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Fictitious domain finite element methods using cut elements: I. A stabilized Lagrange multiplier method

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Abstract

We propose a fictitious domain method where the mesh is cut by the boundary. The primal solution is computed only up to the boundary; the solution itself is defined also by nodes outside the domain, but the weak finite element form only involves those parts of the elements that are located inside the domain. The multipliers are defined as being element–wise constant on the whole (including the extension) of the cut elements in the mesh defining the primal variable. Inf-sup stability is obtained by penalizing the jump of the multiplier over element faces. We consider the case of a polygonal domain with possibly curved boundaries. The method has optimal convergence properties.

Key words: interior penalty, fictitious domain, finite element.

1 Introduction

Problems in fluid mechanics often include moving bodies on which boundary conditions must be applied, or geometries that are obtained from measurements or CAD manufactured. In both these cases meshing of the various objects can become excessivly time consuming and often a penalty method or a fictitious domain method is applied. Unfortunately the volume penalty method, which is simple to implement, is suboptimal, due to the fact that the interface cuts the elements destroying the approximation properties see [10]. The fictitious domain method on the other hand requires a careful construction of the multiplier space. Indeed stability considerations impose that the mesh-size of the multiplier space is at least three times the mesh size of the space for the primal variable, see Girault and Glowinski [7]. Once again when the geometry is complex this becomes leads to a nontrivial boundary meshing problem.

In this paper we propose a very simple solution to this problem in the spirit of Burman and Hansbo [5]. Indeed we define the Lagrange multipliers as the space of functions that are piecewise constant on each of the elements intersected by the boundary of the domain. Stability is then recovered by the introduction of a coarsening operator on the discrete space. Drawing from the ideas of [5] we propose to penalize the jump of the Lagrange multiplier over element faces. This approach has several advantages:

- The primal and the dual variable may be defined on the same computational mesh.
- The penalty is (weakly) consistent and easy to compute since the multiplier is distributed in the interface zone.
- The problem of curved boundaries reduces to a quadrature problem.

Indeed the only remaining difficulty of implementation is the actual integration on the boundary and on parts of elements cut by the boundary. This difficulty however is expected to arise in any optimal order fictitious domain method.

Similar ideas have recently been proposed in the framework of the extended finite element method. In [11] the authors propose a Lagrange multiplier method with an inconsistent penalty term and in [2] using inf-sup stable finite element spaces. Another stabilized approach within the XFEM framework was presented in [9], where a stabilization of the type suggested by Barbosa and Hughes [1] was advocated.

In a companion paper [4], we analyze another approach using Nitsche's method [12] to enforce the Dirichlet boundary conditions, using an interelement penalty on the normal gradient in the interface zone. That method is related to the multiplier method. However, the method proposed herein has some advantages. Firstly it only introduces a single stabilization term on the Lagrange multiplier, secondly when the domain has corners several multiplier degrees of freedom may be defined in the corner cells, giving more freedom for the approximation of the discontinuous fluxes. In the following, we shall use the notation $a \lesssim b$ as a shorthand for $a \leq Cb$, where C is a constant independent of the meshsize and of the boundary position.

2 Model problem

Let Ω be a convex polygonal bounded domain in $\mathbb{R}^2$. The Poisson equation that we propose as a model problem is given by

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega, \\ u &= g \quad \text{on } \Gamma, \end{aligned} \tag{1}$$

where Γ denotes the boundary of the domain Ω, with outward pointing normal $\boldsymbol{n}_\Gamma$; $f \in L^2(\Omega)$ and $g \in H^{\frac{3}{2}}(\Gamma)$ are given. Under these assumptions (1) has a unique solution $u \in H^1(\Omega) \cap H^2(\Omega)$ satisfying $\|u\|_{2,\Omega} \lesssim \|f\|_{0,\Omega}$.

The usual L^2-scalar product on the domain Ω will be denoted by $(\cdot,\cdot)_\Omega$ and on the boundary $\langle \cdot,\cdot \rangle_\Gamma$. We also introduce the discrete norms

$$\|\lambda\|^2_{\frac{1}{2},h,\Gamma} = \langle h^{-1}\lambda, \lambda\rangle_\Gamma \text{ and } \|\lambda\|^2_{-\frac{1}{2},h,\Gamma} = \langle h\lambda, \lambda\rangle_\Gamma,$$

where $h = h(x) > 0$ is a strictly positive weight function. Recall that there holds

$$\langle \lambda, \mu\rangle_\Gamma \leq \|\lambda\|_{-\frac{1}{2},h,\Gamma} \|\mu\|_{\frac{1}{2},h,\Gamma}. \tag{2}$$

We have the following weak formulation: find $u \in H^1_0(\Omega)$ such that

$$(\nabla u, \nabla v)_\Omega = (f, v) - (\nabla \tilde{u}, \nabla v), \quad \forall v \in H^1_0(\Omega), \tag{3}$$

where $\tilde{u} \in H^1(\Omega)$ with $\tilde{u}|_\Gamma = g$.

3 The finite element formulation

We introduce a quasiuniform triangulation $\mathcal{T}_h$, without hanging nodes, but we do not assume that the mesh $\mathcal{T}_h$ is fitted to the boundary of Ω, only that $\Omega \subset \mathcal{T}_h$ and $K \cap \Omega \neq 0$, for all $K \in \mathcal{T}_h$.

We will use the following notation for mesh related quantities. Let h_K be the diameter of K and $h = \max_{K \in \mathcal{T}_h} h_K$.

We make the following assumptions regarding the mesh and the interface.

- A1: We assume that the triangulation is non-degenerate, i.e.,

  $$h_K/\rho_K \leq C \quad \forall K \in \mathcal{T}_h$$

  where h_K is the diameter of K and ρ_K is the diameter of the largest ball contained in K.

