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Abstract

[1] The main purpose of this article is to establish the CKN-type inequalities for all $\alpha \in \mathbb{R}$ and to study the relating matters systematically. Roughly speaking, we shall discuss about the characterizations of the CKN-type inequalities for all $\alpha \in \mathbb{R}$ as the variational problems, the existence and nonexistence of the extremal solutions to these variational problems in proper spaces, the exact values and the assymptotic behaviors of the best constants $S(p,q,\alpha,\beta,n)$ and $C(p,q,\alpha,\beta,n)$, and so on.

In the noncritical case ($\alpha \neq 1-\frac{n}{p}$) we shall establish the CKN-type inequalities represented by

$$\int_{\mathbb{R}^n} |\nabla u|^p |x|^{\alpha p}\, dx \geq S(p,q,\alpha,\beta,n)\left(\int_{\mathbb{R}^n} |u|^q |x|^{\beta q}\, dx\right)^{p/q}$$

for any $u \in W^{1,p}_{\alpha,0}(\mathbb{R}^n)$ with $\nabla u = (\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \ldots, \frac{\partial u}{\partial x_n})$. Here $n \geq 1$, $1 \leq p < +\infty$ and q, α, β are real numbers satisfying the noncritical relation (NCR) given by Definition 3.1.
On the other hand in the critical case ($\alpha = 1-\frac{n}{p}$), the CKN-type inequalities become

$$\int_{B_1} |\nabla u|^p |x|^{p-n}\, dx \geq C(p,q,\alpha,\beta,n)\left(\int_{B_1} \frac{|u|^q}{|x|^n A_1(|x|)^{q+1-\frac{q}{p}}}\, dx\right)^{p/q}$$

for any $u \in W^{1,p}_{\alpha,0}(B_1)$. Here B_1 is a unit ball having its center at the origin, $A_1(t) = \log\frac{R}{t}$ for $R > 1$ if $n \geq 2$, and $A_1(t) = \log\frac{1}{t}$ if $n = 1$, and the parameters should obey the critical relation (CR) given by Definition 3.2.

In these CKN-type inequalities, the presence of weight functions in the both sides prevents us from employing effectively the so-called spherically symmetric rearrangement. Further the invariance of $\mathbb{R}^n$ by the group of dilatations creates some possible loss of compactness. As a result we will see that the existence of extremals and the values of best constants $S(p,q,\alpha,\beta,n)$ and $C(p,q,\alpha,\beta,n)$ and their asymptotic behaviors essentially depend upon the relations among parameters in the inequality.

1 INTRODUCTION

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Key words and phrases. CKN-type inequality, Hardy-Sobolev inequality, Weighted Hardy inequality, Degenerate elliptic equation, Best constant.

We shall begin with recalling the classical weighted Sobolev inequalities (1.1), which are often called the Caffarelli-Kohn-Nirenberg type (the CKN-type inequalities).

There is a positive number $S(p,q,\alpha,\beta,n)$ depending only on p,q,α,β and n such that for any $u \in C_0^\infty(\mathbb{R}^n)$, we have

$$\int_{\mathbb{R}^n} |\nabla u|^p |x|^{\alpha p}\, dx \geq S(p,q,\alpha,\beta,n) \left(\int_{\mathbb{R}^n} |u|^q |x|^{\beta q}\, dx \right)^{p/q}, \tag{1.1}$$

where $\nabla u = (\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \cdots \frac{\partial u}{\partial x_n})$ and $|\nabla u| = \left(\sum_{k=1}^n |\frac{\partial u}{\partial x_k}|^2\right)^{1/2}$.
Here $n \geq 1$, $1 \leq p < +\infty$ and q, α, β are real numbers satisfying

$$\begin{cases} \alpha > 1 - \frac{n}{p}, \\ (1-\alpha+\beta)p < n, \\ 0 \leq 1/p - 1/q = (1-\alpha+\beta)/n, \\ \beta \leq \alpha. \end{cases} \tag{1.2}$$

The main purpose of this article is not only to establish the CKN-type inequalities for all $\alpha \in \mathbb{R}$ but also to study the relating matters systematically. To this end we shall classify the CKN-type inequalities according to the range of the parameter α into the three cases. Namely

Definition 1.1 *The parameter α is said to be subcritical, critical and supercritical if α satisfies $\alpha > 1 - \frac{n}{p}$, $\alpha = 1 - \frac{n}{p}$ and $\alpha < 1 - \frac{n}{p}$ respectively.*

In the classical CKN-type inequalities (1.1), the condition $\beta q > -n$ follows from the conditions (1.2) and hence the weight functions in the both sides are locally integrable on $\mathbb{R}^n$. By this reason these inequalities are classified into the subcritical case of the CKN-type inequalities in this article.

Before we go further into the detail, we give a brief historical review here. As we have already mentioned, these inequalities are the so-called Caffarelli-Kohn-Nirenberg type (the CKN-type inequalities). In fact in [CKN] they established general multiplicative inequalities including these types. In [Ho1] we have also studied these inequalities among more general imbedding theorems on the weighted Sobolev spaces, where the weights are powers of distance from a given closed set F. According to " the smoothness " of the closed set F, we have established various imbedding properties among the weighted Sobolev and Schauder spaces, with a result we have learned that the existence of imbedding operators are essentially dependent upon the asymptotic behavior as $\eta \to 0$ of the tubular neighborhood $F_\eta = \{x \in \mathbb{R}^n : distance(x,F) \leq \eta\}$ of the given closed F. It was also very interesting for us to study further the properties of the imbedding operators obtained there. But for a general F it seemed not easy to study these problems in a detailed way. By this reason, in [Ho2] we restricted ourselves on the simplest case that F consists of a single point, namely, the origin. In this particular case we have studied the relating problems in a various aspect and obtained interesting results such as the exact values of the best constant $S(p,q,\alpha,\beta,n)$ in certain cases, the existence and nonexistence of the extremals and so on.

Recently we have revisited the weighted Hardy-Sobolev inequality in [AH2]. It is easy to see that the classical CKN-type inequalities coincide with them if $\beta = \alpha - 1$. To our surprise it was shown that the weighted Hardy-Sobolev inequalities themselves hold for all $\alpha \in \mathbb{R}$ with some modifications. In fact, even if $\alpha = 1 - \frac{n}{p}$ holds, the sharp inequality of the Hardy type remains valid as long as the whole space $\mathbb{R}^n$ is replaced by a bounded domain containing the origin and the weight functions in the right hand side are replaced by the logarismic ones. Moreover we have successfully improved those weighted Hardy-Sobolev inequalities by finding out sharp missing terms, as a result they turned out to be very useful in many aspects. For the improved inequalities, see Proposition 1.2 below. (For the complete argument and the related applications see [AH2].)

On the other hand, the counterpart in the CKN-type inequalities to the weighted Hardy-Sobolev inequalities in [AH2] seems to be unknown so far. But it seems reasonable for us to expect that the CKN-type inequalities should remain valid for all $\alpha \in \mathbb{R}$ with a similar modification as was performed in the weighted Hardy-Sobolev inequalities. In this spirit we shall establish the CKN type inequalities for all $\alpha \in \mathbb{R}$ and we shall further study them systematically in the present paper. Roughly speaking, we shall discuss about the characterizations of the imbeddings as the variational problems, the existence and nonexistence of the extremal solutions to these variational problems in proper spaces, the exact values and the asymptotic behaviors of the best constants $S(p, q, \alpha, \beta, n)$ and $C(p, q, \alpha, \beta, n)$.

Now we shall explain more precisely the results in the present article. To this end let us introduce the equivalent variational problems to the CKN-type inequalities in both the noncritical and the critical cases.

First in the noncritical case, which consists of the subcritical case ($\alpha > 1 - \frac{n}{p}$) and the supercritical case ($\alpha < 1 - \frac{n}{p}$), the quantity $S(p, q, \alpha, \beta, n)$ in (1.1) is the best constant given by the infimum of

$$E_{p,q,\alpha,\beta,n}(u) = \frac{\int_{\mathbb{R}^n} |\nabla u|^p |x|^{\alpha p} \, dx}{\left(\int_{\mathbb{R}^n} |u|^q |x|^{\beta q} \, dx\right)^{p/q}} \qquad \text{for } u \in W^{1,p}_{\alpha,0}(\mathbb{R}^n) \setminus \{0\}. \tag{1.3}$$

Here the parameters (p, q, α, β, n) are assumed to satisfy the condition (3.1), which is more general than the condition (1.2), and the admissible function spaces $W^{1,p}_{\alpha,0}(\mathbb{R}^n)$ and $L^q(\mathbb{R}^n, |x|^{\beta q})$ are defined in §2. Then it follows from the classical CKN-type inequalities that $S(p, q, \alpha, \beta, n)$ is positive in the subcritical case. (See also [Ho2]). It is worthy mentioning here that there exists an isometric transformation T between the subcritical weighted Sobolev spaces and the supercritical ones such that

$$\begin{cases} T : W^{1,p}_{\alpha,0}(\mathbb{R}^n) \to W^{1,p}_{\overline{\alpha},0}(\mathbb{R}^n) \\ T : L^q(\mathbb{R}^n, |x|^{\beta q}) \to L^q(\mathbb{R}^n, |x|^{\underline{\beta} q}), \end{cases} \tag{1.4}$$

and for any $u \in W^{1,p}_{\alpha,0}(\mathbb{R}^n) \subset L^q(\mathbb{R}^n, |x|^{\beta q})$

$$\begin{cases} ||u||_{W^{1,p}_{\alpha,0}(\mathbb{R}^n)} = ||Tu||_{W^{1,p}_{\overline{\alpha},0}(\mathbb{R}^n)}, \\ ||u||_{L^q(\mathbb{R}^n, |x|^{\beta q})} = ||Tu||_{L^q(\mathbb{R}^n, |x|^{\underline{\beta} q})}, \end{cases} \tag{1.5}$$

where the parameters (p, q, α, β, n) satisfy the subcritical relation (1.2) and the parameters $\overline{\alpha}$ and $\underline{\beta}$ are defined by

$$\overline{\alpha} = 2 - \alpha - \frac{2n}{p} \quad \text{and} \quad \underline{\beta} = -\beta - \frac{2n}{q} \tag{1.6}$$

respectively. Then the parameters $(p, q, \overline{\alpha}, \underline{\beta}, n)$ obviously satisfy the supercritical relation (3.1) with $\alpha = \overline{\alpha} < 1 - \frac{n}{p}$ and $\beta = \underline{\beta}$ from the subcriticality of the parameters (p, q, α, β, n). A reflection map with respect to the unit sphere will define this isometry, which is often called an inversion map. Therefore we immediately see that

$$S(p,q,\alpha,\beta,n) = S(p,q,\overline{\alpha},\underline{\beta},n) \tag{1.7}$$

holds. In view of this fact we mainly explain our results assuming that $\alpha \geq 1-\frac{n}{p}$ from now on. For the reader's convenience we describe not only new results but also some basic known results as the background.

Under the noncritical relation (NCR) with $\alpha > 1-\frac{n}{p}$ on the parameters, let us start from explaining the variational problem in the radial weighted Sobolev space $R^{1,p}_{\alpha,0}(\mathbb{R}^n)$ defind in Definition 2.5. By $S_{rad}(p,q,\alpha,\beta,n)$ we denote the best constant when the infimum of $E_{p,q,\alpha,\beta,n}(u)$ is taken over the radial weighted Sobolev space $R^{1,p}_{\alpha,0}(\mathbb{R}^n)$ which is a proper subspace of $W^{1,p}_{\alpha,0}(\mathbb{R}^n)$. In [Ho2] the best constant $S_{rad}(p,q,\alpha,\beta,n)$ was exactly determined and it was shown that $S_{rad}(p,q,\alpha,\beta,n)$ is always attainted in $R^{1,p}_{\alpha,0}(\mathbb{R}^n)$ provided that $\alpha - 1 < \beta \leq \alpha$. (See Theorem 3.2 for the precise result including the supercritical case.) Moreover we showed that

$$S_{rad}(p,q,\alpha,\beta,n) = S(p,q,\alpha,\beta,n), \quad \text{if } \alpha \leq 0. \tag{1.8}$$

In Theorem 3.6, we shall extend this result for the case that $\max(0, 1-\frac{n}{p}) < \alpha \leq 1-\frac{1}{p}$ with some additional assumptions on α and β by the aid of the rearrangement argument in Appendix 3.

Now we proceed to studying the CKN-type inequalities in the whole space $W^{1,p}_{\alpha,0}(\mathbb{R}^n)$. First we assume that $\alpha - 1 < \beta < \alpha$. Then we shall see that the best constant $S(p,q,\alpha,\beta,n)$ is attained by some elements in $W^{1,p}_{\alpha,0}(\mathbb{R}^n)$. In the study of the existence and nonexistence of extremals of the CKN-type inequalities, as in the study of the Sobolev inequality, the most of difficulty come from the lack of the compactness of the imbedding operators. However, our aim is accomplished by using the multiplicative inequality, the sophisticated compactness Lemma 4.11 and the sharp Fatou's lemma. Moreover we shall give the asymptotic behavior of the best constant as $|\alpha| \to \infty$ and the symmetricity of the extremals. When $p = 2$ and $\alpha > 1-\frac{n}{2}$, these topics were treated by F. Catrina and Z. Wang in [CW1]. They studied the inequality (1.1) with $p = 2$ and $\alpha > 1-\frac{n}{2}$ intensively and obtained interesting results (See also [Ho2] and [CW2]).

If $\beta = \alpha - 1$ in (1.2), then we have $p = q$. Hence in this case the variational problem becomes equivalent to the weighted Hardy-Sobolev inequality, which was already studied in [AH2]. In particular the best constant $S(p,q,\alpha,\alpha-1,n)$ is not attained in the admissible function space $W^{1,p}_{\alpha,0}(\mathbb{R}^n)$. Nevertheless we shall see that the best constant $S(p,q,\alpha,\beta,n)$ is a continuous function on the parameters satisfying the condition (3.1), and as a result it holds that

$$\lim_{\beta \to \alpha-1+0} S(p,q,\alpha,\beta,n) = \Lambda_{n,p,\alpha}, \tag{1.9}$$

where $\Lambda_{n,p,\alpha}$ is the best constant for the weighted Hardy-Sobolev inequality given by (1.17).

If $\alpha = \beta$ in (1.2) holds, then the parameter q becomes the usual critical Sobolev exponent $p^* = \frac{np}{n-p}$ with $1 < p < n$. Then we shall prove the best constant $S(p,q,\alpha,\alpha,n)$ for any $\alpha \geq 0$ satisfies the estimate

$$l(p,\alpha,n) \cdot S(p,p^*,0,0,n) \leq S(p,p^*,\alpha,\alpha,n) \leq S(p,p^*,0,0,n), \tag{1.10}$$

where $l(p,\alpha,n)$ is a positive number defined by (3.12) . If $p=2$ is assumed, then it was already shown in [Ho2] that the best constant satisfies

$$S(2,2^*,\alpha,\alpha,n)=S(2,2^*,0,0,n) \quad \text{for any } \alpha\geq 0 \tag{1.11}$$

and that there is no extremal for the variational problem unless $\alpha=0$, by using the improved weighted Sobolev inequality with a sharp missing term. Here we note that when $p=2$ is assumed, the Euler-Lagrange equations of the related variational problem become linear. Hence we could establish these results making the best of the linearity in spite of their degeneracy at the origin. In the present work we also give a simple alternative proof of theses facts in §4. (See also Remark (4.6).)
On the other hand, if $\alpha<0$, then it follows from (1.8) and the exact value $S_{rad}(p,p^*,\alpha,\alpha,n)$ that

$$S_{rad}(p,p^*,\alpha,\alpha,n)=S(p,p^*,\alpha,\alpha,n)<S(p,p^*,0,0,n). \tag{1.12}$$

Let us proceed to the the critical case ($\alpha=1-\frac{n}{p}$). Then the best constant $C(p,q,\alpha,\beta,n)$ is given by the infimum of the next quantity.

$$E^c_{p,q,\alpha,\beta,n}(u)=\frac{\int_{B_1}|\nabla u|^p|x|^{p-n}\,dx}{\left(\int_{B_1}\frac{|u|^q|x|^{-n}}{A_1(|x|)^{q+1-\frac{q}{p}}}\,dx\right)^{p/q}}, \qquad u\in W^{1,p}_{\alpha,0}(B_1)\setminus\{0\}, \tag{1.13}$$

where B_1 is a unit ball, $A_1(t)=\log\frac{R}{t}$ for a given $R>1$, and the parameters (p,q,α,β,n) should obey the condition (3.4). Since $\beta q=-n$ is assumed, the parameter β is not involved explicitly in the functional $E^c_{p,q,\alpha,\beta,n}(u)$, and the functional $E^c_{p,q,\alpha,\beta,n}(u)$ comes from the weighted Hardy-Sobolev inequality in the critical case (See Proposition 1.2.). Here the unit ball can be replaced by any bounded domain $\Omega\subset\mathbb{R}^n$ containing the origin, but we will see that the best constant $C(p,q,\alpha,\beta,n)$ vanishes if B_1 is replaced by the whole space $\mathbb{R}^n$.

Then we will first establish the CKN-type inequalities in the critical case ($\alpha=1-\frac{n}{p}$) by showing the positivity of the best constant $C(p,q,\alpha,\beta,n)$. This will be accomplished by using the celebrated **Nonlinear potential theory** provided that $1<p<n$. On the other hand, if $p\geq n$, then this idea does not work effectively because the weight functions are too singular to belong to the Muchenhoupt A_p class. The key argument in this case is to develop the theory of the spherically symmetric decreasing rearrangement with respect to a measure having a spherically symmetric monotone weight function as its density instead of the Lebesgue measure.

Secondly we shall solve the variational problem in the radial function space as was done in the noncritical case. As a result the exact value of the best constant $C_{rad}(p,q,\alpha,\beta,n)$ in the radial space will be determined. Apart from the noncritical case, $C_{rad}(p,q,\alpha,\beta,n)$ is achieved only if $n=1$.

Thirdly we shall study the continuity of the best constant $C(p,q,\alpha,\beta,n)$ and show that

$$\lim_{\beta\to\alpha-1+0}C(p,q,\alpha,\beta,n)=\Lambda_{n,p,\alpha} \tag{1.14}$$

holds in this case as well. After all the weighted Hardy-Sobolev inequalities are naturally regarded as the limit of the CKN-type inequalities as $\beta\to\alpha-1+0$ for all $\alpha\in\mathbb{R}$.

Further we shall study the variational problem in a various aspects such as the existence and nonexistence of the extremals, the relation between $C(p,q,\alpha,\beta,n)$

and $C_{rad}(p,q,\alpha,\beta,n)$ and the behavior of $C(p,q,\alpha,\beta,n)$ as $p \to 1+0$, and we have obtained interesting results. For instance, in the critical case we can show that

$$\lim_{p\to 1+0} C(p,q,\alpha,\beta,n) = 0. \tag{1.15}$$

In the noncritical case the CKN-type inequalities remain valid for $p = 1$, and they are considered as a kind of isoperimetric inequalities with weights. Therefore this result means that the isoperimetric inequality does not hold in the critical case. Also we show that if either $p \geq n$ or $\beta = \alpha - 1$, then it holds that

$$C(p,q,\alpha,\beta,n) = C_{rad}(p,q,\alpha,\beta,n). \tag{1.16}$$

In order to emphasize the meaning of this classification of the CKN-type inequalities and our motivation in this paper, let us recall the results on the weighed Hardy-Sobolev inequalities as the necessary background. To this purpose we need the definition of the Hardy-Sobolev best constant.

Definition 1.2 (**The Hardy-Sobolev best constant**) *For $1 < p < +\infty$ we set*

$$\Lambda_{n,p,\alpha} = \begin{cases} \left|\frac{n-p+\alpha p}{p}\right|^p, & \text{if } \alpha \neq 1 - \frac{n}{p}, \\ \left(\frac{p-1}{p}\right)^p, & \text{if } \alpha = 1 - \frac{n}{p}. \end{cases} \tag{1.17}$$

We first review as Proposition 1.1 the classical weighted Hardy-Sobolev inequalities in the noncritical case, and then we also recall as Proposition 1.2 the improved weighted Hardy-Sobolev inequalities with sharp missing terms in [AH2]. It follows from these results that the weighted Hardy-Sobolev inequalities are valid for all $\alpha \in \mathbb{R}$ and Definition1.1 should be natural for us to study the CKN-type inequalities based on the (improved) weighted Hardy-Sobolev inequalities.

Proposition 1.1 *Let $n \geq 1$, $0 \in \Omega$ and Ω is a domain of $\mathbb{R}^n$. Assume that $1 < p < +\infty$ and $\alpha \neq 1 - \frac{n}{p}$. Then we have*

$$\int_\Omega |\nabla u|^p |x|^{\alpha p}\, dx \geq \Lambda_{n,p,\alpha} \int_\Omega \frac{|u(x)|^p}{|x|^p} |x|^{\alpha p}\, dx \tag{1.18}$$

for any $u \in C_0^\infty(\Omega \setminus \{0\})$.

In this inequalities (1.18), the domain Ω may be unbounded and the best constant $\Lambda_{n,p,\alpha}$ is apparently independent of the shape of domains. In particular we can put $\Omega = \mathbb{R}^n$. To state the next one we need more notations.

Definition 1.3 *For a given positive number R, we set for $t > 0$ and $k = 2, 3, \ldots$*

$$A_1(t) := \log \frac{R}{t}, \quad A_k(t) := \log A_{k-1}(t), \quad e_1 := e, \quad e_k := e^{e_{k-1}}. \tag{1.19}$$

Proposition 1.2 *Let $n \geq 1$, $0 \in \Omega$ and Ω is a bounded domain of $\mathbb{R}^n$.*

1. Subcritical case $(\alpha > 1 - \frac{n}{p},\ 1 < p < +\infty)$,

There exist $K = K(n) > 1$ and $C = C(n) > 0$ such that if $R > K \sup_\Omega |x|$ then

$$\begin{aligned} \int_\Omega |\nabla u|^p |x|^{\alpha p}\, dx \geq & \Lambda_{n,p,\alpha} \int_\Omega \frac{|u(x)|^p}{|x|^p} |x|^{\alpha p}\, dx \\ & + C \int_\Omega \frac{|u(x)|^p}{|x|^p} A_1(|x|)^{-2} |x|^{\alpha p}\, dx \end{aligned} \tag{1.20}$$

for any $u \in C_0^\infty(\Omega)$.

