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Finite element approximation of the linear stochastic Cahn-Hilliard equation

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Abstract

The linearized Cahn-Hilliard-Cook equation is discretized in the spatial variables by a standard finite element method. Strong convergence estimates are proved under suitable assumptions on the covariance operator of the Wiener process, which is driving the equation. The backward Euler time stepping is also studied. The analysis is set in a framework based of analytic semigroups. The main part of the work consists of detailed error bounds for the corresponding deterministic equation.

Keywords: Cahn-Hilliard-Cook equation, finite element method, backward Euler method, error estimate, strong convergence

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Ali Mesforush
Göteborg, April 2009

Contents

1 Introduction

In the first part of this work we introduce the Cahn-Hilliard and the Cahn-Hilliard-Cook equation and we formulate the finite element method for these equations. Also we have a short review to semigroups and we consider the Cahn-Hilliard and Cahn-Hilliard-Cook equations in a semigroup approach. At the end of first part we study the linear Cahn-Hilliard-Cook equation in fully discrete case. Finally we will mention some strong convergence estimates.

2 The Cahn-Hilliard equation

In this section we introduce the Cahn-Hilliard equation and an abstract framework based on semigroups of bounded linear operators. We also derive the finite element method for the Cahn-Hilliard equation.

2.1 Introduction

The Cahn-Hilliard equation,

$$u_t = \Delta(-\Delta u + f(u)), \quad x \in \mathbf{R}^n, \ t > 0, \tag{2.1}$$

was proposed in [1] as a simple model for the process of phase separation in a binary alloy at a fixed temperature. The function $f : \mathbf{R} \to \mathbf{R}$ is of "bistable type" with three simple zeroes as shown in Figure 1. A typical model nonlinearity is $f(u) = u^3 - u$. The function u represents the concen-

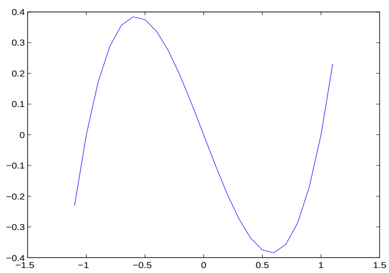

Figure 1: The form of the bistable nonlinearity

tration of one of the two metallic components of alloy. If we assume that the total density is constant, then the composition of the mixture can be expressed by the single function u.

If the alloy is contained in a vessel $\mathcal{D} \subset \mathbf{R}^n$, the equation (2.1) should be supplemented with boundary conditions on the boundary $\partial\mathcal{D}$. These are usually taken to be

$$\frac{\partial u}{\partial n} = 0, \ \frac{\partial(-\Delta u + f(u))}{\partial n} = 0, \quad x \in \partial\mathcal{D},\ t > 0. \tag{2.2}$$

Since $\frac{\partial f(u)}{\partial n} = f'(u)\frac{\partial u}{\partial n} = 0$, the condition (2.2) is equivalent to

$$\frac{\partial u}{\partial n} = 0, \ \frac{\partial \Delta u}{\partial n} = 0, \quad x \in \partial\mathcal{D},\ t > 0.$$

Also we have $u(x,0) = u_0(x)$ as initial condition. So we have the initial boundary value problem

$$\begin{aligned} &u_t + \Delta^2 u = \Delta f(u), && x \in \mathcal{D},\ t > 0, \\ &\frac{\partial u}{\partial n} = 0, \ \frac{\partial \Delta u}{\partial n} = 0, && x \in \partial\mathcal{D},\ t > 0, \\ &u(x,0) = u_0(x), && x \in \mathcal{D}. \end{aligned} \tag{2.3}$$

2.2 Semigroup approach

Definition 2.1 (Semigroup)**.** *Let X be a Banach space with norm $\|\cdot\|$. A family $\{E(t)\}_{t\geq 0}$ of bounded linear operators on X is called a semigroup of bounded linear operators if*

1. $E(0) = I$, (identity operator),
2. $E(t+s) = E(t)E(s)$, $\forall s,t \geq 0$. (semigroup property)

The semigroup is called strongly continuous if

$$\lim_{t\to 0^+} E(t)x = x \quad \forall x \in X.$$

The infinitesimal generator of semigroup is the linear operator G defined by

$$Gx = \lim_{t\to 0^+} \frac{E(t)x - x}{t},$$

its domain of definition $\mathcal{D}(G)$ being the space of all $x \in X$ for which the limit exists. The semigroup can be denoted by $E(t) = \mathrm{e}^{tG}$.

In this work we consider $-\Delta$ with the homogeneous Neumann boundary condition as an unbounded linear operator on $L_2 = L_2(\mathcal{D})$ with standard scalar product $(\cdot,\cdot)$ and norm $\|\cdot\|$. It has eigenvalues $\{\lambda_j\}_{j=0}^{\infty}$ with

$$0 = \lambda_0 < \lambda_1 \leq \cdots \leq \lambda_j \leq \cdots \quad \leq \lambda_j \to \infty,$$

and corresponding orthonormal eigenfunctions $\{\phi_j\}_{j=0}^{\infty}$. Also let H be the subspace of L_2 which is orthogonal to the constants, $H = \{v \in L_2 : (v, 1) = 0\}$, and let P be the orthogonal projection of L_2 onto H. Clearly $Pf = f - \bar{f}$, where $\bar{f} = \frac{1}{|\Omega|}\int_{\mathcal{D}} f \, \mathrm{d}x$. Define the linear operator $A = -\Delta$ with domain of definition

$$\mathcal{D}(A) = \big\{v \in H^2 \cap H : \frac{\partial v}{\partial n} = 0 \text{ on } \partial\mathcal{D}\big\}.$$

By spectral theory we define $\dot{H}^s = \mathcal{D}(A^{s/2})$ with norms $|v|_s = \|A^{s/2}v\|$ for real s. Then e^{-tA^2} can be written as

$$\mathrm{e}^{-tA^2}v = \sum_{j=1}^{\infty} \mathrm{e}^{-t\lambda_j^2}(v, \phi_j)\phi_j.$$

Let $E(t) = \mathrm{e}^{-tA^2}$. The semigroup $\{E(t)\}_{t\geq 0}$ is called semigroup generated by $-A^2$. This is a strongly continuous semigroup. Moreover, it is analytic, meaning that e^{-tA^2} can be extended as a holomorphic function of t. This leads to the important properties in the following lemma.

Lemma 2.2. *If $\{E(t)\}_{t\geq 0}$ is the semigroup generated by $-A^2$, then the following hold*

1. $\|A^\beta E(t)v\| \leq Ct^{-\beta/2}\|v\|, \quad \beta \geq 0,$
2. $\int_0^t \|AE(s)v\|^2 \mathrm{d}s \leq C\|v\|^2.$

In the sequel we will write the equation (2.3) in operator form. By definition of $\mathcal{D}(A)$ and H, equation (2.3) can be written as

$$\begin{aligned} &u_t + A^2 u = -APf(u) \quad t > 0, \\ &u(0) = u_0. \end{aligned} \tag{2.4}$$

which is equivalent to the fixed point equation

$$u(t) = \mathrm{e}^{-tA^2}u_0 - \int_0^t \mathrm{e}^{-(t-s)A^2}APf(u(s))\,\mathrm{d}s.$$

2.3 The finite element method for the Cahn-Hilliard equation

In this section we formulate the finite element method (see [8]) for the Cahn-Hilliard equation. Rewrite (2.3) in the form

$$\begin{aligned} &u_t - \Delta v = 0, && x \in \mathcal{D},\ t > 0, \\ &v = -\Delta u + f(u), && x \in \mathcal{D},\ t > 0, \\ &\frac{\partial u}{\partial n} = 0,\ \frac{\partial v}{\partial n} = 0, && x \in \partial\mathcal{D},\ t > 0, \\ &u(x,0) = u_0(x), && x \in \mathcal{D}. \end{aligned} \tag{2.5}$$

Multiply the first and the second equation of (2.5) by $\phi = \phi(x) \in H^1(\mathcal{D}) = H^1$ and integrate over $\mathcal{D}$. Using Green's formula gives

$$\begin{aligned} &(u_t, \phi) + (\nabla v, \nabla \phi) = 0, && \forall \phi \in H^1, \\ &(v, \phi) = (\nabla u, \nabla \phi) + (f(u), \phi), && \forall \phi \in H^1. \end{aligned} \tag{2.6}$$

So the variational formulation is: Find $u(t), v(t) \in H^1$ such that (2.6) holds and such that $u(x,0) = u_0(x), \text{for}\, x \in \mathcal{D}$.

Let $\tau_h = \{K\}$ denote a triangulation of $\mathcal{D}$ and let S_h denote the continuous piecewise polynomial functions on τ_h. So the finite element problem is: Find $u_h(t), v_h(t) \in S_h$ such that

$$\begin{aligned} &(u_{h,t}, \chi) + (\nabla v_h, \nabla \chi) = 0, && \forall \chi \in S_h,\ t > 0, \\ &(v_h, \chi) = (\nabla u_h, \nabla \chi) + (f(u_h), \chi), && \forall \chi \in S_h,\ t > 0, \\ &u_h(0) = u_{h,0}. \end{aligned} \tag{2.7}$$

Let $\dot{S}_h = \{\chi \in S_h : (\chi, 1) = 0\}$ and define $A_h \colon \dot{S}_h \to \dot{S}_h$ (the discrete Laplacian) by

$$(A_h \chi, \eta) = (\nabla \chi, \nabla \eta) \quad \forall \chi, \eta \in \dot{S}_h \tag{2.8}$$

and $P_h \colon L_2 \to \dot{S}_h$ (the orthogonal projection) such that

$$(P_h f, \chi) = (f, \chi) \quad \forall \chi \in \dot{S}_h. \tag{2.9}$$

Then we can write the equation (2.7) as

$$\begin{aligned} &u_{h,t} + A_h^2 u_h + A_h P_h f(u_h) = 0, \quad t > 0, \\ &u_h(0) = u_{0,h}, \end{aligned} \tag{2.10}$$

which is equivalent to the fixed point equation

$$u_h(t) = \mathrm{e}^{-tA_h^2} u_{0,h} - \int_0^t \mathrm{e}^{-(t-s)A_h^2} A_h P_h f(u_h(s))\,\mathrm{d}s,$$

where

$$\mathrm{e}^{-tA_h^2} v = \sum_{j=1}^{\infty} \mathrm{e}^{-t\lambda_{h,j}^2} (v, \phi_{h,j}) \phi_{h,j},$$

where $(\lambda_{h,j}, \phi_{h,j})$ are the eigenpairs of A_h^2.

