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EXISTENCE AND UNIQUENESS OF THE SOLUTION OF AN INTEGRO-DIFFERENTIAL EQUATION WITH WEAKLY SINGULAR KERNEL

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Abstract. A hyperbolic type integro-differential equation with weakly singular kernel is considered together with mixed homogeneous Dirichlet and non-homogeneous Neumann bounadry conditions. Existence and uniqueness of the solution is proved by means of Galerkin method. Regularity estimates are proved and the limitations of the regularity is discussed.

1. Introduction

We study a model problem, which is a hyperbolic type integro-differential equation with weakly singular kernel. This problem arises as a model for fractional order viscoelasticity. The fractional order viscoelastic model, that is, the linear viscoelastic model with fractional order operators in the constitutive equations, is capable of describing the behavior of many viscoelastic materials by using only a few parameters.

There is an extensive literature regarding well-posedness and numerical treatment for integro-differential equations, see, e.g., [1], [3], [9], [10], [11], [12], [13], [14], [15], [16], and [17]. Existence, uniqueness and regularity of a reformed model has been studied in [12] by means of Fourier series. One may also see [6], where the theory of analytic semigroups is used in terms of interpolation spaces. An abstract Volterra equation, as an abstract model for equations of linear viscoelasticity, has been studied in [4]. In a previous work [9], well-posedness and regularity of the model problem was studied in the framework of the semigroup of linear operators. The drawback of the framework is that this does not admit non-homogeneous Neumann boundary condition. While in practice mixed homogeneous Dirichlet and non-homogeneous Neumann boundary conditions are of special interest. Here we investigate existence, uniqueness and regularity of the solution of the problem (2.5) by means of the Galerkin approximation method.

In the sequel, in §2 we describe the construction of the model and we define a weak (generalized) solution. Then in §3 we study well-posedness, regularity and limitations for higher regularity.

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2. The model problem and weak formulation

Let σ_{ij}, ϵ_{ij} and u_i denote, respectively, the usual stress tensor, strain tensor and displacement vector. We recall that the linear strain tensor is defined by

$$\epsilon_{ij} = \frac{1}{2}\Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\Big).$$

With the decompositions

$$s_{ij} = \sigma_{ij} - \tfrac{1}{3}\sigma_{kk}\delta_{ij}, \quad e_{ij} = \epsilon_{ij} - \tfrac{1}{3}\epsilon_{kk}\delta_{ij},$$

the constitutive equations are formulated in [2] as

$$\begin{aligned} s_{ij}(t) + \tau_1^{\alpha_1} D_t^{\alpha_1} s_{ij}(t) &= 2G_\infty e_{ij}(t) + 2G\tau_1^{\alpha_1} D_t^{\alpha_1} e_{ij}(t), \\ \sigma_{kk}(t) + \tau_2^{\alpha_2} D_t^{\alpha_2} \sigma_{kk}(t) &= 3K_\infty \epsilon_{kk}(t) + 3K\tau_2^{\alpha_2} D_t^{\alpha_2} \epsilon_{kk}(t), \end{aligned} \tag{2.1}$$

with initial conditions

$$s_{ij}(0+) = 2Ge_{ij}(0+), \quad \sigma_{kk}(0+) = 3K\epsilon_{kk}(0+),$$

meaning that the initial response follows Hooke's elastic law. Here G, K are the instantaneous (unrelaxed) moduli, and G_∞, K_∞ are the long-time (relaxed) moduli. Note that we have two relaxation times, $\tau_1, \tau_2 > 0$, and fractional orders of differentiation, α_1, $\alpha_2 \in (0,1)$, where the fractional order derivative is defined by

$$D_t^\alpha f(t) = D_t D_t^{-(1-\alpha)} f(t) = D_t \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} f(s)\, ds.$$

The contitutive equations (2.1) can be solved for σ by means of Laplace transformation, [7]:

$$\begin{aligned} s_{ij}(t) &= 2G\Big(e_{ij}(t) - \frac{G-G_\infty}{G}\int_0^t \theta_1(t-s) e_{ij}(s)\, ds\Big), \\ \sigma_{kk}(t) &= 3K\Big(\epsilon_{kk}(t) - \frac{K-K_\infty}{K}\int_0^t \theta_2(t-s)\epsilon_{kk}(s)\, ds\Big), \end{aligned}$$

where

$$\theta_i(t) = -\frac{d}{dt} E_{\alpha_i}\Big(-\Big(\frac{t}{\tau_i}\Big)^{\alpha_i}\Big), \quad E_{\alpha_i}(t) = \sum_{n=0}^{\infty} \frac{t^n}{\Gamma(1+n\alpha_i)},$$

and E_{α_i} is the Mittag-Leffler function of order α_i. We make the simplifying assumption (synchronous viscoelasticity):

$$\alpha = \alpha_1 = \alpha_2, \quad \tau = \tau_1 = \tau_2, \quad f = f_1 = f_2.$$

Then we may define a parameter γ, a kernel β, and the Lamé constants μ, λ,

$$\gamma = \frac{G-G_\infty}{G} = \frac{K-K_\infty}{K}, \quad \beta(t) = \gamma\theta(t), \quad \mu = G, \quad \lambda = K - \tfrac{2}{3}G,$$

and the constitutive equations become

$$\begin{aligned} \sigma_{ij}(t) &= \Big(2\mu\epsilon_{ij}(t) + \lambda\epsilon_{kk}(t)\delta_{ij}\Big) - \int_0^t \beta(t-s)\Big(2\mu\epsilon_{ij}(s) + \lambda\epsilon_{kk}(s)\delta_{ij}\Big)\, ds, \\ &= (\sigma_0)_{ij}(t) - \int_0^t \beta(t-s)(\sigma_0)_{ij}(s)\, ds. \end{aligned}$$