2. Critical case $(\alpha = 1 - \frac{n}{p},\ 1 < p < +\infty)$,

Then there exist $K = K(n) > 1$ and $C = C(n) > 0$ such that if $R > K \sup_\Omega |x|$ then

$$\begin{aligned}\int_\Omega |\nabla u|^p |x|^{\alpha p}\, dx \geq \Lambda_{n,p,\alpha} &\int_\Omega \frac{|u(x)|^p}{|x|^n} A_1(|x|)^{-p} dx \\ + C &\int_\Omega \frac{|u(x)|^p}{|x|^n} A_1(|x|)^{-p} A_2(|x|)^{-2} dx\end{aligned} \tag{1.21}$$

for any $u \in C_0^\infty(\Omega)$.

3. Supercritical case $(\alpha < 1 - \frac{n}{p},\ 1 < p < +\infty)$,

Then there exist $K = K(n) > 0$ and $C = C(n) > 0$ such that if $R > K \sup_\Omega |x|$ then

$$\begin{aligned}\int_\Omega |\nabla u|^p |x|^{\alpha p}\, dx \geq &\Lambda_{n,p,\alpha} \int_\Omega \frac{|u(x)|^p}{|x|^p} |x|^{\alpha p}\, dx \\ &+ C \int_\Omega \frac{|u(x)|^p}{|x|^p} A_1(|x|)^{-2} |x|^{\alpha p}\, dx\end{aligned} \tag{1.22}$$

for any $u \in C_0^\infty(\Omega \setminus \{0\})$.

In principle we cannot replace a bounded domain Ω by the whole space $\mathbb{R}^n$. We will explain this interesting phenomenon as Theorem 2.1 in §2 in the noncritical case $(\alpha \neq 1 - \frac{n}{p})$. In particular if $p = 2$, then the inequalities (1.20) and (1.22) do not hold for $\Omega = \mathbb{R}^n$. We also give a somewhat weaker but similar result for any $p \in (1, \infty)$ with $p \neq 2$. In the critical case $(\alpha = 1 - \frac{n}{p})$ this phenomenon is also clear because of the presence of the weight $A_1(r) = \log \frac{R}{r}$ with $R \geq K \sup_{x \in \Omega} |x|$. Moreover it follows from Proposition 2.2 that the relative p-capacity of ball $C^p(B_\varepsilon, \mathbb{R}^n; |x|^{p-n})$ is zero and hence no imbedding of the Hardy-Sobolev type is possible. For the detail, see Theorem 3.7 and its Remark 3.8.

For the sake of the convenience of readers, let us put a table of contents.

Contents

2 Function spaces and related properties

In this subsection we shall prepare the necessary function spaces and their fundamental properties.

Definition 2.1 *For $\alpha \in \mathbb{R}$, let $L^p(\Omega, |x|^{p\alpha})$ denote the space of Lebesgue measurable functions, defined on a domain Ω of $\mathbb{R}^n$, for which*

$$||u||_{L^p(\Omega,|x|^{p\alpha})} = \left(\int_\Omega |u|^p |x|^{\alpha p} \, dx \right)^{1/p} < +\infty. \tag{2.1}$$

Definition 2.2 *Let p and α satisfy $1 \le p < +\infty$ and $\alpha \ne 1 - \frac{n}{p}$. Let Ω be a domain of $\mathbb{R}^n$ such that $0 \in \Omega$. Then, by $W^{1,p}_{\alpha,0}(\Omega)$ we denote the completion of $C_0^\infty(\Omega \setminus \{0\})$ with respect to a norm defined by*

$$||u||_{W^{1,p}_{\alpha,0}(\Omega)} = ||\,|\nabla u|\,||_{L^p(\Omega,|x|^{p\alpha})} + ||u||_{L^p(\Omega,|x|^{p(\alpha-1)})}. \tag{2.2}$$

Definition 2.3 *Let p and α satisfy $1 \le p < +\infty$ and $\alpha = 1 - \frac{n}{p}$. Let Ω be a bounded domain of $\mathbb{R}^n$ such that $0 \in \Omega$ and let R be a positive number such that $R >$ diam.(Ω) (a diameter of Ω). Then, by $W^{1,p}_{\alpha,0}(\Omega)$ we denote the completion of $C_0^\infty(\Omega)$ with respect to a norm defined by*

$$||u||_{W^{1,p}_{\alpha,0}(\Omega)} = ||\,|\nabla u|\,||_{L^p(\Omega,|x|^{p\alpha})} + ||u||_{L^p(\Omega,|x|^{-n}A_1(|x|)^{-p})}. \tag{2.3}$$

Here $A_1(r) = \log \frac{R}{t}$.

Remark 2.1 *1. $L^p(\Omega, |x|^{p\alpha})$ and $W^{1,p}_{\alpha,0}(\Omega)$ become Banach spaces with the norm $|| \cdot ||_{L^p(\Omega,|x|^{p\alpha})}$ and $|| \cdot ||_{W^{1,p}_{\alpha,0}(\Omega)}$ respectively.*

2. If $\alpha > -\frac{n}{p}$, then both $C_0^\infty(\Omega)$ and $C_0^\infty(\Omega \setminus \{0\})$ are densely contained in $L^p(\Omega, |x|^{p\alpha})$. On the other hand, if $\alpha \le -\frac{n}{p}$, then only $C_0^\infty(\Omega \setminus \{0\})$ is dense in $L^p(\Omega, |x|^{p\alpha})$.

3. If $\alpha \ne 1 - \frac{n}{p}$ (the noncritical case), then by the Hardy inequalities (1.18), (1.20) and (1.22) the norm for $W^{1,p}_{\alpha,0}(\Omega)$ is equivalent to the norm

$$||\,|\nabla u|\,||_{L^p(\Omega,|x|^{p\alpha})}, \qquad \text{for any } u \in W^{1,p}_{\alpha,0}(\Omega).$$

4. If $\alpha > 1 - \frac{n}{p}$ (the subcritical case), then it is easy to see that both $C_0^\infty(\Omega)$ and $C_0^\infty(\Omega \setminus \{0\})$ are densely contained in $W^{1,p}_{\alpha,0}(\Omega)$.

5. If $\alpha < 1 - \frac{n}{p}$ (the supercritical case), then $C_0^\infty(B \setminus \{0\})$ is clearly dense in $W^{1,p}_{\alpha,0}(B)$ by the definition. $W^{1,p}_{\alpha,0}(\Omega)$, roughly speaking, consists of functions vanishing at the origin, because $|x|^{p(\alpha-1)} \notin L^1_{loc}(\Omega)$.

In the noncritical case ($\alpha \ne 1 - \frac{n}{p}$) we can show the existence of the imbedding operators by the CKN-type inequalities.

Proposition 2.1 *Let p satisfy $1 \le p < +\infty$ and let n satisfy $n \ge 1$. Then we have*

$$W^{1,p}_{\alpha,0}(\mathbb{R}^n) \subset L^q(\mathbb{R}^n, |x|^{\beta q}),$$

provided that

$$\begin{cases} \alpha \ne 1 - \frac{n}{p}, \\ (1 - \alpha + \beta)p < n, \\ 0 \le 1/p - 1/q = (1 - \alpha + \beta)/n \\ \beta \le \alpha. \end{cases} \tag{2.4}$$

Proof: Take and fix a $u \in W^{1,p}_{\alpha,0}(\mathbb{R}^n) \cap C_0^\infty(\mathbb{R}^n \setminus \{0\})$. Then by Minkowski's inequality we have

$$\begin{aligned} &\left(\int_{\mathbb{R}^n} |\nabla(|x|^\alpha u)|^p \, dx\right)^{\frac{1}{p}} \leq \\ &\leq |\alpha| \left(\int_{\mathbb{R}^n} |u|^p |x|^{(\alpha-1)p} \, dx\right)^{\frac{1}{p}} + \left(\int_{\mathbb{R}^n} |\nabla u|^p |x|^{\alpha p} \, dx\right)^{\frac{1}{p}}. \end{aligned} \tag{2.5}$$

Also by the CKN-type inequality in the subcritical case,

$$\begin{aligned} \int_{\mathbb{R}^n} |\nabla(|x|^\alpha u)|^p \, dx &\geq S(p,q,0,\beta-\alpha,n) \left(\int_{\mathbb{R}^n} ||x|^\alpha u|^q |x|^{(\beta-\alpha)q} \, dx\right)^{\frac{p}{q}} \\ &= S(p,q,0,\beta-\alpha,n) \left(\int_{\mathbb{R}^n} |u|^q |x|^{\beta q} \, dx\right)^{\frac{p}{q}}. \end{aligned}$$

Combining these inequalities we have the desired imbedding results. □

Since the space $W^{1,p}_{\alpha,0}(\Omega)$ with $\alpha = 1 - \frac{n}{p}$ plays important role in the critical case, it is helpful for us to prepare somewhat more detailed properties.

Definition 2.4 (A relative capacity of balls with the critical weight) *For $0 \leq \varepsilon < \eta \leq +\infty$, by $C^p(\overline{B_\varepsilon}, B_\eta; |x|^{p-n})$ we denote a relative capacity of a closed ball $\overline{B_\varepsilon} = \{x : |x| \leq \varepsilon\}$ with respect to an open ball $B_\eta = \{x \in \mathbb{R}^n : |x| < \eta\}$ given by*

$$\inf \left[\int_{B_\eta} |\nabla u|^p |x|^{p-n} \, dx : u \in C_0^\infty(B_\eta), u \geq 1 \text{ on } B_\varepsilon\right] \tag{2.6}$$

First we give the weighted relative p-capacity of balls with centers at the origin.

Proposition 2.2 *Assume that $1 < p$ and $0 < \varepsilon < \eta \leq \infty$, then it holds that*

$$C^p(\overline{B_\varepsilon}, B_\eta; |x|^{p-n}) \leq \omega_n \left(\log \frac{\eta}{\varepsilon}\right)^{1-p}. \tag{2.7}$$

In particular we have

$$\begin{cases} C^p(B_\varepsilon, \mathbb{R}^n; |x|^{p-n}) = 0, \\ C^p(\{0\}, B_\eta; |x|^{p-n}) = 0. \end{cases} \tag{2.8}$$

Proof: Let us set

$$\varphi_{\varepsilon,\eta}(r) = \begin{cases} 1, & 0 \leq r \leq \varepsilon, \\ \frac{\log \frac{\eta}{r}}{\log \frac{\eta}{\varepsilon}}, & \varepsilon < r < \eta, \\ 0, & \eta \leq r. \end{cases} \tag{2.9}$$

Then we see that

$$|\partial_r \varphi_{\varepsilon.\eta}(r)| \leq \begin{cases} 0, & 0 < r < \varepsilon, \\ \frac{1}{r \log \frac{\eta}{\varepsilon}}, & \varepsilon < r < \eta, \\ 0, & \eta \leq r. \end{cases} \tag{2.10}$$

Then we have

$$\begin{aligned} &C^p(\overline{B_\varepsilon}, B_\eta; |x|^{p-n}) \\ &\leq \omega_n \left(\log \frac{\eta}{\varepsilon}\right)^{-p} \int_\varepsilon^\eta \frac{1}{r} \, dr \\ &\leq \left(\log \frac{\eta}{\varepsilon}\right)^{1-p} \end{aligned}$$

Since it is an easy task to approximate $\varphi_{\varepsilon,\eta}$ by a sequence of functions $\varphi^m_{\varepsilon,\eta}$ in $C_0^\infty(B_1)$ such as $\varphi^m_{\varepsilon,\eta} = 1$ for $r \in [0,\varepsilon]$ and 0 for $r \in [\eta, +\infty)$, the rest of the proof is now clear. □

Secondly we give the density properties, one of which seems not to be trivial.

Proposition 2.3 *Let p satisfy $1 < p < +\infty$ and let n satisfy $n \geq 1$. Assume that $\alpha = 1 - \frac{n}{p}$ (the critical case). Then we have the followings:*

1. *It holds that $W^{1,p}_{\alpha,0}(B) \subset L^p(B, |x|^{-n}A_1(|x|)^{-p})$, where B is an arbitrary ball with a center being the origin.*

2. *Both $C_0^\infty(B)$ and $C_0^\infty(B \setminus \{0\})$ are densely contained in $W^{1,p}_{\alpha,0}(B)$.*

Remark 2.2 *We should recall that in $W^{1,p}_{\alpha,0}(B)$, $C_0^\infty(B)$ and $C_0^\infty(B \setminus \{0\})$ are dense if $\alpha > 1 - \frac{n}{p}$ and $C_0^\infty(B \setminus \{0\})$ is dense if $\alpha < 1 - \frac{n}{p}$.*

Proof: 1. This property follows from the Hardy inequality of the critical case (1.21).
2. This is a direct consequence of Proposition 2.2. To see this, let us set for a sufficiently small $\varepsilon > 0$ and $1 < m \in \mathbb{N}$

$$\psi^m_\eta(r) = 1 - \varphi_{\frac{\eta}{2^m},\eta}(r). \tag{2.11}$$

Here $\varphi_{\varepsilon,\eta}(r)$ is defined by (2.9), and we assume that $\varphi_{\varepsilon,\eta}(r)$ is smooth without a loss of generality. By Minkowski's inequality we have for any $u \in C_0^\infty(B_1)$

$$\left(\int_{B_1} |\nabla(u(\psi^m_\eta - 1))|^p |x|^{p-n}\, dx\right)^{\frac{1}{p}} \leq \tag{2.12}$$
$$\left(\int_{B_1} |\nabla u|^p (\psi^m_\eta - 1)^p |x|^{p-n}\, dx\right)^{\frac{1}{p}} + \left(\int_{B_1} |\nabla \psi^m_\eta|^p |u|^p |x|^{p-n}\, dx\right)^{\frac{1}{p}}$$

Note that for any $1 < m \in \mathbb{N}$ we have

$$|\psi^m_\eta - 1| \leq \chi_{[0,\eta]}(r), \qquad \text{for } r \geq 0. \tag{2.13}$$

where $\chi_S(r)$ is a characteristic function of the set $S \subset \mathbb{R}$. Since $\psi^m_\eta \leq 1$ and $\psi^m_\eta \to 1$ (a.e.in x uniformly in m) as $\eta \to 0$, from the dominated convergence theorem we see that the first term in the second line goes to 0 as $\eta \to 0$.

As for the second term we see

$$\begin{aligned}\int_{B_1} |\nabla \psi^m_\varepsilon|^p |u|^p |x|^{p-n}\, dx &\leq \omega_n \sup|u| \int_{\frac{\eta}{2^m}}^{\eta} \left(\log \frac{\eta}{\eta 2^{-m}}\right)^{-p} \frac{dr}{r} \\ &= \omega_n \sup|u| m^{1-p} (\log 2)^{1-p} \\ &= C m^{1-p},\end{aligned}$$

where C is a positive number independent of each m. Hence we see that the second term goes to 0 as $m \to \infty$. The assertion is proved. □

We also define a Banach space of radial functions.

Definition 2.5 *For a ball $B_r = \{x \in \mathbb{R}^n : |x| < r\}$ $(0 < r \leq +\infty)$, we set*

$$\begin{cases} R^{1,p}_{\alpha,0}(B_r) = \{u \in W^{1,p}_{\alpha,0}(B_r) : u \textit{ is a radial function }\}, \\ ||u||_{R^{1,p}_{\alpha,0}(B_r)} = ||u||_{W^{1,p}_{\alpha,0}(B_r)}. \end{cases} \tag{2.14}$$

When $p = q$ holds, the CKN-type inequalities are reduced to the Hardy-Sobolev inequality given by Proposition 1.2. It seems to be worth mentioning that the inequalities in Proposition 1.2 do not hold in general if Ω is replaced by the whole domain $\mathbb{R}^n$. In fact we have the following result whose proof will be given in the Appendix 1.

Theorem 2.1 *Let $n \geq 1$, $\Omega = \mathbb{R}^n$ and $\alpha \neq 1 - \frac{n}{p}$.*

1. *Assume that $p = 2$. Then the inequalities (1.20) and (1.22) in Proposition 1.2 do not hold any more.*
2. *Assume that $1 < p < +\infty$. Assume that f is a locally integrable nonnegative function f on $\mathbb{R}^n$ which is not identically 0. Then we have*

$$\inf_{u \in C_0^\infty(\mathbb{R}^n \setminus \{0\}) \setminus \{0\}} \frac{\left(\int_{\mathbb{R}^n} |\nabla u|^p |x|^{\alpha p}\, dx\right)^{\frac{1}{p}} - \Lambda_{n,p,\alpha}^{\frac{1}{p}} \left(\int_{\mathbb{R}^n} |u|^p |x|^{p(\alpha-1)}\, dx\right)^{\frac{1}{p}}}{\left(\int_{\mathbb{R}^n} |u|^p f(x)\, dx\right)^{\frac{1}{p}}} = 0.$$

Remark 2.3 *If $\alpha = 1 - \frac{n}{p}$ (the critical case) and $\Omega = \mathbb{R}^n$, then by the definition of weight function $A_1(x)$ the weighted Hardy inequality does not holds. Moreover it follows from Proposition 2.2 that for any compact set $K \in \mathbb{R}^n$ we have*

$$\inf \left[\int_{\mathbb{R}^n} |\nabla u|^p |x|^{p-n}\, dx : u \in C_0^\infty(\mathbb{R}^n), u \geq 1 \text{ on } K \right] = 0.$$

3 Main Results

3.1 Preliminaries

We shall study the following variational problems. Let p satisfy $1 \leq p < +\infty$ and let n satisfy $n \geq 1$. First we consider the noncritical case($\alpha \neq 1 - \frac{n}{p}$).

Definition 3.1 (The noncritical relation (NCR)) *The parameters p, q, n, α and β are said to satisfy the noncritical relation (NCR) if they satisfy*

$$\begin{cases} \alpha \neq 1 - \frac{n}{p}, \\ (1 - \alpha + \beta)p < n, \\ 0 \leq 1/p - 1/q = (1 - \alpha + \beta)/n, \\ \beta \leq \alpha. \end{cases} \tag{3.1}$$

We remark that under the condition (NCR) β satisfies the conditions

$$\beta = \alpha - 1 + n\left(\frac{1}{p} - \frac{1}{q}\right) \geq \alpha - 1. \tag{3.2}$$

and

$$\beta = \alpha - \left(1 - \frac{n}{p}\right) - \frac{n}{q} \neq -\frac{n}{q}, \tag{3.3}$$

Hence we have $\alpha - 1 \leq \beta \leq \alpha$, and $|x|^{q\beta} \in L^1_{loc}(\mathbb{R}^n)$ if and only if $\alpha > 1 - \frac{n}{p}$ (the subcritical case) is satisfied. We also note that the second condition in (3.1) can be replaced by $q < +\infty$.

Definition 3.2 (The critical relation (CR)) *The parameters p, q, n, α and β are said to satisfy the critical relation (CR) if they satisfy*

$$\begin{cases} \alpha = 1 - \frac{n}{p}, \\ q\beta = -n, \\ p \le q, \\ \beta \le \alpha. \end{cases} \tag{3.4}$$

From the first and second conditions α and q are completely determined by n, p and $\beta(< 0)$. Therefore β or q is the only living parameter in the critical case, and from the remaining conditions they must satisfy

$$\begin{cases} -\frac{n}{p} \le \beta \le 1 - \frac{n}{p}, \quad p \le q \le \frac{np}{n-p} & \text{if } 1 \le p < n, \\ -\frac{n}{p} \le \beta < 0, \quad p \le q < +\infty, & \text{if } p \ge n. \end{cases} \tag{3.5}$$

Let us define

Definition 3.3 *For the noncritical case $\left(\alpha \ne 1 - \frac{n}{p}\right)$ we set*

$$E_{p,q,\alpha,\beta,n}(u) = \frac{\int_{\mathbb{R}^n} |\nabla u|^p |x|^{p\alpha}\, dx}{\left(\int_{\mathbb{R}^n} |u|^q |x|^{\beta q}\, dx\right)^{p/q}}. \quad \left(u \in W^{1,p}_{\alpha,0}(\mathbb{R}^n) \setminus \{0\}\right) \tag{3.6}$$

For the critical case $\left(\alpha = 1 - \frac{n}{p}\right)$ we set

$$E^c_{p,q,\alpha,\beta,n}(u) = \frac{\int_{B_1} |\nabla u|^p |x|^{p-n}\, dx}{\left(\int_{B_1} \frac{|u|^q |x|^{-n}}{A_1(|x|)^{q+1-\frac{q}{p}}}\, dx\right)^{p/q}}. \quad \left(u \in W^{1,p}_{\alpha,0}(B_1) \setminus \{0\}\right) \tag{3.7}$$

Here $A_1(t) = \log \frac{R}{t}$ for a given $R > 1$ and ω_n is the area of $n-1$-dimensional unit sphere.

Remark 3.1 *The functional $E_{p,q,\alpha,\beta,n}(u)$ is clearly invariant under dilations $u_\lambda(x) = u(\lambda x)$. On the other hand the functioanal $E^c_{p,q,\alpha,\beta,n}(u)$ is invariant under dilations $u_\lambda(x) = u(\lambda x)$ up to a constant R in $A_1(|x|)$.*

Definition 3.4 *Under the condition (NCR) we set*

$$S(p,q,\alpha,\beta,n) = \inf_{u \in W^{1,p}_{\alpha,0}(\mathbb{R}^n)\setminus\{0\}} E_{p,q,\alpha,\beta,n}(u) \qquad (P)$$

and

$$S_{rad}(p,q,\alpha,\beta,n) = \inf_{u \in R^{1,p}_{\alpha,0}(\mathbb{R}^n)\setminus\{0\}} E_{p,q,\alpha,\beta,n}(u) \qquad (P_R)$$

Here we remark that in (P_R) we need not assume the condition $\beta \le \alpha$.

Secondly we consider the critical case.

