)).$$

For all k let $\tilde{c}_k[\psi]$ denote the function on $\Delta(L)^\circ$ assuming the value of $c_k[\psi]$ in the nearest lattice point of $\Delta_k(L)$ (or the mean of the values if there are multiple lattice points at equal distance). Then

$$\frac{n!}{k^n} \sum_{\alpha \in \Delta_k(L)} (c_k[\psi](\alpha) - c_k[\varphi](\alpha)) = n! \int_{\Delta_k} (\tilde{c}_k[\psi] - \tilde{c}_k[\varphi]) d\lambda,$$

where Δ_k increases to $\Delta(L)^\circ$. By Propositions 7.5 and 7.8 we can use dominated convergence to conclude that

$$\lim_{k \to \infty} \mathcal{L}_k(\varphi, \psi) = n! \int_{\Delta(L)^\circ} (c[\psi] - c[\varphi]) d\lambda.$$

Combined with Theorem 8.3 this proves the theorem. □

9 Previous results

Some instances of formula (10) are previously known. Here follows three such instances.

9.1 The volume as a relative energy

We consider the case where we let $\varphi = \psi + 1$. It is easy to see that this means that $P(\varphi) - P(\psi) = 1$, thus

$$\mathcal{E}(\varphi, \psi) = \frac{1}{n+1} \int_\Omega \text{MA}_n(P(\varphi), P(\psi)). \tag{18}$$

Furthermore it has been shown by Berman-Boucksom (see e.g. [1]) that for any n-tuple of psh weights ψ_i with minimal singularities it holds that

$$\int_\Omega dd^c\psi_1 \wedge ... \wedge dd^c\psi_n = \text{vol}(L), \tag{19}$$

where Ω denotes the dense Zariski-open set where the weights ψ_i are all locally bounded. Equations (18) and (19) together yields that

$$\mathcal{E}(\varphi, \psi) = \text{vol}(L). \tag{20}$$

Any minimizing section with respect to $\int |.|^2 e^{-k\psi}$ will also minimize the norm

$$\int |.|^2 e^{-k(\psi+1)} = \int |.|^2 e^{-k\varphi}.$$

It follows that $c[\psi] - c[\varphi]$ is identically one. Therefore

$$\int_{\Delta(L)^\circ} (c[\psi] - c[\varphi]) d\lambda = \mathrm{vol}_{\mathbb{R}^n}(\Delta(L)). \tag{21}$$

Equations (20) and (21) and Theorem 6.5 then gives us that

$$\mathrm{vol}(L) = n! \mathrm{vol}_{\mathbb{R}^n}(\Delta(L)).$$

We have thus recovered Theorem A in [9].

9.2 Chebyshev constants and the transfinite diameter

Let K be a regular compact set in $\mathbb{C}$. We let $||.||_K$ denote the norm which takes the supremum of the absolute value on K. Let P_k denote the space of polynomials in z with z^k as highest degree term. Let for any k

$$Y_k(K) := \inf\{||p||_K : p \in P_k\}.$$

One defines the Chebyshev constant $C(K)$ of K as the following limit

$$C(K) := \lim_{k\to\infty} (Y_k(K))^{1/k}.$$

Let $\{x_i\}_{i=1}^k$ be a set of k points in K. Let $d_k(\{x_i\})$ denote the product of their mutual distances, i.e.

$$d_k(\{x_i\}) := \prod_{i<j} |x_i - x_j|.$$

One calls the points $\{x_i\}$ Fekete points if among the set of k-tuples of points in K they maximize the function d_k. Define $T_k(K)$ as $d_k(\{x_i\})$ for any set of Fekete points $\{x_i\}_{i=1}^k$. Then the transfinite diameter $T(K)$ of K is defined as

$$T(K) := \lim_{k\to\infty} (T_k(K))^{1/\binom{k}{2}}.$$

We will now think of $\mathbb{C}$ as imbedded in the complex projective space $\mathbb{P}^1$. Let Z_0, Z_1 be a basis for $H^0(\mathcal{O}(1))$, therefore $[Z_0, Z_1]$ are homogeneous coordinates for $\mathbb{P}^1$. Let

$$z := \frac{Z_1}{Z_0} \qquad \text{and} \qquad w := \frac{Z_0}{Z_1}.$$

Let p denote the point at infinity

$$[0, 1].$$

Then w is a holomorphic coordinate around p, and Z_1 is a local trivialization of the line bundle $\mathcal{O}(1)$ around p. Thus we will identify a section $Z_0^\alpha Z_1^{k-\alpha} \in H^0(\mathcal{O}(k))$ with the polynomial w^α as well as with $z^{k-\alpha}$. This means that the Okounkov body $\Delta(\mathcal{O}(1))$ of $\mathcal{O}(1)$ is the unit interval $[0,1]$ in $\mathbb{R}$. We observe that a section $s \in H^0(\mathcal{O}(k))$ lies in P_i as a polynomial in z if and only if

$$s = w^{k-i} + \text{ higher order terms.}$$

For a section s let $\tilde{s}$ denote the corresponding polynomial in z. Consider the weight $P_K(\ln|Z_0|^2)$. It will be continuous since K is assumed to be regular (see e.g. [1]). Then we have the following lemma.

Lemma 9.1. *For any $\alpha \in [0,1]$, i.e. that lies in the Okounkov body of $\mathcal{O}(1)$, we have that*

$$c[P_K(\ln|Z_0|^2)](\alpha) = 2(1-\alpha)\ln C(K).$$

Proof. By basic properties of the projection operator P_K (see [1]) it holds that for for any section $s \in H^0(\mathcal{O}(k))$

$$\sup_K\{|s|^2 e^{-k\ln|Z_0|^2}\} = \sup_{\mathbb{P}^1}\{|s|^2 e^{-kP_K(\ln|Z_0|^2)}\}. \tag{22}$$

Since the conversion to the z-variable means letting Z_0 be identically one, we also have that

$$\sup_K\{|s|^2 e^{-k\ln|Z_0|^2}\} = \sup_K\{|\tilde{s}|^2\} = ||\tilde{s}||_K^2. \tag{23}$$

We see that $s \in A_{\alpha,k}$ iff $\tilde{s} = z^{k-k\alpha} +$ lower order terms. Hence

$$F[P_K(\ln|Z_0|^2)](k\alpha,k) = 2\ln Y_{k\alpha-k}(K),$$

and

$$c[P_K(\ln|Z_0|^2)](\alpha) = \lim_{k\to\infty}\frac{F[P_K(\ln|Z_0|^2)](k\alpha,k)}{k} = \lim_{k\to\infty}\frac{2}{k}\ln Y_{k\alpha-k}(K) =$$
$$= \lim_{k\to\infty} 2(1-\alpha)\ln(Y_{k-k\alpha}(K))^{k-k\alpha} = 2(1-\alpha)\ln C(K).$$

□

Let K and K' be two regular compact subsets of $\mathbb{C}$. From Theorem 6.5 and Lemma 9.1 we get that

$$\mathcal{E}(P_{K'}(\ln|Z_0|^2), P_K(\ln|Z_0|^2)) = \int_{(0,1)}(c[P_K(\ln|Z_0|^2)] - c[P_{K'}(\ln|Z_0|^2)])d\lambda(\alpha)$$
$$= \int_{(0,1)}(2(1-\alpha)\ln C(K) - 2(1-\alpha)\ln C(K'))\,d\lambda(\alpha) = \ln C(K) - \ln C(K').$$

On the other hand it follows from Corollary A in [1] that

$$\ln T(K) - \ln T(K') = \mathcal{E}(P_{K'}(\ln|Z_0|^2), P_K(\ln|Z_0|^2)). \tag{24}$$

Thus by Theorem 6.5, using Lemma 9.1 and equation (24) we get that

$$\ln T(K) - \ln T(K') = \ln C(K) - \ln C(K').$$

In fact it is easy to check that for the unit disc $D, T(D) = C(D) = 1$, so we recover the classical result in potential theory that the transfinite diameter $T(K)$ and the Chebyshev constant $C(K)$ are equal.

For a thorough exposition on the subject of the transfinite diameter and capacities of compacts in $\mathbb{C}$ we refer the reader to the book [13] by Saff-Totik.

9.3 Invariant weights on toric varieties

Let X be a smooth projective toric variety. We will view X as a compactified $\mathbb{C}^n$, such that the torus action on X via this identification corresponds to the usual torus action on $\mathbb{C}^n$. As is well-known, there is a polytope Δ naturally associated to the embedding $\mathbb{C}^n \subseteq X$. We assume that Δ lies in the non-negative orthant of $\mathbb{R}^n$. There is a line bundle L_Δ with a trivialization on $\mathbb{C}^n$ such that

$$\Delta_k(L_\Delta) = \Delta \cap (\frac{1}{k}\mathbb{Z})^n,$$

and any section $s \in H^0(kL_\Delta)$ can in fact be written as a linear combination of the monomials z^α where

$$\alpha \in k\Delta \cap \mathbb{Z}^n.$$

Let dV be a smooth volume form on X invariant under the torus action. Then it holds that for any torus invariant weight ψ,

$$\int_X z^\alpha \bar{z}^\beta e^{-k\psi} dV = 0$$

when $\alpha \neq \beta$. This follows from Fubini since trivially the monomials are orthogonal with respect to the Lebesgue measure on e.g. tori. Because of this for any torus invariant weight ψ the minimizing sections $t^\psi_{a,k}$ are given by $z^{k\alpha}$, and consequently

$$c_k[\psi, dV](\alpha) = \frac{1}{k} \ln \int_X |z^{k\alpha}|^2 e^{-k\psi} dV.$$

Assume for simplicity that ψ is positive.