3 The stochastic Cahn-Hilliard equation

In this section we introduce some definition and properties about stochastic integrals and stochastic differential equation. For more details you can see [10] and [9].

3.1 Introduction

In this part we introduce the *stochastic differential equation* and in special case we will drive the *stochastic Cahn-Hilliard equation*, also called *the Cahn-Hilliard-Cook equation.*

Definition 3.1. *A U-valued stochastic process, where $U = L_2(\mathcal{D})$ is called a Q-Wiener process if*

- $W(0) = 0$,
- $\{W(t)\}_{t\geq 0}$ *has continuous paths almost surely,*
- $\{W(t)\}_{t\geq 0}$ *has independent increments,*
- *The increments have Gaussian law, that is*

$$P \circ \big(W(t) - W(s)\big)^{-1} = N\big(0, (t-s)Q\big), \quad 0 \leq s \leq t.$$

Definition 3.2 (Hilbert-Schmidt operators)**.** *An operator $T \in L(U, H)$ is Hilbert-Schmidt if $\sum_{k=1}^{\infty} \|Te_k\|^2 < \infty$ for an orthonormal basis $\{e_k\}_{k\in\mathbf{N}}$ in U.*

The Hilbert-Schmidt operators form a linear space denoted by $\mathcal{L}_2(U, H)$ which becomes a Hilbert space with scalar product and norm

$$\langle T, S\rangle_{\mathrm{HS}} = \sum_{k=1}^{\infty} \langle Te_k, Se_k\rangle_H, \ \|T\|_{\mathrm{HS}} = \Big(\sum_{k=1}^{\infty} \|Te_k\|_H^2\Big)^{\frac{1}{2}}.$$

Consider the covariance operator $Q : U \to U$, selfadjoint, positive semidefinite, bounded and linear. Also assume that $W(t)$ is Q-Wiener process. If

$$\mathbf{E}\int_0^t \|T(s)Q^{1/2}\|_{\mathrm{HS}}^2 \,\mathrm{d}s < \infty,$$

then we can define the stochastic integral $\int_0^t T(s)\,\mathrm{d}W(s)$.

One important property the stochastic integral is the *isometry property*:

$$\mathbf{E}\Big\|\int_0^t T(s)\,\mathrm{d}W(s)\Big\|^2 = \mathbf{E}\int_0^t \|T(s)Q^{1/2}\|_{\mathrm{HS}}^2\,\mathrm{d}s. \tag{3.1}$$

Definition 3.3. *Let $\{W(t)\}_{t\in[0,T]}$ be a U-valued Q-Wiener process on the probability space $(\Omega, \mathcal{F}, P)$, adapted to a normal filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$. The stochastic partial differential equation (SPDE) is of the form*

$$\begin{aligned} &\mathrm{d}X(t) = (-AX(t) + F(X(t)))\,\mathrm{d}t + B\,\mathrm{d}W(t), \quad 0 < t < T, \\ &X(0) = \xi, \end{aligned} \tag{3.2}$$

where the following assumptions hold:

1. *A is a linear operator, generating a strongly continuous semigroup of bounded linear operators,*

2. $B \in L(U, H)$,

3. $\{F(X(t))\}_{t\in[0,T]}$ *is a predictable H-valued process with Bochner integrable trajectories,*

4. ξ *is an* $\mathcal{F}_0$*-measurable H-valued random variable.*

Let $U = L_2(\mathcal{D})$ and $H = \{v \in U : (v, 1) = 0\}$. In special case, if we consider A^2 as operator and assume that $F(X(t)) = APf(X(t))$, $B = P$ then the stochastic Cahn-Hilliard equation can be written as

$$\begin{aligned} &\mathrm{d}X(t) + A^2X(t)\,\mathrm{d}t + APf(X(t))\,\mathrm{d}t = P\,\mathrm{d}W(t), \\ &X(0) = PX_0. \end{aligned} \tag{3.3}$$

By using $\{E(t)\}_{t\geq 0} = \{\mathrm{e}^{-tA^2}\}_{t\geq 0}$ as a semigroup generated by $-A^2$, the *mild solution* of (3.3) is formally given by the integral equation

$$X(t) = E(t)PX_0 - \int_0^t E(t-s)APf(X(s))\,\mathrm{d}s + \int_0^t E(t-s)\,P\,\mathrm{d}W(s).$$

For the linear Cahn-Hilliard-Cook equation we have

$$\begin{aligned} &\mathrm{d}X(t) + A^2X(t)\,\mathrm{d}t = P\,\mathrm{d}W(t) \\ &X(0) = PX_0, \end{aligned} \tag{3.4}$$

with mild solution

$$X(t) = E(t)PX_0 + \int_0^t E(t-s)P\,\mathrm{d}W(s).$$

In this work we consider the linear Cahn-Hilliard-Cook equation. The reason why we study the linear equation is that it is a simpled equation serving as a starting point for the study of the nonlinear Cahn-Hilliard-Cook equation. However it should be noted that a linearised Cahn-Hilliard-Cook equation of the form (3.4) but with A^2 replaces by $A^2 + A$ is used in the physics literature [6, 4].

3.2 Finite element method

Assume that $\{\tau_h\}_{0<h<1}$ be a triangulation with mesh size h and $\{S_h\}_{0<h<1}$ is the set of continuous piecewise polynomial functions where $S_h \subset H^1(\mathcal{D})$. Also let A_h and P_h be the same in (2.8) and (2.9). The finite element problem for (3.3) is:

Find $X_h(t) \in \dot{S}_h$ such that

$$\begin{aligned} &\mathrm{d}X_h(t) + A_h^2 X_h(t)\,\mathrm{d}t + A_h P_h f(X_h(s))\,\mathrm{d}t = P_h\,\mathrm{d}W(t),\\ &X_h(0) = P_h X_0, \end{aligned} \tag{3.5}$$

where $P_h W(t)$ is Q_h-Wiener process with $Q_h = P_h Q P_h$. The mild solution is given by the equation

$$X_h(t) = E_h(t)P_hX_0 - \int_0^t E_h(t-s)A_hP_hf(X_h(s))\,\mathrm{d}s + \int_0^t E(t-s)P_h\,\mathrm{d}W(s),$$

where $E_h(t) = \mathrm{e}^{-tA_h^2}$. In the linear case, the finite element problem is

$$\begin{aligned} &\mathrm{d}X_h(t) + A_h^2 X_h(t)\,\mathrm{d}t = P_h\,\mathrm{d}W(t),\\ &X_h(0) = P_h X_0, \end{aligned} \tag{3.6}$$

with mild solution

$$X_h(t) = E(t)P_hX_0 + \int_0^t E(t-s)P_h\,\mathrm{d}W(s).$$

Let $k = \Delta t_n,\ t_n = nk$ and $\Delta W_n = W(t_n) - W(t_{n-1})$. Also consider $\Delta X_{h,n} = X_{h,n} - X_{h,n-1}$ and apply the backward Euler method to (3.6) to get

$$\begin{aligned} &X_{h,n} \in \dot{S}_h,\\ &\Delta X_{h,n} + A_h^2 X_{h,n}\Delta t_n = P_h \Delta W_n,\\ &X_{h,0} = P_h X_0. \end{aligned} \tag{3.7}$$

This implies

$$X_{h,n} - X_{h,n-1} + kA_h^2 X_{h,n} = P_h\Delta W_n.$$

If we set $E_{k,h} = (I + kA_h^2)^{-1}$ we get

$$(I + kA_h^2)X_{h,n} = P_h\Delta W_n + X_{h,n-1}.$$

So

$$X_{h,n} = E_{k,h}P_h\Delta W_n + E_{k,h}X_{h,n-1}.$$

We repeat it for $X_{h,n-1}$, we get

$$\begin{aligned} X_{h,n} &= E_{k,h}P_h\Delta W_n + E_{k,h}\big(E_{k,h}P_h\Delta W_{n-1} + E_{k,h}X_{h,n-2}\big)\\ &= E_{k,h}^2 X_{h,n-2} + E_{k,h}P_h\Delta W_n + E_{k,h}^2 P_h\Delta W_{n-1}\\ &\ \vdots\\ &= E_{k,h}^n X_{h,0} + \sum_{j=1}^n E_{k,h}^{n-j+1} P_h\Delta W_j. \end{aligned}$$

So

$$X_{h,n} = E_{k,h}^n P_h X_0 + \sum_{j=1}^{n} E_{k,h}^{n-j+1} P_h \Delta W_j. \tag{3.8}$$

3.3 Strong convergence estimates

The main results in this work are the following error estimates.

Theorem 3.4. *Let X_h and X be the solutions of* (3.6) *and* (3.4)*. If $X_0 \in L_2(\Omega, \dot{H}^\beta)$ and $\|A^{\frac{\beta-2}{2}} Q^{\frac{1}{2}}\|_{\mathrm{HS}} < \infty$ for some $\beta \in [1, r]$, then for all $t \geq 0$*

$$\begin{aligned}\|X_h(t) - X(t)\|_{L_2(\Omega,H)} \\ \leq Ch^\beta \big(\|X_0\|_{L_2(\Omega,\dot{H}^\beta)} + |\log h| \|A^{\frac{\beta-2}{2}} Q^{\frac{1}{2}}\|_{\mathrm{HS}}\big).\end{aligned}$$

If we consider the fully discrete case we have the following theorem.

Theorem 3.5. *If $X_0 \in L_2(\Omega, \dot{H}^\beta)$ and $\|A^{\frac{\beta-2}{2}} Q^{\frac{1}{2}}\|_{\mathrm{HS}} < \infty$ for some $1 \leq \beta \leq \min(r, 4)$, then*

$$\begin{aligned}&\|X_{h,n}(t) - X(t_n)\|_{L_2(\Omega,H)} \\ &\quad \leq \big(C|\log h| h^\beta + C_{\beta,k} k^{\frac{\beta}{4}}\big)\big(\|X_0\|_{L_2(\Omega,\dot{H}^\beta)} + \|A^{\frac{\beta-2}{2}} Q^{\frac{1}{2}}\|_{\mathrm{HS}}\big).\end{aligned}$$

where $C_{\beta,k} = \frac{C}{4-\beta}$ for $\beta < 4$ and $C_{\beta,k} = C|\log k|$ for $\beta = 4$.

