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Abstract

We present a residual-based a posteriori error estimate in an energy norm of the error in a family of discontinuous Galerkin approximations of linear elasticity problems. The theory is developed in two and three spatial dimensions and general nonconvex polygonal domains are allowed. We also present some illustrating numerical examples.

Key words: discontinuous Galerkin, adaptivity, a posteriori error estimate, elasticity

1 Introduction

In Becker, Hansbo, and Larson [1], we presented a posteriori error estimates for the Laplace equation based on the Helmholtz decomposition of vector fields. This decomposition cannot, however, be used in the case of linearized elasticity. Motivated by the recent work by Beirão da Veiga, Niiranen, and Stenberg [2] on a posteriori error estimates for plates, we here extend the approach of [1] to the case of linear elasticity in two and three spatial dimensions. The proof is based on a Helmholtz type decomposition for tensor fields and thus differs from previous error estimates introduced by Wihler [13] and Houston, Schötzau, and Wihler [10], and by Carstensen [5] and Carstensen and Hu [7], where the error is instead split into a continuous part and a discontinuous part.

The paper is organized as follows: in Section 2 we introduce a model problem and the family of dG methods; in Section 3 we derive our a posteriori estimate; and finally in Section 4 we present some numerical experiments.

2 The Continuous Problem

Consider a domain Ω in $\mathbb{R}^{n_{\mathrm{sd}}}$, $n_{\mathrm{sd}} = 2$ or $n_{\mathrm{sd}} = 3$ with boundary $\Gamma = \Gamma_{\mathrm{D}} \cup \Gamma_{\mathrm{N}}$, $\Gamma_{\mathrm{D}} \cap \Gamma_{\mathrm{N}} = \emptyset$, whose outward pointing normal is denoted $\boldsymbol{n}$. The linear elasticity equations can be written

$$\begin{aligned} -\nabla \cdot \boldsymbol{\sigma}(\boldsymbol{u}) &= \boldsymbol{f} \quad \text{in } \Omega \\ \boldsymbol{u} &= \boldsymbol{g}_{\mathrm{D}} \quad \text{on } \Gamma_{\mathrm{D}} \\ \boldsymbol{n} \cdot \boldsymbol{\sigma}(\boldsymbol{u}) &= \boldsymbol{g}_{\mathrm{N}} \quad \text{on } \Gamma_{\mathrm{N}} \end{aligned} \tag{1}$$

Here, with λ and μ given material data, the stress tensor $\boldsymbol{\sigma}$ is defined by

$$\boldsymbol{\sigma}(\boldsymbol{u}) = 2\mu\, \boldsymbol{\varepsilon}(\boldsymbol{u}) + \lambda\, \nabla \cdot \boldsymbol{u}\, \mathbf{1} \tag{2}$$

where $\boldsymbol{u}$ is the displacement field, $\mathbf{1}$ is the identity tensor,

$$\boldsymbol{\varepsilon}(\boldsymbol{u}) = \frac{1}{2}\left(\nabla \otimes \boldsymbol{u} + (\nabla \otimes \boldsymbol{u})^{\mathrm{T}}\right)$$

is the strain tensor, and $\boldsymbol{f}$, $\boldsymbol{g}_{\mathrm{N}}$, and $\boldsymbol{g}_{\mathrm{D}}$ are given data. We have also used the notation

$$(\nabla \cdot \boldsymbol{\tau})_i = \sum_{j=1}^{n_{\mathrm{sd}}} \frac{\partial \tau_{ij}}{\partial x_j}$$

for the divergence of a tensor field $\boldsymbol{\tau}$.

3 Finite Element Approximation

3.1 Discontinuous Spaces

To define the dG method we introduce a partition $\mathcal{K} = \{K\}$ of Ω and let the mesh function $h : \Omega \to (0, \infty)$ be defined by $h|_K = h_K = \mathrm{diam}(K)$. We let $\mathcal{DP}$ be the space of discontinuous piecewise polynomials defined on $\mathcal{K}$:

$$\mathcal{DP} = \bigoplus_{K \in \mathcal{K}} \mathcal{P}_{p_K}(K) \tag{3}$$

where $\mathcal{P}_{p_K}(K)$ is a space of polynomials of degree p_K defined on the element K. Note that the order of polynomials is allowed to vary from element to element.

3.2 The dG Method

The dG method for (1) is defined by: find $\boldsymbol{u}_h \in [\mathcal{DP}]^{n_{\rm sd}}$ such that

$$a(\boldsymbol{u}, \boldsymbol{v}) = l(\boldsymbol{v}) \quad \text{for all } \boldsymbol{v} \in [\mathcal{DP}]^{n_{\rm sd}} \tag{4}$$

where $a(\cdot,\cdot)$ and $l(\cdot)$ are sums of elementwise defined forms

$$a(\boldsymbol{v}, \boldsymbol{w}) := \sum_{K \in \mathcal{K}} a_K(\boldsymbol{v}, \boldsymbol{w}) \qquad l(\boldsymbol{v}) := \sum_{K \in \mathcal{K}} l_K(\boldsymbol{v}) \tag{5}$$