Lemma 9.2. *For any strictly positive torus invariant weight ψ we have that*

$$c[\psi](\alpha) = \ln \left(\sup_{z \in \mathbb{C}^n} \{|z^\alpha|^2 e^{-\psi(z)}\} \right).$$

Proof. We have that

$$\int_X |z^{k\alpha}|^2 e^{-k\psi} dV \leq dV(X) \sup_X \{|z^{k\alpha}|^2 e^{-k\psi}\} = dV(X) \left(\sup_{z \in X} \{|z^\alpha|^2 e^{-\psi(z)}\} \right)^k,$$

which yieds the inequality

$$c[\psi](\alpha) \leq \ln \left(\sup_{z \in X} \{|z^\alpha|^2 e^{-\psi(z)}\} \right).$$

By the Bernstein-Markov property of dV with respect to ψ we get that

$$\int_X |z^{k\alpha}|^2 e^{-k\psi} dV \geq Ce^{-\varepsilon k} \sup_{z\in X}\{|z^{k\alpha}|^2 e^{-k\psi(z)}\} = Ce^{-\varepsilon k}\left(\sup_{z\in X}\{|z^{\alpha}|^2 e^{-\psi(z)}\}\right)^k.$$

Using Proposition 7.5 it follows from this that

$$c[\psi](\alpha) = \ln\left(\sup_{z\in X}\{|z^{\alpha}|^2 e^{-\psi}\}\right).$$

Since ψ is a weight on L_Δ it obeys certain growth conditions in $\mathbb{C}^n$. In fact for α lying in the interior of $\Delta = \Delta(L_\Delta)$ it holds that

$$\sup_X\{|z^{\alpha}|^2 e^{-\psi(z)}\} = \sup_{z\in\mathbb{C}^n}\{|z^{\alpha}|^2 e^{-\psi(z)}\},$$

and the lemma follows. □

Remark 9.3. *If we do not assume that the weight ψ is strictly positive, the lemma still holds if we in the supremum replace ψ with the projection $P(\psi)$.*

Let Θ denote the map from $\mathbb{C}^n$ to $\mathbb{R}^n$ that maps z to $(\ln|z_1|, ..., \ln|z_n|)$. Since we assumed ψ to be torus invariant, the function $\psi\circ\Theta^{-1}$ is well-defined on $\mathbb{R}^n$. We will denote $\psi\circ\Theta^{-1}$ by ψ_Θ. Since ψ was assumed to be psh, it follows that ψ_Θ will be convex on $\mathbb{R}^n$. Recall the definition of the Legendre transform. Given a convex function g on $\mathbb{R}^n$ the Legendre transform of g, denoted g^*, evaluated in a point $p\in\mathbb{R}^n$ is given by

$$g^*(p) := \sup_{x\in\mathbb{R}^n}\{\langle p, x\rangle - g(x)\}.$$

Observe that

$$\ln\left((|z^{\alpha}|^2 e^{-\psi})\circ\Theta^{-1}(x)\right) = 2\langle\alpha, x\rangle - \psi_\Theta(x). \tag{25}$$

Thus by equation (25) and Lemma 9.2 we get that

$$c[\psi](\alpha) = 2\left(\frac{\psi_\Theta}{2}\right)^*(\alpha).$$

Theorem (6.5) now gives us that for any two invariant weights ψ and φ on L it holds that

$$\mathcal{E}(\psi,\varphi) = 2n!\int_{\Delta^\circ}\left(\frac{\varphi_\Theta}{2}\right)^* - \left(\frac{\psi_\Theta}{2}\right)^* d\lambda.$$

This is a known result in toric geometry.

10 The Chebyshev transform on the zero-fiber

Let us assume that

$$z_1 = 0$$

is a local equation around p for an irreducible variety which we denote by Y. Let $H^0(X|Y, kL)$ denote the image of the restriction map from $H^0(X, kL)$ to $H^0(Y, kL_{|Y})$, and let $\Gamma(X|Y, L)$ denote the semigroup

$$\cup_{k\geq 0} \left(v(H^0(X|Y, kL)) \times \{k\}\right) \subset \mathbb{N}^n.$$

Note that since $z_2, ..., z_n$ are local coordinates on Y, $v(H^0(X|Y, kL)$ will be a set of vectors in $\mathbb{N}^{n-1}$.

Definition 10.1. *The restricted Okounkov body $\Delta_{X|Y}(L)$ is defined as the Okounkov body of the semigroup $\Gamma(X|Y, L)$.*

Lemma 10.2. *If Y is not contained in the augmented base locus $B_+(L)$, then $\Gamma(X|Y, L)$ generates $\mathbb{Z}^n$ as a group.*

This is part of Lemma 2.16 in [9].

Remark 10.3. *The augmented base locus $B_+(L)$ of L is defined as the base locus of any sufficiently small perturbation $L - \varepsilon A$, where A is some ample line bundle. Here we are only interested in the case where L is ample, and then it is easy to see that the augmented base locus $B_+(L)$ always is empty.*

Assume now that L is ample. We will show that the Chebyshev transform $c[\psi]$ can be defined not only in the interior of the Okounkov body but also on the zero fiber,

$$\Delta(L)_0 := \Delta(L) \cap \left(\{0\} \times \mathbb{R}^{n-1}\right).$$

From Theorem 4.24 in [9] we get the following fact,

$$\Delta(L)_0 = \Delta_{X|Y}(L). \tag{26}$$

Note that since the Okounkov body lies in the positive orthant of $\mathbb{R}^n$, $\Delta(L)_0$ is a part of the boundary of $\Delta(L)$, hence the Chebyshev transform of a continuous weight is a priori not defined on the zero-fiber. Nevertheless, we want to show that one can extend the Chebyshev transform to the interior of zero-fiber $\Delta(L)_0$. To do this, we need to know how Γ behaves near this boundary, something which Theorem 2.2 does not tell us anything about.

Lemma 10.4. *Assume L to be ample, and p any point in the interior of $\Delta(L)_0$. Let $\Sigma_{n+1}^{\mathbb{Z}}$ denote the unit simplex in $\mathbb{Z}^{n+1}$, $\Sigma_{n-1}^{\mathbb{R}}$ the unit simplex in $\mathbb{R}^{n-1}$, and let S denote the simplex $\{0\} \times \Sigma_{n-1}^{\mathbb{R}} \times \{0\}$. Then $\Gamma(L)$ contains a translated unit simplex $(\alpha, k) + \Sigma_{n+1}$ such that (kp, k) lies in the interior of the $(n-1)$-simplex*

$$(\alpha, k) + S$$

(i.e interior with respect to the $\mathbb{R}^{n-1}$ topology).

Proof. The augmented base locus of L is empty since L is ample, thus by Lemma 10.2 we may use Lemma 2.3 in combination with equation (26) to reach the conclusion that for large k, there are sections s_k such that (p, k) lies in the interior of $(v(s_k), k) + S$

with respect to the $\mathbb{R}^{n-1}$ topology. We may write L as a difference of two very ample divisors A and B. We may choose B such that $\Delta_1(B)$ contains Σ_n in $\mathbb{Z}^n$, and A such that $\Delta_1(A)$ contains origo. Now

$$kL = B + (kL - B).$$

Since L is ample, for k large we can find sections $s'_k \in H^0(kL - B)$ such that $v(s'_k) = v(s_k)$. We get that

$$(v(s_k), k) + \Sigma_n \subseteq \Gamma(L),$$

by multiplying s'_k by the sections of B corresponding to the points in the unit simplex $\Sigma_n \subseteq \Delta_1(B)$. Also observe that

$$(k+1)L = A + (kL - B).$$

Now by multiplying s'_k with the section of A corresponding to origo in $\Delta_1(A)$ we get

$$(v(s'_k), k) + (0, ..., 0, 1) \subseteq \Gamma(L).$$

Since

$$\Sigma_n \times \{0\} \cup (0, ..., 0, 1) = \Sigma_{n+1}$$

we get

$$(v(s'_k), k) + \Sigma_{n+1} \subseteq \Gamma(L).$$

□

Remark 10.5. *The proof is very close to the proof of Lemma 2.2 in [9], which shows the existence of a unit simplex in $\Gamma(L)$, when L is big. The difference here is that we need to control the position of the unit simplex, but the main trick of writing L as a difference of two very ample divisors is the same.*

Lemma 10.6. *Let p be as in the statement of Lemma 10.4. Then there exists a neighbourhood U of p such that if we denote the intersection $U \cap \Delta(L)$ by $\tilde{U}$, for k large it holds that*

$$(k\tilde{U}, k) \cap \mathbb{Z}^{n+1} \subseteq \Gamma(L).$$

Proof. Let $(\alpha, m) + \Sigma_{n+1}^{\mathbb{Z}} \subseteq \Gamma(L)$ be as in the statement of Lemma 10.4, and let $D^{\mathbb{Z}} \subseteq \Gamma(L)$ denote the set

$$D^{\mathbb{Z}} := (\alpha, m) + \Sigma_n^{\mathbb{Z}} \times \{0\} = (\alpha + \Sigma_n^{\mathbb{Z}}) \times \{m\}.$$

Let also $D^{\mathbb{R}}$ denote the set

$$D^{\mathbb{R}} := (\alpha + \Sigma_n^{\mathbb{R}}) \times \{m\}.$$

Since trivially

$$\underbrace{\Sigma_n^{\mathbb{Z}} + ... + \Sigma_n^{\mathbb{Z}}}_{k} = (k\Sigma_n^{\mathbb{R}}) \cap \mathbb{Z}^n,$$

we have that

$$(kD^{\mathbb{R}}, km) \cap \mathbb{Z}^{n+1} = \underbrace{D^{\mathbb{Z}} + ... + D^{\mathbb{Z}}}_{k} \subseteq \Gamma(L).$$

Therefore the lemma holds when k is a multiple of m. Furthermore, since m and $m+1$ are relatively prime, if k is greater than $m(m+1)$ we can write

$$k = k_1 m + k_2(m+1),$$

where both k_1 and k_2 are non-negative, and $k_2 \leq m$. Thus we consider the set

$$\underbrace{D^{\mathbb{Z}} + ... + D^{\mathbb{Z}}}_{k_1} + k_2(\alpha, m+1) \subseteq \Gamma(L).$$

Because of the bound $k_2 \leq m$, and since $(\alpha, m+1)$ lies on the zero fiber, for a neighbourhood $\tilde{U}$ of p, when k gets large we must have that

$$(k\tilde{U}, k) \cap \mathbb{Z}^{n+1} \subseteq \underbrace{D^{\mathbb{Z}} + ... + D^{\mathbb{Z}}}_{k_1} + k_2(\alpha, m+1) \subseteq \Gamma(L).$$

□

Corollary 10.7. *Assume L is ample, then the chebyshev function $c[\psi]$ is well-defined on the interior of the zero-fiber, $\Delta(L)_0$, and it is continuous and convex on its extended domain $\Delta(L)^\circ \cup \Delta(L)_0^\circ$.*

Proof. The proof goes exactly as for the case of an interior point, now using Lemma 10.6 instead of Theorem 2.2. □

Lemma 10.8. *Assume L is ample, and ψ is a continuous weight. Then for any regular compact set K it holds that the projection $P_K(\psi)$ also is continuous. In particular, since X is regular, $P(\psi)$ is continuous when L is ample.*

Proof. See e.g. [1]. □

We will have use for the Ohsawa-Takegoshi extension theorem. We choose to cite from [3] one version of it .