3.4 Earlier works

In this section we mention to some earlier works about the Cahn-Hilliard-Cook equation.

Da Prato and Debussche [3] considered (3.3) where f is an odd degree polynomial with positive leading coefficient. They have proved the existence and uniqueness of a weak solution to (3.3).

Cardon Weber [2] has done an implicit approximation scheme, based on the finite difference method. Also she has proved that the approximation scheme converges in probability, uniformly in space and time. Kossioris and Zouraris, [7], proved strong convergence for the finite element method for the linear equation in 1-D.

Elder, Rogers and Rashim [4] and Klein and Batrouni [6] have considered linearized Cahn-Hilliard-Cook equation of the form (3.4), with A^2+A instead of A^2 as the operator.

There are a lot of relevant works about the deterministic Cahn-Hilliard equation. Larsson and Elliott [5] have analyzed a finite element method for the Cahn-Hilliard equation in both spatially semidiscrete and completely discrete case based on the backward Euler method. Also they have obtained error bounds of optimal order over a finite time interval. The computations in our work is based on the techniques of this paper.

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FINITE ELEMENT APPROXIMATION OF THE LINEAR STOCHASTIC CAHN-HILLIARD EQUATION

STIG LARSSON[1] AND ALI MESFORUSH

Abstract. The linearized Cahn-Hilliard-Cook equation is discretized in the spatial variables by a standard finite element method. Strong convergence estimates are proved under suitable assumptions on the covariance operator of the Wiener process, which is driving the equation. The backward Euler time stepping is also studied. The analysis is set in a framework based of analytic semigroups. The main part of the work consists of detailed error bounds for the corresponding deterministic equation.

1. Introduction

Let $\mathcal{D}$ be a bounded domain in $\mathbf{R}^d$ for $d \leq 3$ with a sufficiently smooth boundary. The deterministic Cahn-Hilliard equation [2, 3] is

$$\begin{aligned} &u_t - \Delta v = 0, && \text{for } x \in \mathcal{D}, t > 0,\\ &v = -\Delta u + f(u), && \text{for } x \in \mathcal{D}, t > 0,\\ &\frac{\partial u}{\partial n} = 0,\ \frac{\partial \Delta u}{\partial n} = 0, && \text{for } x \in \partial\mathcal{D}, t > 0,\\ &u(\cdot, 0) = u_0, \end{aligned} \tag{1.1}$$

where $u = u(x,t)$, $\Delta = \sum_{i=1}^d \frac{\partial^2}{\partial x_i^2}$, and $u_t = \frac{\partial u}{\partial t}$. In the boundary condition $\frac{\partial}{\partial n}$ denotes the outward normal derivative. A typical f is $f(s) = s^3 - s$.

It is easy to see that a sufficiently smooth solution of (1.1) satisfies conservation of mass

$$\int_{\mathcal{D}} u(x,t)\,\mathrm{d}x = \int_{\mathcal{D}} u_0(x)\,\mathrm{d}x, \quad t \geq 0.$$

Henceforth we assume that the initial datum satisfies $\int_{\mathcal{D}} u_0(x)\,\mathrm{d}x = 0$.

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Let $\|\cdot\|$ and $(\cdot,\cdot)$ denote the usual norm and inner product in $L_2 = L_2(\mathcal{D})$ and let $H^s = H^s(\mathcal{D})$ be the usual Sobolev space with norm $\|\cdot\|_s$. We also let H be the subspace of L_2 which is orthogonal to the constants, i.e., $H = \{v \in L_2 : (v,1) = 0\}$, and let P be the orthogonal projection of L_2 onto H. We define the linear operator $A = -\Delta$ with domain of definition

$$\mathcal{D}(A) = \big\{v \in H^2 \cap H : \frac{\partial v}{\partial n} = 0 \text{ on } \partial\mathcal{D}\big\}.$$

Then A is a selfadjoint, positive definite, densely defined operator on H and (1.1) may be written as an abstract initial value problem to find $u(t) \in H$ such that

$$\begin{aligned} &u_t + A^2 u + APf(u) = 0, \quad t > 0, \\ &u(0) = u_0. \end{aligned} \tag{1.2}$$

By spectral theory we also define $\dot{H}^s = \mathcal{D}(A^{\frac{s}{2}})$ with norms $|v|_s = \|A^{\frac{s}{2}} v\|$ for a real s. It is well known that, for integer $s \geq 0$, $\dot{H}^s$ is a subspace of $H^s \cap H$ characterized by certain boundary conditions and that the norms $|\cdot|_s$ and $\|\cdot\|_s$ are equivalent on $\dot{H}^s$. In particular, we have $\dot{H}^1 = H^1 \cap H$ and the norm $|v|_1 = \|A^{\frac{1}{2}} v\| = \|\nabla v\|$ is equivalent to $\|v\|_1$ on $\dot{H}^1$.

For $v \in H$ we define $\mathrm{e}^{-tA^2} v = \sum_{j=1}^{\infty} \mathrm{e}^{-t\lambda_j^2} (v, \varphi_j)\varphi_j$, where (λ_j, φ_j) are the eigenpairs of A with orthonormal eigenvectors. Let $\{E(t)\}_{t\geq 0} = \{\mathrm{e}^{-tA^2}\}_{t\geq 0}$ be the semigroup generated by $-A^2$.

The stochastic Cahn-Hilliard equation, also called the Cahn-Hilliard-Cook equation [1, 5], is

$$\begin{aligned} &\mathrm{d}X(t) + A^2 X(t)\,\mathrm{d}t + APf(X(t))\,\mathrm{d}t = \mathrm{d}W(t), \quad t > 0, \\ &X(0) = X_0. \end{aligned} \tag{1.3}$$

Let $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}_{t\geq 0})$ be a filtered probability space, let Q be a selfadjoint positive semidefinite bounded linear operator on H, and let $\{W(t)\}_{t\geq 0}$ be an H-valued Wiener process with covariance operator Q. We use the semigroup framework of [11] where (1.3) is given a rigorous meaning in terms of the mild solution which satisfies the integral equation

$$X(t) = E(t)X_0 - \int_0^t E(t-s)APf(X(s))\,\mathrm{d}s + \int_0^t E(t-s)\,\mathrm{d}W(s),$$

where $\int_0^t \mathrm{d}W(s)$ denotes the H-valued Itô integral. Existence and uniqueness of solutions is proved in [6]. In this paper we study numerical approximation of the linear Cahn-Hilliard-Cook equation,

$$\begin{aligned} &\mathrm{d}X + A^2 X\,\mathrm{d}t = \mathrm{d}W, \quad t > 0, \\ &X(0) = X_0, \end{aligned} \tag{1.4}$$

with the mild solution

$$X(t) = E(t)X_0 + \int_0^t E(t-s)\,\mathrm{d}W(s). \tag{1.5}$$

The reason why we study the linearized equation is that it is a first step towards the nonlinear equation (1.3). However, we remark that a linearized equation of the form (1.4), but with A^2 replaced by $A^2 + A$ is studied by numerical simulation in the physics literature [7, 9].

For the approximation of the Cahn-Hilliard equation we follow the framework of [8]. We assume that we have a family $\{S_h\}_{h>0}$ of finite-dimensional approximating subspaces of H^1. We formulate the semidiscrete problem: Find $u_h(t), v_h(t) \in S_h$ such that

$$\begin{aligned} &(u_{h,t}, \chi) + (\nabla v_h, \nabla \chi) = 0, && \forall \chi \in S_h,\ t > 0, \\ &(v_h, \chi) = (\nabla u_h, \nabla \chi) + (f(u_h), \chi), && \forall \chi \in S_h,\ t > 0, \\ &(u_h(0), \chi) = (u_0, \chi), && \forall \chi \in S_h. \end{aligned} \tag{1.6}$$

Let now $\dot{S}_h = \{\chi \in S_h : (\chi, 1) = 0\}$. It is immediate from (1.6) that $u_h(t) \in \dot{S}_h$ if $u_0 \in H$. Therefore u_h can equivalently be obtained from the following equations: Find $u_h(t), w_h(t) \in \dot{S}_h$ such that

$$\begin{aligned} &(u_{h,t}, \chi) + (\nabla w_h, \nabla \chi) = 0, && \forall \chi \in \dot{S}_h,\ t > 0, \\ &(w_h, \chi) = (\nabla u_h, \nabla \chi) + (f(u_h), \chi), && \forall \chi \in \dot{S}_h,\ t > 0, \\ &(u_h(0), \chi) = (u_0, \chi), && \forall \chi \in \dot{S}_h. \end{aligned} \tag{1.7}$$

The relation between w_h and v_h is $w_h = P v_h$. Equivalently we may write this as

$$\begin{aligned} &u_{h,t} + A_h^2 u_h + A_h P_h f(u_h) = 0, \quad t > 0, \\ &u_h(0) = u_{0,h}, \end{aligned} \tag{1.8}$$

where the operator $A_h : \dot{S}_h \to \dot{S}_h$ (the "discrete Laplacian") is defined by

$$(A_h \chi, \eta) = (\nabla \chi, \nabla \eta), \quad \forall \chi, \eta \in \dot{S}_h,$$

and $P_h : L_2 \to \dot{S}_h$ is the orthogonal projection. A_h is selfadjoint and positive definite.