Note that the viscoelastic part of the model contains only three parameters:

$$0 < \gamma < 1, \quad 0 < \alpha < 1, \quad \tau > 0.$$

The kernel is weakly singular:

$$\begin{aligned} \beta(t) &= -\gamma \frac{d}{dt} E_\alpha\Big(-\Big(\frac{t}{\tau}\Big)^\alpha\Big) = \gamma \frac{\alpha}{\tau}\Big(\frac{t}{\tau}\Big)^{-1+\alpha} E'_\alpha\Big(-\Big(\frac{t}{\tau}\Big)^\alpha\Big) \\ &\approx C t^{-1+\alpha}, \quad t \to 0, \end{aligned} \tag{2.2}$$

and we note the properties

$$\begin{aligned} &\beta(t) \geq 0, \\ &\|\beta\|_{L_1(\mathbb{R}^+)} = \int_0^\infty \beta(t)\, dt = \gamma\Big(E_\alpha(0) - E_\alpha(\infty)\Big) = \gamma < 1. \end{aligned} \tag{2.3}$$

The equations of motion now become, (we denote time derivatives with '$\cdot$'):

$$\begin{aligned} &\rho \ddot{u}_i - \sigma_{ij,j} = f_i && \text{in } \Omega, \\ &u_i = 0 && \text{on } \Gamma_{\mathrm{D}}, \\ &\sigma_{ij} n_j = g_i && \text{on } \Gamma_{\mathrm{N}}, \end{aligned} \tag{2.4}$$

where ρ is the (constant) mass density, and f and g represent, respectively, the volume and the surface loads. We let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, be a bounded polygonal domain with boundary $\Gamma_{\mathrm{D}} \cup \Gamma_{\mathrm{N}} = \partial\Omega$, $\Gamma_{\mathrm{D}} \cap \Gamma_{\mathrm{N}} = \varnothing$ and $\mathrm{meas}(\Gamma_{\mathrm{D}}) \neq 0$. We set

$$(Au)_i = -\big(2\mu\epsilon_{ij}(u) + \lambda\epsilon_{kk}(u)\delta_{ij}\big)_{,j},$$

that is, $Au = -\nabla \cdot \sigma_0(u)$, and we write the equations of motion (2.4) in the form

$$\begin{aligned} &\rho\ddot{u}(x,t) + Au(x,t) \\ &\qquad - \int_0^t \beta(t-s) Au(x,s)\, ds = f(x,t) && \text{in } \Omega \times (0,T), \\ &u(x,t) = 0 && \text{on } \Gamma_{\mathrm{D}} \times (0,T), \\ &\sigma(u;x,t) \cdot n = g(x,t) && \text{on } \Gamma_{\mathrm{N}} \times (0,T), \\ &u(x,0) = u^0(x) && \text{in } \Omega, \\ &\dot{u}(x,0) = v^0(x) && \text{in } \Omega. \end{aligned} \tag{2.5}$$

We introduce the function spaces $H = L_2(\Omega)^d$, $H_{\Gamma_{\mathrm{N}}} = L_2(\Gamma_{\mathrm{N}})^d$, and $V = \{v \in H^1(\Omega)^d : v|_{\Gamma_{\mathrm{D}}} = 0\}$. We define the bilinear form (with the usual summation convention)

$$a(u,v) = \int_\Omega \big(2\mu\epsilon_{ij}(u)\epsilon_{ij}(v) + \lambda\epsilon_{ii}(u)\epsilon_{jj}(v)\big)\, dx, \quad \forall u, v \in V,$$

which is coercive on V. We denote the norms in H and $H_{\Gamma_{\mathrm{N}}}$ by $\|\cdot\|$ and $\|\cdot\|_{\Gamma_{\mathrm{N}}}$, respectively, and we equip V with the inner product $a(\cdot,\cdot)$ and norm $\|v\|_V^2 = a(v,v)$.

Now we define a weak solution to be a function $u = u(x,t)$ that satisfies

$$u \in L_2((0,T);V), \quad \dot{u} \in L_2((0,T);H), \quad \ddot{u} \in L_2((0,T);V^*), \tag{2.6}$$

$$\begin{aligned} \rho\langle\ddot{u}(t), v\rangle + a(u(t), v) - \int_0^t \beta(t-s) a(u(s), v)\, ds \\ = (f(t), v) + (g(t), v)_{\Gamma_{\mathrm{N}}}, \quad \forall v \in V, \text{ a.e. } t \in (0,T), \end{aligned} \tag{2.7}$$

$$u(0) = u^0, \quad \dot{u}(0) = v^0. \tag{2.8}$$

Here $(g(t), v)_{\Gamma_{\mathrm{N}}} = \int_{\Gamma_{\mathrm{N}}} g(t) \cdot v \, dS$, and $\langle \cdot, \cdot \rangle$ denotes the pairing of V^* and V. We note that (2.6) implies, by a classical result for Sobolev spaces, that $u \in \mathcal{C}([0,T]; H)$, $\dot{u} \in \mathcal{C}([0,T]; V^*)$ so that the initial conditions (2.8) make sense for $u^0 \in H$, $v^0 \in V^*$.

3. Existence, uniqueness and regularity

In this section we prove existence and uniqueness as well as regularity of a weak solution of (2.5) using Galerkin method, in a similar way as for hyperbolic PDE's in [8], [5]. To this end, we first introduce the Galerkin approximation of a weak solution of (2.5) in a classical way, and we obtain a priori estimates for approximate solutions. These will be used to construct a weak solution and then we will verify uniqueness as well as regularity.