Theorem 10.9. *Let L be a holomorphic line bundle and let S be a divisor. Assume that L and S have metrics Ψ_L and Ψ_S respectively satisfying*

$$dd^c\Psi_L \geq (1+\delta)dd^c\Psi_S + dd^c\Psi_{K_X},$$

where Ψ_{K_X} is some smooth metric on the canonical bundle K_X. Assume also that

$$dd^c\Psi_L \geq dd^c(\Psi_S + \Psi_{K_X}).$$

Then any holomorphic section $\tilde{t}$ of the restriction of L to S extends holomorphically to a section t of L over X satisfying

$$\int_X |t|^2 e^{-\Psi_L}\omega_n \leq C_\delta \int_S |\tilde{t}|^2 e^{-\Psi_L} \frac{dS}{|ds|^2 e^{-\Psi_S}}.$$

Here ω_n is a smooth volume form on X and dS is a smooth volume form on S.

For a proof of this version we refer the reader to [3].

Lemma 10.10. *Suppose L is ample. Let A be an ample line bundle, with a holomorphic section s such that locally $s = z_1$. Also assume that the zero-set of s, which we will denote by Y, is a smooth submanifold. Then for all $\alpha \in \Delta_{X|Y}(L)$ we have that*

$$c_X[\varphi](0,\alpha) = c_Y[P(\varphi)_{|Y}](\alpha). \tag{27}$$

Proof. We may choose $\tilde{z}_1 = z_2, ..., \tilde{z}_{n-1} = z_n$ as holomorphic coordinates on Y around p. We consider the discrete Chebyshev transforms of the restrictions of $P(\varphi)$ and $P(\psi)$ to Y. Since L is ample, by Lemma 10.8 $P(\varphi)$ and $P(\psi)$ are continuous, therefore the restrictions will also be continuous psh-weights on $L_{|Y}$, therefore the Chebyshev transforms $c_Y[P(\varphi)_{|Y}]$ and $c_Y[P(\psi)_{|Y}]$ are well-defined.

We note that if $t \in H^0(X, kL)$ and

$$t = z^{k(0,\alpha)} + \text{higher order terms},$$

the restriction of t to Y will be given by

$$t_{|Y} = \tilde{z}^{k\alpha} + \text{higher order terms}.$$

Furthermore

$$\sup_Y \{|t_{|Y}|^2 e^{-kP(\varphi)}\} \leq \sup_X \{|t|^2 e^{-kP(\varphi)}\}.$$

This gives the inequality

$$c_X[\varphi](0,\alpha) \geq c_Y[P(\varphi)_{|Y}](\alpha),$$

by taking t to be some minimizing section with respect to the supremum norm on X.

For the opposite inequality we use Proposition 7.5 which says that one can use Bernstein-Markov norms to compute the Chebyshev transform.

If $\tilde{t} \in H^0(Y, kL_{|Y})$,

$$\tilde{t} = \tilde{z}^{k\alpha} + \text{higher order terms},$$

then if k is large enough there exists a section $t \in H^0(X, kL)$ such that $t_{|Y} = \tilde{t}$. This is because we assumed L to be ample, so we have extension properties (by e.g. Ohsawa-Takegoshi). We observe that any such extension must look like

$$t = z^{k(0,\alpha)} + \text{higher order terms},$$

because if we had that

$$t = z^{k(\beta_1,\beta)} + \text{higher order terms}$$

with $\beta_1 > 0$, then since all higher order terms also restrict to zero,

$$t_Y = 0,$$

which is a contradiction.

Let Ψ be some smooth strictly positive weight on L. Then for some m

$$dd^c m\Psi > (1+\delta)dd^c\Psi_A + dd^c\Psi_{K_X}$$

and

$$dd^c m\Psi > dd^c\Psi_A + dd^c\Psi_{K_X},$$

where Ψ_A and Ψ_{K_X} are weights on A and K_X respectively. We have that $dd^cP(\varphi) \geq 0$, hence

$$dd^c((k-m)P(\varphi) + m\Psi) > (1+\delta)dd^c\Psi_A + dd^c\Psi_{K_X}$$

and

$$dd^c((k-m)P(\varphi) + m\Psi) > dd^c\Psi_A + dd^c\Psi_{K_X}$$

for all $k > m$. Since $P(\varphi)$ is continuous hence locally bounded, we also have that for some constant C,

$$\Psi - C < P(\varphi) < \Psi + C.$$

We can apply Theorem 10.9 to these weights, and get that for large k, given a $\tilde{t} \in H^0(Y, kL_{|Y})$ there exists an extension $t \in H^0(X, kL)$ such that

$$\int_X |t|^2 e^{-kP(\varphi)}\omega_n \leq e^{mC}\int_X |t|^2 e^{-(k-m)P(\varphi)-m\Psi}d\mu$$
$$\leq e^{mC}C_\delta \int_Y |\tilde{t}|^2 e^{-(k-m)P(\varphi)-m\Psi}d\nu \leq e^{2mC}C_\delta\int_Y |\tilde{t}|^2 e^{-kP(\varphi)}d\nu,$$

where C_δ is constant only depending on δ and $d\nu$ is a smooth volume form on Y. By letting $\tilde{t}$ be the minimizing section with respect to $\int_Y |.|^2 e^{-kP(\varphi)}d\nu$ and using Proposition 7.5 we get that

$$c_X[\varphi](0,\alpha) \leq c_Y[P(\varphi)_{|Y}](\alpha),$$

since

$$\int_X |t|^2 e^{-k\varphi}\omega_n \leq \int_X |t|^2 e^{-kP(\varphi)}\omega_n.$$

□

Proposition 10.11. *Let L, A and Y be as in the statement of Lemma 10.10. Then we have that*

$$\mathcal{E}_Y(P(\varphi)_{|Y}, P(\psi)_{|Y}) = (n-1)!\int_{\Delta(L)_0}(c[\psi] - c[\varphi])(0,\alpha)d\alpha.$$

Proof. The proposition follows from Lemma 10.10 by integration of equality (27) over the interior of the zero-fiber, and Theorem 6.5 which says that

$$\mathcal{E}_Y(P(\varphi)_{|Y}, P(\psi)_{|Y}) = (n-1)!\int_{\Delta(L_{|Y})} c_Y[P(\psi)_{|Y}] - c_Y[P(\varphi)_{|Y}]d\lambda.$$

□

We will cite Proposition 3.7 from [1] which is a recursion formula relating the relative energy and the restricted energy.

Proposition 10.12. *Suppose L is ample, let $s \in H^0(L)$, and let Y be the smooth submanifold defined by s. Let ψ and φ be two continuous weights. Then*

$$(n+1)\mathcal{E}_X(\psi,\varphi) - n\mathcal{E}_Y(P(\psi)_{|Y}, P(\varphi)_{|Y}) =$$
$$= \int_X (\ln|s|^2 - P(\varphi))MA(P(\varphi)) - \int_X (\ln|s|^2 - P(\psi))MA(P(\psi)).$$

In particular, combining Theorem 6.5, Proposition 10.11 and Proposition 10.12 we get the following.

Proposition 10.13. *Let L, s and Y be as in Proposition 10.12. Then it holds that*

$$\int_{\Delta(L)^\circ} (c_X[\varphi] - c_X[\psi])d\lambda_n = \frac{1}{n+1}\int_{\Delta(L)_0^\circ} (c_X[\varphi] - c_X[\psi])d\lambda_{n-1} +$$
$$+\frac{1}{(n+1)!}\int_X (\ln|s|^2 - P(\varphi))MA(P(\varphi)) - \frac{1}{(n+1)!}\int_X (\ln|s|^2 - P(\psi))MA(P(\psi)).$$

11 Directional Chebyshev constants in $\mathbb{C}^n$

In [4] Bloom-Levenberg define what they call directional Chebyshev constants. In this section we will describe how this relates to the Chebyshev transforms we have introduced.

The setting in [4] is as follows. Let $<_1$ be the order on $\mathbb{N}^n$ such that $\alpha <_1 \beta$ if $|\alpha| < |\beta|$, or if $|\alpha| = |\beta|$ and $\alpha <_{\text{lex}} \beta$. Let P_α denote the set of polynomials $p(z_1, ..., z_n)$ in the variables z_i such that

$$p = z^\alpha + \text{ lower order terms.}$$

Observe that here we want lower order terms, and not higher order terms. Let K be a compact set and h an admissible weight function on K. For any $\alpha \in \mathbb{N}^n$ they define the weighted Chebyshev constant $Y_3(\alpha)$ as

$$Y_3(\alpha) := \inf\{\sup_{z\in K}\{|h(z)^{|\alpha|}p(z)|\} : p \in P_\alpha\}.$$

Lemma 2.1 in [4] tells us that the limit

$$\tau^h(K,\theta) := \lim_{\alpha/\deg(\alpha)\to\theta} Y_3(\alpha)^{1/\deg(\alpha)}$$

exists. These limits are called directional Chebyshev constants.