given by

$$\begin{aligned} a_K(\boldsymbol{v}, \boldsymbol{w}) := & (\boldsymbol{\sigma}(\boldsymbol{v}), \boldsymbol{\varepsilon}(\boldsymbol{w}))_K \\ & -(\langle \boldsymbol{n} \cdot \boldsymbol{\sigma}(\boldsymbol{v}) \rangle, \boldsymbol{w})_{\partial K \setminus \Gamma_{\rm N}} \\ & +\alpha([v], \boldsymbol{n} \cdot \boldsymbol{\sigma}(\boldsymbol{w}))_{\partial K \setminus \Gamma_{\rm N}}/2 \\ & +\beta(h^{-1}[\boldsymbol{v}], \boldsymbol{w})_{\partial K \setminus \Gamma_{\rm N}} + \gamma(h^{-1}[\boldsymbol{n} \cdot \boldsymbol{v}], \boldsymbol{n} \cdot \boldsymbol{w})_{\partial K \setminus \Gamma_{\rm N}} \end{aligned} \tag{6}$$

and

$$\begin{aligned} l_K(\boldsymbol{w}) := & (\boldsymbol{f}, \boldsymbol{w})_K + (\boldsymbol{g}_{\rm N}, \boldsymbol{w})_{\partial K \cap \Gamma_{\rm N}} \\ & +\alpha(\boldsymbol{g}_{\rm D}, \boldsymbol{n} \cdot \boldsymbol{\sigma}(\boldsymbol{w}))_{\partial K \cap \Gamma_{\rm D}} \\ & +\beta(\boldsymbol{g}_{\rm D}, h^{-1}\boldsymbol{w})_{\partial K \cap \Gamma_{\rm D}} + \gamma(\boldsymbol{n} \cdot \boldsymbol{g}_{\rm D}, h^{-1}\boldsymbol{n} \cdot \boldsymbol{w})_{\partial K \cap \Gamma_{\rm D}} \end{aligned} \tag{7}$$

where α is a parameter which is typically chosen to be $1, 0$ or -1, while β, and γ are positive real valued penalty parameters. If the penalty parameters are chosen large enough, depending on the choice of α, the bilinear form is coercive on the discrete space $\mathcal{DP}$. Setting $\gamma = 0$ results in the standard dG method and choosing $\beta = \beta_0 \mu$ and $\gamma = \gamma_0 \lambda$ with parameters β_0 and γ_0 we obtain the locking free method presented in [8] and used in the numerical examples below, Section 5. We also refer to [8] for computable lower bounds on the penalty parameters in the case of a symmetric formulation, $\alpha = -1$.

We also employed the notation

$$\langle v \rangle := \begin{cases} (v^+ + v^-)/2 & \text{on } \partial K \setminus \Gamma \\ v^+ & \text{on } \partial K \cap \Gamma \end{cases} \tag{8}$$

and

$$[v] := \begin{cases} v^+ - v^- & \text{on } \partial K \setminus \Gamma \\ v^+ & \text{on } \partial K \cap \Gamma \end{cases} \tag{9}$$

where $v^\pm(x) = \lim_{s\to 0^+} v(x \mp s n_K)$. Further on each edge $E = K^+ \cap K^-$, the mesh parameter h is defined by

$$h := \frac{m(K^+) + m(K^-)}{2m(E)} \tag{10}$$

where $m(\cdot)$ denotes the appropriate Lebesgue measure.

4 An Energy Norm Error Estimate

4.1 Helmholtz Decomposition of Tensor Fields

We shall use the following decomposition of tensor fields in two and three dimensions; a two-dimensional version of the decomposition suitable for analysis of fourth order plate equations was introduced in [2].

Lemma 1 *Let $\boldsymbol{\chi} \in L^2(\Omega, \mathbf{R}^{n_{sd}\times n_{sd}})$ be a second order tensor field. Then there exist $\boldsymbol{z} \in V_0 = \{\boldsymbol{v} \in [H^1(\Omega)]^{n_{sd}} : \ \boldsymbol{v} = 0 \text{ on } \Gamma_D\}$ and $\boldsymbol{\phi} \in [H^1(\Omega)]^2$ in two dimensions, and $\boldsymbol{\phi} := [H(\mathbf{curl}\ , \Omega)]^3$ in three dimensions, such that*

$$\boldsymbol{\chi} = \boldsymbol{\sigma}(\boldsymbol{z}) + \mathbf{Curl}\ \boldsymbol{\phi} \tag{11}$$

where

$$\mathbf{Curl}\ \boldsymbol{\phi} = \begin{bmatrix} -\dfrac{\partial \phi_1}{\partial x_2} & \dfrac{\partial \phi_1}{\partial x_1} \\ -\dfrac{\partial \phi_2}{\partial x_2} & \dfrac{\partial \phi_2}{\partial x_1} \end{bmatrix} \tag{12}$$