We recall (2.2), (2.3) and we define the function

$$\xi(t) = \gamma - \int_0^t \beta(s)\, ds = \int_t^\infty \beta(s)\, ds = \gamma E_\alpha(t), \tag{3.1}$$

and it is easy to see that

$$D_t \xi(t) = -\beta(t) < 0, \quad \xi(0) = \gamma, \quad \lim_{t\to\infty} \xi(t) = 0, \quad 0 < \xi(t) \le \gamma. \tag{3.2}$$

Besides, ξ is a completely monotone function, that is,

$$(-1)^j D_t^j \xi(t) \ge 0, \quad t \in (0,\infty),\ j \in \mathbb{N},$$

since the Mittag-Leffler function E_α, $\alpha \in [0,1]$ is completely monotone. Consequently, an important property of ξ, is that it is a positive type kernel, that is, it is continuous and, for any $T \ge 0$, satisfies

$$\int_0^T \int_0^t \xi(t-s)\phi(t)\phi(s)\, ds\, dt \ge 0, \quad \forall \phi \in \mathcal{C}([0,T]). \tag{3.3}$$

3.1. **Galerkin approximations.** Let $\{(\lambda_j, \varphi_j)\}_{j=1}^\infty$ be the eigenpairs of the weak eigenvalue problem

$$a(\varphi, v) = \lambda(\varphi, v), \quad \forall v \in V. \tag{3.4}$$

It is known that $\{\varphi_j\}_{j=1}^\infty$ can be chosen to be an ON-basis in H and an orthogonal basis for V.

Now, for a fixed positive integer $m \in \mathbb{N}$, we seek a function of the form

$$u_m(t) = \sum_{j=1}^m d_j(t)\varphi_j \tag{3.5}$$

to satisfy

$$\begin{aligned} \rho(\ddot{u}_m(t), \varphi_k) + a(u_m(t), \varphi_k) - \int_0^t \beta(t-s) a(u_m(s), \varphi_k)\, ds \\ = (f(t), \varphi_k) + (g(t), \varphi_k)_{\Gamma_{\mathrm{N}}}, \quad k = 1, \ldots, m,\ t \in (0,T), \end{aligned} \tag{3.6}$$

with initial conditions

$$u_m(0) = \sum_{j=1}^m (u^0, \varphi_j)\varphi_j, \quad \dot{u}_m(0) = \sum_{j=1}^m (v^0, \varphi_j)\varphi_j. \tag{3.7}$$

Theorem 1. *For each $m \in \mathbb{N}$, there exists a unique function u_m of the form (3.5) satisfying (3.6)–(3.7). Moreover, if $u^0 \in V, v^0 \in H$, $f \in L_2((0,T);H)$, $g \in W_1^1((0,T);H^{\Gamma_N})$, there is a constant $C = C(\Omega, \gamma, \rho, T)$ such that,*

$$\begin{aligned} &\|u_m\|_{L_\infty((0,T);V)} + \|\dot{u}_m\|_{L_\infty((0,T);H)} + \|\ddot{u}_m\|_{L_2((0,T);V^*)} \\ &\qquad \le C\big\{\|u^0\|_V + \|v^0\| + \|g\|_{W_1^1((0,T);H^{\Gamma_N})} + \|f\|_{L_2((0,T);H)}\big\}. \end{aligned} \tag{3.8}$$

Proof. Using (3.5) and the fact that $\{\varphi_j\}_{j=1}^\infty$ is an ON-basis for H and a solution of the eigenvalue problem (3.4), we obtain from (3.6) that,

$$\rho\ddot{d}_k(t) + \lambda_k d_k(t) - \lambda_k(\beta * d_k)(t) = f_k(t) + g_k(t), \quad k = 1, \ldots, m,\ t \in (0,T), \tag{3.9}$$

where $*$ denotes the convolution, and $f_k(t) = (f(t), \varphi_k)$, $g_k(t) = (g(t), \varphi_k)_{\Gamma_N}$. This is a system of second order ODE's with the initial conditions

$$d_k(0) = (u^0, \varphi_k), \quad \dot{d}_k(0) = (v^0, \varphi_k), \quad k = 1, \ldots, m. \tag{3.10}$$

The Laplace transform can be used, for example, to find the unique solution of the system. Indeed, the Laplace transform of the Mittag-Leffler function is,

$$\mathcal{L}(E_\alpha(ax^\alpha)) = \frac{s^{\alpha-1}}{s^\alpha - a}, \quad s > |a|^{1/\alpha}.$$

Hence for the kernel β defined in (2.2) we have,

$$\begin{aligned} \mathcal{L}(\beta(t)) &= -\gamma s\mathcal{L}(E_\alpha(-\tau^{-\alpha}t^\alpha)) + \gamma E_\alpha(0) \\ &= -\gamma s\frac{s^{\alpha-1}}{s^\alpha + \tau^{-\alpha}} + \gamma = \gamma - \gamma\frac{s^\alpha}{s^\alpha + \tau^{-\alpha}} = \frac{\gamma}{(\tau s)^\alpha + 1}. \end{aligned} \tag{3.11}$$

Then, taking the Laplace transform of (3.9) we get,

$$\begin{aligned} &\Big(\rho s^2 + \lambda_k - \lambda_k\mathcal{L}(\beta)(s)\Big)\mathcal{L}(d_k)(s) \\ &\qquad = \mathcal{L}(f_k)(s) + \mathcal{L}(g_k)(s) + \rho d_k(0)s + \rho\dot{d}_k(0), \end{aligned} \tag{3.12}$$

where the inverse Laplace transform is computable. Therefore, there is a unique solution for the system (3.9) with the initial conditions (3.10).

Now we prove the a priori estimate (3.8). Since $\beta(t-s) = D_s\xi(t-s)$, by (3.2), we can write (3.6), after partial integration, as

$$\begin{aligned} \rho(\ddot{u}_m(t), \varphi_k) + \tilde{\gamma}a(u_m(t), \varphi_k) + \int_0^t \xi(t-s)a(\dot{u}_m(s), \varphi_k)\,ds \\ = (f(t), \varphi_k) + (g(t), \varphi_k)_{\Gamma_N} \\ - \xi(t)a(u_m(0), \varphi_k), \quad k = 1, \ldots, m,\ t \in (0,T), \end{aligned}$$

where $\tilde{\gamma} = 1 - \gamma$. Then multiplying by $\dot{d}_k(t)$ and summing $k = 1, \ldots, m$, we have,

$$\begin{aligned} \rho(\ddot{u}_m(t), \dot{u}_m(t)) + \tilde{\gamma}a(u_m(t), \dot{u}_m(t)) + \int_0^t \xi(t-s)a(\dot{u}_m(s), \dot{u}_m(t))\,ds \\ = (f(t), \dot{u}_m(t)) + (g(t), \dot{u}_m(t))_{\Gamma_N} \\ - \xi(t)a(u_m(0), \dot{u}_m(t)), \quad t \in (0,T). \end{aligned}$$

