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MIHÁLY KOVACS
STIG LARSSON
FARDIN SAEDPANAH

Department of Mathematical Sciences
Division of Mathematics
CHALMERS UNIVERSITY OF TECHNOLOGY
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Mihály Kovacs, Stig Larsson, and Fardin Saedpanah

Department of Mathematical Sciences
Division of Mathematics
Chalmers University of Technology and University of Gothenburg
SE-412 96 Göteborg, Sweden
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FINITE ELEMENT APPROXIMATION OF THE LINEAR STOCHASTIC WAVE EQUATION WITH ADDITIVE NOISE

MIHÁLY KOVÁCS[1], STIG LARSSON[1,2], AND FARDIN SAEDPANAH

ABSTRACT. Semidiscrete finite element approximation of the linear stochastic wave equation with additive noise is studied in a semigroup framework. Optimal error estimates for the deterministic problem are obtained under minimal regularity assumptions. These are used to prove strong convergence estimates for the stochastic problem. The theory presented here applies to multi-dimensional domains and spatially correlated noise. Numerical examples illustrate the theory.

1. INTRODUCTION

We study the finite element approximation of the linear stochastic wave equation driven by additive noise,

$$\begin{aligned} &\mathrm{d}\dot{u} - \Delta u \,\mathrm{d}t = \mathrm{d}W && \text{in } \mathcal{D} \times (0,\infty), \\ &u = 0 && \text{in } \partial\mathcal{D} \times (0,\infty), \\ &u(\cdot,0) = u_0, \quad \dot{u}(\cdot,0) = v_0 && \text{in } \mathcal{D}, \end{aligned} \tag{1.1}$$

where $\mathcal{D} \subset \mathbb{R}^d$, $d = 1, 2, 3$, is a bounded convex polygonal domain with boundary $\partial\mathcal{D}$, and $\{W(t)\}_{t\geq 0}$ is a $L_2(\mathcal{D})$-valued Wiener process on a filtered probability space $(\Omega, \mathcal{F}, \mathbf{P}, \{\mathcal{F}_t\}_{t\geq 0})$ with respect to the normal filtration $\{\mathcal{F}_t\}_{t\geq 0}$. We let u_0, v_0 be $\mathcal{F}_0$-measurable random variables.

For introduction to the stochastic wave equation and its applications we refer to [1], [6], [14], [16], [23] and the references therein.

The stochastic heat equation and its numerical approximation has been extensively researched in the literature, see, for example, [6], [11], [12], [13], [23], [25], [26], and the references therein. The numerical analysis of the stochastic wave equation is less studied, see [15], [18], [20], [24] for existing results. In particular, these works do not deal with multiple dimensions or correlated noise. This is the purpose of the present work.

We use the semigroup framework of [16] in which the weak solution of (1.1) is represented as a stochastic convolution

$$u(t) = \int_0^t \Lambda^{-1/2} \sin((t-s)\Lambda^{1/2}) \,\mathrm{d}W(s),$$

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where, for simplicity, we have set the initial values $u_0 = v_0 = 0$. Here $\Lambda = -\Delta$ with $D(\Lambda) = H^2(\mathcal{D}) \cap H_0^1(\mathcal{D})$, and $v(t) = \Lambda^{-1/2}\sin(t\Lambda^{1/2})f$ is the solution of

$$\begin{aligned} &\ddot{v} + \Lambda v = 0, \quad t > 0, \\ &v(0) = 0, \quad \dot{v}(0) = f. \end{aligned} \tag{1.2}$$

We show that, if Q denotes the covariance operator of W, and if

$$\|\Lambda^{(\beta-1)/2}Q^{1/2}\|_{\mathrm{HS}} < \infty, \tag{1.3}$$

for some $\beta \geq 0$, then we have spatial regularity of order β,

$$\Big(\mathbf{E}\big(\|u(t)\|_{\dot{H}^\beta}^2\big)\Big)^{1/2} \leq Ct^{1/2}\|\Lambda^{(\beta-1)/2}Q^{1/2}\|_{\mathrm{HS}},$$

where $\dot{H}^\beta = \mathcal{D}(\Lambda^{\beta/2})$. In particular, if $\mathrm{Tr}(Q) = \|Q^{1/2}\|_{\mathrm{HS}}^2 < \infty$ (spatially correlated noise), then we may take $\beta = 1$. On the other hand if $Q = I$ (uncorrelated noise), then $\beta < 1 - d/2$, that is, $\beta < 1/2,\ d = 1$. See Section 3 for details.

We discretize (1.1) in the spatial variables with a standard piecewise linear finite element method, and we show strong convergence estimates in various norms. For example,

$$\Big(\mathbf{E}\big(\|u_h(t) - u(t)\|^2\big)\Big)^{1/2} \leq C(t)h^{\frac{2}{3}\beta}\|\Lambda^{(\beta-1)/2}Q^{1/2}\|_{\mathrm{HS}}, \quad \beta \in [0,3], \tag{1.4}$$

where again $u_0 = v_0 = 0$ and $u_h(t)$ is the approximate solution with maximal meshsize h, see Theorem 5.1.

As a comparison, we recall from [25] that for the stochastic heat equation we have

$$\begin{aligned} \Big(\mathbf{E}\big(\|u(t)\|_{\dot{H}^\beta}^2\big)\Big)^{1/2} &\leq C\|\Lambda^{(\beta-1)/2}Q^{1/2}\|_{\mathrm{HS}}, \quad \beta \geq 0, \\ \Big(\mathbf{E}\big(\|u_h(t) - u(t)\|^2\big)\Big)^{1/2} &\leq Ch^{\beta}\|\Lambda^{(\beta-1)/2}Q^{1/2}\|_{\mathrm{HS}}, \quad \beta \in [0,2]. \end{aligned}$$

Here the order of regularity coincides with the order of convergence.

The main tools for the proof of (1.4) are the Itô-isometry (2.5) and error estimates for the deterministic problem (1.2) with minimal regularity assumptions,

$$\|v_h(t) - v(t)\| \leq C(t)h^2\|f\|_{\dot{H}^2}, \tag{1.5}$$

$$\|v_h(t) - v(t)\| \leq C\|f\|_{\dot{H}^{-1}},$$

and, hence by interpolation, see Corollary 4.3,

$$\|v_h(t) - v(t)\| \leq C(t)h^{\frac{2}{3}\beta}\|f\|_{\dot{H}^\beta}, \quad \beta \in [0,3].$$

As mentioned above, when we specialize to $Q = I,\ d = 1$, we have $\beta < 1/2$ and thus the order of strong convergence is $O(h^\alpha),\ \alpha < 1/3$. This is the same order as in [18], where spatial semi-discretization of the nonlinear stochastic wave equation with a standard difference scheme of uniform meshsize h is considered for $d = 1$ and with space-time white noise ($Q = I$). We note that the order of convergence is less than the order of regularity, which is $\beta < 1/2$. However, it is known that in (1.5), $\|f\|_{\dot{H}^2}$ can not be replaced by $\|f\|_{\dot{H}^{2-\epsilon}}$ for any $\epsilon > 0$, see [19] and Remark 4.4 below. Therefore, $O(h^\alpha),\ \alpha < 1/3$, is the best that one can expect. This explains the discrepancy in the convergence behavior between the heat and wave equations.

In [24] the leap-frog scheme is applied to the nonlinear stochastic wave equation in the unbounded domain $\mathcal{D} = \mathbb{R}$, and a strong convergence rate $O(h^{1/2})$ is proved.

The proofs in both [18] and [24] are based on representation of the exact and approximate solutions by means of Green's functions. The difference in convergence rate between the two is explained by the fact that in $\mathbb{R}$ the Green's functions for the wave equation and the leap-frog scheme coincide at mesh points, see Remark 5.2 for more details.

In summary we may say that we extend the results of [18] to the finite element method in multiple dimensions and correlated noise. But we only consider the linear equation with additive noise. We also explain the discrepancy between [18] and [24]. We plan to address the nonlinear equation $\mathrm{d}\dot{u}-\Delta u\,\mathrm{d}t = f(u)\,\mathrm{d}t+g(u)\,\mathrm{d}W$ in future work.

The paper is organized as follows. In Section 2 some preliminaries are provided and a rigorous meaning to the infinite dimensional Wiener process $\{W(t)\}_{t\geq 0}$ and the stochastic integral are given together with the definition of a weak solution of (1.1). Existence, uniqueness, and regularity of weak solutions are discussed in Section 3. In Section 4 the finite element method for the deterministic problem is formulated and analyzed. The results obtained here are used in Section 5 to derive strong convergence estimates for finite element approximation of the stochastic equation (1.1). Finally, numerical experiments are presented in Section 6 in order to illustrate the theory.

2. Preliminaries

Throughout the paper we use '$\cdot$' to denote the time derivative '$\frac{\partial}{\partial t}$', and C to denote a generic positive constant, not necessarily the same at different occurrences. We refer to [16] and [17] for more details on stochastic integration and for some concepts that we cannot explain here.

Let $\big(U,(\cdot,\cdot)_U\big)$ and $\big(H,(\cdot,\cdot)_H\big)$ be separable Hilbert spaces with corresponding norms $\|\cdot\|_U$ and $\|\cdot\|_H$. We suppress the subscripts when it causes no confusion. Let $\mathcal{L}(U,H)$ denote the space of bounded linear operators from U to H, and $\mathcal{L}_2(U,H)$ the space of Hilbert-Schmidt operators, endowed with norm $\|\cdot\|_{\mathcal{L}_2(U,H)}$. That is, $T\in\mathcal{L}_2(U,H)$ if $T\in\mathcal{L}(U,H)$ and

$$\|T\|^2_{\mathcal{L}_2(U,H)} := \sum_{k=1}^{\infty}\|Te_k\|^2_H < \infty,$$

where $\{e_k\}_{k=1}^{\infty}$ is an arbitrary ON-basis in U. If $H=U$ we write $\mathcal{L}(U)=\mathcal{L}(U,U)$ and $\mathrm{HS}=\mathcal{L}_2(U,U)$. It is well known that if $S\in\mathcal{L}(U)$ and $T\in\mathcal{L}_2(U,H)$, then $TS\in\mathcal{L}_2(U,H)$ and we have the norm inequality

$$\|TS\|_{\mathcal{L}_2(U,H)} \leq \|T\|_{\mathcal{L}_2(U,H)}\|S\|_{\mathcal{L}(U)}. \tag{2.1}$$

Let $(\Omega,\mathcal{F},\mathbf{P})$ be a probability space. We define $L_2(\Omega,H)$ to be the space of H-valued square integrable random variables with norm

$$\|v\|_{L_2(\Omega,H)} = \mathbf{E}(\|v\|_H^2)^{1/2} = \Big(\int_\Omega \|v(\omega)\|^2_H\,\mathrm{d}\mathbf{P}(\omega)\Big)^{1/2},$$

where $\mathbf{E}$ stands for expected value. Let $Q\in\mathcal{L}(U)$ be a selfadjoint, positive semidefinite operator, with $\mathrm{Tr}(Q)<\infty$, where $\mathrm{Tr}(Q)$ denotes the trace of Q. We say that $\{W(t)\}_{t\geq 0}$ is a *U-valued Q-Wiener process with respect to* $\{\mathcal{F}_t\}_{t\geq 0}$ if

(i) $W(0)=0$,

(ii) W has continuous trajectories (almost surely),

(iii) W has independent increments,
(iv) $W(t) - W(s)$, $0 \le s \le t$, is a U-valued Gaussian random variable with zero mean and covariance operator $(t-s)Q$,

and

(v) $\{W(t)\}_{t\ge 0}$ is adapted to $\{\mathcal{F}_t\}_{t\ge 0}$; that is, $W(t)$ is $\mathcal{F}_t$ measurable for all $t \ge 0$;
(vi) the random variable $W(t) - W(s)$ is independent of $\mathcal{F}_s$ for all fixed $s \in [0,t]$.