in two dimensions, and

$$\mathbf{Curl}\ \boldsymbol{\phi} = \begin{bmatrix} \dfrac{\partial \phi_{13}}{\partial x_2} - \dfrac{\partial \phi_{12}}{\partial x_3} & \dfrac{\partial \phi_{11}}{\partial x_3} - \dfrac{\partial \phi_{13}}{\partial x_1} & \dfrac{\partial \phi_{12}}{\partial x_1} - \dfrac{\partial \phi_{11}}{\partial x_2} \\ \dfrac{\partial \phi_{23}}{\partial x_2} - \dfrac{\partial \phi_{22}}{\partial x_3} & \dfrac{\partial \phi_{21}}{\partial x_3} - \dfrac{\partial \phi_{23}}{\partial x_1} & \dfrac{\partial \phi_{22}}{\partial x_1} - \dfrac{\partial \phi_{21}}{\partial x_2} \\ \dfrac{\partial \phi_{33}}{\partial x_2} - \dfrac{\partial \phi_{32}}{\partial x_3} & \dfrac{\partial \phi_{31}}{\partial x_3} - \dfrac{\partial \phi_{33}}{\partial x_1} & \dfrac{\partial \phi_{32}}{\partial x_1} - \dfrac{\partial \phi_{31}}{\partial x_2} \end{bmatrix} \tag{13}$$

in three dimensions, and the following stability estimate holds

$$\|\boldsymbol{z}\|_{H^1(\Omega)} + \|\mathbf{Curl}\ \boldsymbol{\phi}\|_{L_2(\Omega)} \leq C\|\boldsymbol{\chi}\|_{L_2(\Omega)} \tag{14}$$

If $\boldsymbol{\chi}$ is symmetric then **Curl** $\boldsymbol{\phi}$ *is also symmetric.*

PROOF. We let $\boldsymbol{z} \in V_0$ solve the variational problem

$$(\boldsymbol{\sigma}(\boldsymbol{z}), \boldsymbol{\varepsilon}(\boldsymbol{v})) = (\boldsymbol{\chi}, \boldsymbol{\varepsilon}(\boldsymbol{v})) \quad \forall \boldsymbol{v} \in V_0$$

and thus

$$\|\boldsymbol{z}\|_{H^1(\Omega)} \leq C\|\boldsymbol{\chi}\|_{L_2(\Omega)} \tag{15}$$

by Korn's inequality. Then we have $\nabla \cdot (\boldsymbol{\sigma}(\boldsymbol{z}) - \boldsymbol{\chi}) = \mathbf{0}$ in the distributional sense. Thus the divergence of each row $[\boldsymbol{\sigma}(\boldsymbol{z}) - \boldsymbol{\chi}]_i$ of the matrix $\boldsymbol{\sigma}(\boldsymbol{z}) - \boldsymbol{\chi}$ is zero and may be represented in terms of a curl of a function $\phi_i \in H^1(\Omega)$ in two dimensions and $\boldsymbol{\phi}_i = [\phi_{i1}\ \phi_{i2}\ \phi_{i3}] \in H(\mathbf{curl}\,, \Omega)$ in three dimensions. This implies that there exists a $\boldsymbol{\phi} = [\phi_1\ \phi_2]^T \in [H^1(\Omega)]^2$ in two dimensions and $\boldsymbol{\phi} = [\boldsymbol{\phi}_1^T\ \boldsymbol{\phi}_2^T\ \boldsymbol{\phi}_3^T]^T \in [H(\mathbf{curl}\,, \Omega)]^3$ such that

$$\boldsymbol{\chi} - \boldsymbol{\sigma}(\boldsymbol{z}) = \mathbf{Curl}\ \boldsymbol{\phi} \tag{16}$$

and we have the stability estimate

$$\|\mathbf{Curl}\ \boldsymbol{\phi}\|_{L_2(\Omega)} = \|\boldsymbol{\chi} - \boldsymbol{\sigma}(\boldsymbol{z})\|_{L_2(\Omega)} \leq C\|\boldsymbol{\chi}\|_{L_2(\Omega)} \tag{17}$$

which follows from the triangle inequality and (14). Finally, we note that if $\boldsymbol{\chi}$ is symmetric then **Curl** $\boldsymbol{\phi}$ must be symmetric since $\boldsymbol{\sigma}(\boldsymbol{z})$ is symmetric.

4.2 The Main Result

In the following, C denotes a generic constant independent of the meshsize, not necessarily the same at different instances. Introducing the energy norm

$$|||\boldsymbol{v}|||^2 = \sum_{K \in \mathcal{K}} (\boldsymbol{\sigma}(\boldsymbol{v}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_K \tag{18}$$

on $[H^1]^{n_{\text{sd}}} \cup [\mathcal{DP}]^{n_{\text{sd}}}$ and the discrete traction vector defined by

$$\boldsymbol{\Sigma}_n(\boldsymbol{u}_h) := \langle \boldsymbol{n} \cdot \boldsymbol{\sigma}(\boldsymbol{u}_h) \rangle - \beta h^{-1}[\boldsymbol{u}_h] - \gamma h^{-1}\boldsymbol{n}([\boldsymbol{n} \cdot \boldsymbol{u}_h])$$

on $\partial K \setminus \Gamma$,

$$\boldsymbol{\Sigma}_n(\boldsymbol{u}_h) := \boldsymbol{n} \cdot \boldsymbol{\sigma}(\boldsymbol{u}_h) - \beta h^{-1}(\boldsymbol{u}_h - \boldsymbol{g}_{\text{D}}) - \gamma h^{-1}\boldsymbol{n}(\boldsymbol{n} \cdot (\boldsymbol{u}_h - \boldsymbol{g}_{\text{D}}))$$

on $\partial K \cap \Gamma_{\text{D}}$, and

$$\boldsymbol{\Sigma}_n(\boldsymbol{u}_h) := \boldsymbol{g}_{\text{N}}$$

on $\partial K \cap \Gamma_{\text{N}}$, we are ready to formulate our main result.