Then integrating with respect to t, we have,

$$\begin{aligned}\rho\|\dot{u}_m(t)\|^2+\tilde{\gamma}\|u_m(t)\|_V^2+2\int_0^t\int_0^r\xi(r-s)a(\dot{u}_m(s),\dot{u}_m(r))\,ds\,dr\\ =\rho\|\dot{u}_m(0)\|^2+\tilde{\gamma}\|u_m(0)\|_V^2\\ +2\int_0^t(f(r),\dot{u}_m(r))\,dr+2\int_0^t(g(r),\dot{u}_m(r))_{\Gamma_{\mathrm{N}}}\,dr\\ -2\int_0^t\xi(r)a(u_m(0),\dot{u}_m(r))\,dr.\end{aligned}$$

Since ξ is a positive type kernel, by (3.3), the third term of the left hand side is non-negative. Then integration by parts in the last two terms at the right side yields

$$\begin{aligned}&\rho\|\dot{u}_m(t)\|^2+\tilde{\gamma}\|u_m(t)\|_V^2\\ &\quad\le\rho\|\dot{u}_m(0)\|^2+\tilde{\gamma}\|u_m(0)\|_V^2+2\int_0^t(f(r),\dot{u}_m(r))\,dr\\ &\quad-2\int_0^t(\dot{g}(r),u_m(r))_{\Gamma_{\mathrm{N}}}\,dr+2(g(t),u_m(t))_{\Gamma_{\mathrm{N}}}-2(g(0),u_m(0))_{\Gamma_{\mathrm{N}}}\\ &\quad-2\int_0^t\beta(r)a(u_m(0),u_m(r))\,dr\\ &\quad-2\xi(t)a(u_m(0),u_m(t))+2\xi(0)a(u_m(0),u_m(0)),\end{aligned}$$

that using the Cauchy-Schwarz inequality, the trace theorem, $\|\beta\|_{L_1(\mathbb{R}^+)}=\gamma$, and $\xi(0)=\gamma$, we obtain

$$\begin{aligned}&\rho\|\dot{u}_m(t)\|^2+\tilde{\gamma}\|u_m(t)\|_V^2\\ &\quad\le\rho\|\dot{u}_m(0)\|^2+\tilde{\gamma}\|u_m(0)\|_V^2\\ &\quad+2/C_1\max_{0\le r\le t}\|\dot{u}_m(r)\|^2+C_1\Big(\int_0^t\|f(r)\|\,dr\Big)^2\\ &\quad+2C_{Trace}/C_2\max_{0\le r\le t}\|u_m(r)\|_V^2+2C_{Trace}C_2\Big(\int_0^t\|\dot{g}(r)\|_{\Gamma_{\mathrm{N}}}\,dr\Big)^2\\ &\quad+2C_{Trace}/C_3\|u_m(t)\|_V^2+2C_{Trace}C_3\|g(t)\|_{\Gamma_{\mathrm{N}}}^2\\ &\quad+2C_{Trace}/C_4\|u_m(0)\|_V^2+2C_{Trace}C_4\|g(0)\|_{\Gamma_{\mathrm{N}}}^2\\ &\quad+2/C_5\|u_m(0)\|_V^2+2\gamma^2C_5\max_{0\le r\le t}\|u_m(r)\|_V^2\\ &\quad+2/C_6\|u_m(0)\|_V^2+2C_6\xi^2(t)\|u_m(t)\|_V^2+2\gamma\|u_m(0)\|_V^2.\end{aligned}$$

Hence, considering the facts that $C_{Trace}=C(\Omega)$, $\|\dot{u}_m(0)\|\le\|v^0\|$, and $\|u_m(0)\|_V\le\|u^0\|_V$, in a standard way we get,

$$\begin{aligned}&\|\dot{u}_m\|^2_{L_\infty((0,T);H)}+\|u_m\|^2_{L_\infty((0,T);V)}\\ &\quad\le C\big\{\|v^0\|^2+\|u^0\|_V^2+\|g\|^2_{L_\infty((0,T);H^{\Gamma_{\mathrm{N}}})}\\ &\qquad+\|f\|^2_{L_1((0,T);H)}+\|\dot{g}\|^2_{L_1((0,T);H^{\Gamma_{\mathrm{N}}})}\big\},\end{aligned}$$

for some constant $C = C(\Omega, \gamma, \rho, T)$. This, and the facts that $\|f\|_{L_1((0,T);H)} \leq C\|f\|_{L_2((0,T);H)}$, and by Sobolev's inequality

$$\|g\|_{L_\infty((0,T);H^{\Gamma_N})} \leq C\|g\|_{W_1^1((0,T);H^{\Gamma_N})},$$

implies

$$\begin{aligned} &\|\dot{u}_m\|^2_{L_\infty((0,T);H)} + \|u_m\|^2_{L_\infty((0,T);V)} \\ &\quad \leq C\big\{\|v^0\|^2 + \|u^0\|^2_V + \|g\|^2_{W_1^1((0,T);H^{\Gamma_N})} + \|f\|^2_{L_2((0,T);H)}\big\}. \end{aligned} \tag{3.13}$$