It is known, see, e.g., [17, Section 2.1], that for a given Q-Wiener process satisfying (i)–(iv) one can always find a normal filtration $\{\mathcal{F}_t\}_{t\ge 0}$ so that (v)–(vi) holds. Furthermore, $W(t)$ has the orthogonal expansion

$$W(t) = \sum_{j=1}^{\infty} \gamma_j^{1/2} \beta_j(t) e_j, \tag{2.2}$$

where $\{(\gamma_j, e_j)\}_{j=1}^{\infty}$ are the eigenpairs of Q with orthonormal eigenvectors, and $\{\beta_j\}_{j=1}^{\infty}$ is a sequence of real-valued mutually independent standard Brownian motions. We note that the series in (2.2) converges in $L_2(\Omega, U)$, since for $t \ge 0$, we have

$$\begin{aligned} \|W(t)\|_{L_2(\Omega,U)}^2 &= \mathbf{E}\Big(\Big\| \sum_{j=1}^{\infty} \gamma_j^{1/2} e_j \beta_j(t) \Big\|_U^2 \Big) = \sum_{j=1}^{\infty} \gamma_j \mathbf{E}\big(\beta_j(t)\big)^2 \\ &= t \sum_{j=1}^{\infty} \gamma_j = t \operatorname{Tr}(Q) < \infty. \end{aligned} \tag{2.3}$$

We need only a special case of the Itô integral where the integrand is deterministic. If a function $\Phi : [0,\infty) \to L(U,H)$ is strongly measurable and

$$\int_0^t \|\Phi(s) Q^{1/2}\|_{\mathrm{HS}}^2 \, \mathrm{d}s < \infty, \tag{2.4}$$

then the stochastic integral $\int_0^t \Phi(s) \, \mathrm{d}W(s)$ is well defined and Itô's isometry,

$$\mathbf{E}\Big\| \int_0^t \Phi(s) \, \mathrm{d}W(s) \Big\|_{L_2(\Omega,H)}^2 = \int_0^t \|\Phi(s) Q^{1/2}\|_{\mathcal{L}_2(U,H)}^2 \, \mathrm{d}s, \tag{2.5}$$

holds.

More generally, if $Q \in \mathcal{L}(U)$ is a selfadjoint, positive semidefinite operator with eigenpairs $\{(\gamma_j, e_j)\}_{j=1}^{\infty}$, but not trace class, that is, $\operatorname{Tr}(Q) = \infty$, then the series (2.2) does not converge in $L_2(\Omega, U)$. However, it converges in a suitably chosen (usually larger) Hilbert space and the stochastic integral $\int_0^t \Phi(s) \, \mathrm{d}W(s)$ can still be defined and the isometry (2.5) holds, as long as (2.4) is satisfied. In this case W is called a cylindrical Wiener process. In particular, we may have $Q = I$ (the identity operator).

Next we consider the abstract stochastic differential equation

$$\mathrm{d}X(t) = AX(t) \, \mathrm{d}t + B \, \mathrm{d}W(t), \quad t > 0; \quad X(0) = X_0, \tag{2.6}$$

and assume that

(a1) $A : D(A) \subset H \to H$ is the generator of a strongly continuous semigroup (C_0-semigroup) of bounded linear operators $\{E(t)\}_{t\ge 0}$ on H,
(a2) $B \in \mathcal{L}(U,H)$,
(a3) X_0 is an $\mathcal{F}_0$-measurable H-valued random variable.

An H-valued predictable process $\{X(t)\}_{t\geq 0}$ is called a *weak solution* of (2.6), if the trajectories of X are $\mathbf{P}$-a.s. Bochner integrable and, for all $\eta \in D(A^*)$ and all $t \geq 0$,

$$(X(t),\eta) = (X_0,\eta) + \int_0^t (X(s),A^*\eta)\,\mathrm{d}s + \int_0^t (B\,\mathrm{d}W(s),\eta), \quad \mathbf{P}\text{-a.s.} \tag{2.7}$$

3. Abstract framework and regularity

As in the introduction, let $\Lambda = -\Delta$ be the Laplace operator with $D(\Lambda) = H^2(\mathcal{D}) \cap H_0^1(\mathcal{D})$ and let $U = L_2(\mathcal{D})$ with the usual inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$. In order to describe the spatial regularity of functions we introduce the following spaces and norms. Let

$$\dot{H}^\alpha = D(\Lambda^{\alpha/2}), \quad \|v\|_\alpha = \|\Lambda^{\alpha/2}v\| = \Big(\sum_{j=1}^\infty \lambda_j^\alpha (v,\phi_j)^2\Big)^{1/2}, \quad \alpha\in\mathbb{R},\ v\in\dot{H}^\alpha,$$

where $\{(\lambda_j,\phi_j)\}_{j=1}^\infty$ are the eigenpairs of Λ with orthonormal eigenvectors. Then $\dot{H}^\alpha \subset \dot{H}^\beta$ for $\alpha \geq \beta$. It is known that $\dot{H}^0 = U$, $\dot{H}^1 = H_0^1(\mathcal{D})$, $\dot{H}^2 = H^2(\mathcal{D})\cap H_0^1(\mathcal{D})$ with equivalent norms and that $\dot{H}^{-\beta}$ can be identified with the dual space $(\dot{H}^\beta)^*$ for $\beta > 0$, see [22]. We note that the inner product in $\dot{H}^1$ is $(\cdot,\cdot)_1 = (\nabla\cdot,\nabla\cdot)$. We also introduce

$$H^\alpha := \dot{H}^\alpha \times \dot{H}^{\alpha-1}, \quad |||v|||_\alpha^2 := \|v_1\|_\alpha^2 + \|v_2\|_{\alpha-1}^2, \quad \alpha\in\mathbb{R}, \tag{3.1}$$

and set $H = H^0 = \dot{H}^0 \times \dot{H}^{-1}$ with corresponding norm $|||\cdot||| = |||\cdot|||_0$.

Next we write (1.1) as an abstract stochastic differential equation (2.6). To this end, we put $u_1 = u$, $u_2 = \dot{u}$ and note that (1.1) is formally

$$\mathrm{d}\begin{bmatrix}u_1\\u_2\end{bmatrix} = \begin{bmatrix}0 & I\\ -\Lambda & 0\end{bmatrix}\begin{bmatrix}u_1\\u_2\end{bmatrix}\mathrm{d}t + \begin{bmatrix}0\\I\end{bmatrix}\mathrm{d}W.$$

We therefore define

$$A := \begin{bmatrix}0 & I\\ -\Lambda & 0\end{bmatrix}, \quad B := \begin{bmatrix}0\\I\end{bmatrix}, \quad X := \begin{bmatrix}u_1\\u_2\end{bmatrix}, \quad X_0 := \begin{bmatrix}u_0\\v_0\end{bmatrix},$$
$$H := H^0 = \dot{H}^0\times\dot{H}^{-1}, \quad U := \dot{H}^0,$$

with

$$D(A) = \Big\{x\in H : Ax = \begin{bmatrix}x_2\\ -\Lambda x_1\end{bmatrix} \in H = \dot{H}^0\times\dot{H}^{-1}\Big\} = H^1 = \dot{H}^1\times\dot{H}^0.$$

Here Λ is regarded as an operator $\dot{H}^1 \to \dot{H}^{-1}$. The operator A is the generator of a strongly continuous semigroup (C_0-semigroup) $E(t) = \mathrm{e}^{tA}$ on H and

$$E(t) = \mathrm{e}^{tA} = \begin{bmatrix}C(t) & \Lambda^{-1/2}S(t)\\ -\Lambda^{1/2}S(t) & C(t)\end{bmatrix}, \tag{3.2}$$

where $C(t) = \cos(t\Lambda^{1/2})$ and $S(t) = \sin(t\Lambda^{1/2})$ are the so-called cosine and sine operators. For example, using $\{(\lambda_j,\phi_j)\}_{j=1}^\infty$, the eigenpairs of Λ, we have

$$\Lambda^{-1/2}S(t)v = \Lambda^{-1/2}\sin(t\Lambda^{1/2})v = \sum_{j=1}^\infty \lambda_j^{-1/2}\sin(t\lambda_j^{1/2})(v,\phi_j)\phi_j.$$

We also note that $B\in\mathcal{L}(U,H)$ and we let X_0 be an $\mathcal{F}_0$-measurable H-valued random variable to fulfill the assumptions (a1)–(a3). We assume that W is a Q-Wiener process or a cylindrical Wiener process on U. Now (1.1) is set in the form

(2.6), which is given a rigorous meaning by the weak formulation (2.7). Next we consider the existence, uniqueness, and regularity of the weak solution. Recall that we write HS $= \mathcal{L}_2(U,U)$ for the Hilbert-Schmidt operators on U.