- A2: We assume that Γ intersects each element boundary ∂K at most twice and each (open) edge at most once.
- A3: Let $\Gamma_{K,h}$ be the straight line segment connecting the points of intersection between Γ and ∂K. We assume that Γ_K is a function of length on $\Gamma_{K,h}$; in local coordinates

$$\Gamma_{K,h} = \{(\xi, \eta) : 0 < \xi < |\Gamma_{K,h}|, \eta = 0\}$$

and

$$\Gamma_K = \{(\xi, \eta) : 0 < \xi < |\Gamma_{K,h}|, \eta = \delta(\xi)\}.$$

The assumptions A2 and A3 are always fulfilled on sufficiently fine meshes, since $\partial\Omega$ has a bounded number of corners. These assumptions essentially demand that *the boundary is well resolved by the mesh.*

We have the finite element space

$$V^h = \{v \in C^0(\mathcal{T}_h) : v|_K \in P^1(K), \forall K \in \mathcal{T}_h\}.$$

We assume that the boundary Γ consists of N_Γ (possibly curved) sides, $\{\Gamma_{i,h}\}_{i=1}^{N_\Gamma}$, separated by corners. By $G_{h,i} := \{K \in \mathcal{T}_h : K \cap \Gamma_i \neq \emptyset\}$ we denote the set of elements that are intersected by the interface Γ_i. The set of all elements cut by the boundary will be denoted by $G_h = \cup_{i=1}^{n_\Gamma} G_{h,i}$. For an element $K \in G_h$, let $\Gamma_K := \Gamma \cap K$ be the part of Γ in K. The set of faces intersected by the boundary Γ will be denoted by $\mathcal{F}_G$, whereas the set of faces in $G_{h,i}$ intersected by the boundary Γ_i will be denoted by $\mathcal{F}_i$. We define a Lagrange multiplier space for each side Γ_i by assigning a constant function to each element in $G_{h,i}$.

$$W_h^i := \{v_h : \mathrm{dom}(v_h) = G_{h,i};\ v_h|_K \in P_0(K);\ \forall K \in G_{h,i}\}.$$

Let $\mathcal{W}_h := V^h \times \Pi_{i=1}^{N_\Gamma} W_h^i$ and $\lambda_h = \{\lambda_h^i\}$ such that for each i, $\lambda_h^i \in W_h^i$. The finite element discretisation then takes the form: find $(u_h, \lambda_h) \in \mathcal{W}_h$ such that

$$A[(u_h, \lambda_h), (v_h, \mu_h)] + J(\lambda_h, \mu_h) = F(v_h) \quad \forall (v_h, \mu_h) \in \mathcal{W}_h \tag{4}$$

where

$$A[(u_h, \lambda_h), (v_h, \mu_h)] := a_h(u_h, v_h) + b(\lambda_h, v_h) - b(\mu_h, u_h),$$

$$a_h(u_h, v_h) := (\nabla u_h, \nabla v_h)_\Omega,$$

$$b(\lambda_h, v_h) := \sum_{i=1}^{N_\Gamma} \langle \lambda_h^i, v_h \rangle_{\Gamma_i},$$

$$F(v_h) := (f, v_h)_\Omega + \sum_{i=1}^{N_\Gamma} \langle g, \mu_h^i \rangle_{\Gamma_i},$$

and

$$J(\lambda_h, \mu_h) := \sum_{i=1}^{N_\Gamma} j_i(\lambda_h^i, \mu_h^i),$$

where

$$j_i(\lambda_h^i, \mu_h^i) := \sum_{F \in \mathcal{F}_i} \langle \gamma h [\lambda_h^i], [\mu_h^i] \rangle_F.$$

We first state some basic properties regarding the consistency of the formulation and the continuity of the bilinear form $(A + J)[(\cdot, \cdot), (\cdot, \cdot)]$.

Lemma 1 (Galerkin orthogonality) *Let u be the solution of* (1), *u_h the solution of* (4), *and define $\lambda|_\Gamma := -\boldsymbol{n}_\Gamma \cdot \nabla u$. Then there holds*

$$A[(u - u_h, \lambda - \lambda_h), (v_h, \mu_h)] = J(\lambda_h, \mu_h).$$

PROOF. First note that by multiplying (1) by v_h and integrating by parts over Ω we have

$$(\nabla u, \nabla v_h)_\Omega - \langle \boldsymbol{n}_\Gamma \cdot \nabla u, v_h \rangle_\Gamma = (f, v_h)_\Omega. \tag{5}$$

The result now follows by combining (5) and (4) and noting that since $g \in H^{\frac{1}{2}}(\Gamma)$ we also have $\langle u, v_h \rangle_\Gamma = \langle g, v_h \rangle_\Gamma$.

For the subsequent analysis, we will use the triple norm

$$|||(u, \lambda)|||_l^2 := \|\nabla u\|^2 + \|\lambda\|_{-\frac{1}{2},h,\Gamma}^2 + \|u\|_{\frac{1}{2},h,\Gamma}^2 + lJ(\lambda, \lambda), \quad l = 0, 1.$$

This norm is well-defined for $u \in H^1(\Omega)$, $\lambda|_{\Gamma_i} =: \lambda^i \in L^2(\Gamma_i)$, when $l = 0$ and for $(u, \lambda) \in \mathcal{W}_h$ when $l = 1$. By applying the standard Cauchy-Schwarz inequality and (2) we have the following result.

Lemma 2 *Let $\eta \in H^1(\Omega)$, $\nu|_{\Gamma_i} =: \nu^i \in L^2(\Gamma_i)$. Then*

$$A[(\eta, \nu), (v_h, \mu_h)] \leq |||(\eta, \nu)|||_0 \; |||(v_h, \mu_h)|||_0.$$

4 Approximation properties

We need to show that our approximating spaces V_h and W_h^i has optimal approximation properties in norms suitable for the analysis. This follows from some minor modifications of the analysis in [8].

Let $\mathcal{T}_{\Gamma,i}$ be the subset of triangles of $\mathcal{T}$ such that

$$\mathcal{T}_{\Gamma,i} := \{T \in \mathcal{T}_h : T \cap G_{h,i} \neq \emptyset\}.$$

That is all the elements cut by the boundary and the elements sharing a face or a vertex with them. The union of all these mesh partitions will be denoted by

$$\mathcal{T}_\Gamma := \cup_{i=1}^{N_\Gamma} \mathcal{T}_{\Gamma,i}.$$

This is a subset of the mesh of width approximately $2h$. For each i, let us now regroup the elements in $\mathcal{T}_{\Gamma,i}$ in n_i patches, $\{\mathcal{P}_k^i\}_{k=1}^{n_i}$, cutting up the boundary zone in macroelements. Each patch contains a "sufficient" (but uniformly bounded) number of basis functions to construct a patch function $0 \le \varphi_k^i \le 1$ associated to each $\mathcal{P}_k^i$ that is zero on the interior patch boundary (i.e. $\partial\mathcal{P}_k^i \setminus \partial\mathcal{T}_h$), and on triangles that are members in more than one $\mathcal{T}_{\Gamma,i}$ (i.e. containing a corner of the domain or neighbouring to such a triangle) and takes the value 1 on at least one face cut by the interface. The $\mathcal{P}_k^i$ can be constructed so that, with $h_\mathcal{P} := \text{diam}(\mathcal{P}_k^i)$,

- $\exists c_1, c_2 > 0$ such that $c_1 h \le h_\mathcal{P} \le c_2 h$.
- $\exists c_1, c_2 > 0$ $c_1 h \le \displaystyle\int_{\Gamma\cap\mathcal{P}_k^i} \varphi_k^i \, \mathrm{d}s \le c_2 h$.
- $\exists c_1, c_2 > 0$ $c_1 h^{-1} \le \nabla\varphi_k^i \le c_2 h^{-1}$.