Theorem 2 *There holds*

$$|||\boldsymbol{u}-\boldsymbol{u}_h|||^2 \leq C \sum_{K\in\mathcal{K}} \rho_k^2 \tag{19}$$

where the element indicator ρ_K *is defined by*

$$\begin{aligned}\rho_K^2 = {} & h_K^2 \|\boldsymbol{f}+\nabla\cdot\boldsymbol{\sigma}(\boldsymbol{u}_h)_K\|_K^2 \\ & + h_K \|\boldsymbol{\Sigma}_n(\boldsymbol{u}_h)-\boldsymbol{n}\cdot\boldsymbol{\sigma}(\boldsymbol{u}_h)\|^2_{\partial K\setminus\Gamma_D} \\ & + h_K^{-1} \|[\boldsymbol{u}_h]\|^2_{\partial K\setminus\Gamma_N}\end{aligned} \tag{20}$$

PROOF. Letting $\boldsymbol{e}=\boldsymbol{u}-\boldsymbol{u}_h$ be the error and using the decomposition (11) with $\boldsymbol{\chi}=\boldsymbol{\sigma}(\boldsymbol{e})$ and elementwise applied derivatives we obtain

$$\begin{aligned}|||\boldsymbol{e}|||^2 &= \sum_{K\in\mathcal{K}} (\boldsymbol{\sigma}(\boldsymbol{e}),\boldsymbol{\varepsilon}(\boldsymbol{e}))_T \\ &= \sum_{K\in\mathcal{K}} (\boldsymbol{\varepsilon}(\boldsymbol{e}),\boldsymbol{\sigma}(\boldsymbol{z}))_T + \sum_{K\in\mathcal{K}} (\boldsymbol{\varepsilon}(\boldsymbol{e}),\mathbf{Curl}\,\boldsymbol{\phi})_T \\ &= I + II\end{aligned} \tag{21}$$

We proceed with estimates of the two terms.

For the first term I we first eliminate the exact elasticity solution $\boldsymbol{u}$ and use the definition of the finite element method (4) to subtract the continuous Scott-Zhang interpolant $\pi\boldsymbol{z}\in\mathcal{DP}\cap C(\Omega)$, cf. Brenner and Scott [3], as follows

$$\begin{aligned}\sum_{K\in\mathcal{K}} (\boldsymbol{\sigma}(\boldsymbol{e}),\boldsymbol{\varepsilon}(\boldsymbol{z}))_K &= \sum_{K\in\mathcal{K}} (\boldsymbol{\sigma}(\boldsymbol{e}),\boldsymbol{\varepsilon}(\boldsymbol{z}-\pi\boldsymbol{z}))_K \\ &= \sum_{K\in\mathcal{K}} (\boldsymbol{f}+\nabla\cdot\boldsymbol{\sigma}(\boldsymbol{u}_h),\boldsymbol{z}-\pi\boldsymbol{z})_K \\ &\qquad + (\boldsymbol{n}\cdot(\boldsymbol{\sigma}(\boldsymbol{u})-\boldsymbol{\sigma}(\boldsymbol{u}_h)),\boldsymbol{z}-\pi\boldsymbol{z})_{\partial K\setminus\Gamma_D} \\ &= \sum_{K\in\mathcal{K}} (\boldsymbol{f}+\nabla\cdot\boldsymbol{\sigma}(\boldsymbol{u}_h),\boldsymbol{z}-\pi\boldsymbol{z})_K \\ &\qquad + (\boldsymbol{\Sigma}_n(\boldsymbol{u}_h)-\boldsymbol{n}\cdot\boldsymbol{\sigma}(\boldsymbol{u}_h),\boldsymbol{z}-\pi\boldsymbol{z})_{\partial K\setminus\Gamma_D} \\ &= I_1 + I_2\end{aligned}$$

Here we used the fact that by definition $\boldsymbol{\Sigma}_n(\boldsymbol{u}_h)$ is continuous across edges so that

$$0 = \sum_{K\in\mathcal{K}} (\boldsymbol{n}\cdot\boldsymbol{\sigma}(\boldsymbol{u}),\boldsymbol{z}-\pi\boldsymbol{z})_{\partial K\setminus\Gamma_\mathrm{D}} = \sum_{K\in\mathcal{K}} (\boldsymbol{\Sigma}_n(\boldsymbol{u}_h),\boldsymbol{z}-\pi\boldsymbol{z})_{\partial K\setminus\Gamma_\mathrm{D}}$$

The terms I_1 and I_2 may now be directly estimated in a straightforward manner using the Cauchy-Schwartz inequality, the trace inequality

$$\|v\|^2_{L_2(\partial T)} \leq C\left(h_T^{-1}\|v\|^2_{L_2(T)} + h_T\|\nabla v\|^2_{L_2(T)}\right)$$

cf. Thomée [12], standard interpolation error estimates, cf. [3], and finally the stability estimate (14) as follows.