In our setting we wish to view $\mathbb{C}^n$ as an affine space lying in $\mathbb{P}^n$. Also, polynomials in z_i can be interpreted as sections of multiples of the line bundle $\mathcal{O}(1)$ on $\mathbb{P}^n$ in the following sense. Let $Z_0, ..., Z_n$ be a basis for $H^0(\mathcal{O}(1))$ on $\mathbb{P}^n$, and identify them with the homogeneous coordinates $[Z_0, ..., Z_n]$. We can choose

$$p := [1 : 0 : ... : 0]$$

to be our base point, and let $z_i := \frac{Z_i}{Z_0}$ be holomorphic coordinates around p. We also let Z_0 be our local trivialization of the bundle. Given a section $s \in H^0(\mathcal{O}(k))$we represent it as a function in z_i by dividing by a power of Z_0

$$\frac{s}{Z_0^k} = \sum a_\alpha z^\alpha.$$

Therefore we see that

$$Z^{(\alpha_0,\alpha_1,\ldots,\alpha_n)} \mapsto z^{(\alpha_1,\ldots,\alpha_n)}.$$

We could also choose a different set of coordinates. Let

$$q := [0 : \ldots : 0 : 1]$$

be our new base point, and let $w_i := \frac{Z_i}{Z_n}$ be coordinates around q. Let Z_n be the local trivialization around q. Given a section $s \in H^0(\mathcal{O}(k))$ we represent it as a function in w_i by dividing by a power of Z_n

$$\frac{s}{Z_n^k} = \sum b_\alpha w^\alpha.$$

Hence

$$Z^{(\alpha_0,\alpha_1,\ldots,\alpha_n)} \mapsto w^{(\alpha_0,\ldots,\alpha_{n-1})}.$$

To define Chebyshev transforms we need an additive order on $\mathbb{N}^n$. Since the semigroup $\Gamma(\mathcal{O}(1))$ will not depend on the order, we are free to choose any additive order. Let $<_2$ be the order which corresponds to inverting the order $<_1$ with respect to the z_i variables, i.e.

$$(\alpha_0, \ldots, \alpha_{n-1}) <_2 (\beta_0, \ldots, \beta_{n-1})$$

iff

$$(\beta_1, \ldots, \beta_n) <_1 (\alpha_1, \ldots, \alpha_n).$$

Therefore

$$z^{(\alpha_1,\ldots,\alpha_n)} + \text{ lower order terms} = w^{(\alpha_0,\ldots,\alpha_{n-1})} + \text{ higher order terms.} \tag{28}$$

We may identify the weight function h with a metric $h = e^{-\psi/2}$ on $\mathcal{O}(1)$. Consider the weight $P_K(\psi)$. For simplicity assume that K is regular. Since $\mathcal{O}(1)$ is ample from Lemma 10.8 it follows that $P_K(\psi)$ is continuous, therefore the Chebyshev transform $c[P_K(\psi)]$ is well-defined. It is a simple fact that

$$\sup_{z\in K}\{|s(z)|^2 e^{-k\psi(z)}\} = \sup_{z\in\mathbb{P}^n}\{|s(z)|^2 e^{-kP_K(\psi)(z)}\}. \tag{29}$$

Let $\alpha_0 = 0$, and let $k = \sum_1^n \alpha_i$. By (28) we see that $s \in A_{(\alpha_0,\ldots,\alpha_{n-1}),k}$ iff it is on the form

$$z^{(\alpha_1,\ldots,\alpha_n)} + \text{ lower order terms.}$$

By (29) it follows that

$$\ln Y_3(\alpha_1, \ldots, \alpha_n) = F[P_K(\psi)](k\alpha, k).$$

Thus we get that for $\theta = (\theta_1, ..., \theta_n) \in \Sigma^0$

$$c[P_K(\psi)](0, \theta_1, ..., \theta_{n-1}) = 2 \ln \tau^h(\theta_1, ..., \theta_n). \tag{30}$$

Observe that the order $<_2$ we used to defined the Chebyshev transform has the property that $(0, \alpha) <_2 (\beta_1, \beta)$ when $\beta_1 > 0$. It was this property of the lexicographic order we used in the proof of Proposition 10.11. Therefore the theorem holds also for Chebyshev transforms defined using $<_2$ instead of $<_{\text{lex}}$. Let (K', h') be another weighted set in $\mathbb{C}^n$, and let ψ' be the corresponding weight on $\mathcal{O}(1)$ associated to h'. Then integrating (30) gives us that

$$\frac{1}{\text{meas}(\Sigma^0)} \int_{\Sigma^0} \ln \tau^h(K, \theta) - \ln \tau^{h'}(K', \theta) d\theta =$$
$$= \frac{(n-1)!}{2} \int_{\Delta(\mathcal{O}(1))_0} c[P_K(\psi)] - c[P_{K'}(\psi')] d\theta, \tag{31}$$

where $Y := \{Z_0 = 0\}$. Here we used that $\Delta(\mathcal{O}(1))_0$ is a $(n-1)$-dimensional unit simplex, and thus

$$\text{meas}(\Delta(\mathcal{O}(1))_0) = \frac{1}{(n-1)!}.$$

Bloom-Levenberg define a weighted transfinite diameter $d^h(K)$ of K which is given by

$$d^h(K) := \exp\left(\frac{1}{\text{meas}(\Sigma^0)} \int_{\Sigma^0} \ln \tau^h(K, \theta) d\theta\right).$$

There is also another transfinite diameter, $\delta^h(K)$, which is defined as a limit of certain Vandermonde determinants. By Corollary A in [1] we have that

$$\ln \delta^h(K) - \ln \delta^{h'}(K') = \frac{(n+1)}{2n} \mathcal{E}(P_{K'}(\psi'), P_K(\psi)).$$

Then by Theorem 6.5, equation (31) and Proposition **??** we get that

$$\ln \delta^h(K) - \ln \delta^{h'}(K') =$$
$$= \ln d^h(K) - \ln d^{h'}(K') + \frac{1}{n} \int_{\mathbb{P}^n} \frac{1}{2} (\ln |Z_0|^2 - P_K(\psi)) \text{MA}(P_K(\psi)) -$$
$$- \frac{1}{n} \int_{\mathbb{P}^n} \frac{1}{2} (\ln |Z_0|^2 - P_{K'}(\psi')) \text{MA}(P_{K'}(\psi')).$$

In fact, the positive measure $\text{MA}(P_K(\psi))$ has support on K, and $P_K(\psi) = \psi$ a.e. with respect to $\text{MA}(P_K(\psi))$. In the notation of [4], $(\psi - \ln |Z_0|^2)/2$ is denoted Q, and $\text{MA}(P_K(\psi))$ is denoted $(dd^c V^*_{K,Q})^n$. Thus in their notation

$$\ln \delta^h(K) - \ln \delta^{h'}(K') =$$
$$= \ln d^h(K) - \ln d^{h'}(K') - \frac{1}{n} \int_K Q (dd^c V^*_{K,Q})^n + \frac{1}{n} \int_{K'} Q' (dd^c V^*_{K',Q'})^n.$$

For the unit ball B, with $h \equiv 1 \equiv |Z_0|^2$ and therefore $Q_h = 0$, it is straight-forward to show that we have an equality

$$\delta^h(B) = d^h(K).$$

Using this we get that

$$\ln \delta^h(K) = \ln d^h(K) - \frac{1}{n} \int_K Q(dd^c V^*_{K,Q})^n.$$

By taking the exponential we have derived the formula of Theorem 2.7 in [4].

12 Chebyshev transforms of weighted $\mathbb{Q}$- and $\mathbb{R}$-divisors

Because of the homogeneity of Okounkov bodies, one may define the Okounkov body $\Delta(D)$ of any big $\mathbb{Q}$-divisor D. Set

$$\Delta(D) := \frac{1}{p}\Delta(pD)$$

for any p that clears the denominators in D. In [9] Lazarsfeld-Mustaţă show that this mapping of a $\mathbb{Q}$-divisor to its Okounkov body has a continuous extension to the class of $\mathbb{R}$-divisors.

In Proposition 5.10 we saw that Chebyshev transforms also are homogeneous under scaling. Therefore we may define the Chebyshev transform of a $\mathbb{Q}$-divisor D with weight ψ, by letting

$$c[\psi](\alpha) = \frac{1}{p} c[p\psi](p\alpha), \qquad \alpha \in \Delta(D)^\circ, \tag{32}$$

for any p clearing the denominators in D. We wish to show that this can be extended continuously to the class of weighted $\mathbb{R}$-divisors.

We will use the construction introduced in [9]. Let $D_1, ..., D_r$ be divisors such that every divisor is numerically equivalent to a unique sum

$$\sum a_i D_i, \qquad a_i \in \mathbb{Z}.$$

Lazarsfeld-Mustaţă show that for effective divisors the coefficients a_i may be chosen non-negative.

Definition 12.1. *The semigroup of X, $\Gamma(X)$, is defined as*

$$\Gamma(X) := \bigcup_{a \in \mathbb{N}^r} \Big(v(H^0(\mathcal{O}_X(\sum a_i D_i))) \times \{a\} \Big) \subseteq \mathbb{Z}^{n+r},$$

where v stands for the usual valuation,

$$s = z^\alpha + \text{ higher order terms} \qquad \Rightarrow \qquad s \mapsto \alpha.$$

Lemma (4.11) in [9] states that $\Gamma(X)$ generates $\mathbb{Z}^{n+r}$ as a group.

Let $\Sigma(\Gamma(X))$ denote the closed convex cone spanned by $\Gamma(X)$, and let for $a \in \mathbb{N}^r$

$$\Delta(a) := \Sigma(\Gamma(X)) \cap (\mathbb{R}^n \times \{a\}).$$

Theorem (4.5) in [9] states that for any big $\mathbb{Q}$-divisor $D = \sum a_i D_i$,

$$\Delta(a) = \Delta(D), \qquad a = (a_1, ..., a_r).$$

Let for each $1 \leq i \leq r$ ψ_i be a continuous weights on D_i. Then for $a \in \mathbb{N}^r$, $\sum a_i \psi$ is a continuous weight on $\sum a_i D_i$. For an element $(\alpha, a) \in \Gamma(X)$, let $A_{\alpha,a} \subseteq H^0(\sum a_i D_i)$ be the set of sections of the form

$$z^\alpha + \text{ higher order terms.}$$

Definition 12.2. *The discrete global Chebyshev transform $F[\psi_1, ..., \psi_r]$ is defined by*

$$F[\psi_1, ..., \psi_r](\alpha, a) := \inf\{\ln ||s||^2_{\alpha,a} : s \in A_{\alpha,a}\}$$

for $(\alpha, a) \in \Gamma(X)$.