Remark 3.2 *It seems to be worth mentioning that the equality signs in (P) and (P_R) can not be achieved by any function with compact support. For the detail see the Appendix 2.*

Definition 3.5 *Under the condition (CR) we set*

$$C(p,q,\alpha,\beta,n) = \inf_{u \in W^{1,p}_{\alpha,0}(B_1)\setminus\{0\}} E^c_{p,q,\alpha,\beta,n}(u). \qquad (P^c)$$

$$C_{rad}(p,q,\alpha,\beta,n) = \inf_{u \in R^{1,p}_{\alpha,0}(B_1)\setminus\{0\}} E^c_{p,q,\alpha,\beta,n}(u). \qquad (P^c_R)$$

Lastly we prepare the following notation.

Definition 3.6 *Let p satisfy $1 \le p$. Let G be a subset of $\mathbb{R}^2$ such that each points $(\alpha, \beta) \in G$ satisfy $1 - \alpha + \beta = n\left(\frac{1}{p} - \frac{1}{q}\right)$ with some $q \ge p$. Then by $G^\diamondsuit$ we denote the set $\{(\overline{\alpha}, \underline{\beta}); (\alpha, \beta) \in G\}$, where $(\overline{\alpha}, \underline{\beta})$ is given by*

$$\begin{cases} \overline{\alpha} = 2 - \alpha - \frac{2n}{p}, \\ \underline{\beta} = -\beta - \frac{2n}{q}, \end{cases} \tag{3.8}$$

where $1 - \alpha + \beta = n(\frac{1}{p} - \frac{1}{q})$.

Here we see immediately that $(\overline{\alpha}, \underline{\beta})$ is symmetric to (α, β) with respect to a point $\left(1 - \frac{n}{p}, -\frac{n}{q}\right)$ in $\mathbb{R}^2$.

3.2 Results in the noncritical case

First we consider the noncritical case($\alpha \ne 1 - \frac{n}{p}$).

Theorem 3.1 (**The imbedding results**) *Let p satisfy $1 < p < +\infty$ and let n satisfy $n \ge 1$. Assume that the parameters p, q, n, α and β satisfy the noncritical relation (NCR). Then we have $S(p, q, \alpha, \beta, n) > 0$, namely it holds that for any $u \in W^{1,p}_{\alpha,0}(\mathbb{R}^n)$,*

$$\int_{\mathbb{R}^n} |\nabla u|^p |x|^{\alpha p} \, dx \ge S(p, q, \alpha, \beta, n) \left(\int_{\mathbb{R}^n} |u|^q |x|^{\beta p} \, dx \right)^{p/q}. \tag{3.9}$$

Remark 3.3 *When $p = 1$ and $\beta > \alpha - 1$, the inequalities (3.9) remain valid, which are often called " the weighted isoperimetric inequality ".*

Theorem 3.2 (The imbedding results in the radial function space) *Let p satisfy $1 < p < +\infty$ and let n satisfy $n \ge 1$. Assume the noncritical relation (NCR) and $\alpha - 1 < \beta \le \alpha$. Then $S_{rad}(p, q, \alpha, \beta, n)$ is achieved.*

Remark 3.4 *This was accually proved in [Ho2] except for the supercritical case. The exact value is given in Lemma 4.4.*

Theorem 3.3 (**The continuity of the best constant**) *Let p satisfy $1 < p < +\infty$ and let n satisfy $n \ge 1$. Assume that the parameters p, q, n, α and β satisfy the noncritical relation (NCR). Then we have the followings.*

1. *It holds that*
$$S(p, q, \alpha, \beta, n) = S(p, q, \overline{\alpha}, \underline{\beta}, n). \tag{3.10}$$
2. *$S(p, q, \alpha, \beta, n)$ is continuous on q, α and β. In particular we have*
$$\lim_{\beta \to \alpha - 1 + 0} S(p, q, \alpha, \beta, n) = \Lambda_{n,p,\alpha}. \tag{3.11}$$

Remark 3.5 *When $p = 1$ and $\beta > \alpha - 1$, the assertion 1 remains valid as in Theorem 3.1.*

Theorem 3.4 (**Existence and Nonexistence of extremals**) *Let p satisfy $1 < p < +\infty$ and let n satisfy $n \ge 1$. Assume that the parameters p, q, n, α and β satisfy the noncritical relation (NCR). Then we have the followings.*

1. *Assume that $\alpha - 1 < \beta < \alpha$. Then the best constant $S(p, q, \alpha, \beta, n)$ is achieved by some $u \in W^{1,p}_{\alpha,0}(\mathbb{R}^n)$.*

2. *For $\beta = \alpha - 1$, $S(p,q,\alpha,\alpha-1,n) = \Lambda_{n,p,\alpha}$ holds, and $S(p,q,\alpha,\alpha-1,n)$ is not achieved.*

3. *If $p = 2$ and $n > 2$, then $S(2,2^*,\alpha,\alpha,n) = S(2,2^*,0,0,n)$ holds, and $S(2,2^*,\alpha,\alpha,n)$ is not achieved. Here $2^* = \frac{2n}{n-2}$.*

Theorem 3.5 (The asymptotic behavior of the best constant) *Let p satisfy $1 < p < n$ and let n satisfy $n \geq 1$. Assume that the parameters p, q, n, α and β satisfy the noncritical relation (NCR) and assume that either $\alpha \geq 0$ or $\alpha \leq 2(1 - \frac{n}{p})$. Then we have the followings.*

1. *For $\alpha = \beta$, there is a positive number $C(p,\alpha,n)$ such that we have*

$$l(p,\alpha,n) S\left(p,p^*,0,0,n\right) \leq S\left(p,p^*,\alpha,\alpha,n\right) \leq S\left(p,p^*,0,0,n\right).$$

Here $p^* = \frac{np}{n-p}$ and

$$l(p,\alpha,n) = \left(\frac{|n-p+\alpha p|}{|n-p+\alpha p| + |\alpha| p}\right)^p > 0 \tag{3.12}$$

2. *As $|\alpha| \to \infty$ with $\alpha - 1 < \beta \leq \alpha$, we have*

$$S(p,q,\alpha,\beta,n) \geq \left(\Lambda_{n,p,\alpha}\right)^{p(\alpha-\beta)} S(p,p^*,\alpha,\alpha,n)^{1-\alpha+\beta} \tag{3.13}$$

Remark 3.6 *We note that*

$$\lim_{|\alpha|\to\infty} l(p,\alpha,n) = \frac{1}{2^p}.$$

Theorem 3.6 (The symmetricity of the extremals) *Let p satisfy $1 < p < +\infty$ and let n satisfy $n \geq 1$.*

1. *Assume that $p < n$ and $n \geq 2$. Let A be the subset of $\mathbb{R}^2$ given by*

$$\left\{(\alpha,\beta) \in \mathbb{R}^2 ; 1 - \frac{n}{p} < \alpha \leq 0,\ \alpha - 1 \leq \beta \leq \alpha\right\}.$$

Then we have for any (p,q,α,β,n) satisfying (NCR) with $(\alpha,\beta) \in A \cup A^{\diamondsuit}$,

$$S(p,q,\alpha,\beta,n) = S_{rad}(p,q,\alpha,\beta,n).$$

2. *Assume that $n \geq 2$. Let B be the subset of $\mathbb{R}^2$ given by*

$$\left\{(\alpha,\beta) \in \mathbb{R}^2 ; \max\left(0, 1 - \frac{n}{p}\right) < \alpha \leq 1 - \frac{1}{p},\ \alpha - 1 \leq \beta \leq \frac{\alpha(n-p+\alpha p)}{n+\alpha p - np}\right\}.$$

Then we have for any (p,q,α,β,n) satisfying (NCR) with $(\alpha,\beta) \in B \cup B^{\diamondsuit}$,

$$S(p,q,\alpha,\beta,n) = S_{rad}(p,q,\alpha,\beta,n).$$

Remark 3.7 *The condition $\alpha < 1 - \frac{1}{p}$ is needed to have a nonempty set B.*

3.3 Results in the critical case

In the critical case, it follows from Proposition 2.2 that no imbedding inequality of Sobolev type holds in the whole domain $\mathbb{R}^n$. But we will have the following imbedding inequalities in a ball B_1. (See Remark 3.8 just after the theorem.)

Theorem 3.7 (The imbedding results) *Let p satisfy $1 < p < +\infty$. Assume the critical relation (CR). Then if $n \geq 2$, there exists $R > 1$ such that we have $C(p, q, \alpha, \beta, n) > 0$, namely it holds that for any $u \in W^{1,p}_{\alpha,0}(B_1)$,*

$$\int_{B_1} |\nabla u|^p |x|^{p-n}\, dx \geq C(p, q, \alpha, \beta, n) \left(\int_{B_1} \frac{|u|^q}{|x|^n A_1(|x|)^{q+1-\frac{q}{p}}}\, dx \right)^{p/q}. \quad (3.14)$$

Here $A_1(t) = \log \frac{R}{t}$ for $R > 1$ if $n \geq 2$. When $n = 1$, the same inequality holds for $A_1(t) = \log \frac{1}{t}$. Moreover the weight function of the term in the right-hand side is sharp.

Remark 3.8 *1. When $n \geq 2$ and $p \neq q$, we can not replace $R > 1$ by 1.*

2. When $p \geq n$, we shall essentially exploit the decreasing rearrangement method with respect to a positive measure. On the other hand, we shall employ the nonlinear potential theory when $1 < p < n$.

3. If B_1 is replaced by the whole domain $\mathbb{R}^n$, then the inequality does not hold. This fact is easily seen by the capacitary argument. Namely we have proved in Proposition 2.2 that $C^p(\overline{B_\varepsilon}, \mathbb{R}^n; |x|^{p-n}) = 0$ for any $\varepsilon > 0$. Therefore the desired inequality is impossible.

4. If the unit ball B_1 is replaced by any bounded domain Ω containing the origin, then the inequality remains valid with some $R > 1$, and the proof is done in a similar way.

Theorem 3.8 (The imbedding results in the radial function space) *Let p satisfy $1 < p < +\infty$ and let n satisfy $n \geq 1$. Assume the critical relation (CR). Then $C_{rad.}(p, q, \alpha, \beta, n)$ is determined. Moreover $C_{rad}(p, q, \alpha, \beta, n)$ is achieved only if $n = 1$.*

Remark 3.9 *The exact value of $C_{rad.}(p, q, \alpha, \beta, n)$ is given in Lemma 4.18. From the argument in Theorem 3.7, $C(p, q, \alpha, \beta, n) = C_{rad}(p, q, \alpha, \beta, n)$ holds provided that $p \geq n$. On the other hand, from Theorem 3.8 $C_{rad}(p, q, \alpha, \beta, n)$ is achieved only if $n = 1$ and $p > 1$.*

Theorem 3.9 (The continuity of the best constant) *Let p satisfy $1 \leq p < +\infty$ and let R satisfy $R > 1$. Assume the critical relation (CR).*

1. For $p > 1$, $C\,(p, q, \alpha, \beta, n)$ is positive and continuous on $\beta \in [\alpha - 1, \alpha]$ if $1 < p < n$, and on $\beta \in [\alpha - 1, 0)$ if $p \geq n$. In particular we have

$$\lim_{\beta \to \alpha - 1 = -\frac{n}{p}} C(p, q, \alpha, \beta, n) = \Lambda_{n,p,\alpha}. \quad (3.15)$$

2. If $p = 1$, then $C\,(p, q, \alpha, \beta, n) = 0$.

Remark 3.10 *1. It follows from Proposition 1.2 that $C(p, q, \alpha, \alpha - 1, n) = \Lambda_{n,p,\alpha}$ is not achieved because of the existence of the missing term.*

4 Proof of Theorems

4.1 Proof of Theorem 3.1

We shall restate it here for the sake of convenience.

Theorem 3.1(The imbedding results) *Let p satisfy $1 \leq p < +\infty$ and let n satisfy $n \geq 1$. Assume that the parameters p, q, n, α and β satisfy the noncritical relation (NCR). Then we have $S(p, q, \alpha, \beta, n) > 0$, namely it holds that for any $u \in W^{1,p}_{\alpha,0}(\mathbb{R}^n)$,*

$$\int_{\mathbb{R}^n} |\nabla u|^p |x|^{\alpha p}\, dx \geq S(p, q, \alpha, \beta, n) \left(\int_{\mathbb{R}^n} |u|^q |x|^{\beta p}\, dx \right)^{p/q}. \tag{4.1}$$

We shall study the following variational problems under the condition (NCR).

$$S(p, q, \alpha, \beta, n) = \inf_{u \in W^{1,p}_{\alpha,0}(\mathbb{R}^n) \setminus \{0\}} E_{p,q,\alpha,\beta,n}(u), \qquad (P)$$

where

$$E_{p,q,\alpha,\beta,n}(u) = \frac{\int_{\mathbb{R}^n} |\nabla u|^p |x|^{p\alpha}\, dx}{\left(\int_{\mathbb{R}^n} |u|^q |x|^{\beta q}\, dx \right)^{p/q}}, \qquad \left(u \in W^{1,p}_{\alpha,0}(\mathbb{R}^n) \setminus \{0\} \right). \tag{4.2}$$

We need to show that $S(p, q, \alpha, \beta, n) > 0$. If $\alpha = \beta = 0$ and $1 < p < n$, Giorgio Talenti established this result in [Ta1]. If $\alpha > 1 - \frac{n}{p}$ (the subcritical case), then from the classical CKN-type inequality in [CKN] we have the desired result $S(p, q, \alpha, \beta, n) > 0$. Therefore it suffices to study it in the supercritical case.

To this end, we will show that the reflection map $T : W^{1,p}_{\alpha,0}(\mathbb{R}^n) \to W^{1,p}_{\overline{\alpha},0}(\mathbb{R}^n)$ defined by (4.6) becomes isometry, where $\overline{\alpha} = 2 - \alpha - \frac{2n}{p}$ for $\alpha \neq 1 - \frac{n}{p}$. As a result, we will see that $S(p, q, \alpha, \beta, n) = S(p, q, \overline{\alpha}, \underline{\beta}, n)$ provided that (α, β) and $(\overline{\alpha}, \underline{\beta})$ satisfy the relation (4.7) below.

By $(u, v)_{S^{n-1}}$ for $u, v \in L^2(S^{n-1})$ we denote the inner product on $L^2(S^{n-1})$ with the measure dS_ω. By a polar coordinate system $x = (r, \omega), r > 0, \omega \in S^{n-1}$, the Laplacian Δ is represented by $r^{1-n}\partial_r(r^{n-1}\partial_r \cdot) + \frac{\Delta_{S^{n-1}}}{r^2}$. Here $\Delta_{S^{n-1}}$ is the Laplace Beltrami operator on the unit sphere. Then we have

$$|\nabla u|^2 = |\partial_r u|^2 + \frac{1}{r^2} |\Lambda u|^2 \tag{4.3}$$

Here the spherical gradient operator Λ is defined by

$$(-\Delta_{S^{n-1}} u, v)_{S^{n-1}} = (\Lambda u, \Lambda v)_{S^{n-1}} \quad \text{for any } u, v \in C^2(S^{n-1}). \tag{4.4}$$

By B_1 we denote the unit ball $\{x \in \mathbb{R}^n; |x| < 1\}$. Using a polar coordinate system, let us consider a reflection with respect to $S^{n-1} = \partial B_1$ by

$$\mathbb{R}^n \ni (r, \omega) \longrightarrow \left(\frac{1}{r}, \omega \right) \in \mathbb{R}^n, \quad (r > 0). \tag{4.5}$$

Then for $u = u((r, \omega)) \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$, we define its reflection $(Tu)((r, \omega)) \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$ with respect to S^{n-1} by

$$(Tu)((r, \omega)) = u\left(\left(\frac{1}{r}, \omega \right) \right). \tag{4.6}$$

Now we define for $\alpha, \beta \in \mathbb{R}$

$$\begin{cases} \overline{\alpha} = 2 - \alpha - \frac{2n}{p}, \\ \underline{\beta} = -\beta - \frac{2n}{q}. \end{cases} \tag{4.7}$$

Then $(\overline{\alpha}, \underline{\beta})$ is symmetric to (α, β) with respect to a point $\left(1 - \frac{n}{p}, -\frac{n}{q}\right)$ in $\mathbb{R}^2$. Moreover we see that for any $u \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$ and $v((r,\omega)) = u((\frac{1}{r}, \omega))$

$$\begin{aligned}
&\int_{\mathbb{R}^n} |\nabla u|^p |x|^{\alpha p}\, dx \\
&= \int_{S^{n-1}} dS_\omega \int_0^\infty \left(|\partial_r u|^2 + \frac{1}{r^2}|\Lambda u|^2\right)^{\frac{p}{2}} r^{n-1+p\alpha}\, dr \qquad (r = \frac{1}{\rho}) \\
&= \int_{S^{n-1}} dS_\omega \int_0^\infty \left(|(\partial_r u)\big((\frac{1}{\rho},\omega)\big)|^2 + \rho^2 |(\Lambda u)\big((\frac{1}{\rho},\omega)\big)|^2\right)^{\frac{p}{2}} \rho^{-n-1-p\alpha}\, d\rho \\
&= \int_{S^{n-1}} dS_\omega \int_0^\infty \left(|\partial_\rho v|^2 + \frac{1}{\rho^2}|\Lambda v|^2\right)^{\frac{p}{2}} \rho^{2p-n-1-\alpha p}\, d\rho \quad \left(v\big((\rho,\omega)\big) = u\big((\frac{1}{\rho},\omega)\big)\right) \\
&= \int_{\mathbb{R}^n} |\nabla_y v|^p |y|^{p(2-\alpha-\frac{2n}{p})}\, dy, \qquad \text{for } y = (\rho, \omega), \\
&= \int_{\mathbb{R}^n} |\nabla_y v|^p |y|^{p\overline{\alpha}}\, dy.
\end{aligned}$$

Also we have for $v = Tu$ and $\rho = \frac{1}{r}$

$$\begin{aligned}
\int_{\mathbb{R}^n} |u|^q |x|^{\beta q}\, dx &= \int_{S^{n-1}} dS_\omega \int_0^\infty |u|^q r^{n-1+q\beta}\, dr \\
&= \int_{S^{n-1}} dS_\omega \int_0^\infty |v|^q \rho^{-n-\beta q-1}\, d\rho \\
&= \int_{\mathbb{R}^n} |v|^q |y|^{q(-\beta-\frac{2n}{q})}\, dy, \qquad \text{for } y = (\rho,\omega), \\
&= \int_{\mathbb{R}^n} |v|^q |y|^{q\underline{\beta}}\, dy.
\end{aligned}$$

Remark 4.1 *In the Euclidian coordinate we have*

$$\begin{cases} v(y) = u\left(\frac{y}{|y|^2}\right) \quad (y \neq 0) \\ |y|^{2p} |\nabla_y v(y)|^p = |(\nabla_x u)(y)|^p, \quad \textit{for } y = \frac{x}{|x|^2} \end{cases} \tag{4.8}$$

Then by Remark 2.1 and the density argument we have the following. (Review also Remark 2.2 .)

Lemma 4.1 *Let p satisfy $1 \le p < +\infty$ and let n satisfy $n \ge 1$. Suppose that*

$$\begin{cases} \alpha < 1 - \frac{n}{p}, \\ (1 - \alpha + \beta)p < n, \\ 0 \le 1/p - 1/q = (1 - \alpha + \beta)/n \\ \beta \le \alpha. \end{cases} \tag{4.9}$$

Then we have

$$\begin{cases} \overline{\alpha} > 1 - \frac{n}{p} \\ (1 - \overline{\alpha} + \underline{\beta})p < n, \\ 0 \le 1/p - 1/q = (1 - \overline{\alpha} + \underline{\beta})/n \\ \underline{\beta} \le \overline{\alpha}. \end{cases} \tag{4.10}$$

Moreover we have the map $T : W^{1,p}_{\alpha,0}(\mathbb{R}^n) \to W^{1,p}_{\overline{\alpha},0}(\mathbb{R}^n)$ defined by (4.6) is an isometry.

Proof: By the assumptions on α and β, we see that $\beta = n(\frac{1}{p} - \frac{1}{q}) - 1 + \alpha < -\frac{n}{q}$ and hence $\underline{\beta} > -\frac{n}{q}$. In a similar way $\overline{\alpha} > 1 - \frac{n}{p}$ (the subcritical case). Moreover we see that

$$1 - \overline{\alpha} + \underline{\beta} = 1 - \alpha + \beta = n\left(\frac{1}{p} - \frac{1}{q}\right).$$

Since $\underline{\beta} = \overline{\alpha} - 1$ holds if and only if $\beta = \alpha - 1$, we have

$$||u||_{W^{1,p}_{\alpha,0}(\mathbb{R}^n)} = ||v||_{W^{1,p}_{\overline{\alpha},0}(\mathbb{R}^n)}.$$

From Remark 2.1 and Remark 2.2, $C_0^\infty(\mathbb{R}^n \setminus \{0\})$ is densely contained in $W^{1,p}_{\overline{\alpha},0}(\mathbb{R}^n)$, hence the assertion follows from the definition of the spaces. □

End of Proof of Theorem 3.1 From this lemma we see that $S(p,q,\alpha,\beta,n) = S(p,q,\overline{\alpha},\underline{\beta},n)$ provided that (p,q,α,β,n) are assumed to satisfy the noncritical relation $(\overline{\text{N}}\text{CR})$. This proves Theorem 3.1. □

4.2 Proof of Theorem 3.2

In this subsection we shall prove the following.