Term I_1. Using the Cauchy-Schwartz inequality on the sum and scaling with suitable powers of h_T we obtain

$$\begin{aligned} I_1 \leq &\left(\sum_{K\in\mathcal{K}} h_K^2 \|\boldsymbol{f} + \nabla\cdot\boldsymbol{\sigma}(\boldsymbol{u}_h)\|^2_{L_2(K)}\right)^{1/2} \\ &\times\left(\sum_{K\in\mathcal{K}} h_T^{-2}\|\boldsymbol{z} - \pi\boldsymbol{z}\|^2_{L_2(K)}\right)^{1/2} \end{aligned} \tag{22}$$

Next using interpolation error estimates we have

$$\sum_{K\in\mathcal{K}} h_K^{-2}\|\boldsymbol{z} - \pi\boldsymbol{z}\|^2_{L_2(K)} \leq C\|\boldsymbol{z}\|^2_{H^1(\Omega)} \tag{23}$$

Term I_2. Using the Cauchy-Schwartz inequality on the sum with suitable scaling, the trace inequality, and an interpolation error estimate we get

$$\begin{aligned} &\sum_{K\in\mathcal{K}} |(\boldsymbol{\Sigma}_n(\boldsymbol{u}_h) - \boldsymbol{\sigma}(\boldsymbol{u}_h), \boldsymbol{z} - \pi\boldsymbol{z})_{\partial K\setminus\Gamma_{\mathrm{D}}}| \\ &\quad \leq \sum_{K\in\mathcal{K}} \|\boldsymbol{\Sigma}_n(\boldsymbol{u}_h) - \boldsymbol{\sigma}(\boldsymbol{u}_h)\|_{\partial K\setminus\Gamma_{\mathrm{D}}}\|\boldsymbol{z} - \pi\boldsymbol{z}\|_{\partial K\setminus\Gamma_{\mathrm{D}}} \\ &\quad \leq C\left(\sum_{K\in\mathcal{K}} h_K^{1/2}\|\boldsymbol{\Sigma}_n(\boldsymbol{u}_h) - \boldsymbol{\sigma}(\boldsymbol{u}_h)\|_{\partial K\setminus\Gamma_{\mathrm{D}}}\right)\|\boldsymbol{z}\|_{H^1(\Omega)} \end{aligned} \tag{24}$$

Collecting the estimates and using the stability estimate (14) we finally get

$$I \leq C\left(\sum_{K\in\mathcal{K}} \rho_K^2\right)^{1/2}\|\boldsymbol{z}\|_{H^1(\Omega)} \leq C\left(\sum_{K\in\mathcal{K}} \rho_K^2\right)^{1/2} |||\,\boldsymbol{e}||| \tag{25}$$

To estimate the second term II we first use the symmetry of $\mathbf{Curl}\,\boldsymbol{\phi}$ to replace the symmetric gradient $\boldsymbol{\varepsilon}(\boldsymbol{e})$ by the gradient $\nabla\boldsymbol{e}$ and then employ Green's formula to obtain

$$\begin{aligned} II &= \sum_{K \in \mathcal{K}} (\boldsymbol{\varepsilon}(\boldsymbol{e}), \mathbf{Curl}\ \boldsymbol{\phi})_K \\ &= \sum_{K \in \mathcal{K}} (\nabla \boldsymbol{e}, \mathbf{Curl}\ \boldsymbol{\phi})_K \\ &= \sum_{K \in \mathcal{K}} (\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{n} \cdot \mathbf{Curl}\ \boldsymbol{\phi})_{\partial K \setminus \Gamma_{\mathrm{N}}} \end{aligned} \tag{26}$$

since $\nabla \cdot \mathbf{Curl}\ \boldsymbol{\phi} = \mathbf{0}$. Next we note that we can write the right-hand side in the following form

$$\begin{aligned} II &= \sum_{K \in \mathcal{K}} (\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{n} \cdot \mathbf{Curl}\ \boldsymbol{\phi})_{\partial K \setminus \Gamma_{\mathrm{N}}} \\ &= \sum_{K \in \mathcal{K}} (\boldsymbol{w} - \boldsymbol{u}_h, \boldsymbol{n} \cdot \mathbf{Curl}\ \boldsymbol{\phi})_{\partial K \setminus \Gamma_{\mathrm{N}}} \end{aligned} \tag{27}$$

for any continuous function $\boldsymbol{w}$ which equals $\boldsymbol{g}_{\mathrm{D}}$ on Γ_{D}. We can now estimate the contributions from each triangle as follows

$$\begin{aligned} &(\boldsymbol{w} - \boldsymbol{u}_h, \boldsymbol{n} \cdot \mathbf{Curl}\ \boldsymbol{\phi})_{\partial K} \\ &\qquad \leq \|\boldsymbol{u}_h - \boldsymbol{w}\|_{H^{\frac{1}{2}}(\partial K)} \|\boldsymbol{n} \cdot \mathbf{Curl}\ \boldsymbol{\phi}\|_{H^{-\frac{1}{2}}(\partial K)} \end{aligned} \tag{28}$$