Using these patches we define a space of piecewise constant functions on each boundary Γ_i

$$X_h^i = \{x_h : x_h|_{\mathcal{P}_k^i} \in P_0(\mathcal{P}_k^i)\}.$$

We define an H^2-extension E^*u of u on $\mathcal{T}_h$ and the associated (Clément type) interpolation operator $I^* : H^1(\Omega) \to V_h$, defined by $I^*u = C_h E^*u$, with $C_h : H^1(\mathcal{T}_h) \to V_h$ the standard Clément interpolant. Then there holds, for all $v \in H^1(\Omega)$, $\|I^*v\|_{1,\mathcal{T}_h} \lesssim \|v\|_{1,\Omega}$. Following [8], we readily prove that for all $u \in H^2(\Omega)$

$$\|\nabla(u - I^*u)\|_\Omega \le \|\nabla(E^*u - I^*u)\|_{\mathcal{T}_h} \le Ch|u|_{2,\Omega} \tag{6}$$

where the constant C is independent of the position where the interface cuts the mesh. To prove approximation for the Lagrange multiplier we first immerse the mesh $\mathcal{T}_h$ in a larger subdomain $\Omega_\mathcal{T}$, e.g., a cube such that $\text{dist}(\partial\Omega_\mathcal{T}, \Gamma_i) = O(1)$ and $\text{diam}(\Omega) = O(1)$, for all i. For each Γ_i consider the problem: Find $w_{\lambda_i} \in H_0^1(\Omega_\mathcal{T}), \mu \in H^{-\frac{1}{2}}(\Gamma_i)$ such that

$$(\nabla w_{\lambda_i}, \nabla v)_{\Omega_\mathcal{T}} + (\mu, v)_{\Gamma_i} = 0$$
$$(w_{\lambda_i}, y)_{\Gamma_i} = (\lambda_i, y)_{\Gamma_i}$$

for all $v \in H_0^1(\Omega_\mathcal{T})$, $y \in H^{-\frac{1}{2}}(\Gamma_i)$. Since $\lambda \in H^{\frac{1}{2}}(\Gamma_i)$ this problem is well-posed by the Babuska-Neças-Brezzi condition for saddle point problems (see [6, Theorem 2.34, page 100]), and we have the a priori estimate

$$\|w_{\lambda_i}\|_{1,\Omega_\mathcal{T}} + \|\mu\|_{-\frac{1}{2},\Gamma_i} \lesssim \|\lambda_i\|_{\frac{1}{2},\Gamma_i}.$$

For each $G_{h,i}$ define the local projection on piecewise constants $\pi_{0,i} w_{\lambda_i}$ such that

$$\int_K \pi_{0,i} w_{\lambda_i}|_K \, \mathrm{d}x = \int_K w_{\lambda_i} \, \mathrm{d}x$$

for all $K \in G_{h,i}$. Then the following approximation results hold

Lemma 3 *Let $\lambda_i \in H^{\frac{1}{2}}(\Gamma_i)$ then*

$$\|\lambda_i - \pi_{0,i} w_{\lambda_i}\|_{-\frac{1}{2},h,\Gamma_i} \lesssim h\|\lambda\|_{\frac{1}{2},\Gamma_i} \tag{7}$$

and

$$j_i(\pi_{0,i} w_{\lambda_i}, \pi_{0,i} w_{\lambda_i})^{\frac{1}{2}} \lesssim h\|\lambda\|_{\frac{1}{2},\Gamma_i}. \tag{8}$$

PROOF. Consider one element K in $G_{h,i}$. Using that $w_{\lambda_i} = \lambda_i$ on Γ_I and by the trace inequality

$$\|v\|^2_{L_2(\Gamma_i)} \lesssim h^{-1} \|v\|^2_{L_2(K)} + h \|\nabla v\|^2_{L_2(K)}, \quad \forall v \in H^1(K), \tag{9}$$

we have

$$\begin{aligned} \|\lambda_i - \pi_{0,i} w_{\lambda_i}\|_{-\frac{1}{2},h,\Gamma_i} &= \|w_{\lambda_i} - \pi_{0,i} w_{\lambda_i}\|_{-\frac{1}{2},h,\Gamma_i} \\ &\lesssim \left(\sum_{K \in G_{h,i}} \|w_{\lambda_i} - \pi_{0,i} w_{\lambda_i}\|^2_{0,K} + h^2 \|\nabla w_{\lambda_i}\|^2_K \right)^{\frac{1}{2}}. \end{aligned}$$

The claim follows since $\|w_{\lambda_i} - \pi_{0,i} w_{\lambda_i}\|_{0,K} \lesssim h\|\nabla w_{\lambda_i}\|_K$.

Since $w_{\lambda_i} \in H^1(\mathcal{T}_h)$ there holds $j_i(w_{\lambda_i}, w_{\lambda_i}) = 0$. It then follows in a similar fashion as above that

$$\begin{aligned} j_i(\pi_0 w_{\lambda_i}, \pi_0 w_{\lambda_i}) &= j_i(\pi_0 w_{\lambda_i} - w_{\lambda_i}, \pi_0 w_{\lambda_i} - w_{\lambda_i}) \\ &\lesssim \left(\sum_{K \in G_{h,i}} \|w_{\lambda_i} - \pi_{0,i} w_{\lambda_i}\|^2_{0,K} + h^2 \|\nabla w_{\lambda_i}\|^2_K \right)^{\frac{1}{2}} \\ &\lesssim \left(\sum_{K \in G_{h,i}} h^2 \|\nabla w_{\lambda_i}\|^2_K \right)^{\frac{1}{2}}. \end{aligned}$$

Collecting the result (6), Lemma 3, and the trace theorem (cf., e.g., [3]), we have proven the following approximation result.