The finite element approximation of the linearized Cahn-Hilliard-Cook equation (1.4) is: Find $X_h(t) \in \dot{S}_h$ such that,

$$\begin{aligned} &\mathrm{d}X_h + A_h^2 X_h \mathrm{d}t = P_h\,\mathrm{d}W, \quad t > 0, \\ &X_h(0) = P_h X_0. \end{aligned} \tag{1.9}$$

For $v \in \dot{S}_h$ we define $E_h(t)v = \mathrm{e}^{-tA_h^2} v = \sum_{j=1}^\infty \mathrm{e}^{-t\lambda_{h,j}^2} (v, \varphi_{h,j}) \varphi_{h,j}$, where $(\lambda_{h,j}, \varphi_{h,j})$ are the eigenpairs of A_h. Then $\{E_h(t)\}_{t\geq 0}$ is the semigroup generated by $-A_h^2$. The mild solution of (1.9) is

$$X_h(t) = E_h(t)P_hX_0 + \int_0^t E_h(t-s)P_h\,\mathrm{d}W(s). \tag{1.10}$$

Let $k = \Delta t$, $t_n = nk$, $\Delta X_{h,n} = X_{h,n} - X_{h,n-1}$, $\Delta W_n = W(t_n) - W(t_{n-1})$, and apply Euler's method to (1.9) to get

$$\Delta X_{h,n} + A_h^2 X_{h,n}\Delta t = P_h \Delta W_n. \tag{1.11}$$

Set $E_{kh} = (I + kA_h^2)^{-1}$ to obtain a discrete variant of the mild solution

$$X_{h,n} = E_{kh}^n P_h X_0 + \sum_{j=1}^n E_{kh}^{n-j+1} P_h \Delta W_j.$$

In Section 2 we assume that $\{S_h\}_{h>0}$ admits an error estimate of order $\mathcal{O}(h^r)$ as the mesh parameter $h \to 0$ for some integer $r \geq 2$. Then we show error estimates for the semigroup $E_h(t)$ with minimal regularity requirement. More precisely, in Theorem 2.1 we show, for $\beta \in [1, r]$,

$$\begin{aligned} &\|F_h(t)v\| \leq Ch^\beta |v|_\beta, \quad v \in \dot{H}^\beta, \\ &\Big(\int_0^t \|F_h(\tau)v\|^2\,\mathrm{d}\tau\Big)^{\frac12} \leq C|\log h| h^\beta |v|_{\beta-2}, \quad v \in \dot{H}^{\beta-2}, \end{aligned}$$

where $F_h(t) = E_h(t)P_h - E(t)$.

Analogous estimates are obtained for the implicit Euler approximation in Theorem 2.2.

In Section 3 we use these estimates to prove the strong convergence estimates for approximations of the linear Cahn-Hilliard-Cook equation. Let $L_2(\Omega, H)$ define the space of square integrable H-valued random variables with norm

$$\|X\|_{L_2(\Omega,H)} = \Big(\mathbf{E}\big(\|X\|^2\big)\Big)^{\frac12} = \Big(\int_\Omega \|X(\omega)\|^2\,\mathrm{d}P(\omega)\Big)^{\frac12}.$$

Let Q denote the covariance operator of the Wiener process W, and let $\|T\|_{\mathrm{HS}}$ denote the Hilbert-Schmidt norm of bounded linear operators on H. In Theorem 3.1 we study the spatial regularity of the mild solution (1.5) and show

$$\|X(t)\|_{L_2(\Omega,\dot{H}^\beta)} \leq C\big(\|X_0\|_{L_2(\Omega,\dot{H}^\beta)} + \|A^{\frac{\beta-2}{2}}Q^{\frac12}\|_{\mathrm{HS}}\big), \quad \beta \geq 0.$$

Moreover, in Theorem 3.2 we show strong convergence for the mild solution X_h in (1.10):

$$\begin{aligned} &\|X_h(t) - X(t)\|_{L_2(\Omega,H)} \\ &\quad \leq Ch^\beta\big(\|X_0\|_{L_2(\Omega,\dot{H}^\beta)} + |\log h| \|A^{\frac{\beta-2}{2}}Q^{\frac12}\|_{\mathrm{HS}}\big), \quad \beta \in [1, r]. \end{aligned}$$

In Theorem 3.3 for the fully discrete case we obtain similarly, for $\beta \in [1, \min r, 4]$,

$$\begin{aligned}\|X_{h,n}(t) - X(t_n)\|_{L_2(\Omega,H)} \\ \le \big(C|\log h|h^\beta + C_{k,\beta}k^{\frac{\beta}{4}}\big)\big(\|X_0\|_{L_2(\Omega,\dot H^\beta)} + \|A^{\frac{\beta-2}{2}}Q^{\frac12}\|_{\mathrm{HS}}\big),\end{aligned}$$

where $C_{\beta,k} = \frac{C}{4-\beta}$ for $\beta < 4$ and $C_{\beta,k} = C|\log k|$ for $\beta = 4$.

These results require that $\|A^{\frac{\beta-2}{2}}Q^{\frac12}\|_{\mathrm{HS}} < \infty$. In order to see what this means we compute two special cases. For $Q = I$ (spatially uncorrelated noise, or space-time white noise), by using the asymptotics $\lambda_j \sim j^{\frac{2}{d}}$, we have

$$\|A^{\frac{\beta-2}{2}}Q^{\frac12}\|_{\mathrm{HS}}^2 = \|A^{\frac{\beta-2}{2}}\|_{\mathrm{HS}}^2 = \sum_{j=1}^\infty \lambda_j^{\beta-2} \sim \sum_{j=1}^\infty j^{(\beta-2)\frac{2}{d}} < \infty,$$

if $\beta < 2 - \frac{d}{2}$. Hence, for example, $\beta < \frac12$ if $d = 3$. On the other hand, if Q is of trace class, $\mathrm{Tr}(Q) = \|Q^{\frac12}\|_{\mathrm{HS}}^2 < \infty$, then we may take $\beta = 2$.

There are few studies of numerical methods for the Cahn-Hilliard-Cook equation. We are only aware of [4] in which convergence in probability was proved for a difference scheme for the nonlinear equation in multiple dimensions, and [10] where strong convergence was proved for the finite element method for the linear equation in 1-D.

2. Error estimates for the deterministic Cahn-Hilliard equation

We start this section with some necessary inequalities. Let $\{E(t)\}_{t\ge0} = \{\mathrm{e}^{-tA^2}\}_{t\ge0}$ and $\{E_h(t)\}_{t\ge0} = \{\mathrm{e}^{-tA_h^2}\}_{t\ge0}$ be the semigroups generated by $-A^2$ and $-A_h^2$, respectively. By the smoothing property there exist positive constants c, C such that

$$\|A_h^{2\beta}E_h(t)P_hv\| + \|A^{2\beta}E(t)Pv\| \le Ct^{-\beta}\mathrm{e}^{-ct}\|v\|, \quad \beta \ge 0, \tag{2.1}$$

$$\int_0^t \|A_hE_h(s)P_hv\|^2\,\mathrm{d}s + \int_0^t \|AE(s)Pv\|^2\,\mathrm{d}s \le C\|v\|^2. \tag{2.2}$$

Let $R_h : \dot H^1 \to \dot S_h$ be the Ritz projection be defined by

$$(\nabla R_hv, \nabla\chi) = (\nabla v, \nabla\chi), \quad \forall \chi \in \dot S_h.$$

It is clear that $R_h = A_h^{-1}P_hA$. We assume that for some integer $r \ge 2$, we have the error bound

$$\|R_hv - v\| \le Ch^\beta|v|_\beta, \quad v \in \dot H^\beta,\ 1 \le \beta \le r. \tag{2.3}$$

This holds with $r = 2$ for the standard piecewise linear Lagrange finite element method in a bounded convex polynomial domain $\mathcal{D}$. In the next theorem we prove error estimates for the deterministic Cahn-Hilliard equation in the semidiscrete case.

Theorem 2.1. *Set* $F_h(t) = E_h(t)P_h - E(t)$*. Then, for* $1 \le \beta \le r$ *and* $t \ge 0$*, we have*

$$\|F_h(t)v\| \le Ch^\beta |v|_\beta, \quad v \in \dot{H}^\beta, \tag{2.4}$$

$$\Big(\int_0^t \|F_h(\tau)v\|^2 \,\mathrm{d}\tau\Big)^{\frac{1}{2}} \le C|\log h| h^\beta |v|_{\beta-2}, \quad v \in \dot{H}^{\beta-2}. \tag{2.5}$$

Proof. Let $u(t) = E(t)v$, $u_h(t) = E_h(t)P_h v$, and $e(t) = u_h(t) - u(t)$. We want to prove that

$$\|e(t)\| \le Ch^\beta |v|_\beta, \quad v \in \dot{H}^\beta,$$
$$\Big(\int_0^t \|e(\tau)\|^2 \,\mathrm{d}\tau\Big)^{\frac{1}{2}} \le C|\log h| h^\beta |v|_{\beta-2}, \quad v \in \dot{H}^{\beta-2}.$$

Let $G = A^{-1}$ and $G_h = A_h^{-1}P_h$. Apply G to (1.2) with $f(u) \equiv 0$ to get $Gu_t + Au = 0$, and apply G_h^2 to (1.8) with $f(u_h) \equiv 0$ to get $G_h^2 u_{h,t} + u_h = 0$. Hence

$$G_h^2 e_t + e = -G_h^2 u_t - u + G_h(Gu_t + Au) = (G_h A - I)u - G_h(G_h A - I)Gu_t,$$

that is,

$$G_h^2 e_t + e = \rho + G_h \eta, \tag{2.6}$$

where $\rho = (R_h - I)u, \eta = -(R_h - I)Gu_t$. Take the inner product of (2.6) by e_t to get

$$\|G_h e_t\|^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|e\|^2 = (\rho, e_t) + (\eta, G_h e_t),$$

Since $(\eta, G_h e_t) \le \|\eta\|\|G_h e_t\| \le \frac{1}{2}\|\eta\|^2 + \frac{1}{2}\|G_h e_t\|^2$, we obtain

$$\|G_h e_t\|^2 + \frac{\mathrm{d}}{\mathrm{d}t}\|e\|^2 \le 2(\rho, e_t) + \|\eta\|^2.$$

Multiply this inequality by t to get $t\|G_h e_t\|^2 + t\frac{\mathrm{d}}{\mathrm{d}t}\|e\|^2 \le 2t(\rho, e_t) + t\|\eta\|^2$. Note that

$$t\frac{\mathrm{d}}{\mathrm{d}t}\|e\|^2 = \frac{\mathrm{d}}{\mathrm{d}t}\big(t\|e\|^2\big) - \|e\|^2, \quad t(\rho, e_t) = \frac{\mathrm{d}}{\mathrm{d}t}\big(t(\rho, e)\big) - (\rho, e) - t(\rho_t, e),$$

so that

$$t\|G_h e_t\|^2 + \frac{\mathrm{d}}{\mathrm{d}t}\big(t\|e\|^2\big) \le 2\frac{\mathrm{d}}{\mathrm{d}t}\big(t(\rho, e)\big) + 2|(\rho, e)| + 2|t(\rho_t, e)| + t\|\eta\|^2 + \|e\|^2.$$

But

$$|(\rho,e)| \le \|\rho\|\,\|e\| \le \frac{1}{2}\|\rho\|^2 + \frac{1}{2}\|e\|^2,$$
$$|t(\rho_t,e)| \le t\|\rho_t\|\,\|e\| \le \frac{1}{2}t^2\|\rho_t\|^2 + \frac{1}{2}\|e\|^2.$$