Now we need to find a bound for $\ddot{u}_m$. For any fixed $v \in V$ with $\|v\|_V \leq 1$, we write $v = v^1 + v^2$, where $v^1 \in \text{span}\{\varphi_j\}_{j=1}^m$, $v^2 \in \text{span}(\{\varphi_j\}_{j=1}^m)^\perp$. Then from (3.6) we obtain,

$$\begin{aligned} \rho\langle \ddot{u}_m(t), v\rangle = \rho(\ddot{u}_m(t), v^1) = (f(t), v^1) + (g(t), v^1)_{\Gamma_N} - a(u_m(t), v^1) \\ + \int_0^t \beta(t-s) a(u_m(s), v^1)\, ds, \end{aligned}$$

that, using the Cuachy-Schwarz inequality and the trace theorem, implies

$$\begin{aligned} \rho \int_0^T \|\ddot{u}_m(t)\|^2_{V^*}\, dt \leq \int_0^T \Big\{ &\|f(t)\|\|v^1\| + C_{Trace}\|g(t)\|_{\Gamma_N}\|v^1\|_V \\ &+ \|u_m(t)\|_V\|v^1\|_V + \int_0^t \beta(t-s)\|u_m(s)\|_V\|v^1\|_V\, ds\Big\}\, dt. \end{aligned}$$

This, using $\|v^1\|_V \leq 1$ and (3.13), concludes

$$\begin{aligned} \rho\|\ddot{u}_m\|^2_{L_2((0,T);V^*)} &\leq C_\Omega\big\{\|f\|^2_{L_2((0,T);H)} + \|g\|^2_{L_2((0,T);H^{\Gamma_N})}\big\} \\ &\quad + C_{\gamma,T}\|u_m\|^2_{L_\infty((0,T);V)} \\ &\leq C\big\{\|f\|^2_{L_2((0,T);H)} + \|g\|^2_{L_2((0,T);H^{\Gamma_N})} \\ &\quad + \|v^0\|^2 + \|u^0\|^2_V + \|g\|^2_{W_1^1((0,T);H^{\Gamma_N})}\big\}. \end{aligned}$$

Therefore, for some constant $C = C(\Omega, \gamma, \rho, T)$,

$$\begin{aligned} &\|\ddot{u}_m\|^2_{L_2((0,T);V^*)} \\ &\quad \leq C\big\{\|v^0\|^2 + \|u^0\|^2_V + \|g\|^2_{W_1^1((0,T);H^{\Gamma_N})} + \|f\|^2_{L_2((0,T);H)}\big\}. \end{aligned}$$

This and (3.13) imply the estimate (3.8), and the proof is complete. □

3.2. **Existence and uniqueness of the weak solution.** Now, we use Theorem 1 to prove existence and uniqueness of the weak solution of (2.5), that is, a solution of (2.6)–(2.8).

Theorem 2. *If $u^0 \in V$, $v^0 \in H$, $g \in W_1^1((0,T); H^{\Gamma_N})$, $f \in L_2((0,T); H)$, there exists a unique weak solution of* (2.5).

Proof. 1. We note that the estimate (3.8) does not depend on m, so we have

$$\begin{aligned} &\|u_m\|_{L_\infty((0,T);V)} + \|\dot{u}_m\|_{L_\infty((0,T);H)} + \|\ddot{u}_m\|_{L_2((0,T);V^*)} \\ &\qquad \leq K = K(\Omega, \gamma, T, u, v, f, g). \end{aligned}$$

That is,

$$\begin{aligned} &\{u_m\}_{m=1}^\infty \text{ is bounded in } L_\infty((0,T);V) \subset L_2((0,T);V), \\ &\{\dot{u}_m\}_{m=1}^\infty \text{ is bounded in } L_\infty((0,T);H) \subset L_2((0,T);H), \\ &\{\ddot{u}_m\}_{m=1}^\infty \text{ is bounded in } L_2((0,T);V^*). \end{aligned} \tag{3.14}$$

2. First we prove existence. From (3.14) and a classical result in functional analysis, we conclude that the sequences $\{u_m\}_{m=1}^{\infty}$, $\{\dot{u}_m\}_{m=1}^{\infty}$, $\{\ddot{u}_m\}_{m=1}^{\infty}$ are weakly precompact. That is there are subsequences of $\{u_m\}_{m=1}^{\infty}$, $\{\dot{u}_m\}_{m=1}^{\infty}$, $\{\ddot{u}_m\}_{m=1}^{\infty}$, such that,

$$\begin{aligned} u_l &\rightharpoonup u \quad \text{in} \quad L_2((0,T);V), \\ \dot{u}_l &\rightharpoonup \dot{u} \quad \text{in} \quad L_2((0,T);H), \\ \ddot{u}_l &\rightharpoonup \ddot{u} \quad \text{in} \quad L_2((0,T);V^*), \end{aligned} \tag{3.15}$$

where the index l is a replacement of the label of the subsequences and '$\rightharpoonup$' denotes weak convergence. Consequently, (2.6) holds true and we need to verify (2.7) and (2.8). To show (2.7) we fix a positive integer N and we choose $v \in \mathcal{C}([0,T];V)$ of the form

$$v(t) = \sum_{j=1}^{N} h_j(t)\varphi_j. \tag{3.16}$$

Then we take $l \geq N$ and from (3.6) we have

$$\begin{aligned} \int_0^T \Big(\rho\langle \ddot{u}_l, v\rangle + a(u_l, v) - \int_0^t \beta(t-s)a(u_l(s), v)\, ds\Big)\, dt \\ = \int_0^T \big((f,v) + (g,v)_{\Gamma_{\mathrm{N}}}\big)\, dt. \end{aligned} \tag{3.17}$$

This, by (3.15), implies in the limit,

$$\begin{aligned} \int_0^T \Big(\rho\langle \ddot{u}, v\rangle + a(u, v) - \int_0^t \beta(t-s)a(u(s), v)\, ds\Big)\, dt \\ = \int_0^T \big((f,v) + (g,v)_{\Gamma_{\mathrm{N}}}\big)\, dt. \end{aligned} \tag{3.18}$$

Since functions of the form (3.16) are dense in $L_2((0,T);V)$, this equality then holds for all functions $v \in L_2((0,T);V)$, and further it implies (2.7).