Lemma 4 *There holds:*

$$|||(u - I^* u, \lambda - \pi_0 w_\lambda)|||_0 + J(\pi_0 w_\lambda, \pi_0 w_\lambda) \lesssim h|u|_{H^2(\Omega)}.$$

The key result here is the following simple discrete approximation result on the patches $\mathcal{P}_k^i$

Lemma 5

$$\inf_{c_h \in X_h^i} \sum_{\mathcal{P}_k^i} \|h^{\frac{1}{2}}(\lambda_h^i - c_h)\|^2_{\mathcal{P}_k^i \cap \Gamma} \lesssim j_i(\lambda_h^i, \lambda_h^i)$$

PROOF. The proof follows by mapping $\mathcal{P}_k^i \cap \Gamma$ to the unit interval. We note that the jump of the Lagrange multiplier over element edges is a norm on the space of piecewise constants with average value zero on the interval. We conclude by scaling back to physical space and using mesh regularity to trade integrations along the interface for integrations along element faces in the penalty term.

Theorem 6 *The following inf-sup condition holds for the Lagrange-multiplier method: For all (u_h, λ_h) there holds*

$$c|||(u_h, \lambda_h)|||_1 \leq \sup_{(v_h, \mu_h) \in \mathcal{W}_h} \frac{A[(u_h, \lambda_h), (v_h, \mu_h)] + J(\lambda_h, \mu_h)}{|||(v_h, \mu_h)|||_1}.$$

PROOF. First test with $v_h = u_h$ and $\mu_h = \lambda_h$ in (4) to obtain

$$\|\nabla u_h\|^2 + J(\lambda_h, \lambda_h) = A[(u_h, \lambda_h), (u_h, \lambda_h)] + J(\lambda_h, \lambda_h).$$

Next, take $v_h = 0$ and $\mu_h = \mu_h^*$ such that $\mu_h^*|_K = -h_{\mathcal{P}}^{-1} \pi_0 u_h$ for $K \subset \mathcal{P}_k^i$ where $\tilde{\pi}_0 u_h \in X_h^i$ is the patchwise projection defined by

$$\tilde{\pi}_{0,i} u_h|_{\mathcal{P}_k^i} = \frac{1}{|\Gamma_i \cap \mathcal{P}_k^i|} \int_{\Gamma_i \cap \mathcal{P}_k^i} u_h \mathrm{d}s$$

where we recall that $\mathcal{P}_k^i$ denotes the patches introduced earlier. It follows from (4) that

$$\sum_i \|h_{\mathcal{P}}^{-\frac{1}{2}} \pi_0 u_h\|^2_{\Gamma_i} + J(\lambda_h, \mu_h^*) = A[(u_h, \lambda_h), (0, \mu_h^*)] + J(\lambda_h, \mu_h^*).$$

Since the $\mathcal{P}_k^i$ all have size comparable to h we may deduce that

$$J(\mu_h^*, \mu_h^*) \lesssim \sum_i \|h^{-\frac{1}{2}} \pi_0 u_h\|^2_{\Gamma_i} \tag{10}$$

by first applying trace inequalities on each K and then an inverse trace inequality on each of the $\mathcal{P}_k^i$. Similarly,

$$c_1 \sum_i \|h_{\mathcal{P}}^{-\frac{1}{2}} \pi_0 u_h\|_{\Gamma_i}^2 \leq \sum_i \|h^{-\frac{1}{2}} \pi_0 u_h\|_{\Gamma_i}^2.$$

We now use that

$$J(\lambda_h, \mu_h^*) \leq J(\lambda_h, \lambda_h)^{\frac{1}{2}} J(\mu_h^*, \mu_h^*)^{\frac{1}{2}} \leq \frac{1}{\epsilon} J(\lambda_h, \lambda_h) + \frac{\epsilon}{4} J(\mu_h^*, \mu_h^*)$$

for $\epsilon \in \mathbb{R}^+$ to deduce that

$$(c_1 - C\epsilon) \sum_i \|h^{-\frac{1}{2}} \pi_0 u_h\|_{\Gamma_i}^2 - \frac{1}{\epsilon} J(\lambda_h, \lambda_h) \leq A[(u_h, v_h), (0, \mu_h^*)] + J(\lambda_h, \mu_h^*).$$

Since

$$\|h^{-\frac{1}{2}} u_h\|_{\Gamma_i}^2 \lesssim \|h^{-\frac{1}{2}} \pi_0 u_h\|^2 + \|\nabla u_h\|_{\mathcal{T}_{\Gamma,i} \cap \Omega}^2$$

we conclude that

$$C_1 \sum_i \|h^{-\frac{1}{2}} u_h\|_{\Gamma_i}^2 - \frac{1}{\epsilon} J(\lambda_h, \lambda_h) - C_2 \|\nabla u_h\|_{\Omega}^2 \leq A[(u_h, v_h), (0, \mu_h^*)].$$

for some constants C_1 and C_2, choosing ϵ small enough. Finally, to control λ_h we use the properties of the $\mathcal{P}_k^i$ and the corresponding φ_k^i to construct a function such that

I $\xi_h|_{\Omega \setminus \mathcal{T}_\Gamma} = 0$;

II $\int_{\Gamma \cap \mathcal{P}_k^i} \xi_h \mathrm{d}s = \int_{\Gamma \cap \mathcal{P}_k^i} h_{\mathcal{P}} \lambda_h^i \mathrm{d}s$;

III $\xi_h|_{K_c} = 0$ if K_c denotes a triangle containing a corner of the domain.

It follows from standard trace and inverse inequalities that

$$|||\xi_h||| \leq \sum_i \left(\|h^{-1} \xi_h\|_{\mathcal{T}_{\Gamma,i}} + \|\nabla \xi_h\|_{\mathcal{T}_{\Gamma,i}} \right) \lesssim \|\nabla \xi_h\|_{\mathcal{T}_\Gamma}. \tag{11}$$

Since ξ_h by construction is zero on the inner boundary of $\mathcal{T}_\Gamma$ we have

$$\|\nabla \xi_h\|_{\mathcal{T}_{\Gamma_i}} \lesssim \|h^{-\frac{1}{2}} \xi_h\|_{\Gamma_i} \lesssim \|h^{\frac{1}{2}} \lambda_h^i\|_{0,\Gamma_i}. \tag{12}$$

