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Convergence of a mixed discontinuous Galerkin and finite volume scheme for the 3 dimensional Vlasov–Poisson–Fokker–Planck system

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Abstract We construct a numerical scheme for the multi-dimensional Vlasov-Poisson-Fokker-Planck system based on a combined finite volume method for the Poisson equation in spatial domain and streamline-diffusion/ discontinuous Galerkin finite element in time, phase-space variables for the Vlasov-Fokker-Planck part. We derive error estimates with optimal convergence rates.

1 Introduction

In this note we study the approximate solution for the deterministic multi-dimensional Vlasov–Poisson–Fokker-Planck (VPFP) system described below: Given the parameters β and σ and the initial distribution of particle density $f_0(x,v)$, $(x,v) \in \Omega_x \times \mathbf{R}^d \subset \mathbf{R}^d \times \mathbf{R}^d$, $d = 1,2,3$ we seek the evolution of charged plasma particles (ions and electrons), at time t, with a phase space density $f(x,v,t)$ satisfying

$$\begin{cases} \partial_t f + v \cdot \nabla_x f - \nabla_x \varphi \cdot \nabla_v f - \mathrm{div}_v(\beta v f) - \sigma \Delta_v f = S, & \text{in } \Omega \times [0,T], \\ f(x,v,0) = f_0(x,v), & \text{in } \Omega = \mathbf{R}^d \times \mathbf{R}^d, \\ -\Delta_x \varphi = \displaystyle\int_{\mathbf{R}^d} f(x,v,t)\, dv, & \text{in } \mathbf{R}^d \times [0,T], \end{cases} \tag{1}$$

where $\cdot$ denotes the scalar product and S is a source. To construct numerical methods we shall restrict both space and velocity variables x and v to be in bounded domains Ω_x and Ω_v, and provide the equation with a Dirichlet type, inflow boundary conditions. To solve problem (1) the idea is to split the equation system and

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separate Poisson equation from the Vlasov-Fokker-Planck part which are coupled with the potential φ. Thus we reformulate the problem (1) as follows: Given the initial data $f_0(x,v)$, $(x,v) \in \Omega := \Omega_x \times \Omega_v \subset \mathbf{R}^d \times \mathbf{R}^d$, $d = 1,2,3$ find the density function $f(x,v,t)$ in the Dirichlet initial-boundary value problem for the Vlasov-Fokker-Planck equation

$$
\text{(P1)} \begin{cases} \partial_t f + v \cdot \nabla_x f - \nabla_x \varphi \cdot \nabla_v f - \mathrm{div}_v(\beta v f) - \sigma \Delta_v f = S, & \text{in } \Omega \times [0,T], \\ f(x,v,0) = f_0(x,v), & \text{in } \Omega_x \times \Omega_v, \\ f(x,v,t) = 0, & \text{on } \Gamma_G^- \times [0,T], \end{cases} \tag{2}
$$

where $G := (v, -\nabla_x \varphi)$, $\Gamma^- G := \{(x,v) \in \Gamma := \partial\Omega | G \cdot \mathbf{n} < 0\}$, is the inflow boundary and the potential φ satisfies the following problem for the Poisson equation:

$$
\text{(P2)} \qquad \begin{cases} -\Delta_x \varphi = \int_{\Omega_v} f(x,v,t)\, dv, & \text{in } \Omega_x \times [0,T], \\ |\nabla_x \varphi(x,t)| = 0, & \text{on } \partial\Omega_x \times [0,T]. \end{cases} \tag{3}
$$

Now we can solve problem (P2) replacing f by a given function g. Then inserting the corresponding solution φ in (P1) we obtain an equation for f, viz (2). In this way we link the solution f to the given data function g, as say $f = \Lambda[g]$. Now a solution f for the Vlasov-Poisson-Fokker-Planck system is a fixed point of the operator Λ, i.e. $f = \Lambda[f]$, which is obtained by a procedure using a Schauder fixed point theorem. For the discrete version this step can, roughly speaking, be repeated using a Brouwer type fixed point argument, see, e.g. [1] and the reference therein. Positivity, existence, uniqueness and regularity of the solution for the continuous problem are given, e.g. [5]. These results rely on the positivity and boundedness requirement for the second phase-space moment of the initial data: $f_0 \in L_\infty(\mathbf{R}^6) \geq 0$ and $\int_{\mathbf{R}^6}(1 + |x|^2 + |v|^2) f_0\, dxdv < \infty$. Further analytic approaches are given , e.g. by Horst in [11]. For the general mathematical study of equations of this type we refer to studies by J. L. Lions [14]. and Baouendi and Grisvard [4].