Theorem 3.1. *With the above definitions and if* $\|\Lambda^{(\beta-1)/2}Q^{1/2}\|_{\mathrm{HS}} < \infty$ *for some* $\beta \geq 0$, *then* (2.6) *has a unique weak solution, which is given by the variation of constants formula,*

$$X(t) = E(t)X_0 + \int_0^t E(t-s)B\,\mathrm{d}W(s), \quad t \geq 0. \tag{3.3}$$

Moreover,

$$\|X(t)\|_{L_2(\Omega,H^\beta)} \leq C\Big(\|X_0\|_{L_2(\Omega,H^\beta)} + t^{1/2}\|\Lambda^{(\beta-1)/2}Q^{1/2}\|_{\mathrm{HS}}\Big), \quad t \geq 0. \tag{3.4}$$

Proof. To prove that (3.3) is the unique weak solution it is enough to show that, for fixed t,

$$\int_0^t \|E(s)BQ^{1/2}\|^2_{\mathcal{L}_2(U,H)}\,\mathrm{d}s < \infty, \tag{3.5}$$

see [16, Theorem 5.4]. Indeed, with $\{e_k\}_{k=1}^\infty$, an arbitrary ON-basis in U, and for any $\beta \geq 0$, we have

$$\begin{aligned}
\int_0^t \|E(s)BQ^{1/2}\|^2_{\mathcal{L}_2(U,H^\beta)}\,\mathrm{d}s &= \int_0^t \sum_{k=1}^\infty \Big|\Big|\Big|E(s)BQ^{1/2}e_k\Big|\Big|\Big|_\beta^2\,\mathrm{d}s \\
&= \int_0^t \sum_{k=1}^\infty \big\{\big\|\Lambda^{-1/2}S(s)Q^{1/2}e_k\big\|_\beta^2 + \big\|C(s)Q^{1/2}e_k\big\|_{\beta-1}^2\big\}\,\mathrm{d}s \\
&= \int_0^t \big\{\big\|\Lambda^{(\beta-1)/2}S(s)Q^{1/2}\big\|_{\mathrm{HS}}^2 + \big\|\Lambda^{(\beta-1)/2}C(s)Q^{1/2}\big\|_{\mathrm{HS}}^2\big\}\,\mathrm{d}s \\
&\leq 2t\|\Lambda^{(\beta-1)/2}Q^{1/2}\|_{\mathrm{HS}},
\end{aligned} \tag{3.6}$$

where, for the last inequality, we used the fact that the Λ commutes with $C(s)$, $S(s)$ and (2.1) together with the boundedness of the cosine and the sine operators in U. With $\beta = 0$, this implies (3.5), and therefore it implies existence and uniqueness of the weak solution. Finally, (3.4) follows from (3.3), the boundedness of $E(t)$ in H^β, the Itô isometry (2.5), and (3.6):

$$\begin{aligned}
&\|X(t)\|^2_{L_2(\Omega,H^\beta)} \\
&\quad\leq 2\Big(\|E(t)X_0\|^2_{L_2(\Omega,H^\beta)} + \Big\|\int_0^t E(t-s)B\,\mathrm{d}W(s)\Big\|^2_{L_2(\Omega,H^\beta)}\Big) \\
&\quad\leq 2\Big(\|X_0\|^2_{L_2(\Omega,H^\beta)} + \int_0^t \|E(s)BQ^{1/2}\|^2_{\mathcal{L}_2(U,H^\beta)}\,\mathrm{d}s\Big).
\end{aligned}$$

□

Remark 3.2. The parameter β in the condition $\|\Lambda^{(\beta-1)/2}Q^{1/2}\|_{\mathrm{HS}} < \infty$ quantifies the spatial correlation of the noise. We highlight three special cases.

- If Q is of trace class, then $\beta = 1$, because $\|Q^{1/2}\|^2_{\mathrm{HS}} = \mathrm{Tr}(Q) < \infty$.

- If $Q = I$, which corresponds to space-time white noise, then we have $\|\Lambda^{(\beta-1)/2}\|_{\mathrm{HS}} < \infty$ if and only if $d = 1$ and $\beta < 1/2$. Indeed, the eigenvalues of Λ behave asymptotically like $\lambda_j \approx j^{2/d}$, so that

$$\|\Lambda^{(\beta-1)/2}\|_{\mathrm{HS}}^2 = \sum_{j=1}^{\infty} \lambda_j^{\beta-1} \approx \sum_{j=1}^{\infty} j^{2(\beta-1)/d},$$

  and the series converges if and only if $\beta < 1 - d/2$, that is, $d = 1$, $\beta < 1/2$.
- Similarly, if $Q = \Lambda^{-s}$, $s > 0$, then $\beta < 1 + s - d/2$.

Thus, in order to have a positive order of regularity in multiple dimensions ($d > 1$) we need correlated noise.

4. The finite element method for the deterministic problem

In this section we first study the spatially semidiscrete finite element method for the deterministic linear wave equation,

$$\begin{aligned} &\ddot{u} - \Delta u = f && \text{in } \mathcal{D} \times (0, \infty), \\ &u = 0 && \text{on } \partial\mathcal{D} \times (0, \infty) \\ &u(\cdot, 0) = u_0, \quad \dot{u}(\cdot, 0) = v_0 && \text{in } \mathcal{D}, \end{aligned} \tag{4.1}$$

where $\mathcal{D} \in \mathbb{R}^d$, $d = 1, 2, 3$, is a bounded convex polygonal domain with boundary $\partial\mathcal{D}$. Then we specialize to the homogeneous equation and derive error estimates which will used to prove strong convergence of the finite element approximation of the stochastic equation.

4.1. Error estimates for the non-homogeneous problem.

Let $\{\mathcal{T}_h\}$ be a regular family of triangulations of $\mathcal{D}$ with $h_K = \mathrm{diam}(K)$, $h = \max_{K \in \mathcal{T}_h} h_K$, and denote by V_h the space of piecewise linear continuous functions with respect to $\mathcal{T}_h$ which vanish on $\partial\mathcal{D}$. Hence, $V_h \subset H_0^1(\mathcal{D}) = \dot{H}^1$.

The assumption that $\mathcal{D}$ is convex and polygonal guarantees that the triangulations can be exacly fitted to $\partial\mathcal{D}$ and that we have the elliptic regularity $\|v\|_{H^2(\mathcal{D})} \le C\|\Lambda v\|$ for $v \in D(\Lambda)$. We can now quote basic results from the theory of finite elements. We use the norms $\|\cdot\|_s = \|\cdot\|_{\dot{H}^s}$.

For the orthogonal projectors $\mathcal{P}_h : \dot{H}^0 \to V_h$, $\mathcal{R}_h : \dot{H}^1 \to V_h$ defined by

$$(\mathcal{P}_h v, \chi) = (v, \chi), \quad (\nabla \mathcal{R}_h v, \nabla \chi) = (\nabla v, \nabla \chi), \quad \forall \chi \in V_h,$$

we have the following error estimates:

$$\|(\mathcal{R}_h - I)v\|_r \le C h^{s-r} \|v\|_s, \quad r = 0, 1, \ s = 1, 2, \quad v \in \dot{H}^s, \tag{4.2}$$

$$\|(\mathcal{P}_h - I)v\|_r \le C h^{s-r} \|v\|_s, \quad r = -1, 0, \ s = 1, 2, \quad v \in \dot{H}^s. \tag{4.3}$$

If $\{\mathcal{T}_h\}$ is a quasi-uniform family, then $\mathcal{P}_h$ is bounded in $\dot{H}^1$,

$$\|\mathcal{P}_h v\|_1 \le C\|v\|_1, \quad v \in \dot{H}^1. \tag{4.4}$$

Then we have also

$$\|(\mathcal{P}_h - I)v\|_1 \le C h^{s-1} \|v\|_s, \quad s = 1, 2, \quad v \in \dot{H}^s. \tag{4.5}$$

Remark 4.1. We note that the assumption of quasi-uniformity for the validity of (4.4) can be relaxed, see [4], [5], and [7].

We define a discrete variant of the norm $\|\cdot\|_\alpha$:

$$\|v_h\|_{h,\alpha} = \|\Lambda_h^{\alpha/2} v_h\|, \quad v_h \in V_h,\ \alpha \in \mathbb{R},$$

where $\Lambda_h : V_h \to V_h$ is the discrete Laplace operator defined by

$$(\Lambda_h v_h, \chi) = (\nabla v_h, \nabla \chi), \quad \forall \chi \in V_h.$$

It is clear that $\|v_h\|_{1,h} = \|\nabla v_h\| = \|v_h\|_1$ and

$$\|\mathcal{P}_h f\|_{-1,h} \le \|f\|_{-1}, \quad f \in \dot{H}^{-1} \tag{4.6}$$

follows from the calculation

$$\begin{aligned}
\|\Lambda_h^{-\frac{1}{2}} \mathcal{P}_h f\| &= \sup_{v_h \in V_h} \frac{|(\Lambda_h^{-\frac{1}{2}} \mathcal{P}_h f, v_h)|}{\|v_h\|} = \sup_{v_h \in V_h} \frac{|(f, \Lambda_h^{-\frac{1}{2}} v_h)|}{\|v_h\|} \\
&= \sup_{w_h \in V_h} \frac{|(f, w_h)|}{\|\Lambda_h^{\frac{1}{2}} w_h\|} = \sup_{w_h \in V_h} \frac{|(f, w_h)|}{\|w_h\|_1} \\
&\le \sup_{w \in \dot{H}^1} \frac{|(f, w)|}{\|w\|_1} = \|f\|_{-1}.
\end{aligned}$$

With $u_1 = u$, $u_2 = \dot{u}$, the weak form of (4.1) reads: find $u_1(t)$, $u_2(t) \in \dot{H}^1$, such that

$$\begin{aligned}
&(\nabla \dot{u}_1(t), \nabla v_1) - (\nabla u_2(t), \nabla v_1) = 0, \\
&(\dot{u}_2(t), v_2) + (\nabla u_1(t), \nabla v_2) = (f(t), v_2), \qquad \forall v_1, v_2 \in \dot{H}^1,\ t > 0, \\
&u_1(0) = u_0,\ u_2(0) = v_0.
\end{aligned} \tag{4.7}$$

The semidiscrete analogue of (4.7) is then to find $u_{h,1}(t)$, $u_{h,2}(t) \in V_h$ such that

$$\begin{aligned}
&(\nabla \dot{u}_{h,1}(t), \nabla \chi_1) - (\nabla u_{h,2}(t), \nabla \chi_1) = 0, \\
&(\dot{u}_{h,2}(t), \chi_2) + (\nabla u_{h,1}(t), \nabla \chi_2) = (f(t), \chi_2), \qquad \forall \chi_1, \chi_2 \in V_h,\ t > 0, \\
&u_{h,1}(0) = u_{h,0}, \quad u_{h,1}(0) = v_{h,0},
\end{aligned} \tag{4.8}$$

with initial values $u_{h,0}, v_{h,0} \in V_h$.