By property II, we can write

$$\begin{aligned} \langle \lambda_h^i, \xi_h \rangle_{\Gamma_i} &= \langle \pi_0 \lambda_h^i, \xi_h \rangle_{\Gamma_i} + \langle \lambda_h^i - \pi_0 \lambda_h^i, \xi_h \rangle_{\Gamma_i} \\ &= \|h_{\mathcal{P}}^{\frac{1}{2}} \pi_0 \lambda_h^i\|_{\Gamma_i}^2 + \langle \lambda_h^i - \pi_0 \lambda_h^i, \xi_h \rangle_{\Gamma_i}, \\ &\geq c_1 \|h^{\frac{1}{2}} \pi_0 \lambda_h^i\|_{\Gamma_i}^2 + \langle \lambda_h^i - \pi_0 \lambda_h^i, \xi_h \rangle_{\Gamma_i}, \end{aligned}$$

We note that

$$|\langle \lambda_h^i - \pi_0 \lambda_h^i, \xi_h \rangle_{\Gamma_i}| \le \|\lambda_h^i - \pi_0 \lambda_h^i\|_{-\frac{1}{2},h,\Gamma_i} \|\xi_h\|_{\frac{1}{2},h,\Gamma_i}$$

(cf. (2)),

$$\sum_i \|\lambda_h^i - \pi_0 \lambda_h^i\|^2_{-\frac{1}{2},h,\Gamma_i} \lesssim J(\lambda_h, \lambda_h)$$

(by Lemma 5), and

$$\|\xi_h\|_{\frac{1}{2},h,\Gamma_i} \lesssim \|h^{\frac{1}{2}} \lambda_h^i\|_{0,\Gamma_i}$$

(cf. (12)). We thus have

$$\begin{aligned} a_h(u_h, \xi_h) + \langle \lambda_h, \xi_h \rangle_\Gamma \ge & -\frac{1}{\epsilon}\|\nabla u_h\|^2_\Omega - \frac{\epsilon}{4}\|\nabla \xi_h\|^2_{\mathcal{T}_\Gamma} \\ & + \sum_i (\|h^{\frac{1}{2}} \pi_0 \lambda_h^i\|^2_{\Gamma_i} + \langle \lambda_h^i - \pi_0 \lambda_h^i, \xi_h \rangle_{\Gamma_i}) \\ \ge & \left(1 - \frac{c\epsilon}{2}\right) \sum_i \|h^{\frac{1}{2}} \lambda_h^i\|^2 - \frac{1}{\epsilon}\|\nabla u_h\|^2_\Omega - \frac{C}{\epsilon} J(\lambda_h, \lambda_h). \end{aligned}$$

It now follows that by a judicious choice of coefficients a_1 and a_2 we have

$$|||(u_h, \lambda_h)|||_1 \le A[(u_h, \lambda_h), (u_h + a_1 \xi_h, \lambda_h + a_2 \mu_h^*)].$$

The claim then follows after proving the stability estimate

$$|||(w_h, \nu_h)|||_1 \lesssim |||(u_h, \lambda_h)|||_1$$

where $w_h = u_h + a_1 \xi_h$ and $\nu_h = \lambda_h + a_2 \mu_h^*$. This is a consequence of the triangle inequality and the estimates (11), (12) and (10).

5 A priori error estimates

Since $u \in H^2(\Omega)$ there holds that $\lambda|_{\Gamma_i} = -\boldsymbol{n}_{\Gamma_i} \cdot \nabla u \in H^{\frac{1}{2}}(\Gamma_i)$.

Theorem 7 *Let (u_h, λ_h) be the solution of the system (4). Then for all $(v_h, \nu_h) \in \mathcal{W}_h$*

$$|||(u - u_h, \lambda - \lambda_h)|||_0 + J(\lambda_h, \lambda_h)^{\frac{1}{2}} \lesssim |||(u - v_h, \lambda - \nu_h)|||_0 + J(\nu_h, \nu_h)^{\frac{1}{2}}$$

PROOF. By the triangle inequality,

$$\begin{aligned} |||(u - u_h, \lambda - \lambda_h)|||_0 + J(\lambda_h, \lambda_h)^{\frac{1}{2}} \le & |||(u - v_h, \lambda - \nu_h)|||_0 \\ & + |||(u_h - v_h, \lambda_h - \nu_h)|||_0 \\ & + J(\lambda_h - \nu_h, \lambda_h - \nu_h)^{\frac{1}{2}} + J(\nu_h, \nu_h)^{\frac{1}{2}}, \end{aligned}$$

and we only need to consider the discrete quantities $e_{h,u} = v_h - u_h$ and $e_{h,\lambda} = \nu_h - \lambda_h$. It follows from Theorem 6 that

$$c|||(e_{h,u}, e_{h,\lambda})|||_1 \leq \sup_{(v_h,\mu_h)\in\mathcal{W}_h} \frac{A[(e_{h,u}, e_{h,\lambda}), (v_h, \mu_h)] + J(e_{h,\lambda}, \mu_h)}{|||(v_h, \mu_h)|||_1}.$$

By Lemma 1 it now follows that

$$c|||(e_{h,u}, e_{h,\lambda})|||_1 \leq \sup_{(v_h,\mu_h)\in\mathcal{W}_h} \frac{A[(u - v_h, \lambda - \nu_h), (v_h, \mu_h)] + J(\nu_h, \mu_h)}{|||(v_h, \mu_h)|||_1}.$$

We may conclude using the continuity of Lemma 2 and the Cauchy-Schwarz inequality $J(\nu_h, \mu_h) \leq J(\nu_h, \nu_h)^{\frac{1}{2}} J(\mu_h, \mu_h)^{\frac{1}{2}}$.

Corollary 8 *The following energy norm error estimate holds:*

$$|||(u - u_h, \lambda - \lambda_h)|||_0 + J(\lambda_h, \lambda_h)^{\frac{1}{2}} \lesssim h|u|_{H^2(\Omega)}.$$

PROOF. Immediate by choosing $v_h = I^* u$ and $\nu_h = \pi_0 w_\lambda$ in Theorem 7 and applying the approximation result Lemma 4.

For error estimates in L_2–norm, we have the following result.