Lemma 12.3. *$F[\psi_1, ..., \psi_r]$ is subadditive on $\Gamma(X)$.*

Proof. If $s \in H^0(\mathcal{O}_X(\sum a_i D_i))$,

$$s = z^\alpha + \text{ higher order terms,}$$

and $t \in H^0(\mathcal{O}_X(\sum b_i D_i))$,

$$t = z^\beta + \text{ higher order terms,}$$

then $st \in H^0(\mathcal{O}_X(\sum (a_i + b_i) D_i))$ and

$$st = z^{\alpha+\beta} + \text{ higher order terms.}$$

Thus the subadditivity of $F[\psi_1, ..., \psi_r]$ follows exactly as for $F[\psi]$ in Lemma 5.3. □

Lemma 12.4. *$F[\psi_1, ..., \psi_r]$ is locally linearly bounded from below.*

Proof. Let $(\alpha, a) \in \Sigma(\Gamma(X))^\circ$. Let $\psi_{i,p}$ be the trivializations of the weights ψ_i, then

$$\sum a_i \psi_{i,p}$$

is the trivialization of $\sum a_i \psi_i$. Let D be as in the proof of Lemma 5.4, and choose A such that

$$e^{-\sum a_i \psi_{i,p}} > A.$$

Since the inequality

$$e^{-\sum b_i \psi_{i,p}} > A$$

holds for all b in a neighbourhood of a, the lower bound follows as in the proof of Lemma 5.4. □

Definition 12.5. *The global Chebyshev transform $c[\psi_1, ..., \psi_r]$ of the r-tuple $(\psi_1, ..., \psi_r)$ is defined as the convex envelope of $F[\psi_1, ..., \psi_r]$ on $\Sigma(\Gamma(X))^\circ$.*

Proposition 12.6. *For any sequence $(\alpha(k), a(k)) \in \Gamma(X)$ such that $|(\alpha(k), a(k))| \to \infty$ and*

$$\frac{(\alpha(k), a(k))}{|(\alpha(k), a(k))|} \to (p, a) \in \Sigma(\Gamma(X))^\circ$$

it holds that

$$\lim_{k\to\infty} \frac{F[\psi_1, ..., \psi_r](\alpha(k), a(k))}{|(\alpha(k), a(k))|} = c[\psi_1, ..., \psi_r](p, a).$$

Proof. By Lemma 12.3 and Lemma 12.4 we can use Theorem 3.8, which gives us the proposition. □

Proposition 12.7. *For rational a, i.e $a = (a_1, ..., a_r) \in \mathbb{Q}^r$, the global Chebyshev transform $c[\psi_1, ..., \psi_r](p, a)$ coincides with $c\left[\sum a_i\psi_i\right](p)$, where the Chebyshev transform of the $\mathbb{Q}$-divisor $\sum a_iD_i$ as defined by (32).*

Proof. By construction it is clear that for all $(\alpha, a) \in \Gamma(X)$ we have that

$$F[\psi_1, ..., \psi_r](\alpha, ka) = F\left[\sum a_i\psi_i\right](\alpha, k).$$

Choose a sequence $(\alpha(k), ka) \in \Gamma(X)$ such that

$$\lim_{k\to\infty} \frac{(\alpha(k), ka))}{|(\alpha(k), ka))|} = \frac{(p, a)}{|(p, a)|},$$

where we only consider those k such that ka is an integer. Then by Proposition 12.6 we have that

$$c[\psi_1, ..., \psi_r](p, a) = \lim_{k\to\infty} |(p, a)| \frac{F[\psi_1, ..., \psi_r](\alpha(k), ka)}{|(\alpha(k), ka)|} =$$

$$= \lim_{k\to\infty} |(p, a)| \frac{F\left[\sum a_i\psi_i\right](\alpha(k), k)}{|(\alpha(k), ka)|} = \lim_{k\to\infty} \left(\frac{|(p, a)|k}{|(\alpha(k), ka)|}\right) c\left[\sum a_i\psi_i\right](p) =$$

$$= c\left[\sum a_i\psi_i\right](p).$$

□

Now that we have defined the Chebyshev transform for weighted $\mathbb{R}-$divisors we wish to show that the formula of Theorem 6.5 holds true also in this case. First we need some preliminary lemmas.

Lemma 12.8. *The function $\mathcal{E}(t\psi, t\varphi)$ is $(n + 1)$-homogeneous in t for $t > 0$, i.e.*

$$\mathcal{E}(t\psi, t\varphi) = t^{n+1}\mathcal{E}(\psi, \varphi).$$

Proof. For weights with minimal singularities ψ' and φ', by definition of the relative energy we have that

$$\mathcal{E}(t\psi, t\varphi) = \frac{1}{n+1}\int_\Omega (t\psi' - t\varphi')\mathrm{MA}_n(t\psi', t\varphi') =$$

$$= \frac{t^{n+1}}{n+1}\int_\Omega (\psi' - \varphi')\mathrm{MA}_n(\psi', \varphi') = t^{n+1}\mathcal{E}(\psi, \varphi). \tag{33}$$

We also observe that $t\psi'$ is a psh weight on tL iff ψ' is a psh weight on L. Therefore we get that

$$P(t\psi) = tP(\psi). \tag{34}$$

Combining (33) and (34) the lemma follows. □

Lemma 12.9. *Assume that L is ample. Let ψ and ψ' be two continuous weights on L, and let φ and φ' be two continuous weights on some other big line bundle L'. Then the function*

$$\mathcal{E}(\psi + t\varphi, \psi' + t\varphi')$$

is continuous in t for t such that $L + tL'$ is ample.

Proof. We show continuity at $t = 0$. Since L is ample, for some $\varepsilon > 0$

$$L + \varepsilon L'$$

will be ample. Furthermore the relative energy is homogeneous. We may write

$$L + t\varepsilon L'$$

as

$$(1-t)(L + \frac{t}{1-t}(L + \varepsilon L')),$$

thus without loss of generality we can assume that L' is ample. By the cocycle property of the relative energy we have that for any continuous weight $\tilde{\varphi}$ on L'

$$\mathcal{E}(\psi+t\varphi, \psi'+t\varphi') = \mathcal{E}(\psi+t\varphi, \psi+t\tilde{\varphi}) + \mathcal{E}(\psi+t\tilde{\varphi}, \psi'+t\tilde{\varphi}) + \mathcal{E}(\psi'+t\tilde{\varphi}, \psi'+t\varphi').$$

Thus it suffices to consider two special cases. The first where we assume that $\psi = \psi'$. In the second case we instead assume that $\varphi = \varphi'$ and that φ is psh.

First assume that $\psi = \psi'$. Since $\mathcal{E}(\psi, \psi) = 0$, we must show that $\mathcal{E}(\psi+t\varphi, \psi+t\varphi')$ tends to zero when t tends to zero. Lemma 1.12 in [1] tells us that the projection operator is Lipschitz continuous. In our case this means that

$$\sup_X |P(\psi + t\varphi) - P(\psi + t\varphi')| \leq t \sup_X |\varphi - \varphi'|.$$

We get that

$$|\mathcal{E}(\psi + t\varphi, \psi + t\varphi')| =$$

$$= \frac{1}{n+1}|\int_X (P(\psi + t\varphi) - P(\psi + t\varphi'))\mathrm{MA}_n(P(\psi + t\varphi), P(\psi + t\varphi'))| \leq$$

$$\leq t \sup_X |\varphi - \varphi'| \frac{1}{n+1}\int_X \mathrm{MA}_n(P(\psi + t\varphi), P(\psi + t\varphi')) =$$

$$= t \sup_X |\varphi - \varphi'| \mathrm{vol}(L + tL').$$

Since the volume is continuous (see e.g. [1]), we get continuity in this case.

Now we intead assume that $\varphi = \varphi'$ and that φ is psh. We first show right-continuity. Since φ is psh, for all $r \leq t$ we have that

$$P(\psi + r\varphi) + (t - r)\varphi$$

is psh and it is clearly dominated by $\psi + t\varphi$, thus by the definition of the projection operator

$$P(\psi + t\varphi) \geq P(\psi + r\varphi) + (t - r)\varphi.$$

It follows that $P(\psi + t\varphi) - t\varphi$ is increasing in t. Also

$$dd^c(P(\psi + t\varphi) - t\varphi) \geq -tdd^c\varphi,$$

thus by standard results in potential thoery we have that

$$dd^c \lim_{t\to 0}(P(\psi + t\varphi) - t\varphi) \geq 0.$$

This gives us that

$$\lim_{t\to 0}(P(\psi + t\varphi) - t\varphi) = P(\psi).$$

The same holds for

$$P(\psi' + t\varphi) - t\varphi.$$

We now write $P(\psi + t\varphi)$ as

$$(P(\psi + t\varphi) - t\varphi) + t\varphi$$

and $P(\psi' + t\varphi)$ as

$$(P(\psi' + t\varphi) - t\varphi) + t\varphi$$

in the expression for

$$\mathcal{E}(\psi + t\varphi, \psi' + t\varphi)$$

and the right-continuity follows from Theorem 1.6 in [1], which states that mixed Monge-Ampère operators are continuous along pointwise decreasing sequences of psh or quasi-psh weights converging to a weight with minimal singularities. For the left-continuity we use the homogeneity of the relative energy exactly as above to reduce to the case of right-continuity already considered. □

We are now ready to prove our main theorem in the setting of weighted ample $\mathbb{R}$-divisors.