In our error analysis we will use the stability of the slightly more general problem of finding $u_{h,1}(t)$, $u_{h,2}(t) \in V_h$ such that

$$\begin{aligned}
&(\nabla \dot{u}_{h,1}(t), \nabla \chi_1) - (\nabla u_{h,2}(t), \nabla \chi_1) = (\nabla f_1(t), \nabla \chi_1), \\
&(\dot{u}_{h,2}(t), \chi_2) + (\nabla u_{h,1}(t), \nabla \chi_2) = (f_2(t), \chi_2), \qquad \forall \chi_1, \chi_2 \in V_h,\ t > 0, \\
&u_{h,1}(0) = u_{h,0}, \quad u_{h,1}(0) = v_{h,0},
\end{aligned} \tag{4.9}$$

We set $\chi_i = \Lambda_h^\alpha u_{h,i}$, $i = 1, 2$, $\alpha \in \mathbb{R}$, in (4.9) and conclude in a standard way that

$$\begin{aligned}
\|u_{h,1}(t)\|_{h,\alpha+1} + \|u_{h,2}(t)\|_{h,\alpha} \le C\Big\{ &\|u_{h,0}\|_{h,\alpha+1} + \|v_{h,0}\|_{h,\alpha} \\
&+ \int_0^t \|\mathcal{R}_h f_1(s)\|_{h,\alpha+1}\, \mathrm{d}s + \int_0^t \|\mathcal{P}_h f_2(s)\|_{h,\alpha}\, \mathrm{d}s \Big\}.
\end{aligned} \tag{4.10}$$

Next, we obtain optimal order error estimates in $L_\infty([0,\infty), \dot{H}^s)$ with $s = 0, 1$ for $u_{h,1}$ and $s = 0$ for $u_{h,2}$. The regularity requirement is minimal, see Remark 4.6.

Theorem 4.2. *Let u_1, u_2 and $u_{h,1}, u_{h,2}$ be the solutions of* (4.7) *and* (4.8)*, respectively, and set $e_i := u_{h,i} - u_i$, $i = 1, 2$. Then, for $t \geq 0$, we have*

$$\begin{aligned} \|e_1(t)\|_1 &\leq C\{\|u_{h,0} - \mathcal{R}_h u_0\|_1 + \|v_{h,0} - \mathcal{R}_h v_0\|\} \\ &\quad + Ch\Big\{\|u_1(t)\|_2 + \int_0^t \|\dot{u}_2(s)\|_1 \,\mathrm{d}s\Big\}, \end{aligned} \tag{4.11}$$

$$\begin{aligned} \|e_2(t)\| &\leq C\{\|u_{h,0} - \mathcal{R}_h u_0\|_1 + \|v_{h,0} - \mathcal{R}_h v_0\|\} \\ &\quad + Ch^2\Big\{\|u_2(t)\|_2 + \int_0^t \|\dot{u}_2(s)\|_2 \,\mathrm{d}s\Big\}, \end{aligned} \tag{4.12}$$

$$\begin{aligned} \|e_1(t)\| &\leq C\{\|u_{h,0} - \mathcal{R}_h u_0\| + \|v_{h,0} - \mathcal{P}_h v_0\|_{-1}\} \\ &\quad + Ch^2\Big\{\|u_1(t)\|_2 + \int_0^t \|u_2(s)\|_2 \,\mathrm{d}s\Big\}. \end{aligned} \tag{4.13}$$

Proof. We set

$$e_i = \theta_i + \rho_i = (u_{h,i} - \pi_i u) + (\pi_i u_i - u_i), \quad i = 1, 2, \tag{4.14}$$

where π_i will be chosen as $\mathcal{R}_h$ or $\mathcal{P}_h$. By subtraction of (4.7) and (4.8), recalling $V_h \subset \dot{H}^1$, we obtain

$$\begin{aligned} (\nabla \dot{e}_1(t), \nabla \chi_1) - (\nabla e_2(t), \nabla \chi_1) &= 0, \\ (\dot{e}_2(t), \chi_2) + (\nabla e_1(t), \nabla \chi_2) &= 0, \end{aligned} \qquad \forall \chi_1, \chi_2 \in V_h,\ t > 0.$$

Hence,

$$\begin{aligned} (\nabla \dot{\theta}_1, \nabla \chi_1) - (\nabla \theta_2, \nabla \chi_1) &= -(\nabla \dot{\rho}_1, \nabla \chi_1) + (\nabla \rho_2, \nabla \chi_1), \\ (\dot{\theta}_2, \chi_2) + (\nabla \theta_1, \nabla \chi_2) &= -(\dot{\rho}_2, \chi_2) - (\nabla \rho_1, \nabla \chi_2), \end{aligned} \quad \forall \chi_1, \chi_2 \in V_h, t > 0.$$

First, in order to prove the error estimates (4.11) and (4.12), we set

$$\theta_i = u_{h,i} - \mathcal{R}_h u_i, \quad \rho_i = (\mathcal{R}_h - I)u_i, \quad i = 1, 2.$$

By the definitions of the operators $\mathcal{R}_h, \mathcal{P}_h$, we have

$$\begin{aligned} (\nabla \dot{\theta}_1, \nabla \chi_1) - (\nabla \theta_2, \nabla \chi_1) &= 0, \\ (\dot{\theta}_2, \chi_2) + (\nabla \theta_1, \nabla \chi_2) &= -(\dot{\rho}_2, \chi_2), \end{aligned} \qquad \forall \chi_1, \chi_2 \in V_h,\ t > 0,$$

that is, θ_1, θ_2 satisfy (4.9) with $f_1 = 0$, $f_2 = -\dot{\rho}_2$. Therefore, by the stability inequality (4.10) with $\alpha = 0$, we obtain

$$\|\theta_1(t)\|_{h,1} + \|\theta_2(t)\|_{h,0} \leq C\Big\{\|\theta_1(0)\|_{h,1} + \|\theta_2(0)\|_{h,0} + \int_0^t \|\mathcal{P}_h \dot{\rho}_2(s)\|_{h,0} \,\mathrm{d}s\Big\},$$

Recalling (4.14) and that $\|v_h\|_{h,0} = \|v_h\|$ and $\|v_h\|_{h,1} = \|v_h\|_1$, $v_h \in V_h$, we have

$$\begin{aligned} \|e_1(t)\|_1 \leq C\Big\{&\|u_{h,0} - \mathcal{R}_h u_0\|_1 + \|v_{h,0} - \mathcal{R}_h v_0\| \\ &+ \int_0^t \|(\mathcal{R}_h - I)\dot{u}_2(s)\| \,\mathrm{d}s + \|(\mathcal{R}_h - I)u_1(t)\|_1\Big\}, \end{aligned}$$

$$\begin{aligned} \|e_2(t)\| \leq C\Big\{&\|u_{h,0} - \mathcal{R}_h u_0\|_1 + \|v_{h,0} - \mathcal{R}_h v_0\| \\ &+ \int_0^t \|(\mathcal{R}_h - I)\dot{u}_2(s)\| \,\mathrm{d}s + \|(\mathcal{R}_h - I)u_2(t)\|\Big\}. \end{aligned}$$

Using (4.2) we conclude (4.11) and (4.12).

Finally, to prove the error estimates (4.13) we alter the choice of π_i in (4.14) and set

$$\begin{aligned} \theta_1 &= u_{h,1} - \mathcal{R}_h u_1, \quad \rho_1 = (\mathcal{R}_h - I)u_1, \\ \theta_2 &= u_{h,2} - \mathcal{P}_h u_2, \quad \rho_2 = (\mathcal{P}_h - I)u_2. \end{aligned} \tag{4.15}$$

Then, similarly to the previous case,

$$\begin{aligned} (\nabla\dot{\theta}_1, \nabla\chi_1) - (\nabla\theta_2, \nabla\chi_1) &= (\nabla\rho_2, \nabla\chi_1), \\ (\dot{\theta}_2, \chi_2) + (\nabla\theta_1, \nabla\chi_2) &= 0, \end{aligned} \qquad \forall \chi_1, \chi_2 \in V_h, \ t > 0,$$

that is, θ_1, θ_2 satisfy (4.9) with $f_1 = \rho_2$, $f_2 = 0$. Therefore, by the stability inequality (4.10) with $\alpha = -1$, we obtain

$$\begin{aligned} &\|\theta_1(t)\|_{h,0} + \|\theta_2(t)\|_{h,-1} \\ &\qquad \le C\Big\{ \|\theta_1(0)\|_{h,0} + \|\theta_2(0)\|_{h,-1} + \int_0^t \|\mathcal{R}_h\rho_2(s)\|_{h,0}\,\mathrm{d}s \Big\}, \end{aligned}$$

Using (4.6), (4.14), and

$$\|\mathcal{R}_h\rho_2\| = \|\mathcal{P}_h(I - \mathcal{R}_h)u_2\| \le \|(\mathcal{R}_h - I)u_2\|,$$

we have

$$\begin{aligned} \|e_1(t)\| \le C\Big\{ &\|u_{h,0} - \mathcal{R}_h u_0\| + \|v_{h,0} - \mathcal{P}_h v_0\|_{-1} \\ &+ \int_0^t \|(\mathcal{R}_h - I)u_2(s)\|\,\mathrm{d}s + \|(\mathcal{R}_h - I)u_1(t)\| \Big\}. \end{aligned}$$

This proves (4.13). □

4.2. **Error estimates for the homogeneous problem.** Here we specialize to the homogeneous problem

$$\begin{aligned} &\ddot{u}(t) + \Lambda u(t) = 0, \qquad t > 0, \\ &u(0) = u_0, \ \dot{u}(0) = v_0, \end{aligned} \tag{4.16}$$

and express the error estimates in terms of the initial values. Differentiating the equation with respect to t, we obtain in a standard way

$$\|\mathrm{D}_t^r \dot{u}(t)\|_\alpha^2 + \|\mathrm{D}_t^r u(t)\|_{\alpha+1}^2 = \|v_0^r\|_\alpha^2 + \|u_0^r\|_{\alpha+1}^2. \tag{4.17}$$

Here, for $k = 0, 1, \dots,$

$$\begin{aligned} u_0^r &= \Lambda^k u_0, \quad v_0^r = \Lambda^k v_0, \qquad r = 2k, \\ u_0^r &= \Lambda^k v_0, \quad v_0^r = \Lambda^{k+1} u_0, \quad r = 2k+1. \end{aligned}$$