Conventional numerical methods for the Vlasov-Poisson and related equations have been dominated by the particle methods studied, e.g. by Cottet and Raviart [7]; Ganguly, Lee and Victory [9]; and Wollman, Ozizmir and Narasimhan [16]. Filbet has studied a 1-dimensional finite volume scheme for the Vlasov-Poisson [8].

Our study of the VPFP system is, mainly, devoted (see also [1]- [3]) to the construction and analysis of finite element schemes. In this note, however, we study the Poisson part using a finite volume approach. To this end we consider the study of a three dimensional VPFP model ($\Omega_x \subset \mathbf{R}^3$, $\Omega_v \subset \mathbf{R}^3$). As for the discontinuous Galerkin approximation relevant in the VPFP estimates we also refer to the articles by Brezzi, Manzini, Marini and Russo for elliptic problem in [6], and Johnson and Saranen for the Euler and Navier-Stokes equations in [12].

In this note, we give only sketch of the proofs. They can be completed following the techniques in [15] for finite volume, and [1]- [3] and [12] in the finite element cases.

2 The finite Volume Method for Poisson Equation in 3D

The cell-center finite volume (FV) scheme for problem (P2) is given by

$$-\nabla^2_{\mathbf{x}}\varphi = \rho, \quad \text{in} \quad \Omega_{\mathbf{x}} = (0,1)\times(0,1)\times(0,1) \quad |\nabla_x \varphi| = 0, \quad \text{on} \quad \partial\Omega_{\mathbf{x}}, \qquad (4)$$

where $\rho = \int_{\Omega_v} f(x,v,t)\,dv$. Existence uniqueness and regularity studies for (4) are extensions of two-dimensional results in [10]: $\rho \in H^{-1}(\Omega_{\mathbf{x}})$ implies that $\exists!\ \varphi \in H^1_0(\Omega_{\mathbf{x}})$, and for $\rho \in H^s(\Omega_{\mathbf{x}})$, with $-1 \le s <$, $r \neq \pm 1/2$, $\varphi \in H^{s+2}(\Omega_{\mathbf{x}})$. The finite volume scheme can be described as: exploiting divergence from the differential equation, integrating over disjoint "volumes" and using Gauss divergent theorem to convert volume-integrals to counter-integrals, and then discretizing to obtain the approximate solution φ_h, with the mesh size h. Here, the finite volume method is defined on Cartesian product of non-uniform meshes as Petrov-Galerkin method using piecewise trilinear trial functions on *finite element* mesh and piecewise constant test functions on the dual box mesh. The main result of this section is,

Theorem 1. *Then, for $1/2 < s \le 2$ the, respective, optimal finite volume error estimates for general non-uniform and quasi-uniform meshes are given by*

$$\|\varphi - \varphi_h\|_{1,h} \le Ch^s|\varphi|_{H^{s+1}}, \quad \textit{and} \quad \|\varphi - \varphi_h\|_\infty \le Ch^s|\log h||\varphi|_{H^{s+1}}. \qquad (5)$$

whereas the corresponding finite element estimates can be read from the theorem

Theorem 2. *a) For the finite element solution of the Poisson problem (9) with a qusiuniform triangulation we have the error estimate:*

$$\|\varphi - \varphi_h\|_{1,\infty} \le Ch^r|\log h| \times \|\varphi\|_{r+1,\infty}, \qquad r \le 2$$

b) $\forall \varepsilon \in (0,1)$ small, $\exists C_\varepsilon$ such that $\|\varphi - \varphi_h\|_{1,\infty} \ge C_\varepsilon h^{r-\varepsilon}|\log h|$, cf [13].