Theorem 3.2 (The imbedding results in the radial function space) *Let p satisfy $1 < p < +\infty$ and let n satisfy $n \geq 1$. Assume the noncritical relation (NCR) and $\alpha - 1 < \beta \leq \alpha$. Then $S_{rad}(p,q,\alpha,\beta,n)$ is achieved.*

Remark 4.2 *This was already established in [Ho2] except for the supercritical case. Since the proof contains many basic notions and useful results on the radial problems, we shall give a full proof of the theorem. We shall also consider the behavior of the best constant $S_{rad}(p,q,\alpha,\beta,n)$ as $\beta \to \alpha - 1 + 0$.*

Proof: By the symmetricity of the space $W^{1,p}_{\alpha,0}(\mathbb{R}^n)$ between the subcritical case and the supercritical case, it suffices to study the subcritical case ($\alpha > 1 - \frac{n}{p}$). Let us recall the notations. For the noncritical case $\left(\alpha \neq 1 - \frac{n}{p}\right)$ we set for $r = |x|$

$$E_{p,q,\alpha,\beta,n}(v) = \omega_n^{1-\frac{p}{q}} \frac{\int_0^\infty |\partial_r v|^p r^{p\alpha+n-1}\,dr}{\left(\int_0^\infty |v|^q r^{\beta q+n-1}\,dr\right)^{p/q}}, \ \left(v \in R^{1,p}_{\alpha,0}(\mathbb{R}^n) \setminus \{0\}\right), \tag{4.11}$$

$$S_{rad}(p,q,\alpha,\beta,n) = \inf_{v \in R^{1,p}_{\alpha,0}(\mathbb{R}^n)\setminus\{0\}} E_{p,q,\alpha,\beta,n}(v) \qquad (P_R).$$

In the problem (P_R), if we make a change of variables defined by

$$r = \rho^{1/h}, \qquad h = \frac{(1-\alpha+\beta)(n-p+p\alpha)}{n-p(1-\alpha+\beta)} > 0, \tag{4.12}$$

we get an equivalent variational problem $D(p,q,\alpha,\beta,n)$ for $v \in C^1(\mathbb{R}_+)$:

$$S_{rad}(p,q,\alpha,\beta,n) = c(p,q,k)D(p,q,\alpha,\beta,n), \tag{4.13}$$

where

$$D(p,q,\alpha,\beta,n) = \inf_{v \in R^{1,p}_{\alpha,0}(\mathbb{R}^n)\setminus\{0\}} \frac{\int_0^\infty |\partial_\rho v|^p \rho^{\frac{n}{\gamma}-1}\,d\rho}{\left(\int_0^\infty |v|^q \rho^{\frac{n}{\gamma}-1}\,d\rho\right)^{p/q}}, \tag{4.14}$$

$$\begin{cases} c(p,q,h) = \omega_n^{1-p/q} \cdot h^{p-1+p/q}, \\ \gamma = n\left(\frac{1}{p} - \frac{1}{q}\right) = 1 - \alpha + \beta. \end{cases} \tag{4.15}$$

G. Talenti using the notion of Hilbert invariant integral solved this problem. Namely it follows from Lemma 2 in [Ta1] that the infimum is achieved by functions of the form

$$v(r) = [a + b|x|^{\frac{hp}{p-1}}]^{1-\frac{n}{p\gamma}}, \qquad (a > 0, b > 0). \tag{4.16}$$

More precisely we have according to his notation that

Lemma 4.2 *(Talenti) Let m, p, q be real numbers such that*

$$1 < p < m, \quad q = \frac{mp}{m-p}.$$

Let u be any real valued function of real variable r, which is sufficiently smooth on the half-line $(0, \infty)$ (e.g. Lipschitz continuous) and which is such that

$$\int_0^\infty |u'(r)|^p r^{m-1}\, dr < +\infty, \quad u(r) \to 0 \quad \text{if } r \to +\infty.$$

Then

$$J(u) \leq J(\varphi),$$

where

$$J(u) = \frac{\left(\int_0^\infty |u|^q r^{m-1}\, dr\right)^{\frac{1}{q}}}{\left(\int_0^\infty |u'(r)|^p r^{m-1}\, dr\right)^{\frac{1}{p}}},$$

and φ is any function of the form

$$\varphi(r) = (a + br^{p'})^{1-\frac{m}{p}}$$

with a and b positive constants. An easy computation gives

$$J(\varphi) = m^{-\frac{1}{p}} \left(\frac{p-1}{m-p}\right)^{\frac{1}{p'}} \left(\frac{1}{p'} \frac{\Gamma\left(\frac{m}{p}\right)\Gamma\left(\frac{m'}{p}\right)}{\Gamma(m)}\right)^{-\frac{1}{m}},$$

here $p' = \frac{p}{p-1}$

Using this result, the best constant $S_{rad}(p, q, \alpha, \beta, n)$ is given by the next lemma.

Lemma 4.3

$$S_{rad}(p, q, \alpha, \beta, n) = c(p, q, h) J(\varphi)^{-p}$$

for $m = \frac{n}{\gamma} = \frac{pq}{q-p}$.

Then with somewhat more calculations we have the exact value of the best constant.

Lemma 4.4 *Assume that the noncritical relation (NCR) and $\alpha > 1 - \frac{n}{p}$ (The subcritical case). Then we have*

$$\begin{aligned} S_{rad}(p, q, \alpha, \beta, n) = \\ \pi^{\frac{p\gamma}{2}} \cdot n \cdot \left(\frac{n-\gamma p}{p-1}\right)^{p-1} \cdot \left(\frac{n-p+p\alpha}{n-\gamma p}\right)^{p-\frac{p\gamma}{n}} \cdot \left(\frac{2(p-1)}{\gamma p}\right)^{\frac{p\gamma}{n}} \\ \times \left\{\frac{\Gamma(n/\gamma p)\Gamma(n(p-1)/\gamma p)}{\Gamma(n/2)\Gamma(n/\gamma)}\right\}^{\frac{p\gamma}{n}}, \end{aligned} \tag{4.17}$$

where $\gamma = 1 - \alpha + \beta$. In particular if $\alpha = \beta$, then we have

$$S_{rad}(p, q, \alpha, \alpha, n) = S(p, q, 0, 0, n) \cdot \left(\frac{n-p+p\alpha}{n-p}\right)^{p-\frac{p}{n}}. \tag{4.18}$$

End of the proof of Theorem 3.2: This clearly proves the assertion of Theorem 3.2. □

When $\beta = \alpha - 1$ holds, the CKN-type inequalities coincide with the weighted Hardy-Sobolev inequalities which have been established in [AH2]. Let us check this fact shortly. When $\beta \to \alpha - 1 + 0$ or equivalently $q \to p + 0$, we have the following.

Lemma 4.5 *Assume that the noncritical relation (NCR) and $\alpha > 1 - \frac{n}{p}$ (the subcritical case). If β approach to $\alpha - 1$, then we have*

$$\lim_{\beta \to \alpha - 1 + 0} S_{rad}(p, q, \alpha, \beta, n) = \left(\frac{n - p + \alpha p}{p} \right)^p = \Lambda_{n,p,\alpha}. \tag{4.19}$$

Proof: It suffices to show that

$$\lim_{\gamma \to 0} \left\{ \frac{\Gamma(n/\gamma p)\Gamma(n(p-1)/\gamma p)}{\Gamma(n/\gamma)} \right\}^{\frac{p\gamma}{n}} = \frac{1}{p} \left(\frac{p-1}{p} \right)^{p-1}. \tag{4.20}$$

Since $\frac{n}{\gamma p} = \frac{n}{(1-\alpha+\beta)p} = \frac{q}{q-p} = \frac{m}{p}$ holds, the left-hand side equals

$$\begin{aligned} \lim_{m \to +\infty} \left(\int_0^1 t^{\frac{m}{p}-1} (1-t)^{\frac{m}{p'}-1} \, dt \right)^{\frac{p}{m}} &= \lim_{m \to +\infty} \left(\int_0^1 t^{\frac{m}{p}} (1-t)^{\frac{m}{p'}} \, dt \right)^{\frac{p}{m}} \\ &= \max_{0 \le s \le 1} t^{\frac{1}{p}} (1-t)^{\frac{1}{p'}} = \frac{1}{p} \left(\frac{p-1}{p} \right)^{p-1}. \end{aligned}$$

Thus the assertion is proved. □

In a similar way we have the following. The proof is omitted.

Lemma 4.6 *Assume that the noncritical relation (NCR) and $\alpha > 1 - \frac{n}{p}$ (the subcritical case). Then we have*

$$\lim_{p \to 1+0} S_{rad}(p, q, \alpha, \beta, n) = \pi^{\frac{\gamma}{2}} n \left(\frac{n-1+\alpha}{n-\gamma} \right)^{1-\frac{\gamma}{n}} \left(\frac{2}{n\Gamma\left(\frac{n}{2}\right)} \right)^{\frac{\gamma}{n}}. \tag{4.21}$$

The next is an easy consequence of Lemma 4.4, which will be useful later in the proof of Theorem 3.6 in §4.6.

Lemma 4.7 (1) *Assume that $1/p - 1/q = 1/n$ and $1 < p < n$. Then we have*

$$\begin{cases} S(p, q, 0, 0, n) < S_{rad}(p, q, \alpha, \alpha, n), & \text{if } \alpha > 0 \\ S(p, q, 0, 0, n) > S_{rad}(p, q, \alpha, \alpha, n), & \text{if } \alpha < 0 \end{cases} \tag{4.22}$$

(2) *Assume the noncritical relation (NCR) with $\alpha > 1 - \frac{n}{p}$ (The subcritical case). Then we have*

$$S_{rad}(p, q, \alpha, \beta, n) = l^{1-p-p/q} S_{rad}(p, q, 0, \beta - \alpha, n), \tag{4.23}$$

where $\quad l = \frac{n-p}{n-p+\alpha p}$.

4.3 Proof of Theorem 3.3

Let us recall Theorem 3.3.

Theorem 3.3 (The continuity of the best constant) *Let p satisfy $1 < p < +\infty$ and let n satisfy $n \geq 1$. Assume that the parameters p, q, n, α and β satisfy the noncritical relation (NCR). Then we have the followings.*

1. *It holds that*
$$S(p, q, \alpha, \beta, n) = S(p, q, \overline{\alpha}, \underline{\beta}, n). \tag{4.24}$$

2. *$S(p, q, \alpha, \beta, n)$ is continuous on q, α and β. In particular we have*
$$\lim_{\beta \to \alpha - 1 + 0} S(p, q, \alpha, \beta, n) = \Lambda_{n,p,\alpha}. \tag{4.25}$$

Remark 4.3 *We have proved in the previous theorem that $S_R(p, q, \alpha, \beta, n)$ has these properties.*

Proof: 1. This follows from Lemma 4.1.

2. By the assertion 1 it suffices to consider the subcritical case only. Let us recall the notations. For the subcritical case $\left(\alpha > 1 - \frac{n}{p}\right)$ we recall that

$$E_{p,q,\alpha,\beta,n}(u) = \frac{\int_{\mathbb{R}^n} |\nabla u|^p |x|^{p\alpha}\, dx}{\left(\int_{\mathbb{R}^n} |u|^q |x|^{\beta q}\, dx\right)^{p/q}}, \qquad \left(u \in W^{1,p}_{\alpha,0}(\mathbb{R}^n) \setminus \{0\}\right),$$

$$S(p, q, \alpha, \beta, n) = \inf_{v \in W^{1,p}_{\alpha,0}(\mathbb{R}^n) \setminus \{0\}} E_{p,q,\alpha,\beta,n}(u) \qquad (P).$$

In order to make clear the dependency of $S(p, q, \alpha, \beta, n)$ upon the parameters, we perform a simple change of an unknown function $u \in W^{1,p}_{\alpha,0}(\mathbb{R}^n) \cap C_0^\infty(\mathbb{R}^n \setminus \{0\})$ given by
$$u = r^{-\alpha} v, \qquad \text{for } r = |x|. \tag{4.26}$$
Then we see $v \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$ as well. We also see

$$|\nabla u|^p = r^{-\alpha p} \left|\nabla v - \alpha \frac{x}{r^2} v\right|^p = r^{-\alpha p} \left(|\nabla v|^2 - 2\alpha \frac{v}{r}\frac{dv}{dr} + \alpha^2 \frac{v^2}{r^2}\right)^{\frac{p}{2}}. \tag{4.27}$$

Hence we have

$$\left\{ \begin{array}{l} \int_{\mathbb{R}^n} |\nabla u|^p |x|^{\alpha p}\, dx = \int_{\mathbb{R}^n} \left|\nabla v - \alpha \frac{x}{r^2} v\right|^p dx. \\ \int_{\mathbb{R}^n} |u|^q |x|^{\beta q}\, dx = \int_{\mathbb{R}^n} |v|^q r^{q(\frac{n}{p} - \frac{n}{q} - 1)}\, dx. \end{array} \right. \tag{4.28}$$

We define a new norm by

$$|||v|||_\alpha = \left(\int_{\mathbb{R}^n} \left|\nabla v - \alpha \frac{x}{r^2} v\right|^p dx\right)^{\frac{1}{p}}. \tag{4.29}$$

This is clearly a norm on $C_0^\infty(\mathbb{R}^n \setminus \{0\})$. In fact, if $|||v|||_\alpha = 0$, then $\nabla(r^{-\alpha} v) = 0$. Hence v satisfies $\frac{dv}{dr} = \alpha v / r$. Thus we have $v = C r^\alpha$ for some constant C. But $C \equiv 0$, since $v \in C_0^\infty$. By $\widetilde{W}^{1,p}_{\alpha,0}(\mathbb{R}^n)$ we denote the completion of $C_0^\infty(\mathbb{R}^n \setminus \{0\})$ with respect to the norm $|||\cdot|||_\alpha$. Then $\widetilde{W}^{1,p}_{\alpha,0}(\mathbb{R}^n)$ is isometric to $W^{1,p}_{\alpha,0}(\mathbb{R}^n)$ by the linear map $F : \widetilde{W}^{1,p}_{\alpha,0} \ni v \to F(v) = v r^\alpha \in W^{1,p}_{\alpha,0}(\mathbb{R}^n)$. Then we have

$$S(p, q, \alpha, \beta, n) = \inf_{v \in \widetilde{W}^{1,p}_{\alpha,0} \setminus \{0\}} F_{p,q,\alpha,\beta,n}(v) \qquad (P').$$

Here

$$F_{p,q,\alpha,\beta,n}(v) = \frac{\int_{\mathbb{R}^n} \left|\nabla v - \alpha \frac{x}{r^2} v\right|^p \, dx}{\left(\int_{\mathbb{R}^n} |v|^q r^{q(\frac{n}{p} - \frac{n}{q} - 1)} \, dx \right)^{\frac{p}{q}}} \tag{4.30}$$

We prepare some properties of the new norm:

Lemma 4.8 *Let α satisfy $\alpha > 1 - \frac{n}{p}$.*

1. *There is a positive number C such that we have for any $v \in \widetilde{W}^{1,p}_{\alpha,0}(\mathbb{R}^n)$*

$$\left(\int_{\mathbb{R}^n} \frac{|v|^p}{|x|^p} \, dx \right)^{\frac{1}{p}} \le C|||v|||_\alpha. \tag{4.31}$$

Here C is independent of each v.

2. *For any $\alpha' \in \mathbb{R}$ and any $v \in \widetilde{W}^{1,p}_{\alpha,0}(\mathbb{R}^n)$, we have*

$$\big| |||v|||_\alpha - |||v|||_{\alpha'} \big| \le |\alpha - \alpha'| \left(\int_{\mathbb{R}^n} \frac{|v|^p}{|x|^p} \, dx \right)^{\frac{1}{p}}. \tag{4.32}$$

3. $$\left(\int_{\mathbb{R}^n} |v|^{p^*} \, dx \right)^{\frac{1}{p^*}} \le C|||v|||_0. \tag{4.33}$$

Here $p^* = \frac{np}{n-p}$ and C is independent of each v.

Proof: 1. By a transformation $v = r^\alpha u$ and the Hardy inequality, we see

$$\int_{\mathbb{R}^n} \frac{|v|^p}{|x|^p} \, dx = \int_{\mathbb{R}^n} |u|^p |x|^{(\alpha-1)p} \, dx \le C \int_{\mathbb{R}^n} |\nabla u|^p |x|^{\alpha p} \, dx = |||v|||^p_\alpha. \tag{4.34}$$

2. It suffices to prove the inequality for any $v \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$. Then, by a triangle inequality the left-hand side can be estimated from above by

$$\left(\int_{\mathbb{R}^n} \left| \left|\nabla v - \alpha \frac{x}{r^2} v\right| - \left|\nabla v - \alpha' \frac{x}{r^2} v\right| \right|^p \right)^{\frac{1}{p}} \le |\alpha - \alpha'| \left(\int_{\mathbb{R}^n} \frac{|v|^p}{|x|^p} \, dx \right)^{\frac{1}{p}}. \tag{4.35}$$

This proves the assertion.

3. Similarly we see from the Sobolev inequality that

$$\begin{aligned} \left(\int_{\mathbb{R}^n} |v|^{p^*} \, dx \right)^{\frac{1}{p^*}} &= \left(\int_{\mathbb{R}^n} |u|^{p^*} |x|^{\alpha p^*} \, dx \right)^{\frac{1}{p^*}} \\ \le C \left(\int_{\mathbb{R}^n} |\nabla(|x|^\alpha u)|^p \, dx \right)^{\frac{1}{p}} &\le C \left(\int_{\mathbb{R}^n} |\nabla v|^p \, dx \right)^{\frac{1}{p}} = C|||v|||_0. \end{aligned} \tag{4.36}$$

□

Remark 4.4 *From this lemma we see that if $v \in \widetilde{W}^{1,p}_{\alpha,0}(\mathbb{R}^n)$ for some $\alpha > 1 - \frac{n}{p}$, then $v \in \widetilde{W}^{1,p}_{\alpha',0}(\mathbb{R}^n)$ for any $\alpha' \in \mathbb{R}$. Inparticular the norm $|||v|||_\alpha$ is a Lipshitz continuous with respect to α.*

Now we show the continuity of the best constant $S(p, q, \alpha, \beta, n)$ with respect to the parameters (q, α, β) for under the condition (NCR) with $\alpha > 1 - \frac{n}{p}$.

Choose a triplet (q,α,β) and consider a sequence (q_j,α_j,β_j) satisfying (NCR) and $(q_j,\alpha_j,\beta_j)\to(q,\alpha,\beta)$ as $j\to\infty$. First we show that

$$\limsup_{j\to\infty} S(p,q_j,\alpha_j,\beta_j,n)\le S(p,q,\alpha,\beta,n). \tag{4.37}$$

For any $\varepsilon>0$ there is a $v\in C_0^\infty(\mathbb{R}^n\setminus\{0\})$ such that

$$F_{p,q,\alpha,\beta,n}(v)\le S(p,q,\alpha,\beta,n)+\varepsilon.$$

Let us set $\delta_j=\alpha_j-1+\frac{n}{p}$ for $j=1,2,\ldots$. As $(q_j,\alpha_j,\beta_j)\to(q,\alpha,\beta)$, we see $\delta_j\to\delta$ and $v^{q_j}\to v^q$ for all x. Since $v\in C_0^\infty(\mathbb{R}^n\setminus\{0\})$, we have from the dominated convergence theorem

$$\lim_{j\to\infty}\int_{\mathbb{R}^n}|v|^{q_j}r^{q_j(\frac{n}{p}-\frac{n}{q_j}-1)}\,dx=\int_{\mathbb{R}^n}|v|^q r^{q(\frac{n}{p}-\frac{n}{q}-1)}\,dx.$$

Hence for sufficiently large j we have from Lemma 4.8

$$S(p,q_j,\alpha_j,\beta_j,n)\le F_{p,q_j,\alpha_j,\beta_j,n}(v)\le F_{p,q,\alpha,\beta,n}(v)+\varepsilon\le S(p,q,\alpha,\beta,n)+2\varepsilon,$$

and this proves the assertion.

Secondly we will show that $\liminf_{j\to\infty}S(p,q_j,\alpha_j,\beta_j,n)\ge S(p,q,\alpha,\beta,n)$. On the contrary we assume that there is a sequence (q_j,α_j,β_j) such that $(q_j,\alpha_j,\beta_j)\to(q,\alpha,\beta)$ as $j\to\infty$ and

$$\lim_{j\to\infty}S(p,q_j,\alpha_j,\beta_j,n)<S(p,q,\alpha,\beta,n).$$

Then there are $\varepsilon_0>0$ and a sequence of functions $\{v_j\}\subset\widetilde{W}^{1,p}_{\alpha,0}(\mathbb{R}^n)\cap C_0^\infty(\mathbb{R}^n\setminus\{0\})$ such that

$$\int_{\mathbb{R}^n}|v_j|^{q_j}r^{q_j(\frac{n}{p}-\frac{n}{q_j}-1)}\,dx=1$$

and

$$F_{p,q_j,\alpha_j,\beta_j,n}(v_j)<S(p,q,\alpha,\beta,n)-\varepsilon_0$$

Then it follows from Lemma 4.9 below that we obtain

$$S(p,q,\alpha,\beta,n)\le F_{p,q,\alpha,\beta,n}(v_j)\le F_{p,q_j,\alpha_j,\beta_j,n}(v_j)+o(1).$$

Then we reach a contradiction and the continuity of $S(p,q,\alpha,\beta,n)$ on the parameters is now proved. □

The following is an easy variant of Lemma 3.2 in F. Catrina, Z.-Q. Wang [CW1].