We use the notation from Section 3 and we write (4.16) as

$$\begin{aligned} \dot{X}(t) &= AX(t), \quad t > 0, \\ X(0) &= X_0, \end{aligned} \tag{4.18}$$

and we recall that the linear operator A is the generator of a C_0-semigroup $E(t) = \mathrm{e}^{tA}$ given by (3.2). Therefore the solution is $X(t) = E(t)X_0$. The finite element problem is then to find $X_h(t) \in V_h \times V_h$ such that

$$\begin{aligned} \dot{X}_h(t) &= A_h X_h(t), \quad t > 0, \\ X_h(0) &= X_{h,0}, \end{aligned} \tag{4.19}$$

where

$$A_h = \begin{bmatrix} 0 & I \\ -\Lambda_h & 0 \end{bmatrix}, \quad X_h = \begin{bmatrix} u_{h,1} \\ u_{h,2} \end{bmatrix}, \quad X_{h,0} = \begin{bmatrix} u_{h,0} \\ v_{h,0} \end{bmatrix}. \tag{4.20}$$

Similarly to (3.2), it can be shown that A_h generates a C_0-semigroup $E_h(t)$ given by

$$E_h(t) = \mathrm{e}^{tA_h} = \begin{bmatrix} C_h(t) & \Lambda_h^{-1/2} S_h(t) \\ -\Lambda_h^{1/2} S_h(t) & C_h(t) \end{bmatrix} \tag{4.21}$$

with

$$C_h(t) = \cos(t\Lambda_h^{1/2}), \quad S_h(t) = \sin(t\Lambda_h^{1/2}).$$

For example, similarly to the infinite dimensional case, using the eigenpairs of the discrete Laplacian Λ_h, that is $\{(\lambda_{h,j}, \phi_{h,j})\}_{j=1}^{N_h}$, with $N_h = \dim(V_h)$, we have

$$\Lambda_h^{-1/2} \sin(t\Lambda_h^{1/2}) v_h = \sum_{j=1}^{N_h} \lambda_{h,j}^{-1/2} \sin(t\lambda_{h,j}^{1/2})(v_h, \phi_{h,j})\phi_{h,j}, \quad v_h \in V_h.$$

We may now formulate a consequence of Theorem 4.2, which will be used to prove the strong convergence of the finite element approximation of the stochastic wave equation. Recall $|||v|||_\alpha^2 = \|v_1\|_\alpha^2 + \|v_2\|_{\alpha-1}^2$ from (3.1).

Corollary 4.3. *Denote* $X_0 = [u_0, v_0]^T$ *and*

$$F_h(t)X_0 = \big(C_h(t)\mathcal{P}_h - C(t)\big)u_0 + \big(\Lambda_h^{-1/2} S_h(t)\mathcal{P}_h - \Lambda^{-1/2}S(t)\big)v_0, \tag{4.22}$$

$$G_h(t)X_0 = \big(C_h(t)\mathcal{R}_h - C(t)\big)u_0 + \big(\Lambda_h^{-1/2} S_h(t)\mathcal{P}_h - \Lambda^{-1/2}S(t)\big)v_0, \tag{4.23}$$

$$\dot{G}_h(t)X_0 = -\big(\Lambda_h^{1/2} S_h(t)\mathcal{R}_h - \Lambda^{1/2}S(t)\big)u_0 + \big(C_h(t)\mathcal{P}_h - C(t)\big)v_0. \tag{4.24}$$

Then we have

$$\|F_h(t)X_0\| \le C(1+t)h^{\frac{2}{3}\beta}|||X_0|||_\beta, \quad t \ge 0, \quad \beta \in [0,3], \tag{4.25}$$

$$\|G_h(t)X_0\|_1 \le C(1+t)h^{\frac{1}{2}(\beta-1)}|||X_0|||_\beta, \quad t \ge 0, \quad \beta \in [1,3], \tag{4.26}$$

$$\|\dot{G}_h(t)X_0\| \le C(1+t)h^{\frac{2}{3}(\beta-1)}|||X_0|||_\beta, \quad t \ge 0, \quad \beta \in [1,4]. \tag{4.27}$$

Note that F_h and G_h differ only in the choice of initial value: $u_{0,h} = \mathcal{P}_h u_0$ and $u_{0,h} = \mathcal{R}_h u_0$. This is necessary in order to accomodate the lowest order of initial regularity used ($\beta = 0$ and $\beta = 1$).

Proof. We begin with the case $\beta = 0$ of (4.25). By the stability (4.10) with $\alpha = -1$ and its the analogue for the continuous equation, and (4.6), we have

$$\begin{aligned} \|F_h(t)X_0\| &\le \|u_{h,1}(t)\| + \|u_1(t)\| \\ &\le C\big\{\|\mathcal{P}_h u_0\| + \|\mathcal{P}_h v_0\|_{-1,h} + \|u_0\| + \|v_0\|_{-1}\big\} \\ &\le C\big(\|u_0\| + \|v_0\|_{-1}\big) = C|||X_0|||_0. \end{aligned}$$

For the case $\beta = 3$ we use (4.13) with $u_{0,h} = \mathcal{P}_h u_0$ and $v_{0,h} = \mathcal{P}_h v_0$, and (4.17),

$$\begin{aligned}\|F_h(t)X_0\| &= \|e_1(t)\| \\ &\le C\{\|\mathcal{P}_h(I-\mathcal{R}_h)u_0\|\} \\ &\quad + Ch^2\Big\{\|u_1(t)\|_2 + \int_0^t \|u_2(s)\|_2\,\mathrm{d}s\Big\} \\ &\le Ch^2\{\|u_0\|_2 + \|v_0\|_1 + t(\|u_0\|_3 + \|v_0\|_2)\} \\ &\le C(1+t)h^2|||X_0|||_3.\end{aligned}$$

The proof is then completed by interpolation between these cases.

For (4.26) we first use (4.10) with $\alpha = 0$,

$$\begin{aligned}\|G_h(t)X_0\|_1 &\le \|u_{h,1}(t)\|_1 + \|u_1(t)\|_1 \\ &\le C\{\|\mathcal{R}_h u_0\|_1 + \|\mathcal{P}_h v_0\| + \|u_0\|_1 + \|v_0\|\} \\ &\le C\big(\|u_0\|_1 + \|v_0\|\big) = C|||X_0|||_1.\end{aligned}$$

Then we use (4.11) with $u_{0,h} = \mathcal{R}_h u_0$ and $v_{0,h} = \mathcal{P}_h v_0$,

$$\begin{aligned}\|G_h(t)X_0\|_1 &= \|e_1(t)\|_1 \\ &\le C\{\|\mathcal{P}_h(I-\mathcal{R}_h)v_0\|\} + Ch\Big\{\|u_1(t)\|_2 + \int_0^t \|\dot{u}_2(s)\|_1\,\mathrm{d}s\Big\} \\ &\le Ch\{\|u_0\|_2 + \|v_0\|_1 + t(\|u_0\|_3 + \|v_0\|_2)\} \\ &\le C(1+t)h^2|||X_0|||_3.\end{aligned}$$

For (4.27) we apply (4.10) with $\alpha = 0$,

$$\begin{aligned}\|\dot{G}_h(t)X_0\| &\le \|u_{h,2}(t)\| + \|u_2(t)\| \le C\{\|\mathcal{R}_h u_0\|_1 + \|\mathcal{P}_h v_0\| + \|u_0\|_1 + \|v_0\|\} \\ &\le C\big(\|u_0\|_1 + \|v_0\|\big) = C|||X_0|||_1.\end{aligned}$$

Then we use (4.12) with $u_{0,h} = \mathcal{R}_h u_0$ and $v_{0,h} = \mathcal{P}_h v_0$,

$$\begin{aligned}\|\dot{G}_h(t)X_0\| &= \|e_2(t)\| \le Ch^2\Big\{\|u_2(t)\|_2 + \int_0^t \|\dot{u}_2(s)\|_2\,\mathrm{d}s\Big\} \\ &\le Ch\{\|u_0\|_3 + \|v_0\|_2 + t(\|u_0\|_4 + \|v_0\|_3)\} \le C(1+t)h^2|||X_0|||_4.\end{aligned}$$

□

Remark 4.4. The regularity assumption on X_0 in Corollary 4.3 cannot be relaxed. This means that $|||X_0|||_\beta$ can not be replaced by $|||X_0|||_{\beta-\epsilon}$ for any $\epsilon > 0$. This is shown in the lemma below for the periodic problem

$$\begin{aligned}&\ddot{u}(x,t) - u_{xx}(x,t) = 0, && (x,t) \in \mathbb{R}\times(0,\infty), \\ &u(x+2\pi,t) = u(x,t), && (x,t) \in \mathbb{R}\times(0,\infty), \\ &u(x,0) = u_0(x),\ \dot{u}(x,0) = v_0(x), && x \in \mathbb{R}.\end{aligned} \tag{4.28}$$

Lemma 4.5. *Let u be the solution of* (4.28) *and u_h its finite element approximation. Assume that, for some $\beta \ge 0$, there is a constant C such that for all $u_0 \in \dot{H}^\alpha_{per}$, $v_0 \in \dot{H}^{\alpha-1}_{per}$ and $h > 0$,*

$$\|u(t) - u_h(t)\| \le Ch^{\frac{2}{3}\beta}\big(\|u_0\|_{\dot{H}^\alpha_{per}} + \|v_0\|_{\dot{H}^{\alpha-1}_{per}}\big), \quad t \ge 0.$$

Then $\alpha \ge \beta$.

Here $\dot{H}^\alpha_{\rm per}$ stands for the subspace of $\dot{H}^\alpha$ consisting of 2π-periodic functions.

Proof. The proof is adapted from [19]. We omit the details. □

Remark 4.6. Optimal order $L_\infty([0,\infty),\dot{H}^0)$ estimates for the finite element approximation of displacement $u = u_1$ and velocity $\dot{u} = u_2$ were first obtained by [10]. However, the regularity requirement for the initial displacement is not minimal in [10]. This was improved in [3], and in [19] it was shown that the resulting regularity requirement is optimal, see Lemma 4.5 above. The error estimates (4.12) and (4.13) are in agreement with the corresponding ones in [3] and [19]. Furthermore, the proof presented here seems to be more straightforward.

5. The finite element method for the stochastic problem

We now consider the approximation of the stochastic wave equation. The spatially discrete analogue of (2.6) is to find $X_h(t) = (u_{h,1}(t), u_{h,2}(t)) \in V_h \times V_h$ such that

$$\begin{aligned} \mathrm{d}X_h(t) &= A_h X_h(t)\,\mathrm{d}t + \mathcal{P}_h B\,\mathrm{d}W(t), \quad t>0,\\ X_h(0) &= X_{0,h}, \end{aligned} \tag{5.1}$$

where A_h is defined in (4.20). Recall that A_h generates the C_0-semigroup $E_h(t) = \mathrm{e}^{tA_h}$ on V_h given by (4.21), and therefore the unique mild solution of (5.1) is given by

$$X_h(t) = E_h(t)X_{0,h} + \int_0^t E_h(t-s)\mathcal{P}_h B\,\mathrm{d}W(s), \quad t\geq 0. \tag{5.2}$$

Recall $|||v|||_\alpha^2 = \|v_1\|_\alpha^2 + \|v_2\|_{\alpha-1}^2$ from (3.1).