Hence

$$t\|G_h e_t\|^2 + \frac{\mathrm{d}}{\mathrm{d}t}\big(t\|e\|^2\big) \le 2\frac{\mathrm{d}}{\mathrm{d}t}\big(t(\rho,e)\big) + \|\rho\|^2 + t^2\|\rho_t\|^2 + t\|\eta\|^2 + 3\|e\|^2.$$

Integrate over $[0,t]$ and use Young's inequality to get

$$\int_0^t \tau\|G_h e_t\|^2\,\mathrm{d}\tau + t\|e\|^2 \le 2t\|\rho\|^2 + \frac{1}{2}t\|e\|^2 + \int_0^t \|\rho\|^2\,\mathrm{d}\tau + \int_0^t \tau^2\|\rho_t\|^2\,\mathrm{d}\tau + \int_0^t \tau\|\eta\|^2\,\mathrm{d}\tau + 3\int_0^t \|e\|^2\,\mathrm{d}\tau.$$

Hence

$$t\|e\|^2 \le Ct\|\rho\|^2 + C\int_0^t \big(\|\rho\|^2 + \tau^2\|\rho_t\|^2 + \tau\|\eta\|^2 + \|e\|^2\big)\,\mathrm{d}\tau. \tag{2.7}$$

We must bound $\int_0^t \|e\|^2\,\mathrm{d}\tau$. Multiply (2.6) by e to get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|G_h e\|^2 + \|e\|^2 \le \|\rho\|\,\|e\| + \|\eta\|\,\|G_h e\| \le \frac{1}{2}\|\rho\|^2 + \frac{1}{2}\|e\|^2 + \|\eta\|\max_{0\le\tau\le t}\|G_h e\|,$$

so that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|G_h e\|^2 + \|e\|^2 \le \|\rho\|^2 + 2\|\eta\|\max_{0\le\tau\le t}\|G_h e\|. \tag{2.8}$$

Integrate (2.8), note that $G_h e(0) = A_h^{-1}P_h(P_h - I)v = 0$, to get

$$\|G_h e\|^2 + \int_0^t \|e\|^2\,\mathrm{d}\tau \le \int_0^t \|\rho\|^2\,\mathrm{d}\tau + \max_{0\le\tau\le t}\|G_h e\|^2 + \Big(\int_0^t \|\eta\|\,\mathrm{d}\tau\Big)^2.$$

Hence, since t is arbitrary,

$$\int_0^t \|e\|^2\,\mathrm{d}\tau \le \int_0^t \|\rho\|^2\,\mathrm{d}\tau + \Big(\int_0^t \|\eta\|\,\mathrm{d}\tau\Big)^2. \tag{2.9}$$

We insert (2.9) in (2.7) and conclude

$$\begin{aligned} t\|e\|^2 &\le Ct\|\rho\|^2 + C\int_0^t \big(\|\rho\|^2 + \tau^2\|\rho_t\|^2 + \tau\|\eta\|^2\big)\,\mathrm{d}\tau \\ &\quad + C\Big(\int_0^t \|\eta\|\,\mathrm{d}\tau\Big)^2. \end{aligned} \tag{2.10}$$

We compute the terms in the right hand side. With $v \in \dot{H}^\beta$, recalling $\rho = (R_h - I)u$ and using (2.3), we have

$$\|\rho(t)\| \le Ch^\beta |u(t)|_\beta \le Ch^\beta \|E(t)A^{\frac{\beta}{2}} v\| \le Ch^\beta \|A^{\frac{\beta}{2}} v\| \le Ch^\beta |v|_\beta, \tag{2.11}$$

so that,

$$t\|\rho\|^2 \le Ch^{2\beta} t|v|_\beta^2, \quad \int_0^t \|\rho\|^2 \, \mathrm{d}\tau \le Ch^{2\beta} t|v|_\beta^2.$$

Similarly, by (2.1),

$$\|\rho_t(t)\| \le Ch^\beta |u_t(t)|_\beta \le Ch^\beta \|A^2 E(t) A^{\frac{\beta}{2}} v\| \le Ch^\beta t^{-1} |v|_\beta,$$

so that

$$\int_0^t \tau^2 \|\rho_t\|^2 \, \mathrm{d}\tau \le Ch^\beta t|v|_\beta^2. \tag{2.12}$$

Moreover, since $\eta = -(R_h - I)Gu_t$,

$$\|\eta(t)\| \le Ch^\beta |Gu_t(t)|_\beta \le Ch^\beta \|AE(t)A^{\frac{\beta}{2}} v\| \le Ch^\beta t^{-\frac{1}{2}} |v|_\beta,$$

so that

$$\Big(\int_0^t \|\eta\| \, \mathrm{d}\tau \Big)^2 \le Ch^{2\beta} t|v|_\beta^2, \quad \int_0^t \tau \|\eta\|^2 \, \mathrm{d}\tau \le Ch^{2\beta} t|v|_\beta^2.$$

By inserting these in (2.10) we conclude

$$t\|e\|^2 \le Ch^{2\beta} t|v|_\beta^2,$$

which proves (2.4).

To prove (2.5) we recall (2.9) and let $v \in \dot{H}^{\beta-2}$. By using (2.3) and (2.2) we obtain

$$\begin{aligned} \int_0^t \|\rho\|^2 \, \mathrm{d}\tau &\le Ch^{2\beta} \int_0^t |u|_\beta^2 \, \mathrm{d}\tau = Ch^{2\beta} \int_0^t \|AE(\tau)A^{\frac{\beta-2}{2}} v\|^2 \, \mathrm{d}\tau \\ &\le Ch^{2\beta} |v|_{\beta-2}^2. \end{aligned} \tag{2.13}$$

Now we compute $\int_0^t \|\eta\| \, \mathrm{d}\tau$. To this end we assume first $1 < \beta \le r$ and let $1 \le \gamma < \beta$. By using (2.1) and (2.3) we get

$$\begin{aligned} \int_0^t \|\eta\| \, \mathrm{d}\tau &\le Ch^\gamma \int_0^t \|Gu_t\|_\gamma \, \mathrm{d}\tau = Ch^\gamma \int_0^t \|A^{2-\frac{\beta-\gamma}{2}} E(\tau) A^{\frac{\beta-2}{2}} v\| \, \mathrm{d}\tau \\ &\le Ch^\gamma \int_0^t \tau^{-1+\frac{\beta-\gamma}{4}} \mathrm{e}^{-c\tau} \, \mathrm{d}\tau \, |v|_{\beta-2}, \end{aligned}$$

where, since $0 < \beta - \gamma \le r - 1$,

$$\int_0^t \tau^{-1+\frac{\beta-\gamma}{4}} \mathrm{e}^{-c\tau} \, \mathrm{d}\tau = \frac{4}{\beta-\gamma} \int_0^{t^{\frac{\beta-\gamma}{4}}} \mathrm{e}^{-cs^{\frac{4}{\beta-\gamma}}} \, \mathrm{d}s \le \frac{C}{\beta-\gamma} \int_0^\infty \mathrm{e}^{-cs^{\frac{4}{r-1}}} \, \mathrm{d}s.$$

Hence, with C independent of β,

$$\int_0^t \|\eta\| \, \mathrm{d}\tau \le \frac{Ch^\gamma}{\beta-\gamma} |v|_{\beta-2}. \tag{2.14}$$

Now let $\frac{1}{\beta-\gamma} = |\log h| = -\log h$, so $\gamma \to \beta$ as $h \to 0$, and

$$\gamma \log h = (\gamma - \beta + \beta) \log h = 1 + \beta \log h.$$

Therefore we have

$$\frac{h^\gamma}{\beta-\gamma} = |\log h| \mathrm{e}^{\gamma \log h} = |\log h| \mathrm{e}^{1+\beta \log h} \le C |\log h| h^\beta.$$

Put this in (2.14) to get, for $1 < \beta \le r$,

$$\int_0^t \|\eta\| \, \mathrm{d}\tau \le C h^\beta |\log h| |v|_{\beta-2}, \tag{2.15}$$

and hence also for $1 \le \beta \le r$, because C is independent of β. Finally, we put (2.13) and (2.15) in (2.9) to get

$$\Big(\int_0^t \|e\|^2 \, \mathrm{d}\tau \Big)^{\frac{1}{2}} \le C |\log h| h^\beta |v|_{\beta-2},$$

which is (2.5). □

The reason why we assume $\beta \ge 1$ is that in (2.5) wee need at least $v \in \dot{H}^{-1}$ for $E_h(t) P_h v$ to be defined.

Now we turn to the fully discrete case. The backward Euler method applied to

$$\begin{aligned} u_{h,t} + A_h^2 u_h &= 0, \quad t > 0, \\ u_h(0) &= P_h v, \end{aligned}$$

defines $U_n \in \dot{S}_h$ by

$$\begin{aligned} \partial U_n + A_h^2 U_n &= 0, \quad n \ge 1 \\ U_0 &= P_h v, \end{aligned} \tag{2.16}$$

where $\partial U_n = \frac{1}{k}(U_n - U_{n-1})$. Denoting $E_{kh}^n = (I + k A_h^2)^{-n}$, we have $U_n = E_{kh}^n v$ and, similar to (2.1), (2.2),

$$\|E_{kh}^n v\| \le \|v\|, \quad k \sum_{j=1}^n \|A_h E_{kh}^j v\|^2 \le \frac{1}{2} \|v\|^2.$$

To prove this, take the inner product of $\partial U_j + A_h^2 U_j = 0$ by U_j, to get

$$\|U_j\|^2 + k\|A_h U_j\|^2 = (U_j, U_{j-1}) \le \|U_j\|\|U_{j-1}\| \le \frac{1}{2}\|U_j\|^2 + \frac{1}{2}\|U_{j-1}\|^2,$$

which implies that $\|U_i\|^2 - \|U_{j-1}\|^2 + 2k\|A_h U_j\|^2 \le 0$, and hence

$$\|U_n\|^2 + 2k\sum_{j=1}^{n}\|A_h U_j\|^2 \le \|v\|^2.$$

The next theorem provides error estimates in the L_2 norm for the deterministic Cahn-Hilliard equation in the fully discrete case.