Next using the normal trace inequality

$$\|\boldsymbol{n} \cdot \boldsymbol{v}\|_{H^{-\frac{1}{2}}(\partial K)} \leq C \left(\|\boldsymbol{v}\|_{L_2(K)} + h_K \|\mathrm{div}\ \boldsymbol{v}\|_{L_2(K)} \right) \tag{29}$$

see Larson and Målqvist [11], applied to each row of $\mathbf{Curl}\ \boldsymbol{\phi}$, we obtain the following inequality

$$\|\boldsymbol{n} \cdot \mathbf{Curl}\ \boldsymbol{\phi}\|_{H^{-\frac{1}{2}}(\partial K)} \leq C \|\mathbf{Curl}\ \boldsymbol{\phi}\|_{L_2(K)} \tag{30}$$

which together with (27) and (28) gives

$$\begin{aligned} II &\leq C \left(\sum_{K \in \mathcal{K}} \|\boldsymbol{u}_h - \boldsymbol{w}\|^2_{H^{\frac{1}{2}}(\partial T)} \right)^{1/2} \left(\sum_{K \in \mathcal{K}} \|\mathbf{Curl}\ \boldsymbol{\phi}\|^2_{L_2(K)} \right)^{1/2} \\ &\leq C \left(\sum_{K \in \mathcal{K}} \|\boldsymbol{u}_h - \boldsymbol{w}\|^2_{H^{\frac{1}{2}}(\partial K)} \right)^{1/2} |||\, \boldsymbol{e} ||| \end{aligned} \tag{31}$$

Finally, the following inequality holds

$$\inf_{\boldsymbol{w} \in [C(\Omega)]^{n_{\mathrm{sd}}}} \sum_{K \in \mathcal{K}} \|\boldsymbol{u}_h - \boldsymbol{w}\|^2_{H^{\frac{1}{2}}(\partial K)} \leq C \sum_{K \in \mathcal{K}} h_K^{-1} \|[\boldsymbol{u}_h]\|^2_{L_2(\partial K)} \tag{32}$$

see [11] for a detailed proof, and thus we obtain the estimate

$$II \leq C \left(\sum_{K \in \mathcal{K}} h_K^{-1} \|[\boldsymbol{u}_h]\|_{L_2(\partial K)}^2 \right)^{1/2} \||\, \boldsymbol{e} \|| \tag{33}$$

Collecting the estimates of terms I and II, the theorem follows.

Remark 3 *As an alternative the first term can be estimated as follows*

$$\begin{aligned}
\sum_{K \in \mathcal{K}} (\boldsymbol{\sigma}(\boldsymbol{e}), \boldsymbol{\varepsilon}(\boldsymbol{z}))_K &= \sum_{K \in \mathcal{K}} (\boldsymbol{f} + \nabla \cdot \boldsymbol{\sigma}(\boldsymbol{u}_h), \boldsymbol{z})_K \\
&\quad + (\boldsymbol{\Sigma}_n(\boldsymbol{u}_h) - \boldsymbol{n} \cdot \boldsymbol{\sigma}(\boldsymbol{u}_h), \boldsymbol{z})_{\partial K \setminus \Gamma_D} \\
&= \sum_{K \in \mathcal{K}} (\boldsymbol{f} - P_K \boldsymbol{f}, \boldsymbol{z})_K + (P_K \boldsymbol{f} + \nabla \cdot \boldsymbol{\sigma}(\boldsymbol{u}_h), \boldsymbol{z})_K \\
&\quad + (P_{\partial K} \boldsymbol{\Sigma}_n(\boldsymbol{u}_h) - \boldsymbol{n} \cdot \boldsymbol{\sigma}(\boldsymbol{u}_h), \boldsymbol{z})_{\partial K \setminus \Gamma_D} \\
&= \sum_{K \in \mathcal{K}} (\boldsymbol{\Sigma}_K, \boldsymbol{\varepsilon}(\boldsymbol{z}))_K + (\boldsymbol{f} - P_K \boldsymbol{f}, \boldsymbol{z})_K \\
&\quad + (\boldsymbol{g}_N - P_{\partial K} \boldsymbol{g}_N, \boldsymbol{z})_{\partial K \cap \Gamma_N}
\end{aligned}$$

where $\boldsymbol{\Sigma}_K$ *is a matrix with rows* $\boldsymbol{\Sigma}_{K,i} \in BDM_{p_K}(K)$*, and* BDM_{p_K} *denotes the Brezzi–Douglas–Marini space of polynomials of degree* p_K *(restricted to element* K*), cf. [4].*