Lemma 4.9 *Assume that a sequence of functions $\{v_j\}$ are uniformly bounded in $\widetilde{W}^{1,p}_{\alpha,0}(\mathbb{R}^n)$. Then we have*

$$\limsup_{j\to\infty}F_{p,q,\alpha,\beta,n}(v_j)\le\limsup_{j\to\infty}F_{p,q_j,\alpha_j,\beta_j,n}(v_j).$$

Proof: 1. Assume that $q\in(p,p^*)$ for $p^*=\frac{np}{n-p}$. In this case, there is an $\varepsilon>0$ such that we have $p<q-\varepsilon<q<q+\varepsilon<p^*$. Without a loss of generality we assume that $|q_j-q|<\varepsilon$ for all j's. From Lemma 4.8 we have

$$\lim_{j\to\infty}|||v_j|||_\alpha\le\lim_{j\to\infty}\left(|||v_j|||_{\alpha_j}+|\alpha_j-\alpha|\,|||v_j|||_\alpha\right)=\lim_{j\to\infty}|||v_j|||_{\alpha_j}.$$

Therefore it suffices to show that

$$\limsup_{j\to\infty}\int_{\mathbb{R}^n}|v_j|^{q_j}r^{q_j(\frac{n}{p}-\frac{n}{q_j}-1)}\,dx \le \limsup_{j\to\infty}\int_{\mathbb{R}^n}|v_j|^{q}r^{q(\frac{n}{p}-\frac{n}{q}-1)}\,dx$$

Let us set

$$w_j = |v_j|r^{\frac{n}{p}-1}, \qquad \text{for } j = 1,2,\ldots.$$

Then by the mean value theorem we get

$$\int_{\mathbb{R}^n}\left|w_j^{q_j}-w_j^{q}\right|\frac{dx}{r^n} \le |q_j-q|\int_{\mathbb{R}^n} w_j^{\overline{q}_j}\,|\log w_j|\,\frac{dx}{r^n}, \tag{4.38}$$

where $q-\varepsilon < \overline{q}_j < q+\varepsilon$ for $j = 1,2,\cdots$. Noting that for some positive number C we have

$$|\log t| \le C\begin{cases} t^{p^*-(q+\varepsilon)}, & \text{if } 1 < t,\\ t^{p-(q-\varepsilon)}, & \text{if } 0 < t \le 1.\end{cases} \tag{4.39}$$

Then we have

$$\begin{aligned}\int_{\mathbb{R}^n}\left|w_j^{q_j}-w_j^{q}\right|\frac{dx}{r^n} &\le C|q_j-q|\left(\int_{\{w_j>1\}} w_j^{p^*}\frac{dx}{r^n}+\int_{\{w_j\le 1\}} w_j^{p}\frac{dx}{r^n}\right)\\ &\le C|q_j-q|\left(\int_{\{w_j>1\}} v_j^{p^*}\,dx+\int_{\{w_j\le 1\}}\frac{v_j^{p}}{r^p}\,dx\right)\\ &\le C|q_j-q|\cdot(|||v_j|||_0^{p^*}+|||v_j|||_\alpha^{p}).\end{aligned}$$

If $q = p$ holds, then we use

$$\int_{\mathbb{R}^n}(w_j^{q_j}-w_j^{p})\frac{dx}{r^n} \le \int_{\{w_j\ge 1\}}(w_j^{q_j}-w_j^{p})\frac{dx}{r^n}. \tag{4.40}$$

Then by the mean value theorem we can show the assertion. If $q = p^*$ holds, we use

$$\int_{\mathbb{R}^n}(w_j^{q_j}-w_j^{p^*})\frac{dx}{r^n} \le \int_{\{w_j\le 1\}}(w_j^{q_j}-w_j^{p^*})\frac{dx}{r^n}, \tag{4.41}$$

and in a similar way we have the desired estimate. □

4.4 Proof of Theorem 3.4

We shall establish Theorem 3.4 which consists of three assertions.

Theorem 3.4(Existence and Nonexistence of extremals) *Let p satisfy $1 < p < +\infty$ and let n satisfy $n \ge 1$. Assume that the parameters p, q, n, α and β satisfy the noncritical relation (NCR).*

1. *Assume that $\alpha - 1 < \beta < \alpha$. Then the best constant $S(p,q,\alpha,\beta,n)$ is achieved by some $u \in W^{1,p}_{\alpha,0}(\mathbb{R}^n)$.*
2. *For $\beta = \alpha - 1$, $S(p,q,\alpha,\alpha-1,n) = \Lambda_{n,p,\alpha}$ holds, and $S(p,q,\alpha,\alpha-1,n)$ is not achieved.*
3. *If $p = 2$ and $n > 2$, then $S(2,2^*,\alpha,\alpha,n) = S(2,2^*,0,0,n)$ holds, and $S(2,2^*,\alpha,\alpha,n)$ is not achieved. Here $2^* = \frac{2n}{n-2}$.*

In the proof we need three lemmas, which are variants of well-known results. For the sake of the self-contained ness we will give the proofs for them.

Lemma 4.10 (A multiplicative inequality) *Let p and n satisfy $1 < p < +\infty$ and $n \geq 1$ respectively. Let (p, q, n, α, β) satisfy the noncritical relation (NCR). Then we have for any $u \in W^{1,p}_{\alpha,0}(\mathbb{R}^n)$*

$$||u||_{L^q(\mathbb{R}^n,|x|^{\beta q})} \leq ||u||^{1-\varphi}_{L^p(\mathbb{R}^n,|x|^{(\alpha-1)p})} \cdot ||u||^{\varphi}_{L^{p^*}(\mathbb{R}^n,|x|^{\alpha p^*})} \tag{4.42}$$

Here

$$p^* = \frac{np}{n-p}, \quad \varphi = \theta\frac{p^*}{q}, \quad \theta = \frac{q-p}{p^*-p}, \quad \left(1-\theta = \frac{p^*-q}{p^*-p}\right). \tag{4.43}$$

Proof: Noting that $1-\varphi = (1-\theta)\frac{q}{p}$, we see that $0 < \varphi < 1$ and

$$q = (1-\theta)p + \theta p^*. \tag{4.44}$$

Moreover it holds that

$$\beta = (1-\theta)\frac{(\alpha-1)p}{q} + \theta\frac{\alpha p^*}{q}. \tag{4.45}$$

Admitting this for the moment, we shall show the multiplicative inequality. By a Hölder inequality we immediately see that

$$\begin{aligned}
&\left(\int_{\mathbb{R}^n} |u|^q |x|^{\beta q}\, dx\right)^{\frac{1}{q}} = \left(\int_{\mathbb{R}^n} |u|^{(1-\theta)p}|x|^{(1-\theta)(\alpha-1)p}|u|^{\theta p^*}|x|^{\alpha\theta p^*}\, dx\right)^{\frac{1}{q}} \\
&\leq \left(\int_{\mathbb{R}^n} |u|^p|x|^{(\alpha-1)p}\, dx\right)^{\frac{1-\theta}{q}} \cdot \left(\int_{\mathbb{R}^n} |u|^{p^*}|x|^{\alpha p^*}\, dx\right)^{\frac{\theta}{q}} \\
&= \left(\int_{\mathbb{R}^n} |u|^p|x|^{(\alpha-1)p}\, dx\right)^{\frac{1-\varphi}{p}} \cdot \left(\int_{\mathbb{R}^n} |u|^{p^*}|x|^{\alpha p^*}\, dx\right)^{\frac{\varphi}{p^*}}.
\end{aligned}$$

This proves the assertion. Lastly we show the equality (4.45). Using (4.44) the right-hand side equals

$$\begin{aligned}
\alpha - (1-\theta)\frac{p}{q} &= \alpha - \frac{p}{q}\cdot\frac{p^*-q}{p^*-p} \\
&= \alpha - 1 + n\left(\frac{1}{p} - \frac{1}{q}\right) \\
&= \beta.
\end{aligned}$$

Thus we have the desired equality. □

Lemma 4.11 (A Compactness Lemma) *Let p and n satisfy $1 < p < +\infty$ and $n \geq 1$ respectively. Let both $(p, \tilde{q}, n, \alpha, \tilde{\beta})$ and (p, q, n, α, β) satisfy the noncritical relation (NCR). Assume that a sequence of functions $\{u_j\}_{j=1}^{\infty}$ is bounded in $W^{1,p}_{\alpha,0}(\mathbb{R}^n)$. Moreover assume that we have*

$$\sup_{y\in\mathbb{R}^n} \int_{B_{r(y)}(y)} |u_j|^{\tilde{q}}|x|^{\tilde{q}\tilde{\beta}}\, dx \to 0, \qquad \text{as } j \to \infty, \tag{4.46}$$

where $r(y) = \frac{|y|}{4}$ and $B_{r(y)}(y) = \{x \in \mathbb{R}^n : |x-y| < r(y)\}$.

Then, for any $q \in [p, p^*)$ it holds that

$$u_j \to 0 \text{ in } L^q(\mathbb{R}^n, |x|^{q\beta}), \qquad \text{as } j \to \infty.$$

Proof: We shall prove the assertion assuming that $\tilde{q} = p$ $(\beta = \alpha - 1)$ in the first place. We assume that

$$\liminf_{j\to\infty} \sup_{y\in\mathbb{R}^n} \int_{B_{r(y)}(y)} |u_j|^{\tilde{q}} |x|^{\tilde{q}\tilde{\beta}}\, dx = 0. \tag{4.47}$$

Let us set $\varphi = n\left(\frac{1}{p} - \frac{1}{q}\right)$.

First we assume that $\varphi \geq \frac{p}{q}$, which is equivalent to

$$q \geq p + \frac{p^2}{n}.$$

We find a locally finite covering, i.e., $^\exists\{x_k\}$ and $^\exists L \in \mathbb{N}$ such that $^\forall x \in \mathbb{R}^n$ at most L balls of $B_{r(x_k)}(x_k)$ to which x belongs. Then we have

$$\int_{\mathbb{R}^n} |u|^q |x|^{\beta q}\, dx \leq L \sum_{k=1}^{\infty} \int_{B_{r(x_k)}(x_k)} |u|^q |x|^{\beta q}\, dx. \tag{4.48}$$

Le us set $B^k = B_{r(x_k)}(x_k)$. By the multiplicative inequality we obtain

$$\int_{B^k} |u|^q |x|^{\beta q}\, dx \leq ||u||^{(1-\varphi)q}_{L^p_{\alpha-1}(B^k)} \cdot ||u||^{\varphi q}_{L^{p^*}_{\alpha}(B^k)}.$$

By the classical Sobolev inequality, the second factor in the right-hand side can be estimated in the following way;

$$||u||_{L^{p^*}_{\alpha}(B^k)} \leq C||u||_{W^{1,p}_{\alpha}(2B^k)}. \tag{4.49}$$

Here C is a positive number independent of each u and r_k noting that $\varphi q \geq p$, we have

$$\begin{aligned}\int_{B^k} |u|^q |x|^{\beta q}\, dx &\leq C||u||^{(1-\varphi)q}_{L^p_{\alpha-1}(B^k)} \left(\int_{2B^k} (|\nabla u|^p |x|^{\alpha p} + |u|^p |x|^{(\alpha-1)p})\, dx\right)^{\frac{\varphi q}{p}} \\ &\leq C||u||^{(1-\varphi)q}_{L^p_{\alpha-1}(B^k)} \cdot ||u||^{\varphi q - p}_{W^{1,p}_{\alpha,0}(\mathbb{R}^n)} \int_{2B^k} (|\nabla u|^p |x|^{\alpha p} + |u|^p |x|^{(\alpha-1)p})\, dx.\end{aligned}$$

Then we have

$$\begin{aligned}\int_{\mathbb{R}^n} |u|^q |x|^{\beta q}\, dx &\leq L \sum_{k=1}^{\infty} \int_{B_{r(x_k)}(x_k)} |u|^q |x|^{\beta q}\, dx \\ &\leq CL||u||^{\varphi q}_{W^{1,p}_{\alpha,0}(\mathbb{R}^n)} \cdot \sup_k ||u||^{(1-\varphi)q}_{L^p_{\alpha-1}(B^k)}.\end{aligned}$$

Then by (4.47), we conclude that

$$\lim_{j\to\infty} \int_{\mathbb{R}^n} |u_j|^q |x|^{\beta q}\, dx = 0. \tag{4.50}$$

In the next we assume that $\varphi q < p$, namely $p < q < p + \frac{p^2}{n}$. We set $q_0 = p + \frac{p^2}{n}$ and $\beta_0 = \alpha - \frac{p}{n+p}$. Then we see that $\frac{1}{p} - \frac{1}{q_0} = \frac{1-\alpha+\beta_0}{n}$ and $\alpha - 1 < \beta_0 \leq \alpha$, and by the multiplicative inequality we see that for some $\lambda \in (0,1)$

$$||u||_{L^q(\mathbb{R}^n, |x|^{\beta q})} \leq ||u||^{\lambda}_{L^p(\mathbb{R}^n, |x|^{(\alpha-1)p})} \cdot ||u||^{1-\lambda}_{L^{q_0}(\mathbb{R}^n, |x|^{\beta_0 q_0})}.$$

Therefore we have the same conclusion as before.

Lastly we consider the case $p < \tilde{q} < p^*$. If $\tilde{q} \leq q < p^*$, then the assertion follows from

$$||u||_{L^q(\mathbb{R}^n,|x|^{\beta q})} \leq ||u||^{\lambda}_{L^{\tilde{q}}(\mathbb{R}^n,|x|^{\tilde{\beta}\tilde{q}})} \cdot ||u||^{1-\lambda}_{L^{p^*}(\mathbb{R}^n,|x|^{\alpha p^*})}.$$

If $p < q < \tilde{q}$, then we can employ

$$||u||_{L^q(\mathbb{R}^n,|x|^{\beta q})} \leq ||u||^{\lambda}_{L^{p}(\mathbb{R}^n,|x|^{(\alpha-1)p})} \cdot ||u||^{1-\lambda}_{L^{\tilde{q}}(\mathbb{R}^n,|x|^{\tilde{\beta}\tilde{q}})}.$$

□

Lemma 4.12 (A Sharp Fatou's lemma) *Let p and n satisfy $1 < p < +\infty$ and $n \geq 1$ respectively. Let $\{u_j\}_{j=1}^{\infty}$ be a sequence in $W^{1,p}_{\alpha,0}(\mathbb{R}^n)$. Assume that the parameters p, q, n, α and β satisfy the noncritical relation (NCR). Assume that there is a function $u \in W^{1,p}_{\alpha,0}(\mathbb{R}^n)$ such that*

$$\begin{cases} u_j \rightharpoonup u & \text{weakly in } W^{1,p}_{\alpha,0}(\mathbb{R}^n), \\ u_j \to u & \text{almost everywhere.} \end{cases} \tag{4.51}$$

Then we have

$$\int_{\mathbb{R}^n} |\nabla u_j|^p |x|^{\alpha p}\, dx = \int_{\mathbb{R}^n} |\nabla (u_j - u)|^p |x|^{\alpha p}\, dx + \int_{\mathbb{R}^n} |\nabla u|^p |x|^{\alpha p}\, dx + o(1), \tag{4.52}$$

and

$$\int_{\mathbb{R}^n} |u_j|^q |x|^{\beta q}\, dx = \int_{\mathbb{R}^n} |u_j - u|^q |x|^{\beta q}\, dx + \int_{\mathbb{R}^n} |u|^q |x|^{\beta q}\, dx + o(1). \tag{4.53}$$

Proof: Let us set $G(v) = |\nabla v|^p$. Then we see

$$\begin{aligned} &\int_{\mathbb{R}^n} |\nabla u_j|^p |x|^{\alpha p}\, dx - \int_{\mathbb{R}^n} |\nabla (u_j - u)|^p |x|^{\alpha p}\, dx \\ &= -\int_{\mathbb{R}^n} dx \int_0^1 \frac{d}{d\theta} G(u_j - \theta u) |x|^{p\alpha}\, d\theta \\ &= p \int_{\mathbb{R}^n} \int_0^1 |\nabla (u_j - \theta u)|^{p-2} \nabla (u_j - \theta u) \cdot \nabla u |x|^{\alpha p}\, d\theta dx. \end{aligned}$$

Since $u_j - \theta u \in W^{1,p}_{\alpha,0}(\mathbb{R}^n)$, $u_j \to u$ almost everywhere and $u_j \rightharpoonup u$ weakly in $W^{1,p}_{\alpha,0}(\mathbb{R}^n)$ as $j \to \infty$, we see that

$$|\nabla (u_j - \theta u)|^{p-2} \nabla (u_j - \theta u) \rightharpoonup (1-\theta)^{p-1} |\nabla u|^{p-2} \nabla u \qquad \text{weakly in } (L^{p'}(\mathbb{R}^n, |x|^{p\alpha}))^n,$$

as $j \to \infty$. Hence the first identity is now proved. The second identity is known as the sharp Fatou's lemma. For reader's convenience let recall the proof. By Theorem 3.1 we see that the sequence of functions $\{u_j\}$ is uniformly bounded in $L^q(\mathbb{R}^n, |x|^{\beta q})$. By Fatou's lemma we have

$$\int_{\mathbb{R}^n} |u|^q |x|^{\beta q}\, dx \leq \liminf_{j \to \infty} \int_{\mathbb{R}^n} |u_j|^q |x|^{\beta q}\, dx,$$

and hence we see that $u \in L^q(\mathbb{R}^n, |x|^{\beta q})$. For any $\varepsilon > 0$, let us set

$$v_{j,\varepsilon} = \max(0, ||u_j|^q - |u - u_j|^q - |u|^q| - \varepsilon |u_j - u|^q).$$

Since we have for some positive number C_ε

$$||u_j|^q - |u-u_j|^q - |u|^q| \le \varepsilon|u_j - u|^q + C_\varepsilon|u|^q + |u|^q,$$

we have $v_{j,\varepsilon} \le C|u|^q$ for some positive number C and then

$$||u_j|^q - |u-u_j|^q - |u|^q| \le v_{j,\varepsilon} + \varepsilon|u_j - u|^q.$$

By the dominated convergence theorem we have $\int_{\mathbb{R}^n} v_{j,\varepsilon}|x|^{\beta q}\,dx \to 0$ as $j \to \infty$ and

$$\lim_{j\to\infty}\int_{\mathbb{R}^n} ||u_j|^q - |u-u_j|^q - |u|^q|\,|x|^{\beta q}\,dx = 0.$$

This proves the assertion. □

Proof of the assertion 1: Let $\{u_j\} \subset W^{1,p}_{\alpha,0}(\mathbb{R}^n)$ be a minimizing sequence. Then, without a loss of generality we assume that

$$S(p,q,\alpha,\beta,n) = \lim_{j\to\infty}\int_{\mathbb{R}^n} |\nabla u_j|^p|x|^{\alpha p}\,dx \tag{4.54}$$

and

$$\int_{\mathbb{R}^n} |u_j|^q|x|^{\beta q}\,dx = 1, \qquad \text{for } j = 1,2,\cdots. \tag{4.55}$$

Further we can assume that there exists some $u \in W^{1,p}_{\alpha,0}(\mathbb{R}^n)$ such that

$$\begin{cases} u_j \rightharpoonup u & \text{weakly in } W^{1,p}_{\alpha,0}(\mathbb{R}^n), \\ u_j \to u & \text{almost everywhere.} \end{cases} \tag{4.56}$$

Let (p,q,α,β,n) satisfy (NCR). Then, it follows from (4.55) and Lemma 4.11 with $(\tilde{q},\tilde{\beta}) = (q,\beta)$ that there exists a positive number δ such that we have

$$\sup_{y\in\mathbb{R}^n}\int_{B_{r(y)}(y)} |u_j|^q|x|^{q\beta}\,dx \ge \delta > 0, \qquad \text{for } j = 1,2,\cdots. \tag{4.57}$$

Hence there exits a sequence $\{w_j\} \subset \mathbb{R}^n \setminus \{0\}$ such that

$$\int_{B_{r(w_j)}(w_j)} |u_j|^q|x|^{q\beta}\,dx \ge \frac{\delta}{2} > 0, \qquad \text{for } j = 1,2,\cdots. \tag{4.58}$$

If we put $v_j(z) = |w_j|^{\beta+\frac{n}{q}}u_j(|w_j|z)$, then

$$\begin{aligned}\int_{B_{\frac14}(\frac{w_j}{|w_j|})} |v_j(z)|^q|z|^{\beta q}\,dz &= \int_{B_{\frac14}(\frac{w_j}{|w_j|})} |u_j(|w_j|z)|^q|z|^{\beta q}|w_j|^{\beta q+n}\,dz \\ &= \int_{B_{\frac{|w_j|}{4}}(w_j)} |u_j(x)|^q|x|^{\beta q}\,dx \\ &\ge \frac{\delta}{2}.\end{aligned}$$

Since it also holds that

$$\int_{\mathbb{R}^n} |\nabla_z v_j(z)|^p|z|^{\alpha p}\,dz = \int_{\mathbb{R}^n} |\nabla u_j(x)|^p|x|^{\,\alpha p}\,dx,$$

we have $E_{p,q,\alpha,\beta,n}(v_j) = E_{p,q,\alpha,\beta,n}(u_j)$ holds for $j = 1,2,\ldots$. Moreover the functional $E_{p,q,\alpha,\beta,n}(u)$ is invariant under rotations around the origin, we can assume from the beginning that

$$\int_{B_{\frac14}(e_1)} |u_j(x)|^q|x|^{\beta q}\,dz \ge \frac{\delta}{2} \qquad \text{for } j = 1,2,\cdots, \tag{4.59}$$

where $e_1 = (1, 0, \ldots, 0)$. Therefore we see that u is not identically 0. From Lemma 4.12 it follows that

$$S(p,q,\alpha,\beta,n) = \lim_{j\to\infty} \int_{\mathbb{R}^n} |\nabla u_j|^p |x|^{\alpha p}\, dx$$
$$= \int_{\mathbb{R}^n} |\nabla u|^p |x|^{\alpha p}\, dx + \lim_{j\to\infty} \int_{\mathbb{R}^n} |\nabla (u_j - u)|^p |x|^{\alpha p}\, dx$$
$$\geq S(p,q,\alpha,\beta,n) \left(\int_{\mathbb{R}^n} |u|^q |x|^{\beta q}\, dx \right)^{\frac{p}{q}} + S(p,q,\alpha,\beta,n) \lim_{j\to\infty} \left(\int_{\mathbb{R}^n} |u_j - u|^q |x|^{\beta q}\, dx \right)^{\frac{p}{q}}$$
$$= S(p,q,\alpha,\beta,n) \left(\left(\int_{\mathbb{R}^n} |u|^q |x|^{\beta q}\, dx \right)^{\frac{p}{q}} + \left(1 - \int_{\mathbb{R}^n} |u|^q |x|^{\beta q}\, dx \right)^{\frac{p}{q}} \right)$$
$$\geq S(p,q,\alpha,\beta,n).$$

Therefore we can conclude that

$$\int_{\mathbb{R}^n} |u|^q |x|^{\beta q}\, dx = 1 \quad \text{and} \quad \lim_{j\to\infty} \int_{\mathbb{R}^n} |u_j - u|^q |x|^{\beta q}\, dx = 0,$$

since u is not identically 0. □

Proof of the assertion 2: First we note that by the same argument in [AH2; §4.1] the extremal function (if it exists) should be radial. Moreover by the variational principle the extremal function u in the energy space $R^{1,p}_{\alpha,0}(\mathbb{R}^n)$ should satisfy the Euler-Lagrange equation:

$$\begin{cases} -\partial_r \left(r^{\alpha p + n - 1} |\partial_r u|^{p-2} \partial_r u \right) = \Lambda_{n,p,\alpha} |u|^{p-2} u r^{\alpha p - p + n - 1}, \ (0 < r < +\infty), \\ u(+\infty) = 0. \end{cases}$$

Then u should be positive and decreasing by the maximum principle. When $p = 2$, it follows from the theory of ordinary differential equations that the positive solution u has to satisfy $u = O(r^{-\frac{n-2+2\alpha}{2}})$ as $x \to 0$. Hence $u \notin R^{1,2}_{\alpha,0}(\mathbb{R}^n)$. In general, $w = r^{-\frac{n-p+p\alpha}{p}}$ becomes an exact solution to this Euler-Lagrange equation, and one can show that u should have the same singularity as w at the origin, so that $u \notin R^{1,p}_{\alpha,0}(\mathbb{R}^n)$. Though this fact seems to be known, let us give a simple explanation using the existence of sharp remainder. (See [AH2].) To this end, assume that $p \geq 2$ temporalily and $u \in R^{1,p}_{\alpha,0}(\mathbb{R}^n)$ is the positive extremal function. Here we note that the subsequent argument is still valid with minor modifications when $1 < p < 2$. (See the remark just after the proof.)