Theorem 12.10. *For ample $\mathbb{R}$-divisors $\sum a_i D_i$ we have that*

$$\mathcal{E}(\sum a_i\psi_i, \sum a_i\varphi_i) =$$

$$= n! \int_{\Delta(\sum a_i D_i)} (c[\varphi_1, ..., \varphi_r](p, a) - c[\psi_1, ..., \psi_r](p, a))d\lambda(p). \tag{35}$$

Proof. First we show that (35) holds when $a \in \mathbb{Q}^r$. By the homogeneity of the Okounkov body and the Chebyshev transform we have that

$$n! \int_{\Delta(tL)^\circ} (c[t\psi] - c[t\varphi]) d\lambda = t^{n+1} n! \int_{\Delta(L)^\circ} (c[\psi] - c[\varphi]) d\lambda =$$
$$= t^{n+1} \mathcal{E}(\varphi, \psi) = \mathcal{E}(t\varphi, t\psi),$$

where the last equality follows from Lemma 12.8. Then by Proposition 12.7, (35) holds for $a \in \mathbb{Q}^r$. Therefore by the continuity of the relative energy, the continuity of the global Chebyshev transform, and the fact that equation (35) holds for rational a, the proposition follows. □

13 Differentiability of the relative energy

We wish to understand the behaviour of the relative energy $\mathcal{E}(\psi_t, \varphi_t)$ when the weights ψ_t and φ_t vary with t. In [1] Berman-Boucksom study the case where ψ_t and φ_t are weights on a fixed line bundle or more generally a $\mathbb{R}$-divisor. We are interested in the case where the underlying $\mathbb{R}$-divisor is allowed to vary as well. In [9] Lazarsfeld-Mustaţă prove the differentiability of the volume by studying the variation of the Okounkov bodies. Since our Theorem 6.5 and Theorem 12.10 states that the relative energy is given by the integration of the difference of Chebyshev transforms on the Okounkov body, we wish to use the same approach as Lazarsfeld-Mustaţă did in [9]. The situation becomes a bit more involved, since we have to consider not only the variation of the Okounkov bodies but also the variation of the Chebyshev transforms.

In this section we will assume that L is an ample $\mathbb{R}$-divisor.

To account for the variation of the Chebyshev transform when the underlying line bundle changes it becomes necessary to consider not only continuous weights but also weights with singularities. Specifically weights of the form

$$\psi - t \ln |s|^2,$$

where ψ is a continuous weight on L, s is some section of an ample line bundle A, and t is positive. Observe that these weights only have $+\infty$ singularities.

In fact, by general approximation arguments one can show that the results that we have established for continuous weights aslo hold for weights that are lower semicontinuous and only have $+\infty$ singularities. But for completeness we include arguments proving this for $\psi - t \ln |s|^2$.

Let Ψ be some fixed continuous positive weight on A. For any number R we denote by $\ln |s|^2_{+R}$ the weight

$$\ln |s|^2_{+R} := \max(\ln |s|^2, \Psi - R).$$

Lemma 13.1. *For $R \gg 0$ we have that*

$$P(\psi - t \ln |s|^2_{+R}) = P(\psi - r \ln |s|^2).$$

Proof. That

$$P(\psi - t\ln|s|^2_{+R}) \leq P(\psi - t\ln|s|^2)$$

is clear since

$$\psi - t\ln|s|^2_{+R} \leq \psi - t\ln|s|^2.$$

$P(\psi - t\ln|s|^2)$ is psh, therefore upper semicontinuous by definition, which means that it is locally bounded from above. Thus locally we can find $R \gg 0$ such that

$$\psi - t(\Psi - R) \geq P(\psi - t\ln|s|^2).$$

But we have assumed that our manifold X is compact, so there exists an R such that $\psi - t(\Psi - R)$ dominates $P(\psi - t\ln|s|^2)$ on the whole of X. The same must be true for $\psi - t\ln|s|^2_{+R}$. By definition $P(\psi - t\ln|s|^2_{+R})$ dominates all psh weights less or equal to $\psi - t\ln|s|^2_{+R}$, in particular it must dominate $P(\psi - r\ln|s|^2)$. □

Lemma 13.2. *If L is integral, i.e. a line bundle, then for $R \gg 0$ such that*

$$P(\psi - t\ln|s|^2_{+R}) = P(\psi - t\ln|s|^2),$$

we have that $F[\psi - t\ln|s|^2_{+R}] = F[\psi - t\ln|s|^2]$.

Proof. This follows the fact that for all weights φ and all sections s it holds that

$$\sup_{x\in X}\{|s(x)|^2 e^{-\varphi(x)}\} = \sup_{x\in X}\{|s(x)|^2 e^{-P(\varphi)(x)}\},$$

see e.g. [1]. □

From Lemma 13.2 it follows that the Chebyshev transform $c[\psi - t\ln|s|^2]$ is well-defined, also for $\mathbb{R}-$divisors, and that Proposition 5.6 holds in this case. The formula for the relative energy as the integral of Chebyshev transforms will also still hold.

Proposition 13.3. *For any continuous weight φ on $L - tA$ it holds that*

$$\mathcal{E}(\psi - t\ln|s|^2, \varphi) = \tag{36}$$

$$= n!\int_{\Delta(L-tA)^{\circ}} c[\varphi] - c[\psi - t\ln|s|^2]d\lambda. \tag{37}$$

Proof. For integral L, choose an $R \gg 0$ such that

$$P(\psi - t\ln|s|^2_{+R}) = P(\psi - t\ln|s|^2).$$

Then (36) follows in this case from Theorem 6.5 and Lemma 13.2. By homogeneity (36) holds for rational L, and by continuity for arbitrary ample $\mathbb{R}$-divisors. □

Theorem B in [1] states that the relative energy is differentiable when the weights correspond to a fixed big line bundle. By the comment in the beginning of section 3 in [1] this holds more generally for big $(1,1)$ cohomology classes, e.g. $\mathbb{R}$-divisors. We thus have the following.

Theorem 13.4. *Let ψ_t be a smooth family of weights on a big $\mathbb{R}$-divisor D, and φ any psh-weight with minimal singularities. Then the function*

$$f(t) := \mathcal{E}(\psi_t, \varphi)$$

is differentiable, and

$$f'(0) = \int_\Omega u MA(P(\psi_0)),$$

where $u = \frac{d}{dt}_{|0}\psi_t$.

We also need to consider the case where

$$\psi_t = \psi_0 + t(\Phi - \ln |s|^2),$$

where Φ is some continuous weight on A.

Lemma 13.5. *For every ε there exists a $R \gg 0$ such that*

$$P(\psi_0 + t(\Phi - \ln |s|^2_{+R})) = P(\psi_0 + t(\Phi - \ln |s|^2))$$

for $t \geq \varepsilon$.

Proof. Recall that $\ln |s|^2_{+R}$ was defined as $\max\{\Psi - R, \ln |s|^2\}$ for some continuous weight Ψ on A. That

$$P(\psi_0 + t(\Phi - \ln |s|^2_{+R})) \leq P(\psi_0 + t(\Phi - \ln |s|^2))$$

is clear since

$$\psi_0 + t(\Phi - \ln |s|^2_{+R}) \leq \psi_0 + t(\Phi - \ln |s|^2)$$

and the projection operator is monotone. When

$$R \geq \frac{P(\psi_0 + t(\Phi - \ln |s|^2)) - \psi_0 - t\Phi}{t} + \Psi$$

we get that

$$P(\psi_0 + t(\Phi - \ln |s|^2_{+R})) = P(\psi_0 + t(\Phi - \ln |s|^2))$$

because for such R

$$\psi_0 + t(\Phi - \ln |s|^2_{+R}) \geq \psi - t(\Psi - R) \geq P(\psi_0 + t(\Phi - \ln |s|^2))$$

and the same is true for the projection. By the homogeneity of the projection operator we have that

$$\frac{P(\psi_0 + t(\Phi - \ln |s|^2)) - \psi_0 - t\Phi}{t} + \Psi = P(\frac{\psi_0}{t} + \Phi - \ln |s|^2) - \frac{\psi_0}{t} - \Phi + \Psi.$$

We also have that for $t > r$

$$P(\frac{\psi_0}{t} + \Phi - \ln |s|^2) - \frac{\psi_0}{t} \leq P(\frac{\psi_0}{t} + \Phi - \ln |s|^2) - \frac{P(\psi_0)}{t} \leq$$
$$\leq P(\frac{\psi_0}{r} + \Phi - \ln |s|^2) - \frac{P(\psi_0)}{r}$$

by the same arguments as in the proof of Lemma 12.9. $P(\frac{\psi_0}{r} + \Phi - \ln|s|^2)$ is psh and therefore upper semicontinuous, and since L is ample, $P(\psi_0)$ is continuous. This yields that

$$P(\frac{\psi_0}{r} + \Phi - \ln|s|^2) - \frac{P(\psi_0)}{r} - \Phi + \Psi$$

is an upper semicontinuous function on the compact space X, so it has an upper bound. The lemma follows by setting $r = 1/\varepsilon$ and choosing R larger than

$$P(\frac{\psi_0}{r} + \Phi - \ln|s|^2) - \frac{P(\psi_0)}{r} - \Phi + \Psi.$$

□

We state and prove a slight variation of Lemma 1.3 in [2].

Lemma 13.6. *Let f_k be a sequence of concave functions on the unit interval, and let g be a function on $[0,1]$ such that f_k converges to g pointwise. It follows that*

$$g'(0) \leq \liminf_{k\to\infty} f_k'(0).$$

Proof. Since f_k is concave we have that

$$f_k(0) + f_k'(0)t \geq f_k(t)$$

hence

$$\liminf_{k\to\infty} t f_k'(0) \geq g(t) - g(0).$$

The lemma follows by letting t tend to zero. □

We now prove that Theorem 13.4 holds true also in our singular setting.