Theorem 2.2. *Set $F_n = E_{kh}^n P_h - E(t_n)$. Then, for $1 \le \beta \le \min(r,4)$ and $n \ge 1$, we have*

$$\|F_n v\| \le C(h^\beta + k^{\frac{\beta}{4}})|v|_\beta, \quad v \in \dot{H}^\beta, \tag{2.17}$$

$$\Big(k\sum_{j=1}^{n}\|F_j v\|^2\Big)^{\frac{1}{2}} \le \big(C|\log h|h^\beta + C_{\beta,k}k^{\frac{\beta}{4}}\big)|v|_{\beta-2}, \quad v \in \dot{H}^{\beta-2}, \tag{2.18}$$

where $C_{\beta,k} = \frac{C}{4-\beta}$ for $\beta < 4$ and $C_{\beta,k} = C|\log k|$ for $\beta = 4$.

Proof. Let G and G_h be as in the proof of Theorem 2.1. With $e_n = U_n - u_n$, we get

$$G_h^2 \partial e_n + e_n = \rho_n + G_h\eta_n + G_h\delta_n, \tag{2.19}$$

where $u_n = u(t_n)$, $u_{t,n} = u_t(t_n)$ and

$$\rho_n = (R_h - I)u_n, \quad \eta_n = -(R_h - I)G\partial u_n, \quad \delta_n = -G(\partial u_n - u_{t,n}).$$

Multiply (2.19) by ∂e_n and note that

$$(\eta_n, G_h\partial e_n) \le \|\eta_n\|^2 + \frac{1}{4}\|G_h\partial e_n\|^2,\ (\delta_n, G_h\partial e_n) \le \|\delta_n\|^2 + \frac{1}{4}\|G_h\partial e_n\|^2,$$

to get

$$\|G_h\partial e_n\|^2 + 2(e_n, \partial e_n) \le 2(\rho_n, \partial e_n) + 2\|\eta_n\|^2 + 2\|\delta_n\|^2. \tag{2.20}$$

We have the following identities

$$\partial(a_n b_n) = (\partial a_n)b_n + a_{n-1}(\partial b_n) \tag{2.21}$$

$$= (\partial a_n)b_n + a_n(\partial b_n) - k(\partial a_n)(\partial b_n). \tag{2.22}$$

By using (2.22) we have

$$2(e_n, \partial e_n) = \partial\|e_n\|^2 + k\|\partial e_n\|^2,$$
$$(\rho_n, \partial e_n) = \partial(\rho_n, e_n) - (\partial\rho_n, e_n) + k(\partial\rho_n, \partial e_n).$$

Put these in (2.20) and cancel $k\|\partial e_n\|^2$ to get

$$\|G_h\partial e_n\|^2 + \partial\|e_n\|^2 \le 2\partial(\rho_n, e_n) - 2(\partial\rho_n, e_n) + k\|\partial\rho_n\|^2 + 2\|\eta_n\|^2 + 2\|\delta_n\|^2.$$

Multiply this by t_{n-1} and note that $k \le t_{n-1}$ for $n \ge 2$, so that we have for $n \ge 1$

$$\begin{aligned} t_{n-1}\|G_h\partial e_n\|^2 + t_{n-1}\partial\|e_n\|^2 &\le 2t_{n-1}\partial(\rho_n, e_n) - 2t_{n-1}(\partial\rho_n, e_n) + t_{n-1}^2\|\partial\rho_n\|^2 \\ &\quad + 2t_{n-1}\|\eta_n\|^2 + 2t_{n-1}\|\delta_n\|^2. \end{aligned} \tag{2.23}$$

By (2.21) we have

$$\begin{aligned} t_{n-1}\partial\|e_n\|^2 &= \partial(t_n\|e_n\|^2) - \|e_n\|^2, \\ 2t_{n-1}\partial(\rho_n, e_n) &= 2\partial(t_n(\rho_n, e_n)) - 2(\rho_n, e_n). \end{aligned}$$

Put these in (2.23) to get

$$\begin{aligned} t_{n-1}\|G_h\partial e_n\|^2 + \partial(t_n\|e_n\|^2) &\le C\big(\partial(t_n(\rho_n, e_n)) + \|\rho_n\|^2 + t_{n-1}^2\|\partial\rho_n\|^2 + \|e_n\|^2\big) \\ &\quad + C\big(t_{n-1}\|\eta_n\|^2 + t_{n-1}\|\delta_n\|^2\big). \end{aligned} \tag{2.24}$$

Note that

$$k\sum_{j=1}^{n}\partial\big(t_j\|e_j\|^2\big) = t_n\|e_n\|^2,, \quad k\sum_{j=1}^{n}\partial\big(t_j(\rho_j, e_j)\big) = t_n(\rho_n, e_n) \tag{2.25}$$

By summation in (2.24) and using (2.25) we get

$$\begin{aligned} k\sum_{j=1}^{n} t_{j-1}\|G_h\partial e_j\|^2 + t_n\|e_n\|^2 &\le Ct_n\|\rho_n\|^2 \\ &\quad + Ck\sum_{j=1}^{n}\big(\|\rho_j\|^2 + t_{j-1}^2\|\partial\rho_j\|^2 + \|e_j\|^2\big) \\ &\quad + Ck\sum_{j=1}^{n}\big(t_{j-1}\|\eta_j\|^2 + t_{j-1}\|\delta_j\|^2\big). \end{aligned} \tag{2.26}$$

Now we estimate $k\sum_{j=1}^{n}\|e_j\|^2$. Take the inner product of (2.19) by e_n to get

$$2(G_h^2\partial e_n, e_n) + \|e_n\|^2 \le \|\rho_n\|^2 + 2\big(\|\eta_n\| + \|\delta_n\|\big)\|G_h e_n\|. \tag{2.27}$$

By (2.22) we have

$$2(G_h^2\partial e_n, e_n) = 2(\partial G_h e_n, G_h e_n) = \partial\|G_h e_n\|^2 + k\|\partial G_h e_n\|^2. \tag{2.28}$$

By summation in (2.27) and using $G_h e_0 = 0$, we get

$$\|G_h e_n\|^2 + k\sum_{j=1}^{n}\|e_j\|^2 \le k\sum_{j=1}^{n}\|\rho_j\|^2 + \frac{1}{2}\max_{j\le n}\|G_h e_j\|^2 + 2\Big(k\sum_{j=1}^{n}\big(\|\eta_j\| + \|\delta_j\|\big)\Big)^2.$$

Hence

$$k\sum_{j=1}^{n}\|e_j\|^2 \le k\sum_{j=1}^{n}\|\rho_j\|^2 + 2\Big(k\sum_{j=1}^{n}\big(\|\eta_j\| + \|\delta_j\|\big)\Big)^2. \tag{2.29}$$

By putting (2.29) in (2.26) we get

$$\begin{aligned} t_n\|e_n\|^2 \le\, & Ct_n\|\rho_n\|^2 \\ & + Ck\sum_{j=1}^{n}\Big(\|\rho_j\|^2 + t_{j-1}^2\|\partial\rho_j\|^2 + t_{j-1}\|\eta_j\|^2 + t_{j-1}\|\delta_j\|^2\Big) \\ & + C\Big(k\sum_{j=1}^{n}\big(\|\eta_j\| + \|\delta_j\|\big)\Big)^2. \end{aligned} \tag{2.30}$$

Now we compute the terms in the right hand side. With $v \in \dot{H}^\beta$ we have by (2.11),

$$\|\rho_n\|^2 \le Ch^{2\beta}|v|_\beta^2, \quad k\sum_{j=1}^{n}\|\rho_j\|^2 \le Ch^{2\beta}t_n|v|_\beta^2. \tag{2.31}$$

By using the Cauchy-Schwartz inequality we have

$$\begin{aligned} k\sum_{j=1}^{n} t_{j-1}^2\|\partial\rho_j\|^2 &= k\sum_{j=2}^{n} t_{j-1}^2\Big\|\frac{1}{k}\int_{t_{j-1}}^{t_j}\rho_t\,\mathrm{d}\tau\Big\|^2 \\ &\le \sum_{j=2}^{n}\Big(t_{j-1}^2\frac{1}{k}\int_{t_{j-1}}^{t_j}\tau^{-2}\mathrm{d}\tau\int_{t_{j-1}}^{t_j}\tau^2\|\rho_t(\tau)\|^2\,\mathrm{d}\tau\Big), \\ &\le \int_0^{t_n}\tau^2\|\rho_t\|^2\mathrm{d}\tau. \end{aligned}$$

Hence, by (2.12),

$$k\sum_{j=1}^{n} t_{j-1}^2\|\partial\rho_j\|^2 \le Ch^{2\beta}t_n|v|_\beta^2. \tag{2.32}$$

By using (2.3) and (2.1) we have

$$\begin{aligned}\|\eta_j\| &\le Ch^\beta|G\partial u_j|_\beta \le \frac{Ch^\beta}{k}\Big\|\int_{t_{j-1}}^{t_j} AE(\tau)A^{\frac{\beta}{2}}v\,\mathrm{d}\tau\Big\| \\ &\le \frac{Ch^\beta}{k}\int_{t_{j-1}}^{t_j}\tau^{-\frac12}\,\mathrm{d}\tau\|A^{\frac{\beta}{2}}v\| \le \frac{Ch^\beta}{k}(\sqrt{t_j}-\sqrt{t_{j-1}})|v|_\beta \le \frac{Ch^\beta}{\sqrt{t_j}}|v|_\beta.\end{aligned}$$

So

$$k\sum_{j=1}^n t_{j-1}\|\eta_j\|^2 \le Ch^{2\beta}t_n|v|_\beta^2, \quad k\sum_{j=1}^n\|\eta_j\| \le Ch^\beta t_n^{\frac12}|v|_\beta. \tag{2.33}$$

By using (2.1) we have, for $j\ge 2$,

$$\begin{aligned}\|\delta_j\| &\le \Big\|\frac1k\int_{t_{j-1}}^{t_j}(\tau-t_{j-1})Gu_{tt}(\tau)\mathrm{d}\tau\Big\| \le \int_{t_{j-1}}^{t_j}\|A^{3-\frac{\beta}{2}}E(\tau)A^{\frac{\beta}{2}}v\|\,\mathrm{d}\tau \\ &\le C\int_{t_{j-1}}^{t_j}\tau^{\frac{-6+\beta}{4}}\,\mathrm{d}\tau|v|_\beta,\end{aligned}$$

so that, by Hölder's inequality with $p=\frac{4}{\beta}$ and $q=\frac{4}{4-\beta}$, $1\le\beta<4$,

$$\begin{aligned}\int_{t_{j-1}}^{t_j}\tau^{\frac{-6+\beta}{4}}\,\mathrm{d}\tau &\le Ck^{\frac{\beta}{4}}\Big(\int_{t_{j-1}}^{t_j}\big(\tau^{\frac{-6+\beta}{4}}\big)^{\frac{4}{4-\beta}}\,\mathrm{d}\tau\Big)^{\frac{4-\beta}{4}} \\ &\le Ck^{\frac{\beta}{4}}\Big(\frac{\beta-4}{2}\big(t_{j-1}^{-\frac{2}{4-\beta}}-t_j^{-\frac{2}{4-\beta}}\big)\Big)^{\frac{4-\beta}{4}} \\ &\le Ck^{\frac{\beta}{4}}t_{j-1}^{-\frac12}.\end{aligned}$$