Theorem 5.1. *Let $X_0 = [u_0, v_0]^T$ and let $X = [u_1, u_2]^T$ and $X_h = [u_{h,1}, u_{h,2}]^T$ be given by (3.3) and (5.2), respectively. Then, the following estimates hold for $t \geq 0$, where $C(t)$ is an increasing function.*
If $u_{0,h} = \mathcal{P}_h u_0$, $v_{0,h} = \mathcal{P}_h v_0$, and $\beta \in [0,3]$, then

$$\|u_{h,1}(t) - u_1(t)\|_{L_2(\Omega,\dot{H}^0)} \leq C(t)h^{\frac{2}{3}\beta}\big\{\|X_0\|_{L_2(\Omega,H^\beta)} + \|\Lambda^{\frac12(\beta-1)}Q^{1/2}\|_{\rm HS}\big\}. \tag{5.3}$$

If $u_{0,h} = \mathcal{R}_h u_0$, $v_{0,h} = \mathcal{P}_h v_0$, and $\beta \in [1,3]$, then

$$\|u_{h,1}(t) - u_1(t)\|_{L_2(\Omega,\dot{H}^1)} \leq C(t)h^{\frac{1}{2}(\beta-1)}\big\{\|X_0\|_{L_2(\Omega,H^\beta)} + \|\Lambda^{\frac12(\beta-1)}Q^{1/2}\|_{\rm HS}\big\}. \tag{5.4}$$

If $u_{0,h} = \mathcal{R}_h u_0$, $v_{0,h} = \mathcal{P}_h v_0$, and $\beta \in [1,4]$, then

$$\|u_{h,2}(t) - u_2(t)\|_{L_2(\Omega,\dot{H}^0)} \leq C(t)h^{\frac{2}{3}(\beta-1)}\big\{\|X_0\|_{L_2(\Omega,H^\beta)} + \|\Lambda^{\frac12(\beta-1)}Q^{1/2}\|_{\rm HS}\big\}. \tag{5.5}$$

The discrete initial values ($u_{0,h} = \mathcal{R}_h u_0$, or $u_{0,h} = \mathcal{R}_h u_0$, and $v_{0,h} = \mathcal{P}_h v_0$) and the regularity of the initial values ($X_0 \in H^\beta$) are chosen so that the corresponding rates of convergence match those of the stochastic convolution terms. Of course, other choices are possible with different convergence rates that can be derived from Theorem 4.2.

Proof. We prove (5.3); the proofs of the other estimates are similar.

In addition to F_h defined in (4.22) we introduce

$$K_h(t)f = \big(\Lambda_h^{-1/2} S_h(t)\mathcal{P}_h - \Lambda^{-1/2} S(t)\big)f \tag{5.6}$$

and deduce from (4.25) with $u_0 = 0$ that

$$\|K_h(t)f\| \le C(1+t)h^{\frac{2}{3}\beta}\|f\|_{\beta-1}. \tag{5.7}$$

Then we have

$$u_{h,1}(t) - u_1(t) = F_h(t)X_0 + \int_0^t K_h(t-s)\,\mathrm{d}W(s).$$

By Itô's isometry (2.5),

$$\begin{aligned}
\|u_{h,1}(t) - u_1(t)\|_{L_2(\Omega,U)} &\le \|F_h(t)X_0\|_{L_2(\Omega,U)} + \Big\| \int_0^t K_h(t-s)\,\mathrm{d}W(s)\Big\|_{L_2(\Omega,U)} \\
&= \|F_h(t)X_0\|_{L_2(\Omega,U)} + \Big(\int_0^t \|K_h(s)Q^{1/2}\|_{\mathrm{HS}}^2\,\mathrm{d}s\Big)^{1/2} \\
&= I + II.
\end{aligned}$$

From (4.25) it follows that

$$I^2 = \mathbf{E}\big(\|F_h(t)X_0\|^2\big) \le C(t)h^{\frac{4}{3}\beta}\mathbf{E}\big(|||X_0|||_\beta^2\big).$$

Recalling the definition of the Hilbert-Schmidt norm from Section 2, using an orthonormal basis $\{e_k\}_{k=1}^\infty$ in $U = \dot{H}^0$, we obtain

$$II^2 = \sum_{k=1}^\infty \int_0^t \|K_h(s)Q^{1/2}e_k\|^2\,\mathrm{d}s.$$

Finally, by setting $f = Q^{1/2}e_k$ in (5.7), we conclude that

$$II^2 \le C(t)th^{\frac{4}{3}\beta}\sum_{k=1}^\infty \|Q^{1/2}e_k\|_{\beta-1}^2 = C(t)h^{\frac{4}{3}\beta}\|\Lambda^{(\beta-1)/2}Q^{1/2}\|_{\mathrm{HS}}^2,$$

which completes the proof of (5.3). □

Remark 5.2. Let consider the one dimensional case with space-time white noise, that is, when $d = 1$, $Q = I$. Then $\beta < 1/2$ (see Remark 3.2) and the convergence rate in (5.3) is $O(h^\alpha)$, $\alpha < 1/3$, which is in agreement with [18], while $O(h^{1/2})$ was shown for the leap-frog scheme in [24]. The reason why a higher rate of convergence is obtained in [24] is that the Green's functions of the continuous and the discrete equations coincide at the mesh points.

Another example of a numerical scheme where this happens is Galerkin's method with

$$V_h = \operatorname{span}\{\mathrm{e}^{inx} : |n| \le 1/h\},$$

see [19, Remark 2]. Then instead of (4.25) we would have

$$\|F_h(t)X_0\| \le Ch^\beta |||X_0|||_\beta, \quad t \ge 0,$$

and, under the assumptions of (5.3),

$$\|u_{h,1}(t) - u_1(t)\|_{L_2(\Omega,U)} \le Ch^\beta\big\{\|X_0\|_{L_2(\Omega,H^\beta)} + \|\Lambda^{(\beta-1)/2}Q^{1/2}\|_{\mathrm{HS}}\big\}.$$

This yields the optimal order $O(h^\alpha)$, $\alpha < 1/2$, for $Q = I$.

The error estimates in Theorem 4.2, and therefore in Corollary 4.3 and Theorem 5.1, can be extended to higher order finite element methods. The reason is that the error estimates for the elliptic and the orthogonal projections in (4.2) and (4.3), respectively, as well as the stability inequality (4.10) hold for higher order finite element spaces V_h consisting of continuous piecewise polynomials of order at most $k \geq 1$. This means that in case of highly correlated noise, one might expect higher order of strong convergence when using a higher order finite element method. In this case the counterpart of Theorem 5.1 reads as follows.

Theorem 5.3. *Let $X_0 = [u_0, v_0]^T$ and let $X = [u_1, u_2]^T$ and $X_h = [u_{h,1}, u_{h,2}]^T$ be given by* (3.3) *and* (5.2), *respectively, where the finite element spaces V_h consist of continuous piecewise polynomials of order at most $k \geq 1$. Then, the following estimates hold for $t \geq 0$, where $C(t)$ is an increasing function.*
If $u_{0,h} = \mathcal{P}_h u_0$, $v_{0,h} = \mathcal{P}_h v_0$, and $\beta \in [0, k+2]$, then

$$\|u_{h,1}(t) - u_1(t)\|_{L_2(\Omega,\dot{H}^0)} \leq C(t) h^{\frac{k+1}{k+2}\beta} \{\|X_0\|_{L_2(\Omega,H^\beta)} + \|\Lambda^{\frac{1}{2}(\beta-1)} Q^{1/2}\|_{\mathrm{HS}}\}.$$

If $u_{0,h} = \mathcal{R}_h u_0$, $v_{0,h} = \mathcal{P}_h v_0$, and $\beta \in [1, k+2]$, then

$$\begin{aligned}\|u_{h,1}(t) - u_1(t)\|_{L_2(\Omega,\dot{H}^1)} \\ \leq C(t) h^{\frac{k}{k+1}(\beta-1)} \{\|X_0\|_{L_2(\Omega,H^\beta)} + \|\Lambda^{\frac{1}{2}(\beta-1)} Q^{1/2}\|_{\mathrm{HS}}\}.\end{aligned}$$

If $u_{0,h} = \mathcal{R}_h u_0$, $v_{0,h} = \mathcal{P}_h v_0$, and $\beta \in [1, k+3]$, then

$$\begin{aligned}\|u_{h,2}(t) - u_2(t)\|_{L_2(\Omega,\dot{H}^0)} \\ \leq C(t) h^{\frac{k+1}{k+2}(\beta-1)} \{\|X_0\|_{L_2(\Omega,H^\beta)} + \|\Lambda^{\frac{1}{2}(\beta-1)} Q^{1/2}\|_{\mathrm{HS}}\}.\end{aligned}$$

6. Numerical experiments

In this section we demonstrate the order of strong convergence of the finite element method for the linear stochastic wave equation LSWE (1.1) by numerical examples. To this end, the backward Euler method is used for time stepping and some computational analysis on the approximation of the stochastic convolution is reviewed, see [26].