5 Numerical examples

5.1 A problem with known exact solution

We take a problem from Carstensen et al. [6] which solves $\nabla \cdot \boldsymbol{\sigma} = 0$ on the domain shown in Figure 1, with exact solution in polar coordinates given by

$$\begin{aligned}
u_r &= \frac{r^\alpha}{2\mu}(-(\alpha+1)\cos((\alpha+1)\theta) + (C_2 - (\alpha+1))C_1 \cos((\alpha-1)\theta)) \\
u_\theta &= \frac{r^\alpha}{2\mu}((\alpha+1)\sin((\alpha+1)\theta) + (C_2 + \alpha - 1)C_1 \sin((\alpha-1)\theta))
\end{aligned}$$

where $C_1 = -\cos((\alpha+1)\omega)/\cos((\alpha-1)\omega)$, $C_2 = 2(\lambda+2\mu)/(\lambda+\mu)$, $\omega = 3\pi/4$, and $\alpha = 0.54448373678246\ldots$ solves $\alpha \sin(2\omega) + \sin(2\omega\alpha) = 0$. Here, r is the length of the radius vector, and θ is the angle between the radius vector and the horizontal axis. The problem was solved with Youngs modulus $E = 10000$, Poisson's ratio $\nu = 0.3$, and $\beta_0 = \gamma_0 = 20$. In Figure 2 we show the effect of using the a posteriori error estimate as a guide to refining the mesh. Using an edge bisection method, we refined the elements having the 25% largest element indicators according to (20) in each successive refinement step. In Figure 4 we show the ratio between the exact broken energy norm error and the corresponding a posteriori estimate. The ratio stays almost constant, as expected, indicating the sharpness of the estimate.

5.2 A problem with known locking problems

The second problem is known as Cook's membrane problem and is chosen for its severe locking tendencies. In Figure 5 we give the computational domain, and in Figure 6 the result of the adaptive process on a deformed mesh. The data were $E = 50$, $\nu = 0.499$, and $\boldsymbol{f} = (0, -1)$. At $x = 0$ we set $\boldsymbol{u} = 0$, for the rest of the boundary we used Neumann conditions with $\boldsymbol{g}_{\mathrm{N}} = 0$. The parameters β_0 and γ_0 were chosen as in the previous example. We remark that the source of the locking problem for Cook's membrane lies in the upper left corner, where the solution has to accommodate a kink. Low order conforming elements typically fail in this respect. We show, in Figure 7, the effect of using discontinuous approximations using a zoom at the upper left corner. By allowing for a limited inter-element sliding, the solution can accommodate without losing the elementwise incompressibility and normal continuity required by the equilibrium and constitutive equations.

6 Concluding remarks

We have proposed a new approach to a posteriori error estimation for discontinuous Galerkin approximations of vector–valued problems, based on a Helmholtz–type decomposition of tensor fields. Though the analysis is done for completely discontinuous approximations, the analysis carries over, with minor modifications, to the computationally more efficient nonconforming, non-locking, element proposed in [9].

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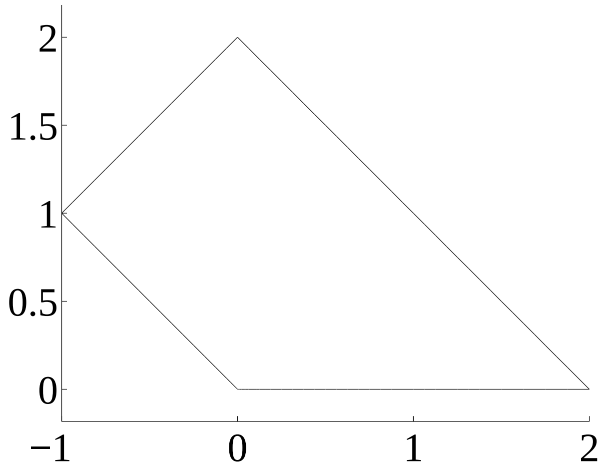

Fig. 1. Computational domain for example 1.

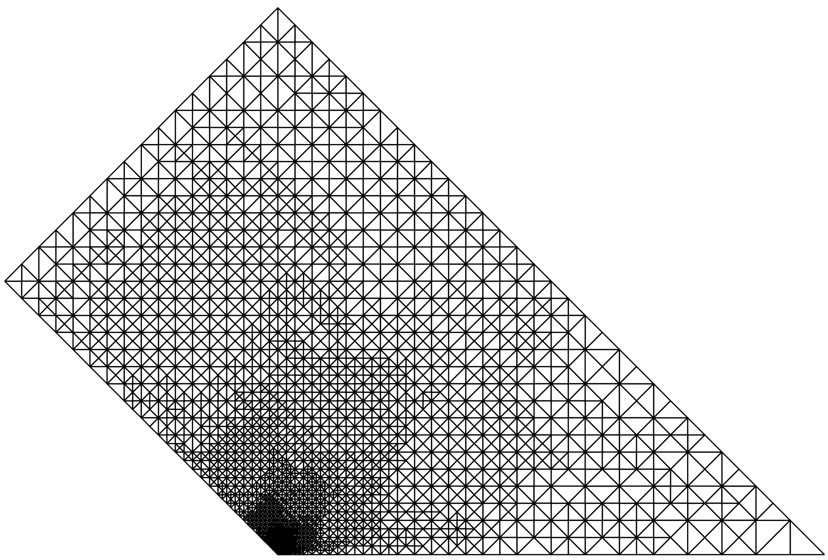

Fig. 2. Last adapted mesh in a sequence .