Now, we need to show that u satisfies the initial conditions (2.8). Let $v \in \mathcal{C}^2([0,T];V)$ be any function which is zero in a neighborhood of T (or simply $v(T) = \dot{v}(T) = 0$). Then by partial integration in (3.17) we have

$$\begin{aligned} \int_0^T \Big(\rho\langle u_l, \ddot{v}\rangle + a(u_l, v) - \int_0^t \beta(t-s)a(u_l(s), v)\, ds\Big)\, dt \\ = \int_0^T \big((f,v) + (g,v)_{\Gamma_{\mathrm{N}}}\big)\, dt - \rho(u_l(0), \dot{v}(0)) + \rho(\dot{u}_l(0), v(0)), \end{aligned}$$

so that, recalling (3.15) and (3.7), in the limit we conclude,

$$\begin{aligned} \int_0^T \Big(\rho\langle u, \ddot{v}\rangle + a(u, v) - \int_0^t \beta(t-s)a(u(s), v)\, ds\Big)\, dt \\ = \int_0^T \big((f,v) + (g,v)_{\Gamma_{\mathrm{N}}}\big)\, dt - \rho(u^0, \dot{v}(0)) + \rho(v^0, v(0)). \end{aligned}$$

On the other hand integration by parts in (3.18) gives,

$$\int_0^T \Big(\rho\langle u,\ddot{v}\rangle + a(u,v) - \int_0^t \beta(t-s)a(u(s),v)\,ds\Big)\,dt$$
$$= \int_0^T \big((f,v)+(g,v)_{\Gamma_{\mathrm{N}}}\big)\,dt - \rho(u(0),\dot{v}(0)) + \rho(v(0),v(0)).$$

Compairing the last two identities we conclude (2.8), since $v(0)$, $\dot{v}(0)$ are arbitrary. Hence u is a weak solution of (2.5).

3. It remains to prove uniqueness. To this end, we show that $u=0$ is the only solution of (2.6)–(2.8) for $u^0=v^0=f=g=0$. Let us fix $r\in[0,T]$ and define

$$v(t) = \begin{cases} \displaystyle\int_t^r u(\omega)\,d\omega & 0\le t\le r,\\ 0 & r\le t\le T.\end{cases}$$

We note that

$$v(t)\in V,\quad v(r)=0,\quad \dot{v}(t)=-u(t). \tag{3.19}$$

Then inserting v in (2.7) and integrating with respect to t, we have

$$\int_0^r \rho\langle \ddot{u},v\rangle\,dt + \int_0^r a(u,v)\,dt - \int_0^r\int_0^t \beta(t-s)a(u(s),v(t))\,ds\,dt = 0. \tag{3.20}$$

For the last term we obtain

$$\begin{aligned}
-\int_0^r\int_0^t \beta(t-s)a(u(s),v(t))\,ds\,dt &= \int_0^r\int_s^r D_t\xi(t-s)a(u(s),v(t))\,dt\,ds\\
&= \int_0^r \xi(r-s)a(u(s),v(r))\,ds\\
&\quad - \int_0^r \xi(0)a(u(s),v(s))\,ds\\
&\quad - \int_0^r\int_s^r \xi(t-s)a(u(s),\dot{v}(t))\,dt\,ds\\
&= -\gamma\int_0^r a(u(s),v(s))\,ds\\
&\quad + \int_0^r\int_0^t \xi(t-s)a(u(s),u(t))\,ds\,dt,
\end{aligned}$$

where we changed the order of integrals and we used integration by parts, $\xi(0)=\gamma$ from (3.2) and $v(r)=0$ from (3.19). Therefore integration by parts in the first term of (3.20), recalling $\tilde{\gamma}=1-\gamma$, yields

$$-\rho\int_0^r(\dot{u},\dot{v})\,dt + \tilde{\gamma}\int_0^r a(u,v)\,dt + \int_0^r\int_0^t \xi(t-s)a(u(s),u(t))\,ds\,dt = 0.$$

This, using $\dot{v}=-u$ from (3.19), implies

$$\begin{aligned}
&\rho\|u(r)\|^2 - \rho\|u(0)\|^2 - \tilde{\gamma}\|v(r)\|_V^2 + \tilde{\gamma}\|v(0)\|_V^2\\
&\qquad + 2\int_0^r\int_0^t \xi(t-s)a(u(s),u(t))\,ds\,dt = 0.
\end{aligned}$$

Consequently, recalling (3.3), $v(r)=0$, and $u(0)=0$, we have $u=0$ a.e., and this completes the proof. □

3.3. **Regularity.** Here we study the regularity of the unique weak solution of (2.5), that is, a solution of (2.6)–(2.8).

Corollary 1. *If $u^0 \in V, v^0 \in H, g \in W^1_1((0,T);H^{\Gamma_N})$, and $f \in L_2((0,T);H)$, then for the unique solution u of* (2.6)–(2.8) *we have*

$$u \in L_\infty((0,T);V), \quad \dot{u} \in L_\infty((0,T);H), \quad \ddot{u} \in L_2((0,T);V^*). \tag{3.21}$$

Moreover we have the estimate

$$\begin{aligned} &\|u\|_{L_\infty((0,T);V)} + \|\dot{u}\|_{L_\infty((0,T);H)} + \|\ddot{u}\|_{L_2((0,T);V^*)} \\ &\qquad \le C\big\{\|u^0\|_V + \|v^0\| + \|g\|_{W^1_1((0,T);H^{\Gamma_N})} + \|f\|_{L_2((0,T);H)}\big\}. \end{aligned} \tag{3.22}$$

Proof. It is known that if $u_m \rightharpoonup u$, then

$$\|u\| \le \lim_{m\to\infty} \inf \|u_m\|. \tag{3.23}$$

Then, recalling (3.15) and the a priori estimates (3.8), we conclude (3.21) and (3.22). □