6.1. **Computational analysis.** First recall the matrix form of (5.1),

$$\begin{bmatrix} \mathrm{d}u_{h,1}(t) \\ \mathrm{d}u_{h,2}(t) \end{bmatrix} = \begin{bmatrix} 0 & I \\ -\Lambda_h & 0 \end{bmatrix} \begin{bmatrix} u_{h,1}(t) \\ u_{h,2}(t) \end{bmatrix} \mathrm{d}t + \begin{bmatrix} 0 \\ \mathcal{P}_h \, \mathrm{d}W(t) \end{bmatrix}, \tag{6.1}$$

Let $0 = t_0 < t_1 < \cdots < t_{N_t} = T_N$, be a uniform partition of the time interval $[0, T_N]$ with time step $k = 1/N_t$ and time subintervals $I_n = (t_{n-1}, t_n)$, $n = 1, 2, \cdots, N_t$. Then the backward Euler method is formulated as, for $n = 1, 2, \cdots, N_t$,

$$\begin{bmatrix} U_1^n \\ U_2^n \end{bmatrix} - \begin{bmatrix} U_1^{n-1} \\ U_2^{n-1} \end{bmatrix} = \begin{bmatrix} 0 & kI \\ -k\Lambda_h & 0 \end{bmatrix} \begin{bmatrix} U_1^n \\ U_2^n \end{bmatrix} + \begin{bmatrix} 0 \\ \mathcal{P}_h \, \Delta W^n \end{bmatrix}. \tag{6.2}$$

Here $U_i^n \in V_h$ is an approximation of $u_i(\cdot, t_n)$, $i = 1, 2$, and $[U_1^0, U_2^0]^T = \xi_h$. We multiply (6.2) by

$$\begin{bmatrix} \Lambda_h & 0 \\ 0 & I \end{bmatrix}$$

to take advantage of the resulting skew-symmetric structure, see Subsection 6.3, and rearrange, to obtain, for $n = 1, 2, \cdots, N_t$,

$$\begin{bmatrix} \Lambda_h & -k\Lambda_h \\ k\Lambda_h & I \end{bmatrix} \begin{bmatrix} U_1^n \\ U_2^n \end{bmatrix} = \begin{bmatrix} \Lambda_h & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} U_1^{n-1} \\ U_2^{n-1} \end{bmatrix} + \begin{bmatrix} 0 \\ \mathcal{P}_h \, \Delta W^n \end{bmatrix}. \tag{6.3}$$

For some other ways of approximating the noise and the stochastic integrals we refer to, for example, [2] and [8].

Recalling the Fourier expansion (2.2) of W, we have, for all $\chi \in V_h$,

$$\Big(\mathcal{P}_h \, \Delta W^n, \chi\Big) = \sum_{j=1}^{\infty} \gamma_j^{1/2} \Delta\beta_j^n (e_j, \chi) \approx \sum_{j=1}^{J} \gamma_j^{1/2} \Delta\beta_j^n (e_j, \chi), \tag{6.4}$$

where we truncated the sum to J terms. Recall that $\{\beta_j(t)\}_{j=1}^J$ are mutually independent standard real-valued Brownian motions, and that the increments in (6.4) are

$$\Delta\beta_j^n = \beta_j(t_n) - \beta_j(t_{n-1}) \sim \sqrt{k}\mathcal{N}(0,1), \tag{6.5}$$

that is, real-valued Gaussian random variables with 0 mean and variance k. We also note that $\gamma_j = 1$ for the white noise.

Recalling the semidiscrete solution u_h from (5.2), we denote by u_h^J the semidiscrete solution obtained by using the truncated noise; that is,

$$u_h^J(t) = E_h(t) X_{0,h} + \sum_{j=1}^{J} \gamma_j^{1/2} \int_0^t E_h(t-s) \mathcal{P}_h B e_j \, \mathrm{d}\beta_j(s). \tag{6.6}$$

The following lemma shows, that under some assumptions on the triangulation and the covariance operator Q, it is enough to take $J \geq N_h$ with $N_h = \dim(V_h)$ in order to preserve the order of the FEM.

Lemma 6.1. *Let u_h^J and u_h be defined by* (6.6) *and* (5.2)*, respectively. Assume that Λ and Q have a common orthonormal basis of eigenfunctions $\{e_j\}_{j=1}^\infty$ and that V_h, with dimension N_h, is defined on a family of quasi-uniform triangulations $\{\mathcal{T}_h\}$ of $\mathcal{D}$. Then for $J \geq N_h$ the following estimates hold, where $C(t)$ is an increasing function.*

If $\|\Lambda^{(\beta-1)/2} Q^{1/2}\|_{\mathrm{HS}} < \infty$ for some $\beta \in [0,3]$, then,

$$\|u_{h,1}^J(t) - u_{h,1}(t)\|_{L_2(\Omega, \dot{H}^0)} \leq C(t) h^{\frac{2}{3}\beta} \|\Lambda^{(\beta-1)/2} Q^{1/2}\|_{\mathrm{HS}}.$$

If $\|\Lambda^{(\beta-1)/2} Q^{1/2}\|_{\mathrm{HS}} < \infty$ for some $\beta \in [1,3]$, then,

$$\|u_{h,1}^J(t) - u_{h,1}(t)\|_{L_2(\Omega, \dot{H}^1)} \leq C(t) h^{\frac{1}{2}(\beta-1)} \|\Lambda^{(\beta-1)/2} Q^{1/2}\|_{\mathrm{HS}}.$$

If $\|\Lambda^{(\beta-1)/2} Q^{1/2}\|_{\mathrm{HS}} < \infty$ for some $\beta \in [1,4]$, then,

$$\|u_{h,2}^J(t) - u_{h,2}(t)\|_{L_2(\Omega, \dot{H}^0)} \leq C(t) h^{\frac{2}{3}(\beta-1)} \|\Lambda^{(\beta-1)/2} Q^{1/2}\|_{\mathrm{HS}}.$$

Proof. We prove the second estimate; the others are proved similarly. From (5.2) and (6.6) it follows that

$$u_{h,1}^J(t) - u_{h,1}(t) = \sum_{j=J+1}^{\infty} \gamma_j^{1/2} \int_0^t \Lambda_h^{-1/2} S_h(t-s) \mathcal{P}_h e_j \, \mathrm{d}\beta_j(s).$$

By Itô's isometry (2.5), the independence of β_j's and recalling the error operator from (5.6), we have

$$\begin{aligned}\|u^J_{h,1}(t)-u_{h,1}(t)\|^2_{L_2(\Omega,\dot H^1)} &= \sum_{j=J+1}^{\infty}\gamma_j\int_0^t\|\Lambda_h^{-1/2}S_h(s)\mathcal{P}_he_j\|_1^2\,\mathrm{d}s\\ &\le 2\sum_{j=J+1}^{\infty}\gamma_j\int_0^t\|\Lambda^{-1/2}S(s)e_j\|_1^2\,\mathrm{d}s\\ &\quad+2\sum_{j=J+1}^{\infty}\gamma_j\int_0^t\|K_h(s)e_j\|_1^2\,\mathrm{d}s\\ &= I+II.\end{aligned}$$

Let λ_j denote the eigenvalues of Λ corresponding to e_j. Then

$$\|\Lambda^{-1/2}\sin(s\Lambda^{1/2})e_j\|_1^2=\sin^2(s\lambda_j^{1/2}).$$

Thus,

$$\begin{aligned}I&=2\sum_{j=J+1}^{\infty}\gamma_j\int_0^t\|\Lambda^{-1/2}\sin(s\Lambda^{1/2})e_j\|_1^2\,\mathrm{d}s\\ &=2\sum_{j=J+1}^{\infty}\gamma_j\int_0^t\sin^2(s\lambda_j^{1/2})\,\mathrm{d}s\\ &\le 2t\sum_{j=J+1}^{\infty}\gamma_j\le 2t\sum_{j=J+1}^{\infty}\lambda_j^{-(\beta-1)}(\lambda_j^{\beta-1}\gamma_j)\\ &\le 2t\lambda_{J+1}^{-(\beta-1)}\sum_{j=J+1}^{\infty}\lambda_j^{\beta-1}\gamma_j\le 2t\lambda_{J+1}^{-(\beta-1)}\|\Lambda^{(\beta-1)/2}Q^{1/2}\|^2_{\mathrm{HS}}.\end{aligned}$$

For II, by (4.26) with $u_0=0$, $v_0=e_j$, we have

$$\begin{aligned}II&\le C(t)h^{\beta-1}\sum_{j=J+1}^{\infty}\gamma_j\int_0^t\|e_j\|^2_{\beta-1}\,\mathrm{d}s\\ &=C(t)h^{\beta-1}\sum_{j=J+1}^{\infty}\gamma_j\|e_j\|^2_{\beta-1}\le C(t)h^{\beta-1}\|\Lambda^{(\beta-1)/2}Q^{1/2}\|^2_{\mathrm{HS}}.\end{aligned}$$

Hence the proof is completed by the fact that, for a quasi-uniform family of triangulations, we have $N_h\approx h^{-d}$ and therefore,

$$\lambda_{J+1}^{-1}\le CJ^{-2/d}\le CN_h^{-2/d}\le Ch^2.$$

□

Remark 6.2. In practice Q and Λ do not have a common orthonormal basis of eigenfunctions and the eigenfunctions of Q are not known explicitly. In this case, one has to solve the eigenvalue problem $Qu=\lambda u$ on S_h in order to represent $\mathcal{P}_hW$. Computationally this could be very expensive if Q is given by an integral operator. However, if the kernel is smooth then this can be done more efficiently, see [21]. Furthermore, similarly to the parabolic case [13], it is enough to keep $J<N_h$ terms, for suitable J depending on the kernel, in the expansion of $\mathcal{P}_hW$.

6.2. **Numerical example.** For the numerical experiments, we consider the LSWE in one spatial dimension,

$$(6.7)\qquad \begin{aligned} &\mathrm{d}\dot{u} - \Delta u \mathrm{d}t = \mathrm{d}W, && (x,t) \in (0,1)\times(0,1),\\ &u(0,t) = u(1,t) = 0, && t \in (0,1),\\ &u(x,0) = \cos(\pi(x-1/2)),\ u_t(x,0) = 0, && x \in (0,1). \end{aligned}$$

Clearly, there is no exact solution available from a numerical viewpoint as even the solution of the deterministic problem is given as an infinite Fourier series expansion (see, e.g., [9]). Therefore we take the exact solution to be a finite element approximation on a very fine mesh with mesh size h_{exact} to approximate $u = u(x,1)$, using the backward Euler method (6.3) for time stepping with a small fixed time step k. We note that we chose the time step k according to $k \leq h^2$, since the rate of convergence of the fully discrete (6.3) for the deterministic problem is $O(k + h^2)$.

Applying the time stepping (6.3) to (6.7) we obtain the discrete system

$$(6.8)\qquad \Sigma X^n = \Xi X^{n-1} + b,$$

where $b = [0, b_2]^T$ and b_2 is computed using (6.4). We note that for the deterministic problem $b = 0$, the expected rate of convergence in the L_2-norm for both the displacement $u = u_1$ and the velocity $\dot{u} = u_2$ is 2 by (4.13) and (4.12), respectively, see Figure 1.