For $\varepsilon > 0$ let us set $u_\varepsilon = \max(0, u - \varepsilon)$. Then we see that $u_e \to u$ in $W^{1,p}_{\alpha,0}(\mathbb{R}^n)$ as $\varepsilon \to 0$. Therefore we can assume that u is compactly supported without a loss of generality. Now we set $u = r^{-\delta} v$ for $\delta = \frac{n-p+\alpha p}{p} = \Lambda^{\frac{1}{p}}_{n,p,\alpha}$. Since $v(0) = 0$, we have

$$\int_0^\infty |\partial_r u|^p r^{n-1+\alpha p}\, dr = \delta^p \int_0^\infty v^p r^{-\delta p - p} \left| 1 - \frac{r \partial_r v}{\delta v} \right|^p r^{n-1+\alpha p}\, dr$$
$$\geq \delta^p \int_0^\infty v^p \frac{1}{r}\, dr - \delta^{p-1} p \int_0^\infty v^{p-1} \partial_r v\, dr + C \int_0^\infty v^{p-2} (\partial_r v)^2 r\, dr$$
$$= \Lambda_{n,p,\alpha} \int_0^\infty u^p r^{n-1+\alpha(p-1)}\, dr + C \int_0^1 v^{p-2} (\partial_r v)^2 r\, dr,$$

where C is a positive number independent of an each u. Since u is the extremal, the last integrand has to vanish. Hence u should coincide with $r^{-\delta}$ up to constants in a neighborhood of the origin, therefore $u \notin R^{1,p}_{\alpha,0}(\mathbb{R}^n)$. □

Remark 4.5 *If $p \in (1,2)$ in this argument, then one can employ the next elementary lemma with $M = 1$ instead of Taylor's expansion.*

Lemma 4.13 *For $1 < p \leq 2$ and $M \geq 1$, we have*

$$|1+X|^p - 1 - pX \geq c(p) \begin{cases} M^{p-2}X^2, & |X| \leq M, \\ |X|^p, & |X| \geq M. \end{cases} \tag{4.60}$$

Here $c(p)$ is a positive number independent of each X and $M \geq 1$.

Proof. *When $X > -1$, this follows from Taylor expansion. If we choose $c(p)$ sufficiently small, then it remains valid for $X \leq -1$.* □

Proof of the assertion 3: This fact was already known in the author's paper [Ho2] as the assertion (1) of Theorem 3.4, which was based on the spherically symmetric decreasing rearrangement of a nonnegative function and the existence of missing terms in the weighted Hardy-Sobolev inequalities. For reader's convenience we shall give a somewhat simpler proof here.

Let us prepare an elementary lemma borrowing the idea from [Ho2].

Lemma 4.14 *Under the condition (NCR) with $\alpha = \beta$, we have*

$$S(p, p^*, \alpha, \alpha, n) \leq S(p, p^*, 0, 0, n).$$

Proof of Lemma: Let us set, for $y \in \mathbb{R}^n \setminus \{0\}$,

$$S(y) = \inf\{\int_{\mathbb{R}^n} |\nabla u|^p |y|^{p\alpha}\, dx \,:\, \int_{\mathbb{R}^n} |u|^{p^*} |y|^{\alpha p^*}\, dx = 1, u \in C_0^\infty(\mathbb{R}_x^n)\}. \tag{4.61}$$

Then one can check easily that if we replace u by $\varepsilon^{-n/p^*} u(\cdot - y/\varepsilon), \varepsilon > 0, p^* = np/(n-p)$ and let ε tend to 0, we have $S(p, p^*, \alpha, \alpha, n) \leq S(y)$. On the other hand, it holds

$$S(y) = \inf\left\{\int_{\mathbb{R}^n} |\nabla u|^p\, dx \,:\, \int_{\mathbb{R}^n} |u|^{p^*}\, dx = 1\right\} = S(p, p^*, 0, 0, n). \tag{4.62}$$

So that we see $S(p, p^*, \alpha, \alpha, n) \leq S(p, p^*, 0, 0, n)$. □

In the next we shall show the opposite inequality assuming that $p = 2$. For $\delta = \alpha + \frac{n}{2} - 1 > 0$ and $u \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$,which is not identically 0, let us set

$$\begin{cases} u = r^{-\delta} v, \\ a_2(\delta, v) = |\nabla v|^2 - 2\delta \frac{\partial v}{\partial r}\frac{v}{r} + \delta^2 \frac{v^2}{r^2}. \end{cases} \tag{4.63}$$

Then we have

$$S(2, 2^*, \alpha, \alpha, n) = \inf_{v \in C_0^\infty(\mathbb{R}^n \setminus \{0\}) \setminus \{0\}} \frac{A_2(\delta, v)}{\left(\int_{\mathbb{R}^n} v^{2^*} |x|^{-n}\, dx\right)^{\frac{2}{2^*}}} \tag{4.64}$$

Here

$$A_2(\delta, v) = \int_{S^{n-1}} d\omega \int_0^\infty a_2(\delta, v) r dr. \tag{4.65}$$

By differentiating this with respect to α we have

$$\begin{aligned}
\frac{d}{d\alpha}A_2(\delta, v) &= \int_{S^{n-1}} d\omega \int_0^\infty \frac{d}{d\alpha} a_2(\delta, v) r dr \\
&= \int_{S^{n-1}} d\omega \int_0^\infty \left(-2v\frac{\partial v}{\partial r} + 2\delta\frac{v^2}{r}\right) dr, \\
&= 2\delta \int_{S^{n-1}} d\omega \int_0^\infty \left(\frac{v^2}{r}\right) dr \\
&= 2\delta \int_{\mathbb{R}^n} |u|^2 |x|^{2\alpha}\, dx > 0.
\end{aligned}$$

Here we used the integration by parts noting that $v(0) = 0$. Therefore we see that $A_2(\delta, v)$ is increasing with respect to α for any test function $v \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$, and hence we have $S(2, 2^*, \alpha, \alpha, n) \geq S(2, 2^*, 0, 0, n)$. □

Remark 4.6 *In [Ho2] we used a variant of the Hardy-Sobolev inequality with a missing term instead of the last argument.*

$$S(2, 2^*, 0, 0, n)\left(\int_{\mathbb{R}^n} |u|^{2^*} |x|^{\alpha 2^*}\, dx\right)^{2/2^*} + \alpha(\alpha + n - 2) \int_{\mathbb{R}^n} u^2 |x|^{2(\alpha-1)}\, dx$$
$$\leq \int_{\mathbb{R}^n} |\nabla u|^2 |x|^{2\alpha}\, dx. \tag{4.66}$$

Here $1/2 - 1/2^* = 1/n$, $n > 2$. *This clearly implies* $S(2, 2^*, \alpha, \alpha, n) \geq S(2, 2^*, 0, 0, n)$.

4.5 Proof of Theorem 3.5

We shall establish the asymptotic behavior of the best constant.

Theorem 3.5 (The asymptotic behavior of the best constant) *Let* p *satisfy* $1 < p < n$ *and let* n *satisfy* $n \geq 1$. *Assume that the parameters* p, q, n, α *and* β *satisfy the noncritical relation (NCR) and assume that either* $\alpha \geq 0$ *or* $\alpha \leq 2(1 - \frac{n}{p})$. *Then we have the followings.*

1. *For* $\alpha = \beta$, *there is a positive number* $C(p, \alpha, n)$ *such that we have*

$$l(p, \alpha, n) S(p, p^*, 0, 0, n) \leq S(p, p^*, \alpha, \alpha, n) \leq S(p, p^*, 0, 0, n). \tag{4.67}$$

   Here $p^* = \frac{np}{n-p}$ *and*

$$l(p, \alpha, n) = \left(\frac{|n - p + \alpha p|}{|n - p + \alpha p| + |\alpha| p}\right)^p > 0 \tag{4.68}$$

2. *As* $|\alpha| \to \infty$ *with* $\alpha - 1 < \beta \leq \alpha$, *we have*

$$S(p, q, \alpha, \beta, n) \geq (\Lambda_{n,p,\alpha})^{p(\alpha-\beta)} S(p, p^*, \alpha, \alpha, n)^{1-\alpha+\beta} \tag{4.69}$$

Proof of the assertion 1: It follows from Lemma 4.14 that the second inequality in (4.67) holds. Let us set $\mu(\alpha) = S(p, p^*, \alpha, \alpha, n)$. Then for any $\varepsilon > 0$ there exists a $u \in C_0^\infty(\mathbb{R}^n)$ such that $\int_{\mathbb{R}^n} |u|^{p^*} |x|^{\alpha p^*}\, dx = 1$ and $\mu(\alpha) \leq$

$\int_{\mathbb{R}^n} |\nabla u|^p |x|^{\alpha p}\, dx \leq \mu(\alpha) + \varepsilon$. Then it follows from the classical Sobolev inequality and the weighted Hardy-Sobolev inequality that we have

$$S(p,p^*,0,0,n)^{\frac{1}{p}} \leq \left(\int_{\mathbb{R}^n} |\nabla(u|x|^\alpha)|^p\, dx\right)^{\frac{1}{p}} \tag{4.70}$$
$$\leq \left(\int_{\mathbb{R}^n} |\nabla u|^p |x|^{\alpha p}\, dx\right)^{\frac{1}{p}} + |\alpha| \left(\int_{\mathbb{R}^n} |u|^p |x|^{(\alpha-1)p}\, dx\right)^{\frac{1}{p}}$$
$$\leq \left(\int_{\mathbb{R}^n} |\nabla u|^p |x|^{\alpha p}\, dx\right)^{\frac{1}{p}} + \frac{p|\alpha|}{|n-p+p\alpha|} \left(\int_{\mathbb{R}^n} |\nabla u|^p |x|^{\alpha p}\, dx\right)^{\frac{1}{p}}$$

Then we immediately have

$$\int_{\mathbb{R}^n} |\nabla u|^p |x|^{\alpha p}\, dx \geq S((p,p^*,0,0,n) \left(\frac{|n-p+\alpha p|}{|n-p+\alpha p| + |\alpha| p}\right)^p \tag{4.71}$$
$$= l(p,\alpha,n) S(p,p^*,0,0,n)$$

Since ε is any positive number we see

$$\mu(\alpha) \geq l(p,\alpha,n) S((p,p^*,0,0,n).$$

□

Remark 4.7 *1. Using the same argument in the proof of the existence of extremals in Theorem 3.4 one can show that $S(p,p^*,\alpha,\alpha,n)$ is achieved in $W^{1,p}_{\alpha,0}(\mathbb{R}^n)$ if $S(p,p^*,\alpha,\alpha,n) < S(p,p^*,0,0,n)$ holds. In fact we see from the second line of the inequalities (4.70) that the minimizing sequence $\{u_j\} \subset W^{1,p}_{\alpha,0}(\mathbb{R}^n)$ for a fixed α satisfies*

$$\int_{\mathbb{R}^n} |u_j|^p |x|^{(\alpha-1)p}\, dx \geq C, \qquad (j = 1,2,3,\cdots)$$

for some positive number C. Therefore it follows from Lemma 4.11 that it contains a convergent subsequence to an nontrivial limit $u \in W^{1,p}_{\alpha,0}(\mathbb{R}^n)$.

2. When $2(1-n/p) < \alpha < 0$, we have seen $S(p,q,\alpha,\beta,n) = S_{rad}(p,q,\alpha,\beta,n)$.

Proof of the assertion 2: It suffices to assume that $\alpha > \beta > \alpha - 1$ and $\alpha > 1 - \frac{n}{p}$. By the assertion 1 of Theorem 3.4 we have an extremal function $u \in W^{1,p}_{\alpha,0}(\mathbb{R}^n)$ which satisfies $\int_{\mathbb{R}^n} |\nabla u|^p |x|^{p\alpha}\, dx = S(p,q,\alpha,\beta,n)$ and $\int_{\mathbb{R}^n} |u|^q |x|^{\beta q}\, dx = 1$. Then it follows from Lemma 4.10 that

$$1 = ||u||_{L^q(\mathbb{R}^n,|x|^{\beta q})} \leq ||u||^{1-\varphi}_{L^p(\mathbb{R}^n,|x|^{(\alpha-1)p})} \cdot ||u||^{\varphi}_{L^{p^*}(\mathbb{R}^n,|x|^{\alpha p^*})}$$
$$\leq ||u||^{1-\varphi}_{L^p(\mathbb{R}^n,|x|^{(\alpha-1)p})} \cdot S(p,p^*,\alpha,\alpha,n)^{-\frac{\varphi}{p}} \cdot ||\nabla u||^{\varphi}_{L^p(\mathbb{R}^n,|x|^{\alpha p})}$$

where

$$p^* = \frac{np}{n-p}, \qquad \varphi = \theta\frac{p^*}{q}, \qquad \theta = \frac{q-p}{p^*-p}, \qquad 1-\theta = \frac{p^*-q}{p^*-p}.$$

In order to obtain the second inequality we used

$$S(p,p^*,\alpha,\alpha,n)||u||^p_{L^{p^*}(\mathbb{R}^n,|x|^{\alpha p^*})} \leq ||\nabla u||^p_{L^p(\mathbb{R}^n,|x|^{\alpha p})}.$$

Then we get

$$||u||^{1-\varphi}_{L^p(\mathbb{R}^n,|x|^{(\alpha-1)p})} \geq S(p,p^*,\alpha,\alpha,n)^{\frac{\varphi}{p}} \cdot ||\nabla u||^{-\varphi}_{L^p(\mathbb{R}^n,|x|^{\alpha p})}.$$

Since $S(p,q,\alpha,\beta,n)$ is achieved by u, we have

$$||u||^p_{L^p(\mathbb{R}^n,|x|^{(\alpha-1)p})} \geq S(p,p^*,\alpha,\alpha,n)^{\frac{\varphi}{1-\varphi}} \cdot S(p,q,\alpha,\beta,n)^{-\frac{\varphi}{1-\varphi}}.$$

By the weighted Hardy-Sobolev inequality we see that

$$||u||^p_{L^p(\mathbb{R}^n,|x|^{(\alpha-1)p})} \leq \Lambda^{-1}_{n,p,\alpha}||\nabla u||^p_{L^p(\mathbb{R}^n,|x|^{\alpha p})} = \left(\frac{p}{n-p+\alpha p}\right)^p S(p,q,\alpha,\beta,n).$$

Then we have

$$S(p,q,\alpha,\beta,n) \geq \left(\frac{n-p+\alpha p}{p}\right)^{p(1-\varphi)} S(p,p^*,\alpha,\alpha,n)^{\varphi}.$$

Noting that $1-\varphi = \alpha - \beta$, we have the desired estimate.

□

4.6 Proof of Theorem 3.6

In this subsection we shall prove the results on the symmetricity of the extremals according to Definition 3.6

Theorem 3.6 (The symmetricity of the extremals) *Let p satisfy $1 < p < +\infty$ and let n satisfy $n \geq 2$.*

1. *Assume that $p < n$. Let A be the subset of $\mathbb{R}^2$ given by*

$$\left\{(\alpha,\beta) \in \mathbb{R}^2; 1-\frac{n}{p} < \alpha \leq 0,\ \alpha - 1 \leq \beta \leq \alpha\right\}.$$

Then we have for any (p,q,α,β,n) satisfying (NCR) with $(\alpha,\beta) \in A \cup A^{\diamondsuit}$,

$$S(p,q,\alpha,\beta,n) = S_{rad}(p,q,\alpha,\beta,n).$$

2. *Let B be the subset of $\mathbb{R}^2$ given by*

$$\left\{(\alpha,\beta) \in \mathbb{R}^2; \max\left(0, 1-\frac{n}{p}\right) < \alpha \leq 1-\frac{1}{p},\ \alpha - 1 \leq \beta \leq \frac{\alpha(n-p+\alpha p)}{n+\alpha p - np}\right\}.$$

Then we have for any (p,q,α,β,n) satisfying (NCR) with $(\alpha,\beta) \in B \cup B^{\diamondsuit}$,

$$S(p,q,\alpha,\beta,n) = S_{rad}(p,q,\alpha,\beta,n).$$

Proof of the assertion 1: By the assertion 1 of Theorem 3.3, it suffices to show $S(p,q,\alpha,\beta,n) = S_{rad}(p,q,\alpha,\beta,n)$ for any (p,q,α,β,n) satisfying (NCR) with $(\alpha,\beta) \in A$ (the subcritical case). The following lemma is proved in [Ho2].

Lemma 4.15 *Assume that (p,q,α,β,n) satisfies (NCR) and $(\alpha,\beta) \in A$. Then*

$$S(p,q,\alpha,\beta,n) = S_{rad}(p,q,\alpha,\beta,n). \tag{4.72}$$

Proof of Lemma 4.15 By a polar coordinate system, we rewrite (P) to obtain

$$S(p,q,\alpha,\beta,n) = \inf\Bigg[\int_{S^{n-1}}\int_0^{\infty} (|\partial_r u|^2 + \frac{|\Lambda u|^2}{r^2})^{p/2}\, r^{\alpha p+n-1}\, dr dS_{\omega} :$$
$$u \in W^{1,p}_{\alpha,0}(\mathbb{R}^n), \int_{S^{n-1}}\int_0^{\infty} |u|^q r^{\beta q+n-1}\, dr dS_{\omega} = 1\Bigg],$$

where S_ω is a $n-1$-dimensional Lebesgue measure and Λ is a usual spherical gradient operator on the unit sphere S^{n-1}. Making a change of variables defined by

$$r = \rho^l, \qquad l = \frac{n-p}{n-p+\alpha p}, \tag{4.73}$$

we have

$$\inf\Big[l^{1-p-p/q}\int_{S^{n-1}}\int_0^\infty (|\partial_\rho v|^2 + l^2\frac{|\Lambda v|^2}{\rho^2})^{p/2}\,\rho^{n-1}\,d\rho dS_\omega : \qquad (P')$$
$$u \in W^{1,p}_{\alpha,0}(\mathbb{R}^n), \quad \int_{S^{n-1}}\int_0^\infty |v|^q\rho^{(n-p)q/p-1}\,d\rho dS_\omega = 1\Big],$$

where $v((\rho,\omega)) = l^{-1/q}u((\rho^l,\omega))$. Since $\alpha \le 0$ by the assumption, we see $l \ge 1$. Therefore we see

$$S(p,q,\alpha,\beta,n) \ge l^{1-p-p/q}\cdot S(p,q,0,\beta-\alpha,n), \tag{4.74}$$

where we used $(n-p)q/p-1 = q(\beta-\alpha)+n-1$. Since $\beta-\alpha \le 0$, the spherically symmetric decreasing rearrangement of u leads us to

$$S(p,q,0,\beta-\alpha,n) = S_{rad}(p,q,0,\beta-\alpha,n). \tag{4.75}$$

Therefore the assertion follows from Lemma 4.7 (the assertion (2)). $\square$.

Proof of the assertion 2: It suffices to assume that $\alpha > 1-\frac{n}{p}$ (the subcritical case). We need to introduce a spherically symmetric decreasing rearrangement of function with respect to a spherically symmetric weight functions.

Definition 4.1 (An admissible weight function) *A positive function f on $\mathbb{R}^n$ is said to be admissible if and only if it is spherically symmetric, monotonically decreasing, locally integrable in $\mathbb{R}^n$ and continuous in $\mathbb{R}^n\setminus\{0\}$.*

Then we define a spherically symmetric decreasing rearrangement of function u with respect to an admissible weight $f(x)$. Let us set for any measurable set $A\subset\mathbb{R}^n$,

$$\mu(A) = \int_A f(x)\,dx. \tag{4.76}$$

Then we define for any nonnegative, bounded integrable u on $\mathbb{R}^n$ and $t\ge 0$

$$\mu_u(t) = \mu(\{x\in\mathbb{R}^n : u(x) > t\}) = \int_{\{u(x)>t\}} f(x)\,dx. \tag{4.77}$$

Finally we define

$$u^*(x) = \sup\{t : \mu_u(t) > \mu(B_{|x|}(0))\}, \tag{4.78}$$

where $B_{|x|}(0)$ is a unit ball centered at the origin with a radius $|x|$. We need a fundamental lemma involving an isometric inequality with weights, which will be established in Appendix 3.