It is known from the theory of the elliptic operators, that global higher spatial regularity can not be obtained with mixed boundary conditions. Therefore we specialize to the homogeneous Dirichlet boundary condition, that is $\Gamma_{\rm N} = \varnothing$, and assume that the polygonal domain Ω is convex. We recall the usual Sobolev space $H^2 = H^2(\Omega)$ and we note that here $V = H^1_0(\Omega)$. We then use the extension of the operator $Au = -\nabla \cdot \sigma_0(u)$ to an abstract operator A with $\mathcal{D}(A) = H^2(\Omega)^d \cap V$ such that $a(u,v) = (Au,v)$ for sufficiently smooth u, v. We note that, the elliptic regularity holds, that is,

$$\|u\|_{H^2} \le C\|Au\|, \quad u \in H^2(\Omega)^d \cap V. \tag{3.24}$$

Theorem 3. *We assume that $\Gamma_N = \varnothing$. If $u^0 \in H^2$, $v^0 \in V$, and $\dot{f} \in L_2((0,T);H)$, then for the unique solution u of* (2.6)–(2.8) *we have*

$$\begin{aligned} &u \in L_\infty((0,T);H^2), \quad \dot{u} \in L_\infty((0,T);V), \\ &\ddot{u} \in L_\infty((0,T);H), \quad \dddot{u} \in L_2((0,T);V^*). \end{aligned} \tag{3.25}$$

Moreover we have the estimate

$$\begin{aligned} &\|u\|_{L_\infty((0,T);H^2)} + \|\dot{u}\|_{L_\infty((0,T);V)} + \|\ddot{u}\|_{L_\infty((0,T);H)} + \|\dddot{u}\|_{L_2((0,T);V^*)} \\ &\qquad \le C\big\{\|u^0\|_{H^2} + \|v^0\|_V + \|f\|_{H^1((0,T);H)}\big\}. \end{aligned} \tag{3.26}$$

Proof. Writing

$$\int_0^t \beta(t-s)a(u_m(s),\varphi_k)\,ds = \int_0^t \beta(s)a(u_m(t-s),\varphi_k)\,ds,$$

differentiating (3.6) with respect to time, and writing $\underline{v} = \dot{v}$, we have

$$\begin{aligned} &\rho(\underline{\ddot{u}}_m(t),\varphi_k) + a(\underline{u}_m(t),\varphi_k) - \int_0^t \beta(t-s)a(\underline{u}_m(s),\varphi_k)\,ds \\ &\qquad = (\dot{f}(t),\varphi_k) + \beta(t)a(u_m(0),\varphi_k), \quad k = 1,\,\ldots,\,m,\ t \in (0,T), \end{aligned} \tag{3.27}$$

with the initial conditions

$$
\begin{aligned}
\underline{u}_m(0) = \dot{u}_m(0) &= \sum_{j=1}^{m} (v^0, \varphi_j)\varphi_j, \\
\dot{\underline{u}}_m(0) = \ddot{u}_m(0) &= \sum_{j=1}^{m} \big(f(0) - Au_m(0), \varphi_j\big)\varphi_j.
\end{aligned} \tag{3.28}
$$

Then, using $\beta(t-s) = D_s\xi(t-s)$ from (3.2) and partial integration, we have

$$
\begin{aligned}
\rho(\ddot{\underline{u}}_m(t), \varphi_k) + \tilde{\gamma} a(\underline{u}_m(t), \varphi_k) + \int_0^t \xi(t-s) a(\dot{\underline{u}}_m(s), \varphi_k)\, ds \\
= (\dot{f}(t), \varphi_k) + \beta(t) a(u_m(0), \varphi_k) \\
- \xi(t) a(\underline{u}_m(0), \varphi_k), \quad k = 1, \ldots, m,\ t \in (0,T).
\end{aligned}
$$

Now, multiplying by $\ddot{d}_k(t)$ and summing $k = 1, \ldots, m$, we have

$$
\begin{aligned}
\rho(\ddot{\underline{u}}_m(t), \dot{\underline{u}}_m(t)) + \tilde{\gamma} a(\underline{u}_m(t), \dot{\underline{u}}_m(t)) + \int_0^t \xi(t-s) a(\dot{\underline{u}}_m(s), \dot{\underline{u}}_m(t))\, ds \\
= (\dot{f}(t), \dot{\underline{u}}_m(t)) + \beta(t) a(u_m(0), \dot{\underline{u}}_m(t)) \\
- \xi(t) a(\underline{u}_m(0), \dot{\underline{u}}_m(t)), \quad t \in (0,T).
\end{aligned}
$$

Integrating over $[0,t]$ and partial integration in the last term, we have

$$
\begin{aligned}
\rho\|\dot{\underline{u}}_m(t)\|^2 + \tilde{\gamma}\|\underline{u}_m(t)\|_V^2 \\
\le \rho\|\dot{\underline{u}}_m(0)\|^2 + \tilde{\gamma}\|\underline{u}_m(0)\|_V^2 \\
+ 2\int_0^t (\dot{f}(r), \dot{\underline{u}}_m(r))\, dr + 2\int_0^t \beta(r) a(u_m(0), \dot{\underline{u}}_m(r))\, dr \\
- 2\int_0^t \beta(r) a(\underline{u}_m(0), \underline{u}_m(r))\, dr \\
- 2\xi(t) a(\underline{u}_m(0), \underline{u}_m(t)) + 2\xi(0) a(\underline{u}_m(0), \underline{u}_m(0)),
\end{aligned}
$$

that using the Cauchy-Schwarz inequality, the trace theorem, $\|\beta\|_{L_1(\mathbb{R}^+)} = \gamma$, and $\xi(0) = \gamma$, we obtain

$$
\begin{aligned}
\rho\|\dot{\underline{u}}_m(t)\|^2 + \tilde{\gamma}\|\underline{u}_m(t)\|_V^2 \\
\le \rho\|\dot{\underline{u}}_m(0)\|^2 + (1+\gamma)\|\underline{u}_m(0)\|_V^2 \\
+ 2/C_1 \max_{0\le r\le t} \|\dot{\underline{u}}_m(r)\|^2 + C_1\Big(\int_0^t \|\dot{f}(r)\|\, dr\Big)^2 \\
+ 2/C_2\|u_m(0)\|_{H^2}^2 + 2\gamma^2 C_2 \max_{0\le r\le t} \|\dot{\underline{u}}_m(r)\|^2 \\
+ 2/C_3\|\underline{u}_m(0)\|_V^2 + 2\gamma^2 C_3 \max_{0\le r\le t} \|\underline{u}_m(r)\|_V^2 \\
+ 2/C_4\|\underline{u}_m(0)\|_V^2 + 2\gamma^2 C_4\|\underline{u}_m(t)\|_V^2.
\end{aligned}
$$