The same result is obtained with $\beta=4$. For $j=1$ we have

$$\begin{aligned}\|\delta_1\| &\le \Big\|\frac1k\int_0^k\tau Gu_{tt}(\tau)\,\mathrm{d}\tau\Big\| \le C\frac1k\int_0^k\tau^{\frac{-2+\beta}{4}}\,\mathrm{d}\tau|v|_\beta \\ &\le C\frac{4}{2+\beta}k^{\frac{-2+\beta}{4}}|v|_\beta \le Ck^{\frac{\beta}{4}}t_1^{-\frac12}|v|_\beta\end{aligned}$$

So we have, for $j\ge1$,

$$\|\delta_j\| \le Ck^{\frac{\beta}{4}}t_j^{-\frac12}|v|_\beta.$$

Hence

$$k\sum_{j=1}^n\|\delta_j\| \le ck^{\frac{\beta}{4}}t_n^{\frac12}|v|_\beta, \quad k\sum_{j=1}^n t_{j-1}\|\delta_j\|^2 \le Ck^{\frac{\beta}{2}}t_n|v|_\beta^2. \tag{2.34}$$

Put (2.31), (2.32), (2.33), and (2.34) in (2.30), to get

$$\|e_n\| \le C(h^\beta + k^{\frac{\beta}{4}})|v|_\beta.$$

This completes the proof (2.17).

To prove (2.18) we recall (2.29) and let $v \in \dot{H}^{\beta-2}$. We know that $k\sum_{j=1}^{n}\|\rho_j\|^2 = k\|\rho_1\|^2 + k\sum_{j=2}^{n}\|\rho_j\|^2$, where by (2.1)

$$k\|\rho_1\|^2 \le kCh^{2\beta}\|AE(k)A^{\frac{\beta-1}{2}}v\|^2 \le Ch^{2\beta}|v|_{\beta-2},$$

and

$$\begin{aligned}
k\sum_{j=2}^{n}\|\rho_j\|^2 &= \sum_{j=2}^{n}\int_{t_{j-1}}^{t_j}\Big\|\rho(s)+\int_s^{t_j}\rho_t(\tau)\,\mathrm{d}\tau\Big\|^2\,\mathrm{d}s\\
&\le 2\sum_{j=2}^{n}\int_{t_{j-1}}^{t_j}\|\rho(s)\|^2\,\mathrm{d}s + 2\sum_{j=2}^{n}\int_{t_{j-1}}^{t_j}\Big\|\int_s^{t_j}\rho_t(\tau)\,\mathrm{d}\tau\Big\|^2\,\mathrm{d}s\\
&\le 2\int_{t_1}^{t_n}\|\rho(s)\|^2\,\mathrm{d}s + 2\sum_{j=2}^{n}\int_{t_{j-1}}^{t_j}(t_j-s)\int_{t_{j-1}}^{t_j}\|\rho_t(\tau)\|^2\,\mathrm{d}\tau\,\mathrm{d}s\\
&\le 2\int_0^{t_n}\|\rho\|^2\,\mathrm{d}\tau + 2k\int_{t_1}^{t_n}\tau\|\rho_t\|^2\,\mathrm{d}\tau,
\end{aligned}$$

since $t_j - s \le k \le \tau$ and where, by (2.13),

$$\int_0^{t_n}\|\rho\|^2\,\mathrm{d}\tau \le Ch^{2\beta}|v|_{\beta-2}^2,$$

and

$$\begin{aligned}
k\int_{t_1}^{t_n}\tau\|\rho_t\|^2\,\mathrm{d}\tau &\le Ch^{2\beta}k\int_{t_1}^{t_n}\tau\|A^3E(\tau)A^{\frac{\beta-2}{2}}v\|^2\,\mathrm{d}\tau\\
&\le Ch^{2\beta}k\int_k^{t_n}\tau^{-2}\,\mathrm{d}\tau|v|_{\beta-2}^2\\
&\le Ch^{2\beta}k(k^{-1}-t_n^{-1}) \le Ch^{2\beta}|v|_{\beta-2}^2.
\end{aligned}$$

So

$$k\sum_{j=1}^{n}\|\rho_j\|^2 \le Ch^{2\beta}|v|_{\beta-2}^2. \tag{2.35}$$

Now we compute $k\sum_{j=1}^{n}\|\eta_j\|$. Recall that $\eta_j = -(R_h-I)G\partial u_j$ and $\eta = -(R_h-I)Gu_t$, so

$$\begin{aligned}
\|\eta_j\| &= \Big\|(R_h-I)G\frac{1}{k}\int_{t_{j-1}}^{t_j}u_t\,\mathrm{d}\tau\Big\| \le \frac{1}{k}\int_{t_{j-1}}^{t_j}\|(R_h-I)Gu_t\|\,\mathrm{d}\tau\\
&\le \frac{1}{k}\int_{t_{j-1}}^{t_j}\|\eta\|\,\mathrm{d}\tau,
\end{aligned}$$

and hence by (2.15) we have

$$k\sum_{j=1}^{n}\|\eta_j\| \le \int_0^{t_n}\|\eta\|\,\mathrm{d}\tau \le Ch^{\beta}|\log h||v|_{\beta-2}. \tag{2.36}$$

For computing $k\sum_{j=1}^{n}\|\delta_j\|$ we use (2.1) and obtain for $1\le\beta<4$,

$$\begin{aligned}\|\delta_j\| &= \frac{1}{k}\int_{t_{j-1}}^{t_j}(\tau-t_{j-1})\|Gu_{tt}(\tau)\|\,\mathrm{d}\tau \le \int_{t_{j-1}}^{t_j}\|A^{4-\frac{\beta}{2}}E(\tau)A^{\frac{\beta-2}{2}}v\|\,\mathrm{d}\tau\\ &\le C\int_{t_{j-1}}^{t_j}\tau^{-2+\frac{\beta}{4}}\,\mathrm{d}\tau|v|_{\beta-2}.\end{aligned}$$

Hence

$$\begin{aligned}k\sum_{j=2}^{n}\|\delta_j\| = k\sum_{j=2}^{n}\|\delta_j\| &\le Ck\int_k^{t_n}\tau^{-2+\frac{\beta}{4}}\,\mathrm{d}\tau|v|_{\beta-2}\\ &\le Ck\frac{4}{4-\beta}\Big(k^{-1+\frac{\beta}{4}}-t_n^{-1+\frac{\beta}{4}}\Big)|v|_{\beta-2}\\ &\le \frac{C}{4-\beta}k^{\frac{\beta}{4}}|v|_{\beta-2},\end{aligned}$$

and

$$\begin{aligned}k\|\delta_1\| &\le \int_0^k\tau\|Gu_{tt}(\tau)\|\,\mathrm{d}\tau \le \int_0^k\tau\|A^{4-\frac{\beta}{2}}E(\tau)A^{\frac{\beta-2}{2}}v\|\,\mathrm{d}\tau\\ &\le C\int_0^k\tau^{\frac{\beta}{4}-1}\,\mathrm{d}\tau|v|_{\beta-2} \le \frac{C}{4-\beta}k^{\frac{\beta}{4}}|v|_{\beta-2}.\end{aligned}$$

Therefore, for $1\le\beta<4$,

$$k\sum_{j=1}^{n}\|\delta_j\| \le \frac{C}{4-\beta}k^{\frac{\beta}{4}}|v|_{\beta-2}.$$

If we put $\frac{1}{4-\beta}=|\log k|$, we also have

$$\begin{aligned}k\sum_{j=1}^{n}\|\delta_j\| &\le \frac{C}{4-\beta}k^{1-\frac{4-\beta}{4}}|v|_{\beta-2} = C|\log k|k\mathrm{e}^{-\frac{4-\beta}{4}\log k}|v|_{\beta-2}\\ &\le Ck|\log k||v|_{\beta-2} = C|\log k||v|_{\beta-2}\end{aligned}$$

Therefore, for $1\le\beta\le4$, we have

$$k\sum_{j=1}^{n}\|\delta_j\| \le C_{\beta,k}k^{\frac{\beta}{4}}|v|_{\beta-2}. \tag{2.37}$$

where $C_{\beta,k} = \frac{C}{4-\beta}$ for $\beta < 4$ and $C_{\beta,k} = C|\log k|$ for $\beta = 4$. Finally we put (2.35), (2.36) and (2.37) in (2.29), to get

$$\Big(k\sum_{j=1}^{n}\|e_j\|^2\Big)^{\frac12} \le \Big(Ch^{\beta}|\log h| + C_{\beta,k}k^{\frac{\beta}{4}}\Big)|v|_{\beta-2}.$$

□

3. Finite element method for the Cahn-Hilliard-Cook equation

Consider The Cahn-Hilliard-Cook equation (1.4) with mild solution

$$X(t) = E(t)X_0 + \int_0^t E(t-s)\,\mathrm{d}W(s). \tag{3.1}$$

We recall the isometry of the Itô integral

$$\mathbf{E}\Big\|\int_0^t B(s)\,\mathrm{d}W(s)\Big\|^2 = \mathbf{E}\int_0^t \|B(s)Q^{\frac12}\|_{\mathrm{HS}}^2\,\mathrm{d}s, \tag{3.2}$$

where the Hilbert-Schmidt norm is defined by

$$\|T\|_{\mathrm{HS}}^2 = \sum_{l=1}^{\infty}\|T\varphi_l\|^2,$$

where $\{\varphi_l\}_{l=1}^{\infty}$ is an arbitrary O.N-basis for H. In the next theorem we consider the regularity of the mild solution (3.1).