Lemma 9 *There holds*

$$\|u - u_h\|_{L^2(\Omega)} \lesssim h^2 |u|_{H^2(\Omega)}.$$

PROOF. Consider the dual problem

$$\begin{aligned} -\Delta z &= u - u_h \text{ in } \Omega \\ z &= 0 \text{ on } \partial\Omega. \end{aligned} \tag{13}$$

Under the assumptions on Ω we have

$$\|z\|_{H^2(\Omega)} \lesssim \|u - u_h\|_\Omega. \tag{14}$$

We will use the notation $\lambda_z := -\boldsymbol{n}_\Gamma \cdot \nabla z$ on the boundary Γ. Multiplying the first equation of equation (13) by $u - u_h$ and integrating over Ω yields

$$\|u - u_h\|_\Omega = (\nabla(u - u_h), \nabla z) + \langle u - u_h, \lambda_z \rangle_\Gamma .$$

We now apply the Galerkin orthogonality (1) with $v_h = I^* z$ and $\mu_h = \pi_0 w_{\lambda_z}$ to get

$$\|u - u_h\|_\Omega^2 = A[(u - u_h, \lambda - \lambda_h), (z - I^* z, \lambda_z - \pi_0 w_{\lambda_z})] - J(\lambda_h, \pi_0 w_{\lambda_z}).$$

Proceeding by the continuity of Lemma 2 and the Cauchy-Schwarz inequality applied to $J(\cdot,\cdot)$ leads to

$$\begin{aligned}\|u-u_h\|_\Omega^2 \leq {} & |||(u-u_h,\lambda-\lambda_h)|||_0\ |||(z-I^*z,\lambda_z-\pi_0 w_{\lambda_z})|||_0\\ & +J(\lambda_h,\lambda_h)^{\frac{1}{2}}J(\pi_0 w_{\lambda_z},\pi_0 w_{\lambda_z})^{\frac{1}{2}}.\end{aligned}$$

We now apply the approximation property of Lemma 4, the trace theorem for λ_z, and the stability (14).

$$\begin{aligned}\|u-u_h\|_\Omega^2 &\lesssim \left(|||(u-u_h,\lambda-\lambda_h)|||_0+J(\lambda_h,\lambda_h)^{1/2}\right)h|z|_{H^2(\Omega)}\\ &\lesssim \left(|||(u-u_h,\lambda-\lambda_h)|||_0+J(\lambda_h,\lambda_h)^{1/2}\right)h\|u-u_h\|_\Omega.\end{aligned}$$

We conclude by applying Lemma 8.

6 Numerical results

6.1 A problem with smooth solution

We consider a radially symmetric solution on a disc with radius $r_0 = 0.5$. With r the length of the radius vector, we use $f = r$ to obtain the exact solution $u = u = (r_0^3 - r^3)/9$. The stabilization parameter was set to $\gamma = 10$. In Fig. 1 we show the obtained convergence rates in $L_2(\Omega)-$ and $H^1(\Omega)-$norms, which are optimal. An elevation of the solution is given in Fig. 2, and the error interpolated on the mesh is shown in Fig. 3. We note that the largest contribution to the error occurs at the boundary, as expected. Finally, in Fig. 4 we show the effect of choosing $\gamma = 0$. The solution is then unstable due to violation of the inf-sup condition. Note that the interface is not symmetrically placed in the mesh.

6.2 A problem with non-smooth solution

For our second example we choose Ω to be an L-shaped domain

$$\Omega := \{[-1,1]\times[-1,1]\}\setminus\{[0,1]\times[-1,0]\} \tag{15}$$

with exact solution

$$u(r,\theta) := r^{2/3}\sin(2\theta/3),$$

where (r,θ) denote the polar coordinates. The fictitious boundary is put at $x \in [0,1]$, $y = 0$; the problem is solved on half the domain using a symmetry line (Neumann bounday conditions) at $x = -y$. Dirichlet data are enforced at $x = 1$ and at $y = 1$. For this kind of domain, $r^{2/3}$ is a typical singularity, located in the origin.

For this problem, there holds

$$u \in H^1(\Omega), \quad u \notin H^2(\Omega),$$

therefore we cannot expect linear convergence in the energy norm or quadratic convergence in the L_2-norm. It is possible to show that $u \in H^{5/3-\epsilon}(\Omega)$, $\forall \epsilon > 0$, therefore the convergence rate $2/3 - \epsilon$ in the H^1_0-norm, is optimal. The theoretical convergence rate agrees with the numerical convergence rate ≈ 0.66, as shown in Figure 5. In Fig. 6, and the error interpolated on the mesh is shown in Fig. 7. We note that the largest contribution to the error occurs at close to the singularity.

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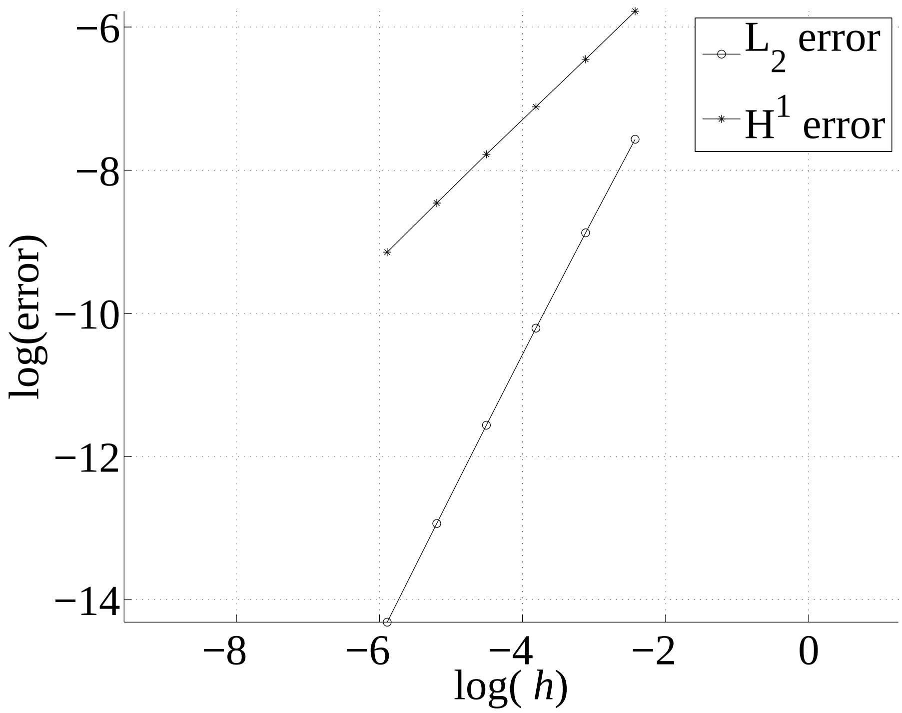