Lemma 13.7. *The function*

$$f(t) := \mathcal{E}(\psi_0 + t(\Phi - \ln|s|^2), \varphi)$$

is right-differentiable at zero and

$$\frac{d}{dt}_{|0+} f(t) = \int_\Omega (\Phi - \ln|s|^2) MA(P(\psi_0)).$$

Proof. Let us denote $\Phi - \ln|s|^2$ by u, and let

$$u_k := \Phi - \ln|s|^2_{+k}.$$

Let f_k denote the functions

$$f_k(t) := \mathcal{E}(\psi_0 + t u_k, \varphi).$$

By e.g. [1] the functions f_k are concave, and by Theorem 13.4 they are differentiable. By Lemma 13.5 we get that for any $\varepsilon > 0$ there exists a k such that $f = f_k$ on $(\varepsilon, 1)$. Therefore it follows that f is concave and that

$$f_k \to f$$

pointwise. Since f is concave it is right-differentiable. We also have that

$$f_k'(0) = \int_\Omega u_k \mathrm{MA}(P(\psi_0))$$

by Theorem 13.4. Thus from Lemma 13.6 we get that

$$f'(0) \leq \int_\Omega u\mathrm{MA}(P(\psi_0)).$$

Since f is concave the derivative is decreasig, for all $\varepsilon > 0$

$$f'(0) \geq f'(\varepsilon) = \lim_{k\to\infty} \int_\Omega u_k \mathrm{MA}(P(\psi_0 + \varepsilon u_k)) = \int_\Omega u\mathrm{MA}(P(\psi_0 + \varepsilon u)),$$

where the last step follows by monotone convergence since

$$\mathrm{MA}(P(\psi_0 + \varepsilon u_k)) = \mathrm{MA}(P(\psi_0 + \varepsilon u))$$

for large k by Lemma 13.5. The projection operator is 1-Lipschitz continuous, therefore we get that $P(\psi_0 + \varepsilon u_k)$ will converge to $P(\psi_0)$ uniformly. By Theorem 1.6 in [1] the Monge-Ampère operator is continuous along sequences of psh weights with minimal singularities converging uniformly, hence

$$\lim_{\varepsilon\to 0} \int_\Omega u\mathrm{MA}(P(\psi_0 + \varepsilon u)) = \int_\Omega u\mathrm{MA}(P(\psi_0)),$$

and the lemma follows. □

We will also need an integration by parts formula involving $\ln|s|^2$.

Lemma 13.8. *Let φ and φ' be continuous weights on an ample $\mathbb{R}$-divisor L. Let ψ be a continuous psh weight on an ample line bundle A, and let $s \in H^0(A)$ be a section such that its zero set variety Y is a smooth submanifold. Then it holds that*

$$\int_X (\psi - \ln|s|^2) dd^c(P(\varphi) - P(\varphi')) \wedge MA_{n-1}(P(\varphi), P(\varphi')) =$$

$$= \int_X (P(\varphi) - P(\varphi')) dd^c\psi \wedge MA_{n-1}(P(\varphi), P(\varphi')) - n\mathcal{E}_Y(P(\varphi)_{|Y}, P(\varphi')_{|Y}).$$

Proof. The lemma will follow by the Lelong-Poincaré formula as soon as we establish that

$$\int_X (\psi - \ln|s|^2) dd^c(P(\varphi) - P(\varphi')) \wedge \mathrm{MA}_{n-1}(P(\varphi), P(\varphi')) =$$

$$= \int_X (P(\varphi) - P(\varphi')) dd^c(\psi - \ln|s|^2) \wedge \mathrm{MA}_{n-1}(P(\varphi), P(\varphi')),$$

which is an integration by parts formula. By Theorem 1.7 in [1] we may integrate by parts when the functions are differences of quasi-psh weights with minimal singularities. We denote by u_k the quasi-psh weight with minimal singularities $\psi - \ln|s|^2_{+k}$ and get that

$$\int_X u_k dd^c(P(\varphi) - P(\varphi')) \wedge \mathrm{MA}_{n-1}(P(\varphi), P(\varphi')) =$$
$$= \int_X (P(\varphi) - P(\varphi'))dd^c u_k \wedge \mathrm{MA}_{n-1}(P(\varphi), P(\varphi')).$$

Since $P(\varphi)$ and $P(\varphi')$ are both continuous, by the Chern-Levine-Nirenberg inequalities (see e.g. [6]) we get that

$$\int_X |(\psi - \ln|s|^2)| dd^c P(\varphi) \wedge \mathrm{MA}_{n-1}(P(\varphi), P(\varphi')) \leq C \int_X |(\psi - \ln|s|^2)| dV$$

and

$$\int_X |(\psi - \ln|s|^2)| dd^c P(\varphi') \wedge \mathrm{MA}_{n-1}(P(\varphi), P(\varphi')) \leq C' \int_X |(\psi - \ln|s|^2)| dV$$

for some constants C and C' and some smooth volume form dV. By standard results $\ln|s|^2$ is locally integrable, thus both integrals are finite. This means that we can use monotone convergence to conclude that the LHS will converge to

$$\int_X (\psi - \ln|s|^2) dd^c(P(\varphi) - P(\varphi')) \wedge \mathrm{MA}_{n-1}(P(\varphi), P(\varphi'))$$

when k goes to infinity. A special case of Proposition 4.9 in [6], chapter 3, is that monotone convergence for Monge-Ampère expressions holds when one of the terms has analytic singularities and the others are locally bounded. By this it follows that the LHS will converge to

$$\int_X (P(\varphi) - P(\varphi'))dd^c(\psi - \ln|s|^2) \wedge \mathrm{MA}_{n-1}(P(\varphi), P(\varphi')),$$

and we are done. □

Assume that we have chosen our coordinates $z_1, ..., z_n$ centered at p such that

$$z_1 = 0$$

is a local equation for an irreducible variety Y. Assume also that Y is the zero-set of a holomorphic section $s \in H^0(A)$ of an ample line bundle A. Then by Theorem 4.24 in [9] the Okounkov bodies of L and $L + tA$ with respect to these coordinates are related in the following way

$$\Delta(L) = (\Delta(L + tA) - te_1) \cap (\mathbb{R}_+)^n.$$

There is also correspondence between the Chebyshev transforms of weights on L and $L + tA$.

Proposition 13.9. *Let A and s be as above. Suppose also that we have chosen the holomorphic coordinates so that $z_1 = s$ locally. Then for $a \geq r$ it holds that*

$$c_L[\psi](a,\alpha) - c_L[\varphi](a,\alpha) = \\ = c_{L-rA}[\psi - r\ln|s|^2](a-r,\alpha) - c_{L-rA}[\varphi - r\ln|s|^2](a-r,\alpha). \tag{38}$$

Proof. First assume that L is integral. Since we have that locally $s = z_1$, for $t \in H^0(kL)$,

$$t = z^{k(a,\alpha)} + \text{higher order terms},$$

if and only if

$$\frac{t}{s^{rk}} = z^{k(a-r,\alpha)} + \text{higher order terms}.$$

We also have that

$$\sup_{x\in X}\{|t(x)|^2 e^{-k\varphi(x)}\} = \sup_{x\in X}\{\frac{|t(x)|^2}{|s^{rk}(x)|^2} e^{-k(\varphi(x)-r\ln|s(x)|^2)}\}.$$

Thus (38) holds for integral L. By the homogeneity and continuity of the Chebyshev transform it will therefore hold for ample $\mathbb{R}$-divisors. □

We are now ready to state and prove our generalization of Theorem 13.4 in the ample setting, where the underlying $\mathbb{R}$-divisor is allowed to vary within the ample cone.

Theorem 13.10. *Let A_i, $i = 1, ..., m$ be a finite collection of ample line bundles, and for each i let φ_i and φ_i' be two continuous weights on A_i. Let O denote the open cone in $\mathbb{R}^d$ such that $a \in O$ iff $\sum a_i A_i$ is an ample $\mathbb{R}$-divisor. Then the function*

$$f(a) := \mathcal{E}_{\sum a_i A_i}(\sum a_i\varphi_i, \sum a_i\varphi_i')$$

is $\mathcal{C}^1$ on O.

Proof. Let a be a point in O, and denote $\sum a_i A_i$ by L. Denote $\sum a_i\varphi$ by φ and $\sum a_i\varphi_i'$ by φ'. We want to calculate the partial derivatives of F at a. Thus we let L' be an ample line bundle, let ψ and ψ' be two continuous metrics on L' and we consider the function

$$f(t) := \mathcal{E}_{L+tL'}(\varphi + t\psi, \varphi' + t\psi').$$

We claim that f is differentiable at $t = 0$, and that the derivative varies continuously with L, φ and φ'.

We may assume that L' has a non-trivial section s such that $Y := \{s = 0\}$ is a smooth manifold, since otherwise because of the homogeneity we may just as well consider some large multiple of L' instead. We choose local holomorphic coordinates such that $z_1 = s$. Recall that the Okounkov bodies of L and $L + tL'$ are related in the following way

$$\Delta(L) = (\Delta(L + tL') - te_1) \cap (\mathbb{R}_+)^n. \tag{39}$$

Let $\Delta(L)_r$ denote the fiber over r of the projection of the Okounkov body down to the first coordinate, i.e.

$$\Delta(L)_r := \Delta(L) \cap (\{r\} \times \mathbb{R}^{n-1}).$$

Then one may write equation (39) as

$$\Delta(L + tL') = \cup_{0 \leq r \leq t} \Delta(L + tL')_r \cup (\Delta(L) + te_1). \tag{40}$$

Furthermore the energy is given by integration of the Chebyshev transforms over the Okounkov bodies. Using (40) and Proposition 13.9 we get that

$$\begin{aligned}
&\mathcal{E}_{L+tL'}(\varphi + t\psi_t, \varphi' + t\psi'_t) = \\
&= n! \int_{\Delta(L+tL')^\circ} c[\varphi' + t\psi'_t] - c[\varphi + t\psi_t] d\lambda = \\
&= n! \int_{r=0}^{t} \int_{\Delta(L+tL')^\circ_r} c[\varphi' + t\psi'_t](r,\alpha) - c[\varphi + t\psi_t](r,\alpha) d\alpha dr + \\
&+ n! \int_{\Delta(L)^\circ} c[\varphi' + t(\psi'_t - \ln|s|^2)] - c[\varphi + t(\psi_t - \ln|s|^2)] dp = \\
&= n! \int_{r=0}^{t} \int_{\Delta(L+tL')^\circ_r} c[\varphi' + t\psi'_t](r,\alpha) - c[\varphi + t\psi_t](r,\alpha) d\alpha dr + \\
&+ \mathcal{E}_L(\varphi + t(\psi_t - \ln|s|^2), \varphi' + t(\psi'_t - \ln|s|^2)).
\end{aligned}$$

Hence by Theorem 13.4 and the fundamental theorem of calculus it follows that this function is right-differentiable. We also want to calculate the right-derivative.