If $\{\lambda_j\}_{j=1}^{\infty}$ are the eigenvalues of Λ, and we set $Q = \Lambda^{-s}$, $s \in \mathbb{R}$, then

$$\|\Lambda^{(\beta-1)/2} Q^{1/2}\|_{\mathrm{HS}}^2 = \|\Lambda^{(\beta-s-1)/2}\|_{\mathrm{HS}}^2 = \sum_{j=1}^{\infty} \lambda_j^{\beta-s-1} \approx \sum_{j=1}^{\infty} j^{\frac{2}{d}(\beta-s-1)},$$

which is finite if and only if $\beta < 1+s-d/2$ with d being the dimension of the domain $\mathcal{D}$. In our example (6.7), where $d = 1$, we consider two different choices for the noise. First, we consider space-time white noise corresponding to $s = 0$ and hence $\beta < 1/2$ and then a correlated noise corresponding $s = -1$ and hence $\beta < 3/2$. We note that since the eigenfunctions of Λ are given as $e_j = \sqrt{2}\sin(j\pi x)$, $j \geq 1$, (e_j, χ) can be computed exactly for $\chi = \varphi_i$, $i = 1, \cdots, N_h$, with $\{\varphi_i\}_{i=1}^{N_h}$ being a basis in V_h. Thus, in the case of space-time white noise, we do not expect convergence for the finite element approximation of velocity $u_{h,2}$ by (5.5), but we expect the rate of convergence to be 1/3 for displacement $u_{h,1}$ by (5.3). These are confirmed by Figure 2. In the second case, the expected rate of strong convergence is 1 and 1/3 for displacement and velocity by (5.3) and (5.5), respectively, as Figure 3 also confirms. We note that we have used a uniform spatial mesh and therefore with $Q = \Lambda^s$, the assumptions of Lemma 6.1 are fulfilled.

6.3. **Comments on numerical linear algebra.** On each time level the linear system (6.8) has to be solved. This can simply be done by the backslash operator "\" in Matlab, but it can be performed faster if instead we perform a minimum degree permutation of the coefficeint matrix Σ and then use the "LU" factorization of the permuted Σ. The coefficient matrix Σ in (6.8) is skew-symmetric, which implies that, in particular, $\Sigma_{ij} \neq 0$ if $\Sigma_{ji} \neq 0$. This means that the command "symamd" in Matlab can be used. The algorithm for solving the linear system

(6.8) is performed in the following steps, with obvious notations,

$$\begin{aligned}(P_{\mathrm{s}}\Sigma P_{\mathrm{s}}^T)P_{\mathrm{s}}X^n &= P_{\mathrm{s}}(\Xi X^{n-1}+b)\\ \hat{\Sigma}\hat{X}^n &= \hat{b}\\ L_{\mathrm{lu}}U_{\mathrm{lu}}\hat{X}^n &= P_{\mathrm{lu}}\hat{b}\\ \hat{X}^n &= U_{\mathrm{lu}}\backslash\big(L_{\mathrm{lu}}\backslash(P_{\mathrm{lu}}\hat{b})\big)\\ X^n &= P_{\mathrm{s}}^{-1}\hat{X}^n,\end{aligned}$$

where $P_{\mathrm{s}} = \mathrm{symamd}(\Sigma)$, and $[P_{\mathrm{lu}}, L_{\mathrm{lu}}, U_{lu}] = \mathrm{lu}\,(\hat{X}^n)$ in Matlab. With $h_{\mathrm{exact}} = 2^{-7}$ and $k = h_{\mathrm{exact}}^2$, the computation time for each realization, that is, the computation time of generating the Brownian motion, computing the exact solution and the approximated solutions with mesh sizes $h = 2^{-1}$ to $h = 2^{-5}$, takes approximatly 40 seconds with "\" while it takes 4 seconds with minimum degree permutation. The reason for this can be seen in Figure 4 and Figure 5, where the structure and the number of nonzero entries in the "LU" factorization of Σ and $\hat{\Sigma}$ are shown. An AMD Opteron computer with 15 Gigabytes RAM memory and 2.2 GHz CPU has been used for these experiments.

Remark 6.3. One might consider two ways to compute the vector b in (6.8). Either using matrix-matrix multiplication, that is, we need to generate the increments (6.5) at once in a big $N_t \times N_h$ matrix, or using vector-matrix multiplications that means we need to generate the increments (6.5) in a loop and each time in a vector $1 \times N_h$. We used the first idea since it is faster and there was enough memory for the computations. However, the size of the matrix of the increments, and hence the memory usage, grows considerably when refining the mesh and taking smaller time steps. For example, with $N_h = 2^7$ and $N_t = 2^{14}$, in our experiments 256 Mbytes RAM was needed for storing the increment matrix, while for $N_h = 2^8$ and $N_t = 2^{16}$, we needed almost 2 Gbytes. In the latter case we used the second approach, that is vector matrix multiplications, and the computation time for each realization took about 6 seconds.

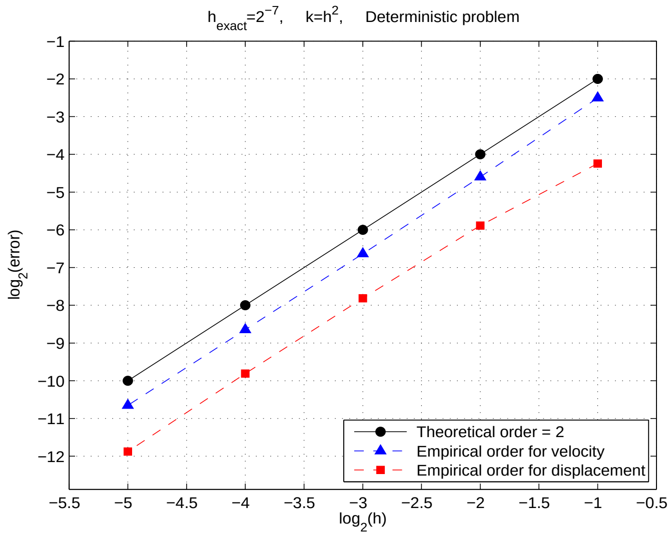

FIGURE 1. Deterministic problem: the order of strong convergence in the L_2-norm is 2 for both the displacement u (dashed-square) and the velocity $\dot{u}$ (dashed-triangle).

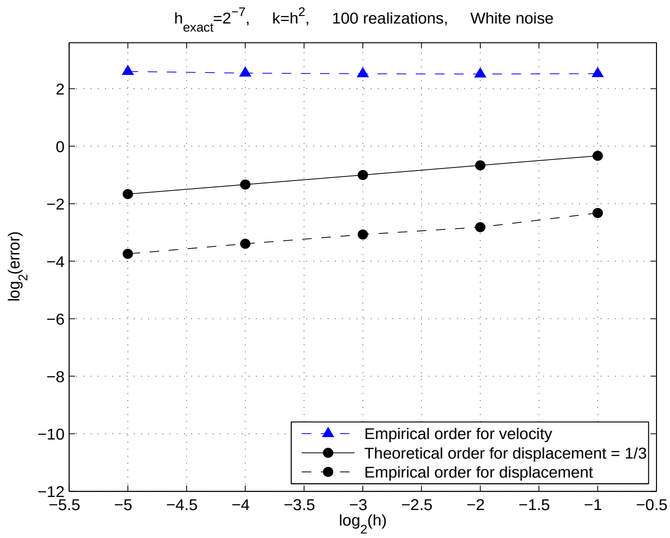

FIGURE 2. LSWE with white noise: the order of strong convergence in the L_2-norm is $1/3$ for the displacement u (dashed-circle); but there is no convergence for the velocity $\dot{u}$ (dashed-triangle).

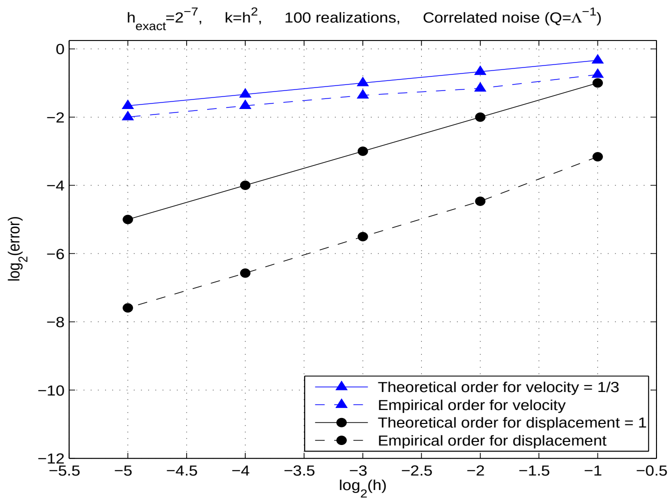

FIGURE 3. LSWE with correlated noise $Q = \Lambda^{-1}$: the order of strong convergence in the L_2-norm is 1 for the displacement u (dashed-circle), and 1/3 for the velocity $\dot{u}$ (dashed-triangle).

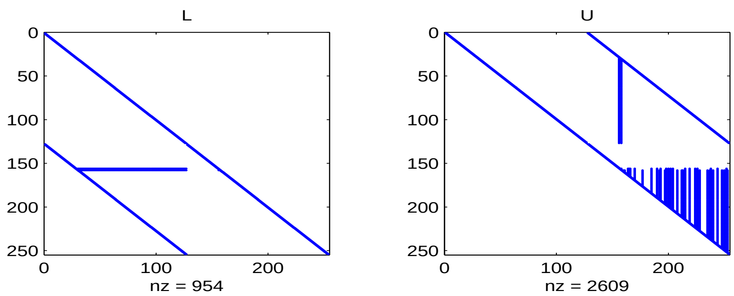

FIGURE 4. Structure and number of nonzero elements of LU(Σ)

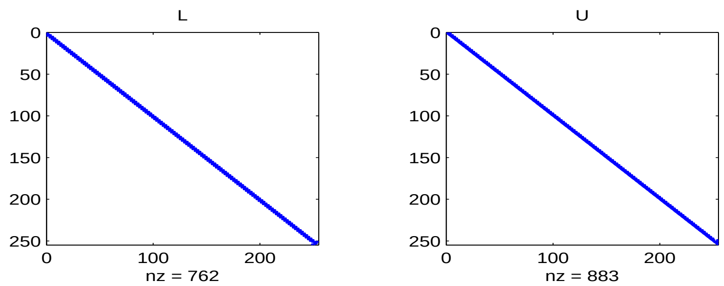

FIGURE 5. Structure and number of nonzero elements of LU$(\hat{\Sigma})$.

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Department of Mathematics and Statistics, University of Otago, P. O. Box 56, Dunedin, New Zealand
E-mail address: mihaly@chalmers.se

Department of Mathematical Sciences, Chalmers University of Technology and University of Gothenburg, SE–412 96 Göteborg, Sweden
E-mail address: stig@chalmers.se

Department of Mathematical Sciences, Chalmers University of Technology and University of Gothenburg, SE–412 96 Göteborg, Sweden
E-mail address: fardin@chalmers.se