Lemma 4.16 *Let $p\ge 1$. Assume that a weight f is admissible. Then for any nonnegative $u, v\in C_0^\infty(\mathbb{R}^n)$, we have the followings:*

1. *For almost all $t\in[0,\infty)$*

$$\int_{\{u^*=t\}} dH^{n-1} \le \int_{\{u=t\}} dH^{n-1} \tag{4.79}$$

2. For $g(x) = f(x)^{1-p}$,

$$\int_{\mathbb{R}^n} |\nabla u^*|^p g(x)\, dx \le \int_{\mathbb{R}^n} |\nabla u|^p g(x)\, dx. \tag{4.80}$$

3.

$$\int_{\mathbb{R}^n} u \cdot v f(x)\, dx \le \int_{\mathbb{R}^n} u^* \cdot v^* f(x)\, dx. \tag{4.81}$$

The proof is given in Appendix 3. □

Proof of Theorem 3.6: Let us set $f(x) = |x|^{-\frac{\alpha p}{p-1}}$. By u^* we denote the spherically symmetric decreasing rearrangement of a nonnegative $u \in C_0^\infty(\mathbb{R}^n) \setminus \{0\}$ with respect to an admissible weight $|x|^{-\frac{\alpha p}{p-1}}$. We have to show

$$E_{p,q,\alpha,\beta,n}(u) \ge E_{p,q,\alpha,\beta,n}(u^*),$$

where

$$\left\{ (\alpha, \beta) \in \mathbb{R}^2 ; \max\left(0, 1 - \frac{n}{p}\right) < \alpha \le 1 - \frac{1}{p},\ \alpha - 1 \le \beta \le \frac{\alpha(n - p + \alpha p)}{n + \alpha p - np} \right\}.$$

From the assertion 2 of Lemma 4.16 it suffices to prove that

$$\int_{\mathbb{R}^n} u^q |x|^{\beta q}\, dx \le \int_{\mathbb{R}^n} u^{*q} |x|^{\beta q}\, dx.$$

This fact follows from the assertion 3. In fact we see that

$$\begin{aligned}
&\int_{\mathbb{R}^n} u^q |x|^{\beta q}\, dx = \int_{\mathbb{R}^n} u^q (|x|^{\beta q} |x|^{\frac{\alpha p}{p-1}}) |x|^{-\frac{\alpha p}{p-1}}\, dx \\
&\le \int_{\mathbb{R}^n} u^{*q} (|x|^{\beta q} |x|^{\frac{\alpha p}{p-1}})^* |x|^{-\frac{\alpha p}{p-1}}\, dx \\
&= \int_{\mathbb{R}^n} u^{*q} (|x|^{\beta q} |x|^{\frac{\alpha p}{p-1}}) |x|^{-\frac{\alpha p}{p-1}}\, dx \\
&= \int_{\mathbb{R}^n} u^{*q} |x|^{\beta q}\, dx
\end{aligned}$$

Here we note that $\beta q + \frac{\alpha p}{p-1} \le 0$ provided that $\beta \le \frac{\alpha(n-p+\alpha p)}{n+\alpha p - np}$. Then $|x|^{\beta q}|x|^{\frac{\alpha p}{p-1}}$ is monotonically decreasing and can be approximated uniformly by a sequence of decreasing functions in $C_0^\infty(\mathbb{R}^n)$. Therefore we can apply Lemma 4.16 noting that $(|x|^{\beta q}|x|^{\frac{\alpha p}{p-1}})^* = |x|^{\beta q}|x|^{\frac{\alpha p}{p-1}}$ holds.

□

4.7 Proof of Theorem 3.7

Now we proceed to the proof of the results in the critical case. First we shall establish the imbedding theorem.

Theorem 3.7 (The imbedding results) *Let p satisfy $1 < p < +\infty$. Assume the critical relation (CR). Then if $n \ge 2$, there exists $R > 1$ such that we have $C(p, q, \alpha, \beta, n) > 0$, namely it holds that for any $u \in W^{1,p}_{\alpha,0}(B_1)$,*

$$\int_{B_1} |\nabla u|^p |x|^{p-n}\, dx \ge C(p, q, \alpha, \beta, n) \left(\int_{B_1} \frac{|u|^q}{|x|^n A_1(|x|)^{q+1-\frac{q}{p}}}\, dx \right)^{p/q}. \tag{4.82}$$

Here $A_1(t) = \log \frac{R}{t}$ for $R > 1$ if $n \ge 2$. When $n = 1$, the same inequality holds for $A_1(t) = \log \frac{1}{t}$. Moreover the weight function of the term in the right-hand side is sharp.

Remark 4.8 *1. When $n \geq 2$ and $p \neq q$, we can not replace $R > 1$ by 1.*

2. When $p \geq n$, we shall exploit the decreasing rearrangement method with respect to a positive measure. On the other hand, we shall employ the nonlinear potential theory when $1 < p < n$.

Proof: When $p = q$ holds, then the inequality of the assertion is known as the weighted Hardy-Sobolev inequality with the critical weight, which was established in [AH2] with a sharp missing term. Therefore we assume that $p > q$.

First we treat the case that $p \geq n$. In this case we can reduce the problem to the corresponding 1-dimensional one by the rearrangement argument used in the previous section. Let us set $f(x) = |x|^{\frac{n-p}{p-1}}$. Then f is clearly admissible in the sense of Definition 4.1. Therefore it suffices to check the denominator of $E^c_{p,q,\alpha,\beta,n}(u)$ is nondecreasing under the method of rearrangement with respect to an admissible weight. Namely we have to show that

$$\int_{B_1} \frac{u^q}{|x|^n A_1(|x|)^{q+1-\frac{q}{p}}}\, dx \leq \int_{B_1} \frac{u^{*q}}{|x|^n A_1(|x|)^{q+1-\frac{q}{p}}}\, dx.$$

By virtue of the assertion (3) of Lemma 4.16 and the approximation argument as before, it suffices to show that

$$|x|^{-n+\frac{p-n}{p-1}} A_1(|x|)^{-q-1+\frac{q}{p}}$$

is nonincreasing for a large $R > 0$. If $n = 1$, then this is clear. So we assume that $n \geq 2$. To this end we set $h(r) = r^A \left(\log \frac{R}{r}\right)^B$ for $A = -n + \frac{p-n}{p-1}$ and $B = -q - 1 + \frac{q}{p}$. Then the condition becomes for a large $R > 1$

$$h'(r) = Ar^{A-1} \left(\log \frac{R}{r}\right)^{B-1} \left(\log \frac{R}{r} - \frac{A}{B}\right) \leq 0.$$

Since $A < 0$ and $0 < r \leq 1$, this is equivalent to

$$R \geq e^{\frac{B}{A}} = \exp\left(\frac{(p-1)(pq+p-q)}{p^2(n-1)}\right) \equiv R_0.$$

Therefore the assertion follows. The problem is reduced to the next one.

$$C_{rad}(p,q,\alpha,\beta,n) = \inf_{u \in R^{1,p}_{\alpha,0}(B_1)\setminus\{0\}} E^c_{p,q,\alpha,\beta,n}(u). \qquad (P^c_R)$$

Here

$$E^c_{p,q,\alpha,\beta,n}(v) = \omega_n^{1-\frac{p}{q}} \frac{\int_0^1 |\partial_r v|^p r^{p-1}\, dr}{\left(\int_0^1 \frac{|v|^q}{rA_1(r)^{q+1-\frac{q}{p}}}\, dr\right)^{p/q}}. \quad \left(v \in R^{1,p}_{\alpha,0}(B_1) \setminus \{0\}\right)$$

Here $A_1(t) = \log \frac{R}{t}$ for a given $R \geq R_0 > 1$ and ω_n is the area of $n-1$-dimensional unit sphere. We note that when $n = 1$, we can put $R = 1$ from the beginning. In the next step we will reduce this problem to the one of Talenti type on $\overline{\mathbb{R}_+} = \{r : 0 \leq r < +\infty\}$ with $R = 1$.

Let us recall for $q > p$

$$D(p,q,\alpha,\beta,n) = \inf_{u \in C_0^\infty(\overline{\mathbb{R}_+})\setminus\{0\}, u \geq 0} \frac{\int_0^\infty |\partial_t u|^p t^{\frac{n}{\gamma}-1}\, dt}{\left(\int_0^\infty u^q t^{\frac{n}{\gamma}-1}\, dt\right)^{\frac{p}{q}}}, \qquad (4.83)$$

where $\gamma = n\left(\frac{1}{p} - \frac{1}{q}\right) > 0$. Then we can show the following result using the method of change variables and Talenti's lemma 4.2.

Lemma 4.17 *Let us set*

$$k = \left(1 - \frac{1}{p}\right)\left(\frac{q}{p} - 1\right) > 0.$$

Then it holds that:

$$C_{rad}(p,q,\alpha,\beta,n) = k^{p-1+\frac{p}{q}}\omega_n^{1-\frac{p}{q}} D(p,q,\alpha,\beta,n). \tag{4.84}$$

Moreover if and only if $n = 1$, $C_{rad}(p,q,\alpha,\beta,n)$ is attained by some function in $R^{1,p}_{\alpha,0}(B_1)$.

Proof of Lemma 4.17: In the functional $E^c_{p,q,\alpha,\beta,n}(u)$ for $u \in C^\infty([0,1))$ with $u(1) = 0$, we perform a change of variable defined by

$$\log \frac{R}{r} = t \qquad (r = Re^{-t})$$

to obtain

$$E^c_{p,q,\alpha,\beta,n}(u) = \omega_n^{1-\frac{p}{q}} \frac{\int_{\log R}^\infty |\partial_t v|^p \, dt}{\left(\int_{\log R}^\infty |v|^q t^{-q-1+\frac{q}{p}} \, dt\right)^{\frac{p}{q}}}, \tag{4.85}$$

where $v(t) = u(Re^{-t}) \in C^\infty([\log R, \infty)) \setminus \{0\}$ with $v(\log R) = 0$. Then we use a change of variable defined by

$$t = \rho^{-\frac{1}{k}}.$$

Thus the right-hand side becomes

$$k^{p-1+\frac{p}{q}}\omega_n^{1-\frac{p}{q}} \frac{\int_0^{(\log R)^{-k}} |\partial_\rho w|^p \rho^{\frac{n}{\gamma}-1} \, d\rho}{\left(\int_0^{(\log R)^{-k}} |w|^q \rho^{\frac{n}{\gamma}-1} \, d\rho\right)^{p/q}}, \tag{4.86}$$

where $w(\rho) = v(\rho^{-\frac{1}{k}}) \in C^\infty([0, (\log R)^{-k})) \setminus \{0\}$ with $w((\log R)^{-k}) = 0$.

Here we note that if $n = 1$, then we have $R = 1$. Hence in this case we have

$$E^c_{p,q,\alpha,\beta,n}(u) = k^{p-1+\frac{p}{q}}\omega_n^{1-\frac{p}{q}} \frac{\int_0^\infty |\partial_t w|^p t^{\frac{n}{\gamma}-1} \, dt}{\int_0^\infty w^q t^{\frac{n}{\gamma}-1} \, dt},$$

where $w \in C^\infty([0,\infty)) \setminus \{0\}$ with $w(\infty) = 0$. Therefore the assertion is proved. Now we assume that $n \geq 2$ and set

$$\rho = \frac{t}{M} \qquad \text{for any } M > 0.$$

By the homogeneity we have

$$k^{p-1+\frac{p}{q}}\omega_n^{1-\frac{p}{q}} \frac{\int_0^{M(\log R)^{-k}} |\partial_t u|^p t^{\frac{n}{\gamma}-1} \, dt}{\left(\int_0^{M(\log R)^{-k}} |u|^q t^{\frac{n}{\gamma}-1} \, dt\right)^{q/p}} \tag{4.87}$$

where $u(t) = w(\frac{t}{M}) \in C^\infty([0, M(\log R)^{-k}) \setminus \{0\}$ with $u(M(\log R)^{-k}) = 0$. Since an admissible function space becomes rich as letting $M \to \infty$, we see that $C_{rad}(p,q,\alpha,\beta,n) \geq k^{p-1+\frac{p}{q}}\omega_n^{1-\frac{p}{q}} D(p,q,\alpha,\beta,n)$. On the other hand $u \in$

$C_0^\infty(\overline{\mathbb{R}_+})$ is approximated by a sequence of compactly supported functions $u_\varepsilon = \max(u-\varepsilon, 0)$ as $\varepsilon \to 0$, and hence we have the equality. $\square$

From Lemma 4.2 $D(p,q,\alpha,\beta,n)$ is attained by functions of the form

$$(a + br^{\frac{p}{p-1}})^{-\frac{p}{q-p}}$$

with a and b positive constants. Hence we can get the exact value of $D(p,q,\alpha,\beta,n)$ and $C_{rad}(p,q,\alpha,\beta,n)$ with somewhat more calculations.

Lemma 4.18

$$\begin{aligned} C_{rad}(p,q,\alpha,\beta,n) &= 2^{\frac{p\gamma}{n}} \pi^{\frac{p\gamma}{2}} \cdot \frac{q(p-1)}{p} \left(\frac{q\gamma}{n}\right)^{-\frac{p\gamma}{n}} \left\{\frac{\Gamma(n/\gamma p)\Gamma(n(p-1)/\gamma p)}{\Gamma(n/2)\Gamma(n/\gamma)}\right\}^{\frac{p\gamma}{n}}, \\ &= \left(\frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}\right)^{1-\frac{p}{q}} \left(\frac{q-p}{p}\right)^{\frac{q}{p}-1} \frac{q(p-1)}{p} \left(\frac{\Gamma\left(\frac{q}{q-p}\right)\Gamma\left(\frac{q(p-1)}{q-p}\right)}{\Gamma\left(\frac{pq}{q-p}\right)}\right)^{\frac{q-p}{q}}. \end{aligned}$$

Remark 4.9 *Here we note that $C_{rad}(p,q,\alpha,\beta,n)$ is completely determined by (p,q) for a fixed n and continuous except for the diagonal set (p,p). It is not difficult to calculate directly the limit of $C_{rad}(p,q,\alpha,\beta,n)$ as $q \to p+0$ (or equivalently $\beta \to \alpha - 1$), namely*

$$\lim_{\beta \to \alpha - 1 + 0} C_{rad}(p,q,\alpha,\beta,n) = \left(\frac{p-1}{p}\right)^p. \tag{4.88}$$

Moreover, if $\alpha > 1 - \frac{n}{p}$ (the subcritical case), we have

$$\begin{aligned} S_{rad}(p,q,\alpha,\beta,n) &= C_{rad}(p,q,\alpha,\beta,n) \times \left(\frac{h}{k}\right)^{p-\frac{p\gamma}{n}} \\ &= C_{rad}(p,q,\alpha,\beta,n) \times \left(\frac{n-p+p\alpha}{p-1}\right)^{p-\frac{p\gamma}{n}}, \end{aligned} \tag{4.89}$$

where $h = \frac{\gamma(n-p+\alpha p)}{n-p\gamma}$. Therefore the property (4.88) can be derived from this relation and the continuity of $S_{rad}(p,q,\alpha,\beta,n)$ as well.

Secondly we establish the assertion when $1 < p < n$. In this case we shall employ the so-called nonlinear potential theory. We need more notations. Let us set

$$\mu(x) = |x|^{-n}\left(\log\frac{R}{|x|}\right)^{-q-1+\frac{q}{p}}, \qquad \text{for } R > 1. \tag{4.90}$$

and

$$\omega = \begin{cases} |x|^{p-n}, & |x| \le 1, \\ 1 & |x| > 1. \end{cases} \tag{4.91}$$

Then we easily see that ω belongs to Muckenhoupt's A_p class, namely, there exists a positive number C such that for any ball $B_r(x)$ with a center x and a radius r we have

$$\frac{1}{|B_r(x)|}\int_{B_r(x)} \omega(y)\,dy \left(\frac{1}{|B_r(x)|}\int_{B_r(x)} \omega(y)^{-\frac{1}{p-1}}\,dy\right)^{p-1} < \infty. \tag{4.92}$$

We also set

$$J_x(r) = \int_r^\infty t^{\frac{p-n}{p-1}} \left(\frac{1}{|B_t(x)|}\int_{B_t(x)} \omega(y)^{\frac{-1}{p-1}}\,dy\right)\frac{dt}{t}. \tag{4.93}$$

Let us recall the appropriate results due to R. Adams in [Ad].

Proposition 4.1 *Assume that $1 < p < q < \infty$. Then the following two assertions are equivalent to each other.*

1. *There exists a positive number C such that for any $x \in \mathbb{R}^n$ and any $r > 0$ we have*
$$\mu(B_r(x)) \leq C J_x(r)^{-\frac{q}{p'}}, \qquad p' = \frac{p}{p-1}. \tag{4.94}$$

2. *There exists a positive number C such that for any $f \in L^p(\mathbb{R}^n, \omega)$ we have*
$$||I_1 * f||_{L^q(\mathbb{R}^n,\mu)} \leq C||f||_{L^p(\mathbb{R}^n,\omega)}, \tag{4.95}$$
where
$$I_1 * f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-1}}\, dy. \tag{4.96}$$

End of the proof of Theorem 3.7: Since we see that for any $u \in C_0^\infty(\mathbb{R}^n)$
$$|u(x)| \leq C \int_{\mathbb{R}^n} \frac{|\nabla u(y)|}{|x-y|^{n-1}}\, dy,$$
it follows from(4.96) with $f(y) = |\nabla u(y)|$ that the assertion in the case $1 < p < n$ is established. Hence it suffices to check the condition given by (4.94). Since this condition follows from rather elemental calculations, we only check it assuming that $x = 0$ below. If $x \neq 0$, then the weight functions have no singularity, and hence the proof is much easier. Now $J_0(r)$ is estimated as follows.

$$\begin{aligned} J_0(r) &\leq \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (4.97)\\ &\leq C\int_r^\infty t^{\frac{p-n}{p-1}-n}\left(\int_0^{\min(t,1)} r^{-\frac{p-n}{p-1}+n-1}\, dr + \int_{\min(t,1)}^t r^{n-1}\, dr\right)\frac{dt}{t}\\ &\leq C'\int_r^\infty t^{\frac{p-n}{p-1}-n}\left(\min(t,1)^{-\frac{p-n}{p-1}+n} + (t^n - \min(t,1)^n)\right)\frac{dt}{t}. \end{aligned}$$

If $0 < r \leq 1$, the we have for some positive numbers C' and C''
$$J_0(r) \leq C'\left(\int_r^1 \frac{dt}{t} + \int_1^\infty t^{\frac{p-n}{p-1}}\frac{dt}{t}\right) \leq C''\left(\log\frac{1}{r} + 1\right). \tag{4.98}$$
On the other hand, if $r > 1$, then we have
$$J_0(r) \leq C'\left(\int_r^\infty t^{\frac{p-n}{p-1}}\frac{dt}{t}\right) \leq C'' r^{\frac{p-n}{p-1}}. \tag{4.99}$$
We also have
$$\begin{aligned} \mu(B_r(0)) &= \int_{B_r(0)} \frac{dx}{\left(\log\frac{R}{r}\right)^{q+1-\frac{q}{p}}} \qquad\qquad (4.100)\\ &= C\left(\log\frac{R}{r}\right)^{-q+\frac{q}{p}} = C\left(\log\frac{R}{r}\right)^{-\frac{q}{p'}} \end{aligned}$$

Therefore the have the desired estimate (4.94). Lastly we shall show the sharpness of our inequalities. To this end it suffices to establish the next proposition.

Proposition 4.2 *Let p satisfy $1 < p < +\infty$. Assume the critical relation (CR). Then we have:*

1. *The weight function in the right-hand side of (4.82) is sharp.*
2. *Assume that $R = 1$. Then $C(p, q, \alpha, \beta, n) = 0$ if $n \geq 2$ and $p \neq q$.*

Proof: The first assertion is shown similarly in the proof of the sharpness of the weighted Hardy-Sobolev inequalities. In fact, if $p = q$ holds, then the sharpness including the remainder terms follows from Proposition 2.2 . We employ the test functions of the form

$$\varphi_\varepsilon(x) = f_\varepsilon(x) A_1(r)^a,$$

where $0 < \varepsilon < 1$, $R > 1$, $1 - \frac{1}{p} > a$, $A_1(r) = \log \frac{R}{r}$, and f_ε is smooth function having its support in $B_\varepsilon(0)$ and $f_\varepsilon = 1$ in $B_{\frac{\varepsilon}{2}}(0)$. Noting that $(a-1)p + 1 < 0$, we have for some positive numbers C and C'

$$\int_{B_1(0)} |\nabla \varphi_\varepsilon|^p |x|^{p-n}\, dx = \omega_n \int_0^\varepsilon |\nabla \varphi_\varepsilon|^p r^{p-1}\, dr$$

$$\geq C \int_0^{\varepsilon/2} \left(\frac{d}{dr} A_1(r)^a\right)^p r^{p-1}\, dr = C' \left(\log \frac{R}{\varepsilon}\right)^{(a-1)p+1}$$

and

$$\left(\int_{B_1(0)} |\varphi_\varepsilon|^q |x|^{-n} A_1(r)^{\frac{q}{p}-q-1}\, dx\right)^{p/q} = \left(\omega_n \int_0^\varepsilon r^{-1} A_1(r)^{aq+\frac{q}{p}-q-1}\, dr\right)^{p/q}$$

$$\leq C \left(\log \frac{R}{\varepsilon}\right)^{(a-1)p+1}.$$

Hence we see that the power of the weight function A_1 is best possible.