Fig. 1. Convergence in the smooth case

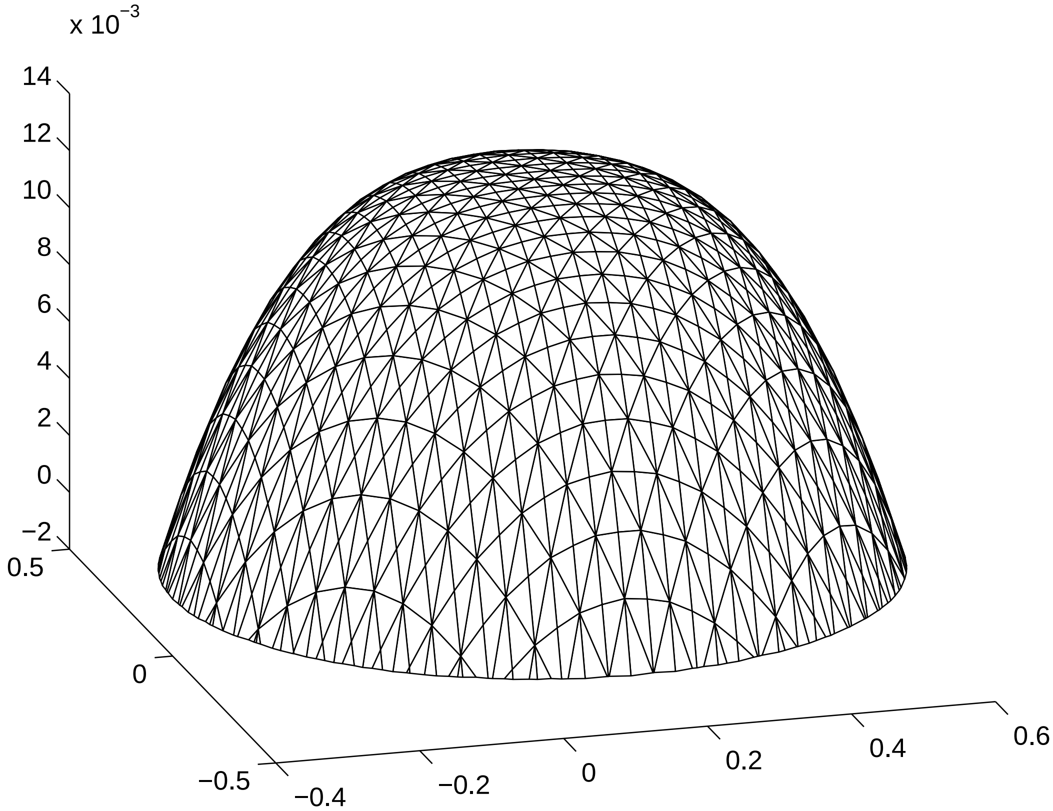

Fig. 2. Elevation of the smooth solution

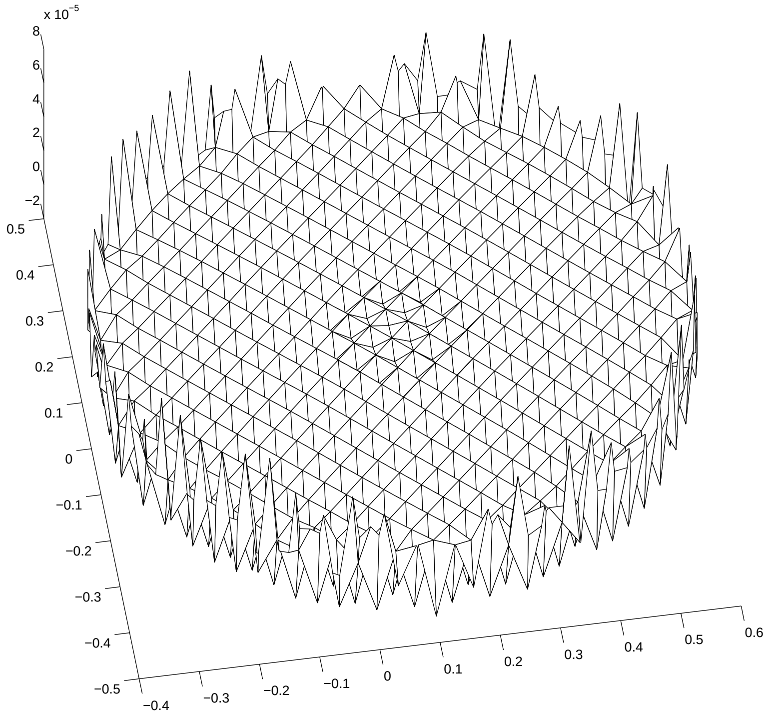

Fig. 3. Elevation of the smooth (interpolated) error

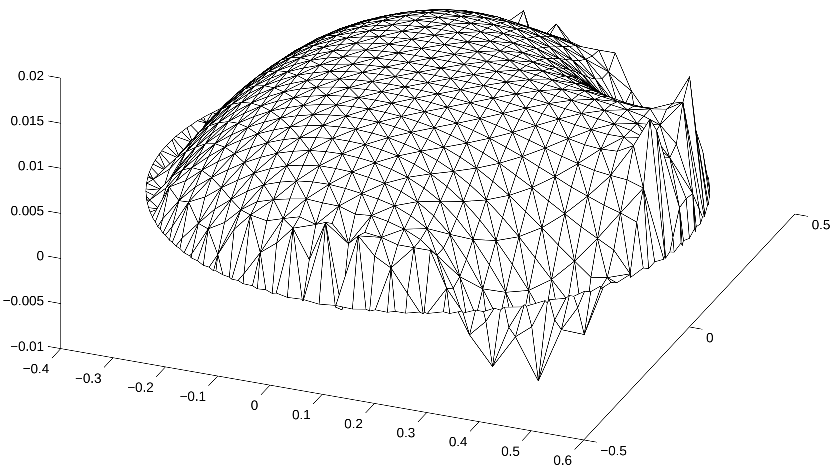

Fig. 4. Unstable solution for $\gamma = 0$