We get that

$$\begin{aligned}
&\frac{d}{dt}_{|0+} \mathcal{E}_{L+tL'}(\varphi + t\psi_t, \varphi' + t\psi'_t) = \\
&= n! \int_{\Delta(L)^\circ_0} c[\varphi'](0,\alpha) - c[\varphi](0,\alpha) d\alpha + \\
&+ \frac{d}{dt}_{|0+} \mathcal{E}_L(\varphi + t(\psi_t - \ln|s|^2), \varphi' + t(\psi'_t - \ln|s|^2)) = \\
&= n\mathcal{E}_Y(P(\varphi')_{|Y}, P(\varphi)_{|Y}) + \frac{d}{dt}_{|0+} \mathcal{E}_L(\varphi + t(\psi_t - \ln|s|^2), \varphi' + t(\psi'_t - \ln|s|^2)),
\end{aligned}$$

using Proposition 10.11 in the last step. Since in the second term the divisor L does not change with t, we may use Theorem 13.4. Also, because of the cocycle property of the relative energy, we only need to consider two cases, one where $\varphi = \varphi'$, and the other one where we let $\varphi \neq \varphi'$ but instead assume that $\psi_t = \psi'_t = \psi$ is some fixed smooth positive metric on L'.

First assume that $\varphi=\varphi'$. The first term disappears and we get that

$$\frac{d}{dt}_{|0+}\mathcal{E}_{L+tL'}(\varphi+t\psi_t,\varphi+t\psi_t')=$$
$$=\frac{d}{dt}_{|0}\mathcal{E}_L(\varphi+t(\psi_t-\ln|s|^2),\varphi+t(\psi_t'-\ln|s|^2))=$$
$$=\int_X(\psi_0-\ln|s|^2)\mathrm{MA}(P(\varphi))-\int_X(\psi_0'-\ln|s|^2)\mathrm{MA}(P(\varphi))=$$
$$=\int_X(\psi_0-\psi_0')\mathrm{MA}(P(\varphi)). \tag{41}$$

Here we used Lemma 13.7.

By Theorem 1.6 in [1] this term depends continuously on the weight φ.

Now let $\varphi\neq\varphi'$ but instead assume that $\psi_t=\psi_t'=\psi$ is some fixed smooth positive metric on L'. Then we have that

$$\frac{d}{dt}_{|0+}\mathcal{E}_{L+tL'}(\varphi+t\psi,\varphi'+t\psi)=$$
$$=n\mathcal{E}_Y(P(\varphi)_{|Y},P(\varphi')_{|Y})+ \tag{42}$$
$$+\frac{d}{dt}_{|0}\mathcal{E}_L(\varphi+t(\psi-\ln|s|^2),\varphi'+t(\psi-\ln|s|^2))=$$
$$=n\mathcal{E}_Y(P(\varphi)_{|Y},P(\varphi')_{|Y})+\int_X(\psi-\ln|s|^2)\mathrm{MA}(P(\varphi))-$$
$$-\int_X(\psi-\ln|s|^2)\mathrm{MA}(P(\varphi'))=$$
$$=n\mathcal{E}_Y(P(\varphi)_{|Y},P(\varphi')_{|Y})+$$
$$+\int_X(\psi-\ln|s|^2)dd^c(P(\varphi)-P(\varphi'))\wedge\mathrm{MA}_{n-1}(P(\varphi),P(\varphi'))=$$
$$=\int_X(P(\varphi)-P(\varphi'))dd^c\psi\wedge\mathrm{MA}_{n-1}(P(\varphi),P(\varphi')). \tag{43}$$

In the last step we used Lemma 13.8.

This will also depend continuously on the pair (φ,φ') exactly as in Lemma 12.9.

By definition a $\mathbb{R}-$divisor can be written as a finite positive sum of ample line bundles, thus since we have shown that the relative energy is continuously partially right-differentiable in the ample integral directions it follows that the function f is right-differentible when L' is any ample $\mathbb{R}$-divisor. Since the derivatives we have calculated for ample line bundles are linear, the same formulas hold for arbitrary $\mathbb{R}$-divisors.

Now we consider the question of left-differentiability By Lemma 12.8 the relative energy is $(n+1)-$homogeneous. For some possibly large k $kL-L'$ is ample. Because of the homogeneity of the relative energy, without loss of generality, we may assume that $L-L'$ is ample, otherwise just change L to kL. Also

$$\frac{1}{1-t}(L-tL')=L+\frac{t}{1-t}(L-L').$$

Using this and the homogeneity we get that

$$\mathcal{E}_{L-tL'}(\varphi - t\psi_t, \varphi' - t\psi'_t) =$$
$$= (1-t)^{n+1}\mathcal{E}_{L+\frac{t}{1-t}(L-L')}(\varphi + \frac{t}{1-t}(\varphi - \psi_t), \varphi' + \frac{t}{1-t}(\varphi' - \psi'_t)). \tag{44}$$

The left-differentiability thus follows from the previous case by equation (44) and the chain rule.

To show the differentiability of f then, we only need to calculate the left-derivative to make sure it coincides with the right-derivative. Recall that because of the cocycle property we only needed to consider two cases. First assume that $\varphi = \varphi'$. Equations (44) and (41) now yields that

$$\frac{d}{dt}_{|0-}\mathcal{E}_{L+tL'}(\varphi + t\psi_t, \varphi + t\psi'_t) = -\frac{d}{dt}_{|0+}\mathcal{E}_{L-tL'}(\varphi - t\psi_t, \varphi - t\psi'_t) =$$
$$-\frac{d}{dt}_{|0+}(1-t)^{n+1}\mathcal{E}_{L+\frac{t}{1-t}(L-L')}(\varphi + \frac{t}{1-t}(\varphi - \psi_t), \varphi + \frac{t}{1-t}(\varphi - \psi'_t)) =$$
$$= -\frac{d}{dt}_{|0+}\mathcal{E}_{L+\frac{t}{1-t}(L-L')}(\varphi + \frac{t}{1-t}(\varphi - \psi_t), \varphi + \frac{t}{1-t}(\varphi - \psi'_t)) =$$
$$-\int_X((\varphi - \psi_0) - (\varphi - \psi'_0))\mathrm{MA}(P(\varphi)) = \int_X(\psi_0 - \psi'_0)\mathrm{MA}(P(\varphi)) =$$
$$= \frac{d}{dt}_{|0+}\mathcal{E}_{L+tL'}(\varphi + t\psi_t, \varphi + t\psi'_t).$$

Now let $\varphi \neq \varphi'$ but instead assume that $\psi_t = \psi'_t = \psi$ is some smooth positive weight on L'. By the cocycle property we may also assume that φ and $\varphi - \psi$ are smooth and positive. By equation (42) we get that

$$\frac{d}{dt}_{|0-}\mathcal{E}_{L+tL'}(\varphi + t\psi, \varphi' + t\psi) =$$
$$= -\frac{d}{dt}_{|0+}(1-t)^{n+1}\mathcal{E}_{L+\frac{t}{1-t}(L-L')}(\varphi + \frac{t}{1-t}(\varphi - \psi), \varphi' + \frac{t}{1-t}(\varphi' - \psi)) =$$
$$= (n+1)\mathcal{E}_L(\varphi.\varphi') -$$
$$-\frac{d}{dt}_{|0+}\mathcal{E}_{L+\frac{t}{1-t}(L-L')}(\varphi + \frac{t}{1-t}(\varphi - \psi), \varphi' + \frac{t}{1-t}(\varphi' - \psi)) =$$
$$= (n+1)\mathcal{E}_L(\varphi.\varphi') - \int_X(P(\varphi) - P(\varphi'))dd^c(\varphi - \psi)\wedge \mathrm{MA}_{n-1}(P(\varphi), P(\varphi')) -$$
$$-\int_X((\varphi - \psi) - (\varphi' - \psi))\mathrm{MA}(P(\varphi')) =$$
$$= \int_X(P(\varphi) - P(\varphi'))dd^c(\psi)\wedge \mathrm{MA}_{n-1}(P(\varphi), P(\varphi')) =$$
$$= \frac{d}{dt}_{|0+}\mathcal{E}_{L+tL'}(\varphi + t\psi, \varphi' + t\psi).$$

We used that $\varphi' = P(\varphi')$ a.e. with respect to $\mathrm{MA}(P(\varphi'))$ (see e.g. [1]). We also used the observation that

$$dd^c\varphi \wedge \mathrm{MA}_{n-1}(P(\varphi), P(\varphi')) + \mathrm{MA}(P(\varphi')) = \mathrm{MA}_n(P(\varphi), P(\varphi')),$$

and that by definition

$$\int (P(\varphi) - P(\varphi'))\mathrm{MA}_n(P(\varphi), P(\varphi')) = (n+1)\mathcal{E}_L(\varphi, \varphi').$$

The differentiability of f follows, and we saw that the derivative depended continuously on L, φ and φ'. Hence the function F is $\mathcal{C}^1$ on O. □

Note that in the special case where $\psi_t = \psi_0 + t\Psi$ and $\varphi_t = \varphi_0 + t\Psi$ for some fixed positive weight Ψ on L', our calculations show that

$$f'(0) = \sum_{j=0}^{n-1} \int_X (P(\psi_0) - P(\varphi_0)) dd^c\Psi \wedge (dd^c P(\psi_0))^j \wedge (dd^c P(\varphi_0))^{n-j-1}.$$

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