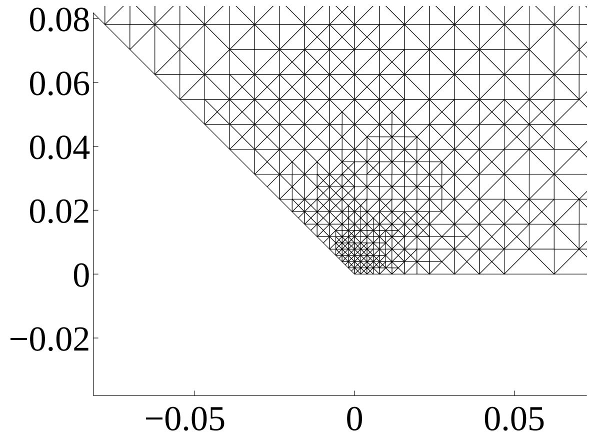

Fig. 3. Zoom at the origin of Fig. 2.

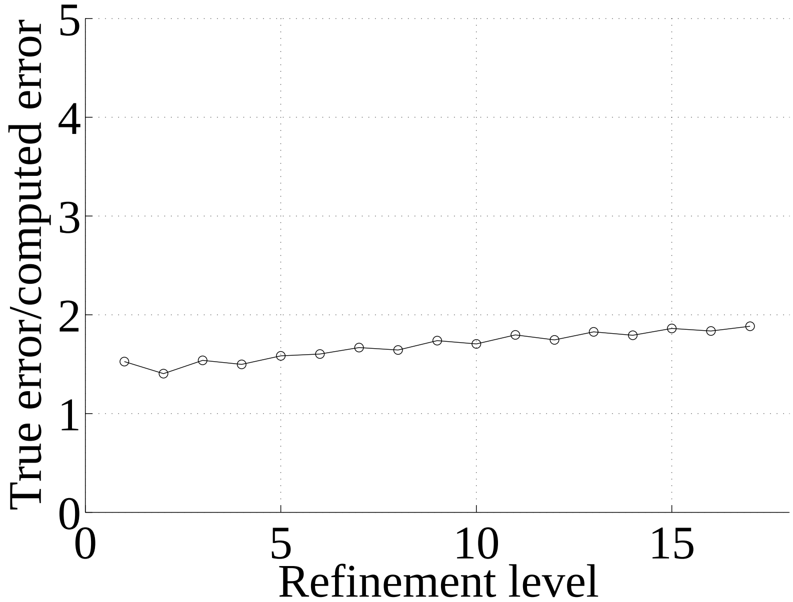

Fig. 4. Effectivity index.

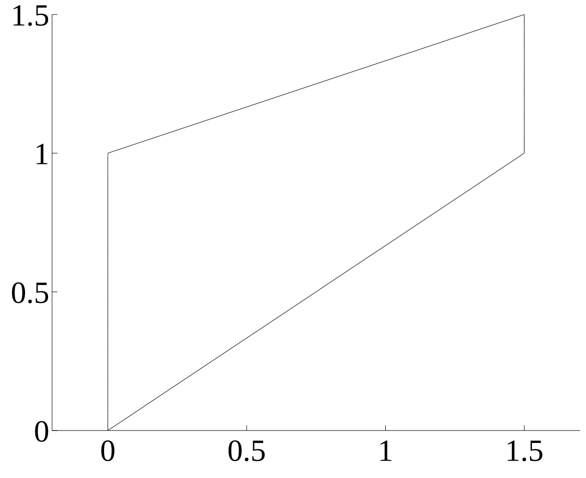

Fig. 5. Computational domain for example 2.

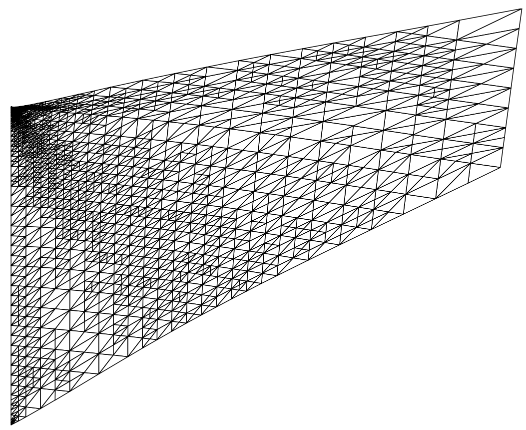

Fig. 6. Adapted mesh showing displacements.

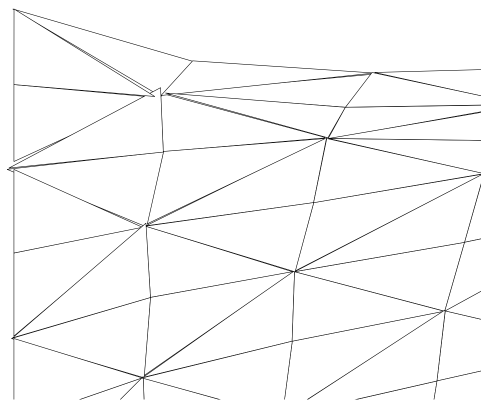

Fig. 7. Zoom of the top left displacement pattern.