Secondly we show the assertion 2. Let U be a small neighborhood of $(1, 0, \dots, 0) \in \partial B_1 \subset \mathbb{R}^n$. Without a loss of generarity we can assume that by a diffeomorphism ξ, $U \cap B_1$ and $U \cap \partial B_1$ are mapped onto $V = \{y = (y', y_n) \in \mathbb{R}^n_+ : y' \in \mathbb{R}^{n-1}, y_n > 0, |y| < 1\}$ and $\overline{V} \cap \{y_n = 0\}$ respectively. We assume that $R = 1$, and hence $\log \frac{1}{|x|}$ vanishes on the sphere ∂B_1. Therefore the ratio

$$E^c_{p,q,\alpha,\beta,n}(u) \qquad \text{with } R = 1$$

is equivalent to

$$\frac{\int_V \left((\partial_{y_n} v)^2 + |\nabla_{y'} v|^2\right)^{\frac{p}{2}}\, dy}{\left(\int_V |v|^q y_n^{-q-1+\frac{q}{p}}\, dy\right)^{\frac{p}{q}}} \tag{4.101}$$

provided that $u \in C_0^\infty(U)$ and $v(y) = u(\xi^{-1}(y)) \in C_0^\infty(V)$. Now we input a test function defined by

$$v(y) = \varphi y_n^m$$

with $m > 0$ and $\varphi \in C_0^\infty(V)$ satisfies that $\varphi = 1$ if $|y| < \varepsilon$, 0 if $|y| > 2\varepsilon$ and $|\nabla \varphi| \leq \frac{C}{\varepsilon}$ for a sufficiently small $\varepsilon > 0$ and a positive number C. Then the denominator satisfies

$$\left(\int_V |v|^q y_n^{-q-1+\frac{q}{p}}\, dy\right)^{\frac{p}{q}} \geq \left(\int_{|y| \leq \varepsilon} y_n^{mq-q-1+\frac{q}{p}}\, dy\right)^{\frac{p}{q}} = O(\varepsilon^{mp-p+1+\frac{p}{q}n-\frac{p}{q}}),$$

and the numerator satisfies

$$\int_V \left((\partial_{y_n} v)^2 + |\nabla_{y'} v|^2\right)^{\frac{p}{2}}\, dy \leq C \left(\int_V \varphi^p (m y_n^{m-1})^p\, dx + \int_V |\nabla \varphi|^p y_n^{mp}\, dy\right)$$

$$= O(\varepsilon^{pm-p+n})$$

Hence we see that (4.101) is estimated from above by

$$C\varepsilon^{(n-1)(1-\frac{p}{q})}.$$

Therefore the quantity $C(p, q, \alpha, \beta, n) = 0$ unless $n = 1$ or $p = q$. $\square$

4.8 Proof of Theorem 3.8

Theorem 3.8 (The imbedding results in the radial function space) *Let p satisfy $1 < p < +\infty$ and let n satisfy $n \geq 1$. Assume the critical relation (CR). Then $C_{rad.}(p, q, \alpha, \beta, n)$ is determined. Moreover $C_{rad}(p, q, \alpha, \beta, n)$ is achieved only if $n = 1$.*

Proof: As is discussed in the previous section, the variational problem (P^c_R) is attained only when $n = 1$. More precisely it was shown that this problem is equivalent to the next one.

$$C_{rad}(p, q, \alpha, \beta, n) = \inf_{u \in R^{1,p}_{\alpha,0}(B_1)\setminus\{0\}} E^c_{p,q,\alpha,\beta,n}(u). \qquad (P^c_R)$$

Here

$$E^c_{p,q,\alpha,\beta,n}(v) = \omega_n^{1-\frac{p}{q}} \frac{\int_0^1 |\partial_r v|^p r^{p-1}\, dr}{\left(\int_0^1 \frac{|v|^q}{rA_1(r)^{q+1-\frac{q}{p}}}\, dr\right)^{q/p}}. \quad \left(v \in R^{1,p}_{\alpha,0}(B_1) \setminus \{0\}\right),$$

where $A_1(t) = \log \frac{R}{t}$ for $R > 1$ if $n > 1$ and $R = 1$ if $n = 1$. Then we showed that

$$C_{rad}(p, q, \alpha, \beta, n) = k^{p-1+\frac{p}{q}} \omega_n^{1-\frac{p}{q}} D(p, q, \alpha, \beta, n),$$

where $\gamma = n\left(\frac{1}{p} - \frac{1}{q}\right) > 0$, $k = \left(1 - \frac{1}{p}\right)\left(\frac{q}{p} - 1\right) > 0$ and

$$D(p, q, \alpha, \beta, n) = \inf_{u \in C_0^\infty(\overline{\mathbb{R}_+})\setminus\{0\}, u \geq 0} \frac{\int_0^\infty |\partial_t u|^p t^{\frac{n}{\gamma}-1}\, dt}{\left(\int_0^\infty u^q t^{\frac{n}{\gamma}-1}\, dt\right)^{\frac{p}{q}}}$$

Moreover if and only if $n = 1$, $C_{rad}(p, q, \alpha, \beta, n)$ is attained by some function in $R^{1,p}_{\alpha,0}(B_1)$. □

4.9 Proof of Theorem 3.9

We shall establish the following.

Theorem 3.9 (The continuity of the best constant) *Let p satisfy $1 \leq p < +\infty$ and let R satisfy $R > 1$. Assume the critical relation (CR).*

1. *For $p > 1$, $C(p, q, \alpha, \beta, n)$ is positive and continuous on $\beta \in [\alpha - 1, \alpha]$, if $1 < p < n$, and on $\beta \in [\alpha - 1, 0)$, if $p \geq n$. In particular we have*

$$\lim_{\beta \to \alpha - 1 + 0} C(p, q, \alpha, \beta, n) = \Lambda_{n,p,\alpha}. \tag{4.102}$$

2. *If $p = 1$, then $C(p, q, \alpha, \beta, n) = 0$.*

Proof of Theorem 3.8: The assertion (1) can be shown in the same way as the argument in the proof of the continuity of $S(p, q, \alpha, \beta, n)$. In particular in the radial case the property (4.102) follows directly from Lemma 4.18 and Remark 4.9. (See also (4.20).)

Now we assume that $p = 1$. Then the inequality becomes

$$\int_0^1 |\partial_r u|\, dr \geq C(p, q, \alpha, \beta, n) \left(\int_0^1 |u|^q \left(\log \frac{R}{r}\right)^{-1} dr\right)^{\frac{1}{q}}$$

for any radial function $u \in C_0^\infty([0,1))$. They by a change of variable $t = \log\frac{R}{r}$, it becomes

$$\int_{\log R}^{\infty} |\partial_t v|\, dt \geq C(p,q,\alpha,\beta,n) \left(\int_{\log R}^{\infty} |v|^q t^{-1}\, dt \right)^{\frac{1}{q}}$$

for any $v \in C^\infty(\log R, \infty)$ with $v(\log R) = 0$ and $\lim_{t\to\infty} |v(t)| < +\infty$. Since $\frac{1}{t}$ is not integrable on $[\log R, \infty]$, this does not holds for functions v such that v is monotone increasing and $v(\infty) = 1$.

5 Appendices

5.1 Appendix 1

In this section we shall explain that the inequalities in Proposition 1.2 do not hold in general if Ω is replaced by the whole domain $\mathbb{R}^n$. In fact we have

Theorem 2.1 *Let $n \geq 1$, $\Omega = \mathbb{R}^n$ and $\alpha \neq 1 - \frac{n}{p}$.*

1. *Assume that $p = 2$. Then the inequalities (1.20) and (1.22) in Proposition 1.2 do not hold any more.*
2. *Assume that $1 < p < +\infty$. Assume that f is a locally integrable nonnegative function f on $\mathbb{R}^n$ which is not identically 0. Then we have*

$$\inf_{u \in C_0^\infty(\mathbb{R}^n \setminus \{0\}) \setminus \{0\}} \frac{\left(\int_{\mathbb{R}^n} |\nabla u|^p |x|^{\alpha p}\, dx\right)^{\frac{1}{p}} - \Lambda_{n,p,\alpha}^{\frac{1}{p}} \left(\int_{\mathbb{R}^n} |u|^p |x|^{p(\alpha-1)}\, dx\right)^{\frac{1}{p}}}{\left(\int_{\mathbb{R}^n} |u|^p f(x)\, dx\right)^{\frac{1}{p}}} = 0.$$

Proof: We consider the subcritical case ($\alpha > 1 - \frac{n}{p}$) only, since the argument is quite similar in the other case. On the contrary, assume that there is a locally integrable nonnegative function f on $\mathbb{R}^n$ such that $f > 0$ in some neighborhood of the origin, we have the Hardy-Sobolev inequality with a remainder. If $p \neq 2$, then it holds that

$$\begin{aligned} &\left(\int_{\mathbb{R}^n} |\nabla u|^p |x|^{\alpha p}\, dx \right)^{\frac{1}{p}} \\ &\geq \Lambda_{n,p,\alpha}^{\frac{1}{p}} \left(\int_{\mathbb{R}^n} |u|^p |x|^{p(\alpha-1)}\, dx \right)^{\frac{1}{p}} + \left(\int_{\mathbb{R}^n} |u|^p f(x)\, dx \right)^{\frac{1}{p}} \end{aligned} \tag{5.1}$$

for any $u \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$.
If $p = 2$, then we assume that

$$\int_{\mathbb{R}^n} |\nabla u|^2 |x|^{2\alpha}\, dx \geq \Lambda_{n,2,\alpha} \int_{\mathbb{R}^n} |u|^2 |x|^{2(\alpha-1)}\, dx + \int_{\mathbb{R}^n} |u|^2 f(x)\, dx. \tag{5.2}$$

Now we assume that u is a radial function and let us set $u = r^{-\delta} v$, for $u \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$, $\delta = \alpha - 1 + \frac{n}{p} > 0$ and $r = |x|$. Then we have for any $p > 1$

$$\begin{aligned} &\left(\int_{\mathbb{R}^n} \left| \nabla v - \delta \frac{xv}{r^2} \right|^p r^{p-n}\, dr \right)^{\frac{1}{p}} \\ &\geq \Lambda_{n,p,\alpha}^{\frac{1}{p}} \left(\int_{\mathbb{R}^n} |v|^p r^{-n}\, dx \right)^{\frac{1}{p}} + \left(\int_{\mathbb{R}^n} |v|^p r^{-\delta p} f(x)\, dx \right)^{\frac{1}{p}}. \end{aligned}$$

By the Minkowski inequality and the fact $\delta^p = \Lambda_{n,p,\alpha}$ we have for any $v \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$

$$\left(\int_{\mathbb{R}^n} |\nabla v|^p r^{p-n}\, dx\right)^{\frac{1}{p}} \geq \left(\int_{\mathbb{R}^n} |v|^p r^{-\delta p} f(x)\, dx\right)^{\frac{1}{p}}. \tag{5.3}$$

When $p = 2$, we have the same conclusion by the aid of integrations by parts. Without a loss of generality we assume that $fr^{-\delta p}$ is bounded and integrable. By the definition of $W^{1,p}_{1-\frac{n}{p},0}(\mathbb{R}^n)$ (the critical case) we see that (5.3) holds for any $v \in W^{1,p}_{1-\frac{n}{p},0}(\mathbb{R}^n)$ (the critical case). From Proposition 2.3 we see that both $C_0^\infty(\mathbb{R}^n)$ and $C_0^\infty(\mathbb{R}^n \setminus \{0\})$ are densely contained in $W^{1,p}_{1-\frac{n}{p},0}(\mathbb{R}^n)$. Therefore (5.3) also holds for any $v \in C_0^\infty(\mathbb{R}^n)$. But this is impossible because of Proposition 2.3. In fact for any ball B_ε such as $\int_{B_\varepsilon} fr^{-\delta p}\, dx > 0$, we have $C^p(\overline{B}_\varepsilon, \mathbb{R}^n; |x|^{p-n}) = 0$. □

5.2 Appendix 2

In this section we shall review that the equality signs in the variational problems P and P_R given by Definition 3.4 can not be achieved by any function with compact support. On the contrary, we assume that there exists an extremal u having the support in a ball $B_r = \{x \in \mathbb{R}^n : |x| < r\}$, namely, the infimum is attained by u. Here we may assume u is nonnegative. Moreover it has to satisfy the Euler Lagrange equation in distribution sense;

$$-\operatorname{div}(|x|^{\alpha p}|\nabla u|^{p-2}\nabla u) = \lambda |x|^{\beta q} u^{q-1}, \quad \text{in } B_r \tag{5.4}$$

$$u|_{\partial B_r} = 0, \qquad u > 0 \qquad \text{in } B_r. \tag{5.5}$$

Here $\lambda > 0$ is a Lagrange multiplier. Then it follows from the next lemma that u have to vanish almost everywhere in B_r.

Lemma 5.1 *(Pohozaev identity)Let p, q, n, α and β satisfy $1 < p$, $0 \leq 1/p - 1/q \leq (1-\alpha+\beta)/n$, $(1-\alpha+\beta)p < n$ and $\beta > -n/q$. Assume that $u \in W^{1,p}_{\alpha,0}(\mathbb{R}^n)$ satisfy the equation (5.4) with Dirichlet boundary condition (5.5) in distribution sense. Then it holds that*

$$\begin{aligned}\lambda[1-\alpha+\beta-n(1/p-1/q)] &\int_{B_r} |x|^{\beta q} u^q\, dx \\ &= (1-1/p) \int_{\partial B_r} |x|^{\alpha p}(x,\nu)|\nabla u|^p\, dS_\omega,\end{aligned} \tag{5.6}$$

where ν is the unit outer normal to ∂B_r and S_ω is the $(n-1)$-dimensional Lebesgue measure, and $W^{1,p}_{\alpha,0}(\mathbb{R}^n)$ is defined by Definition 2.4.

When $1/p - 1/q > (1-\alpha+\beta)/n$ it follows immediately from (5.6) that $u = 0$. When $1/p - 1/q = (1-\alpha+\beta)/n$, we deduce from (5.6) that $\frac{\partial u}{\partial \nu} = 0$ on ∂B_r, and then by (5.4)

$$0 = -\int_{B_r} \operatorname{div}(|x|^{\alpha p}|\nabla u|^{p-2}\nabla u)\, dx = \lambda \int_{B_r} |x|^{\beta q} u^{q-1}\, dx, \tag{5.7}$$

thus $u = 0$.

Proof of Lemma 5.1 By a standard argument of regularization, we may assume that u is smooth. Then the equality is established by the computation of $\operatorname{div} P$ and an integration by parts for

$$P = |x|^{\alpha p}|\nabla u|^{p-2}(\nabla u, x)\nabla u. \tag{5.8}$$

For the precise see [EG3; Prop.13], [GV] and [PO]. □

5.3 Appendix 3

We shall introduce a spherically symmetric decreasing rearrangement of function with respect to an admissible weight function. Let us recall the definition of an Admissible weight function.

Definition 4.1(An admissible weight function) *A positive function f on $\mathbb{R}^n$ is said to be admissible if and only if it is spherically symmetric, monotonically decreasing, locally integrable in $\mathbb{R}^n$ and continuous in $\mathbb{R}^n \setminus \{0\}$.*

Then we define a spherically symmetric decreasing rearrangement of function u with respect to an admissible weight $f(x)$. Let us set for any measurable set $A \subset \mathbb{R}^n$,

$$\mu(A) = \int_A f(x)\,dx. \tag{5.9}$$

Then we define for any nonnegative, bounded integrable u on $\mathbb{R}^n$ and $t \geq 0$

$$\mu_u(t) = \mu(\{x \in \mathbb{R}^n : u(x) > t\}) = \int_{\{u(x)>t\}} f(x)\,dx. \tag{5.10}$$

Finally we define

$$u^*(x) = \sup\{t : \mu_u(t) > \mu(B_{|x|}(0))\}, \tag{5.11}$$

where $B_{|x|}(0)$ is a unit ball centered at the origin with a radius $|x|$. Since both u and u^* have the same distribution function $\mu(t)$ we see that

Lemma 5.2 *Foy any nonnegative, bounded and integrable function u on $\mathbb{R}^n$, it holds that*

$$\int_{\mathbb{R}^n} u^p f(x)\,dx = \int_{\mathbb{R}^n} u^{*p} f(x)\,dx.$$

Proof: By the definition we have

$$\int_{\mathbb{R}^n} u^p f(x)\,dx = \int_0^\infty pt^{p-1}\,dt \int_{\{u>t\}} f(x)\,dx = \int_0^\infty \mu_u(t)\,d(t^p)$$
$$= \int_0^\infty \mu_{u^*}(t)\,d(t^p) = \int_0^\infty pt^{p-1}\,dt \int_{\{u^*>t\}} f(x)\,dx = \int_{\mathbb{R}^n} u^{*p} f(x)\,dx.$$

□

Now we recall the coarea formula.

Lemma 5.3 *For any $u \in C_0^\infty(\mathbb{R}^n)$ and any bounded measurable function Φ on $\mathbb{R}^n$ we have*

$$\int_{\mathbb{R}^n} |\nabla u|^p \Phi(x)\,dx = \int_{-\infty}^\infty dt \int_{\{u=s\}} |\nabla u|^{p-1} \Phi\, dH^{n-1}. \tag{5.12}$$

Here H^{n-1} is the $n-1$-dimensional Hausdorff measure.

Remark 5.1 *This formula is also valid under the assumption that u is Lipschitz continuous and ∇u is integrable.*

In this formula, assuming that $u \in C_0^\infty(\mathbb{R}^n)$ is nonnegative and

$$\Phi = \begin{cases} \frac{\chi_{\{u>t\}}(x)}{|\nabla u|^p} f(x), & \text{if } \nabla u \neq 0, \\ 0 & \text{if } \nabla u = 0. \end{cases}$$

we have

$$\mu_u(t) = \mu(\{u > t\} \cap \{\nabla u = 0\}) + \int_t^\infty dt \int_{\{u=s\}} \frac{f(x)}{|\nabla u|} \, dH^{n-1}. \tag{5.13}$$

By Sard's lemma the set of critical values of $u \in C_0^\infty(\mathbb{R}^n)$ has a vanishing measure, hence we have

$$\mu_u'(t) = -\int_{\{u=t\}} \frac{f(x)}{|\nabla u|} \, dH^{n-1}, \qquad \text{for almost all } t \in (0,\infty) \tag{5.14}$$

Remark 5.2 *When u is not so smooth, we can not employ Sard's lemma. But a somewhat weaker result holds provided that the formula (5.13) is valid, namely*

$$\mu_u'(t) \geq -\int_{\{u=t\}} \frac{f(x)}{|\nabla u|} \, dH^{n-1}, \qquad \textit{for almost all } t \in (0,\infty). \tag{5.15}$$

Now we replace u by its rearrangement u^* in (5.13). We recall that both u and u^* share the same distribution function, and u^* is at least Lipschitz continuous as a rearrangement of a smooth u. Therefore we have

$$\mu_u'(t) = -\frac{\int_{\{u^*=t\}} f \, dH^{n-1}}{|\nabla u^*|_{|\{u^*=t\}}}, \qquad \text{for almost all } t \in (0,\infty). \tag{5.16}$$

Then we have

Lemma 5.4 *Assume that a weight f is admissible. Then for any nonnegative $u, v \in C_0^\infty(\mathbb{R}^n)$, we have the followings:*

1. *For almost all $t \in [0,\infty)$*

$$\int_{\{u^*=t\}} dH^{n-1} \leq \int_{\{u=t\}} dH^{n-1} \tag{5.17}$$

2. *For $g(x) = f(x)^{1-p}$,*

$$\int_{\mathbb{R}^n} |\nabla u^*|^p g(x) \, dx \leq \int_{\mathbb{R}^n} |\nabla u|^p g(x) \, dx. \tag{5.18}$$

3.

$$\int_{\mathbb{R}^n} u \cdot v f(x) \, dx \leq \int_{\mathbb{R}^n} u^* \cdot v^* f(x) \, dx. \tag{5.19}$$

Proof of Lemma 5.4. Assertion 1. Let A be any Borel set such as $0 < |A| < \infty$. By A^* we denote the rearrangement of A with respect to an admissible weight f, namely, A^* ia a ball centered at the origin satisfying

$$\mu(A) = \mu(A^*). \tag{5.20}$$

Let r be a positive number such that

$$|B_r(0)| = |A|. \tag{5.21}$$

Since

$$\int_{B_r(0)} f(x) \, dx \geq \int_A f(x) \, dx, \tag{5.22}$$

we see that

$$A^* \subset B_r(0). \tag{5.23}$$

Therefore we conclude that

$$\int_{\partial A} dH^{n-1} \geq \int_{\partial B_r(0)} dH^{n-1} \geq \int_{\partial A^*} dH^{n-1}. \tag{5.24}$$

Since A is an arbitrary Borel set, the assertion follows immediately.
Assertion 2. Since u^* is Lipschitz continuous, we can employ the coarea formula. Then we have

$$\begin{aligned}
&\int_{\mathbb{R}^n} |\nabla u^*|^p g(x)\, dx = \int_0^\infty dt \int_{\{u^*=t\}} |\nabla u^*|^{p-1} g(x)\, dH^{n-1} \\
&= \int_0^\infty dt \frac{\left(\int_{\{u^*=t\}} dH^{n-1}\right)^p}{\left(\int_{\{u^*=t\}} \frac{g(x)^{-\frac{1}{p-1}}}{|\nabla u^*|} dH^{n-1}\right)^{p-1}} \\
&= \int_0^\infty dt \frac{\left(\int_{\{u^*=t\}} dH^{n-1}\right)^p}{\left(\int_{\{u^*=t\}} \frac{f(x)}{|\nabla u^*|} dH^{n-1}\right)^{p-1}} \\
&\leq \int_0^\infty dt \frac{\left(\int_{\{u=t\}} dH^{n-1}\right)^p}{(-\mu'(t))^{p-1}} \qquad \text{(Assertion 1)} \\
&\leq \int_0^\infty dt \int_{\{u=t\}} |\nabla u|^{p-1} g(x)\, dH^{n-1} \qquad \text{(Hölder inequality)} \\
&= \int_{\mathbb{R}^n} |\nabla u|^p g(x)\, dx.
\end{aligned}$$

Clearly this proves the assertion.
Assertion 3. By the layer cake representation of functions, the desired inequality becomes

$$\begin{aligned}
&\int_0^\infty \int_0^\infty \int_{\mathbb{R}^n} \chi_{\{u>t\}}(x)\chi_{\{v>s\}}(x) f(x)\, dxdsdt \\
&\qquad \leq \int_0^\infty \int_0^\infty \int_{\mathbb{R}^n} \chi_{\{u^*>t\}}(x)\chi_{\{v^*>s\}}(x) f(x)\, dxdsdt.
\end{aligned}$$

The it suffices to show the next inequality for any bounded Borel sets A and B.

$$\int_{\mathbb{R}^n} \chi_A(x)\chi_B(x)\, d\mu \leq \int_{\mathbb{R}^n} \chi_{A^*}(x)\chi_{B^*}(x)\, d\mu, \qquad \text{for } d\mu = f(x)dx. \tag{5.25}$$

We may assume that $\mu(A) \leq \mu(B)$. Then we see that $A^* \subset B^*$. Therefore we have

$$\mu(A \cap B) \leq \mu(A) = \mu(A^*) = \mu(A^* \cap B^*). \tag{5.26}$$

This proves the assertion. □

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