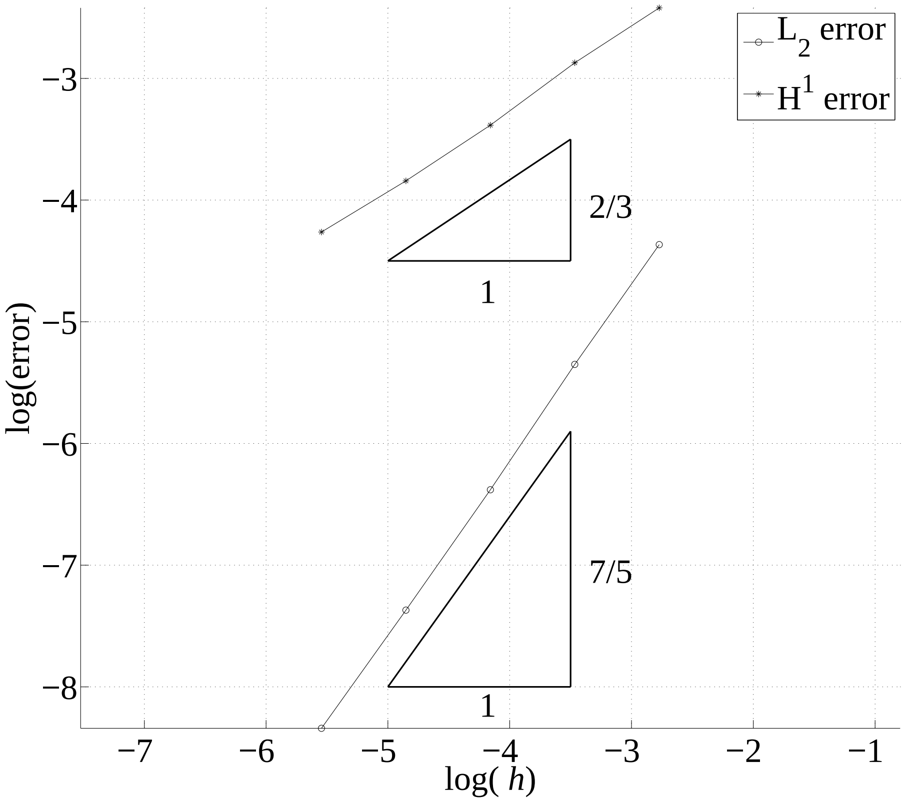

Fig. 5. Convergence in the non-smooth case

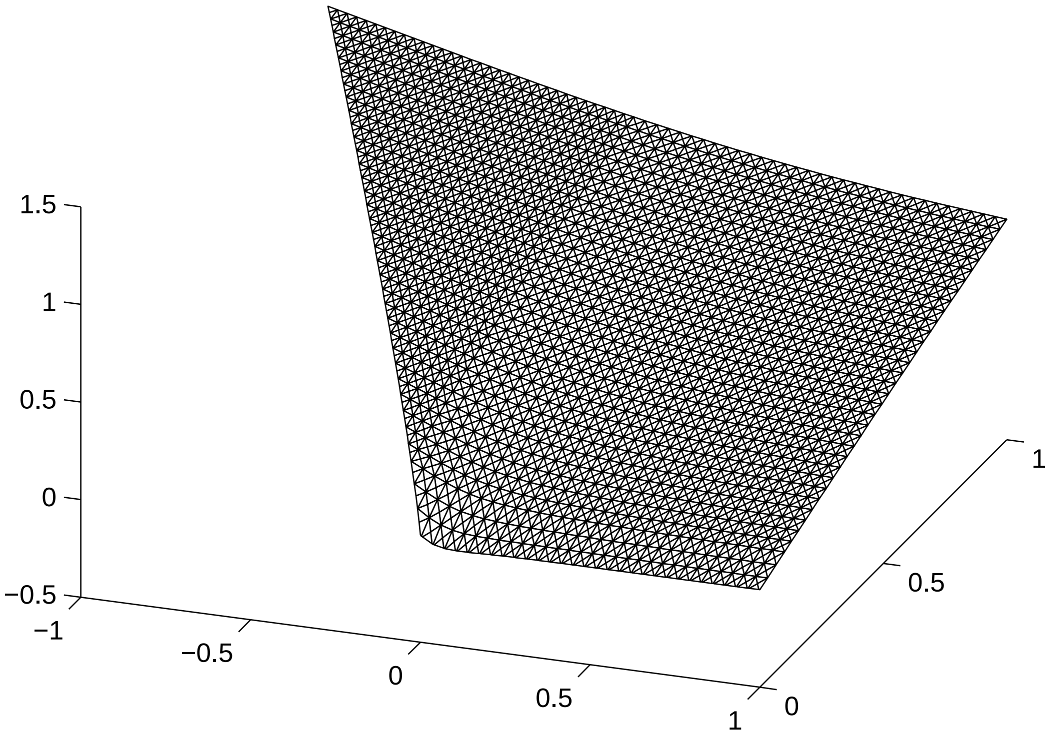

Fig. 6. Elevation of the non-smooth solution

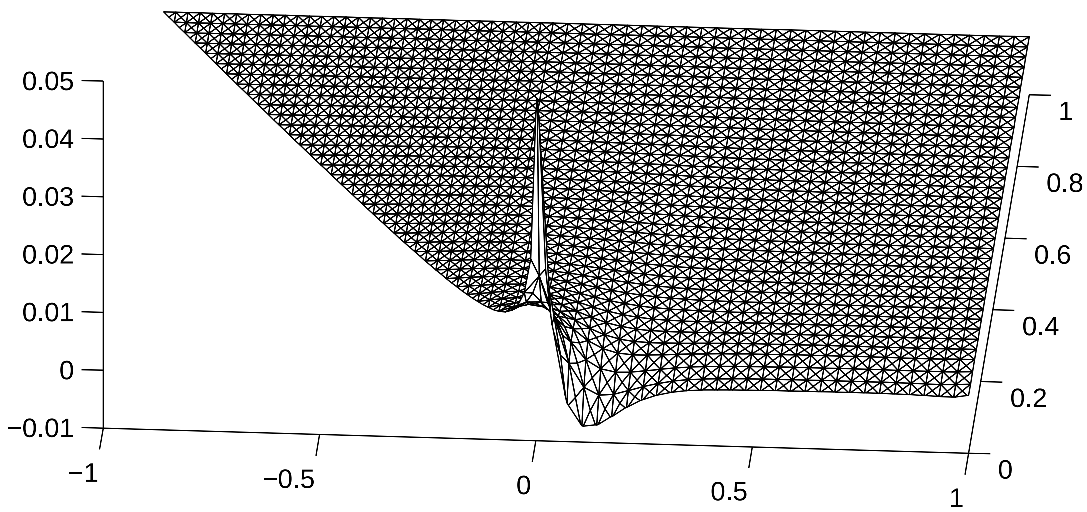

Fig. 7. Elevation of the non-smooth (interpolated) error