Theorem 3.1. *Let $X(t)$ be the mild solution* (3.1). *If $X_0 \in L_2(\Omega, \dot H^{\beta})$ and $\|A^{\frac{\beta-2}{2}}Q^{\frac12}\|_{\mathrm{HS}} < \infty$ for some $\beta \ge 0$, then*

$$\|X(t)\|_{L_2(\Omega,\dot H^{\beta})} \le C\Big(\|X_0\|_{L_2(\Omega,\dot H^{\beta})} + \|A^{\frac{\beta-2}{2}}Q^{\frac12}\|_{\mathrm{HS}}\Big), \quad t \ge 0.$$

Proof. By using the isometry (3.2), the definition of the Hilbert-Schmidt norm, and (2.1), (2.2) we get

$$\begin{aligned}
\|X(t)\|_{L_2(\Omega,\dot H^{\beta})}^2 &= \mathbf{E}\Big|E(t)X_0 + \int_0^t E(t-s)\,\mathrm{d}W(s)\Big|_{\beta}^2 \\
&\le C\Big(\mathbf{E}\big|E(t)X_0\big|_{\beta}^2 + \mathbf{E}\Big\|\int_0^t A^{\frac{\beta}{2}}E(t-s)\,\mathrm{d}W(s)\Big\|^2\Big) \\
&\le C\Big(\|X_0\|_{L_2(\Omega,\dot H^{\beta})}^2 + \int_0^t \|A^{\frac{\beta}{2}}E(s)Q^{\frac12}\|_{\mathrm{HS}}^2\,\mathrm{d}s\Big) \\
&\le C\Big(\|X_0\|_{L_2(\Omega,\dot H^{\beta})}^2 + \sum_{l=1}^{\infty}\|A^{\frac{\beta-2}{2}}Q^{\frac12}\varphi_l\|^2\Big) \\
&\le C\Big(\|X_0\|_{L_2(\Omega,\dot H^{\beta})}^2 + \|A^{\frac{\beta-2}{2}}Q^{\frac12}\|_{\mathrm{HS}}^2\Big).
\end{aligned}$$

□

The finite element problem for Cahn-Hilliard-Cook equation is: Find $X_h(t) \in S_h$ such that

$$\begin{aligned} &\mathrm{d}X_h + A_h^2 X_h \,\mathrm{d}t = P_h \,\mathrm{d}W, \\ &X_h(0) = P_h X_0. \end{aligned} \tag{3.3}$$

So the mild solution with can be written as

$$X_h(t) = E_h(t) P_h X_0 + \int_0^t E_h(t-s) P_h \,\mathrm{d}W(s). \tag{3.4}$$

Theorem 3.2. *Let X_h and X be the mild solutions* (3.4) *and* (3.1)*. If $X_0 \in L_2(\Omega, \dot{H}^\beta)$ and $\|A^{\frac{\beta-2}{2}} Q^{\frac{1}{2}}\|_{\mathrm{HS}} < \infty$ for some $\beta \in [1, r]$, then for all $t \geq 0$*

$$\begin{aligned} \|X_h(t) - X(t)\|_{L_2(\Omega,H)} & \\ &\leq C h^\beta \big(\|X_0\|_{L_2(\Omega,\dot{H}^\beta)} + |\log h| \|A^{\frac{\beta-2}{2}} Q^{\frac{1}{2}}\|_{\mathrm{HS}} \big). \end{aligned}$$

Proof. Use (3.1) and (3.4) and set $F_h(t) = E_h(t) P_h - E(t)$ to get

$$\|X_h(t) - X(t)\|_{L_2(\Omega,H)} \leq \|e_1(t)\|_{L_2(\Omega,H)} + \|e_2(t)\|_{L_2(\Omega,H)}, \tag{3.5}$$

where $e_1(t) = F_h(t) X_0$ and $e_2(t) = \int_0^t F_h(t-s) \,\mathrm{d}W(s)$. By using Theorem 2.1 we get

$$\|e_1(t)\|_{L_2(\Omega,H)} = \big(\mathbf{E} \|F_h(t) X_0\|^2 \big)^{\frac{1}{2}} \leq C h^\beta \big(\mathbf{E} |X_0|_\beta^2 \big)^{\frac{1}{2}} = C h^\beta \|X_0\|_{L_2(\Omega,\dot{H}^\beta)}.$$

For the second term we use the isometry (3.2), the definition of Hilbert-Schmidt norm and Theorem 2.1,

$$\begin{aligned} \|e_2(t)\|_{L_2(\Omega,H)}^2 &= \mathbf{E}\Big(\Big\| \int_0^t F_h(t-s) \,\mathrm{d}W(s) \Big\|^2 \Big) \\ &= \int_0^t \|F_h(t-s) Q^{\frac{1}{2}}\|_{\mathrm{HS}}^2 \,\mathrm{d}s \\ &= \sum_{l=1}^\infty \int_0^t \|F_h(s) Q^{\frac{1}{2}} \varphi_l\|^2 \,\mathrm{d}s \\ &\leq C |\log h|^2 h^{2\beta} \sum_{l=1}^\infty |Q^{\frac{1}{2}} \varphi_l|_{\beta-2}^2 \\ &= C |\log h|^2 h^{2\beta} \|A^{(\beta-2)/2} Q^{\frac{1}{2}}\|_{\mathrm{HS}}^2. \end{aligned}$$

□

Now we consider the fully discrete Cahn-Hilliard-Cook equation (1.11) with mild solution

$$X_{h,n} = E_{kh}^n P_h X_0 + \sum_{j=1}^{n} E_{kh}^{n-j+1} P_h \Delta W_j, \tag{3.6}$$

where $E_{kh} = (I + kA_h^2)^{-1}$.

Theorem 3.3. *Let $X_{h,n}$ and X be given by (3.5) and (3.1) . If $X_0 \in L_2(\Omega, \dot{H}^\beta)$ and $\|A^{\frac{\beta-2}{2}} Q^{\frac{1}{2}}\|_{\mathrm{HS}} < \infty$ for some $\beta \in [1, \min(r, 4)]$, then*

$$\begin{aligned}\|X_{h,n}(t) - X(t_n)\|_{L_2(\Omega,H)} \\ \le \big(C|\log h| h^\beta + C_{\beta,k} k^{\frac{\beta}{4}}\big)\big(\|X_0\|_{L_2(\Omega,\dot{H}^\beta)} + \|A^{\frac{\beta-2}{2}} Q^{\frac{1}{2}}\|_{\mathrm{HS}}\big),\end{aligned}$$

where $C_{\beta,k} = \frac{C}{4-\beta}$ for $\beta < 4$ and $C_{\beta,k} = C|\log k|$ for $\beta = 4$.

Proof. By using (3.1) and (3.6) we get, with $F_n = E_{kh}^n P_h - E(t_n)$,

$$\begin{aligned} e_n &= F_n X_0 + \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} F_{n-j+1} \,\mathrm{d}W(s) \\ &\quad + \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \big(E(t_n - t_{j-1}) - E(t_n - s)\big) \,\mathrm{d}W(s) \\ &= e_{n,1} + e_{n,2} + e_{n,3} \end{aligned}$$

By using Theorem 2.2 we have

$$\|e_{n,1}\|_{L_2(\Omega,H)} = \big(\mathbf{E}\|F_n X_0\|^2\big)^{\frac{1}{2}} \le C(h^\beta + k^{\frac{\beta}{4}})\|X_0\|_{L_2(\Omega,\dot{H}^\beta)}. \tag{3.7}$$

By using the isometry (3.2) and Theorem 2.2 we get

$$\begin{aligned} \|e_{n,2}\|_{L_2(\Omega,H)}^2 &= \mathbf{E}\Big(\Big\|\sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} F_{n-j+1} \,\mathrm{d}W(s)\Big\|^2\Big) \\ &= \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \|F_{n-j+1} Q^{\frac{1}{2}}\|_{\mathrm{HS}}^2 \,\mathrm{d}s \\ &= k \sum_{l=1}^{\infty} \sum_{j=1}^{n} \|F_{n-j+1} Q^{\frac{1}{2}} \varphi_l\|^2 \\ &\le \sum_{l=1}^{\infty} \big(C|\log h| h^\beta + C_{\beta,k} k^{\frac{\beta}{4}}\big)^2 |Q^{\frac{1}{2}} \varphi_l|_{\beta-2}^2 \\ &= \big(C|\log h| h^\beta + C_{\beta,k} k^{\frac{\beta}{4}}\big)^2 \|A^{\frac{\beta-2}{2}} Q^{\frac{1}{2}}\|_{\mathrm{HS}}^2. \end{aligned}$$

By using isometry property (3.2) again we have

$$\begin{aligned}
&\|e_{n,3}\|^2_{L_2(\Omega,H)}\\
&\le \mathbf{E}\Big(\Big\|\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}\big(E(t_n-t_{j-1})-E(t_n-s)\big)\,\mathrm{d}W(s)\Big\|^2\Big)\\
&=\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}\|(E(t_n-t_{j-1})-E(t_n-s))Q^{\frac12}\|^2_{\mathrm{HS}}\,\mathrm{d}s\\
&=\sum_{l=1}^{\infty}\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}\|A^{-\frac{\beta}{2}}(E(s-t_{j-1})-I)AE(t_n-s)A^{\frac{\beta-2}{2}}Q^{\frac12}\varphi_l\|^2\,\mathrm{d}s.
\end{aligned}$$

Using the well-known inequality

$$\|A^{\frac{-\beta}{2}}\big(E(t)-I\big)w\|\le Ct^{\frac{\beta}{4}}\|w\|,$$

with $t=s-t_j, w=AE(t_n-s)A^{\frac{\beta-2}{2}}Q^{\frac12}\varphi_l$, together with (2.2), we get

$$\begin{aligned}
\|e_{n,3}\|^2_{L_2(\Omega,H)} &\le Ck^{\frac{\beta}{2}}\sum_{l=1}^{\infty}\int_0^{t_n}\|AE(t_n-s)A^{\frac{\beta-2}{2}}Q^{\frac12}\varphi_l\|^2\,\mathrm{d}s\\
&\le Ck^{\frac{\beta}{2}}\sum_{l=1}^{\infty}\|A^{\frac{\beta-2}{2}}Q^{\frac12}\varphi_l\|^2=Ck^{\frac{\beta}{2}}\|A^{\frac{\beta-2}{2}}Q^{\frac12}\|^2_{\mathrm{HS}}.
\end{aligned}$$

Putting these together proves the desired result. □

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