This implies, for some constant C,

$$
\begin{aligned}
\|\dot{\underline{u}}_m\|_{L_\infty((0,T);H)}^2 + \|\underline{u}_m\|_{L_\infty((0,T);V)}^2 \\
\le C\big\{\|\dot{\underline{u}}_m(0)\|^2 + \|\underline{u}_m(0)\|_V^2 + \|u_m(0)\|_{H^2}^2 + \|\dot{f}\|_{L_1((0,T);H)}^2\big\}.
\end{aligned}
$$

Then recalling $\underline{u} = \dot{u}$, the initial data from (3.28), and using

$$\|u_m(0)\|_{H^2} \le \|u^0\|_{H^2}, \quad \|\dot{u}_m(0)\|_V \le \|v^0\|_V,$$

we have

$$\begin{aligned} \|\ddot{u}_m\|^2_{L_\infty((0,T);H)} + \|\dot{u}_m\|^2_{L_\infty((0,T);V)} \\ \le C\big\{\|u^0\|^2_{H^2} + \|v^0\|^2_V + \|f(0)\|^2 + \|\dot{f}\|^2_{L_1((0,T);H)}\big\}. \end{aligned} \tag{3.29}$$

We now find a bound for $\|u_m(t)\|_{H^2}$. We recall the eigenvalue problem (3.4) with eigenpairs $\{(\lambda_j, \varphi_j)\}_{j=1}^\infty$, $a(u,v) = (Au, v)$. Then we multiply (3.6) by $\lambda_k d_k(t)$ and add for $k = 1, \ldots, m$ to obtain

$$a(u_m, Au_m) = (f - \rho\ddot{u}_m, Au_m) + \int_0^t \beta(t-s) a(u_m(s), Au_m(t))\, ds.$$

This implies

$$\begin{aligned} \|Au_m(t)\|^2 \le \frac{C}{\epsilon}\Big(\|f(t)\|^2 + \rho\|\ddot{u}_m(t)\|^2\Big) + \epsilon\|Au_m(t)\|^2 \\ + \gamma \max_{0\le r\le t} \|Au_m(s)\|^2, \end{aligned}$$

that gives us, by elliptic regularity (3.24),

$$\|u_m\|^2_{L_\infty((0,T);H^2)} \le C\Big(\|f\|^2_{L_\infty((0,T);H)} + \|\ddot{u}_m\|^2_{L_\infty((0,T);H)}\Big).$$

From this and (3.29) we conclude

$$\begin{aligned} \|\ddot{u}_m\|^2_{L_\infty((0,T);H)} + \|\dot{u}_m\|^2_{L_\infty((0,T);V)} + \|u_m\|^2_{L_\infty((0,T);H^2)} \\ \le C\big\{\|u^0\|^2_{H^2} + \|v^0\|^2_V + \|f\|^2_{L_\infty((0,T);H)} + \|\dot{f}\|^2_{L_1((0,T);H)}\big\}, \end{aligned}$$

that using $\|f\|_{L_\infty((0,T);H)} \le C\|f\|_{W^1_1((0,T);H)}$, by Sobolev inequality, we have

$$\begin{aligned} \|\ddot{u}_m\|^2_{L_\infty((0,T);H)} + \|\dot{u}_m\|^2_{L_\infty((0,T);V)} + \|u_m\|^2_{L_\infty((0,T);H^2)} \\ \le C\big\{\|u^0\|^2_{H^2} + \|v^0\|^2_V + \|f\|^2_{W^1_1((0,T);H)}\big\} \\ \le C\big\{\|u^0\|^2_{H^2} + \|v^0\|^2_V + \|f\|^2_{H^1((0,T);H)}\big\}. \end{aligned}$$

Finally from (3.27) , similar to the proof of Theorem 1, we obtain

$$\|\dddot{u}_m\|^2_{L_2((0,T);V^*)} \le C\big\{\|u^0\|^2_{H^2} + \|v^0\|^2_V + \|f\|^2_{H^1((0,T);H)}\big\}.$$

The last two estimates then, in the limit, imply the desired estimate (3.26), and the proof is now complete. □

Remark 1. If we continue differentiation in time to investigate more regularity, we obtain from (3.27)

$$\begin{aligned} \rho(\dddot{\underline{u}}_m(t), \varphi_k) + a(\dot{\underline{u}}_m(t), \varphi_k) - \int_0^t \beta(t-s) a(\dot{\underline{u}}_m(s), \varphi_k)\, ds \\ = (\ddot{f}(t), \varphi_k) + \dot{\beta}(t) a(u_m(0), \varphi_k) \\ + \beta(t) a(\underline{u}_m(0), \varphi_k), \quad k = 1, \ldots, m,\ t \in (0,T), \end{aligned}$$

Further the term $\dot{\beta}(t)a(u_m(0), \varphi_k)$ leads to

$$\dot{\beta}(t) a(u_m(0), \ddot{\underline{u}}_m(t)).$$

But $\dot{\beta}$ is not integrable. Besides, after integration in time, we can not use partial integration to transfer one time derivative from $\dot{\beta}$ to $\ddot{\underline{u}}_m(t)$, since there is not enough regularity to handle $\dddot{\underline{u}}_m$. This means that we can not get more regularity with weakly singular kernel β. This also indicates that with smoother kernel we can get higher regularity in case of homogeneous Dirichlet boundary condition under the appropriate assumption on the data, that is, more regularity and compatibility.

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