Note that $s = 2$ in Theorem 1 corresponds to $r = 1$ in Theorem 2, where the two L_∞ estimates are coinciding. To derive the finite volume formula we consider the Cartesian mesh

$$\begin{aligned}
I^h_x &:= \{x_i :\ i = 0,1,\dots,I; \quad x_0 = 0, \quad x_i - x_{i-1} = h_i; \quad x_I = 1\},\\
I^h_y &:= \{y_j :\ j = 0,1,\dots,J; \quad y_0 = 0, \quad y_j - y_{j-1} = k_j; \quad y_J = 1\},\\
I^\ell_z &:= \{z_n :\ n = 0,1,\dots,N; \quad z_0 = 0, \quad z_n - z_{n-1} = \ell_n; \quad z_N = 1\}.
\end{aligned}$$

With each (x_i, y_j, z_n) we associate the finite volume box:

$$\omega_{lmn} = \left(x_{i-1/2}, x_{i+1/2}\right) \times \left(y_{j-1/2}, y_{j+1/2}\right) \times \left(z_{n-1/2}, z_{n+1/2}\right).$$

Now we choose *central finite volume boxes* inside each 27-points stencil element:

$$\begin{cases}
x_{i-1/2} = x_i - h_i/2, & x_{i+1/2} = x_i - h_{(i+1)}/2, & \hbar_i = \frac{h_i + h_{i+1}}{2}\\
y_{j-1/2} = y_j - k_j/2, & y_{j+1/2} = y_j - k_{(j+1)}/2, & \bar{k}_j = \frac{k_j + k_{j+1}}{2},\\
z_{n-1/2} = z_n - \ell_n/2, & z_{n+1/2} = z_n - \ell_{(n+1)}/2, & \bar{\ell}_n = \frac{\ell_n + \ell_{n+1}}{2},
\end{cases}$$

and define, $\forall\,\tau < 1/2$, the characteristic function:

$$\chi_{ijn} = \mathrm{Char}\Big[\Big(-\frac{h_{i+1}}{2},\frac{h_i}{2}\Big)\times\Big(-\frac{k_{j+1}}{2},\frac{k_j}{2}\Big)\Big]\times\Big(-\frac{\ell_{n+1}}{2},\frac{\ell_n}{2}\Big)\Big]\in H^{\tau}(\mathbf{R}^3).$$

For finite volume approximation let $\rho\in H^s(\Omega_{\mathbf{x}})$, $r>-1/2$. Extend ρ to $\mathbf{R^3}$ preserving its Sobolev class. Thus, we may define using three dimensional convolutions, $\chi_{ijn}*\rho$, which is continuous in $\mathbf{R^3}$ and

$$\frac{1}{|\omega_{ijn}|}\int_{\partial\omega_{ijn}}\frac{\partial\varphi}{\partial\mathbf{n}}\,ds=\frac{1}{|\omega_{ijn}|}\Big(\chi_{ijn}*\rho\Big)(x_i,y_j,z_n) \tag{6}$$

Further, recalling that $\rho\in L^1_{loc}(\Omega_{\mathbf{x}})$ we may write

$$\frac{1}{|\omega_{ijn}|}\int_{\partial\omega_{ijn}}\frac{\partial\varphi}{\partial\mathbf{n}}\,ds=\frac{1}{\hbar_i\bar{k}_j\bar{\ell}_n}\int_{x_{i-1/2}}^{x_{i+1/2}}\int_{y_{j-1/2}}^{y_{j+1/2}}\int_{z_{n-1/2}}^{z_{n+1/2}}\rho(x,y,z)\,dxdydz. \tag{7}$$

Now we let $\mathscr{V}_h$ be the set of piecewise bilinear functions defined on the box $\Omega_{\mathbf{x}}$ induced by $\bar{\Omega}^h_{\mathbf{x}}$, i.e. $\mathscr{V}_h^\circ=\{F\in\mathscr{V}_h\big|F=0\quad\text{on}\quad\partial\Omega_{\mathbf{x}}\}$.

Definition 1. The finite volume approximation of the solution φ for the Poisson equation (9): $\varphi_h\in\mathscr{V}_h^\circ$ is defined (implicitly) through the following algorithm:

$$-\frac{1}{\hbar_i\bar{k}_j\bar{\ell}_n}\int_{\partial\omega_{ijn}}\frac{\partial\varphi_h}{\partial\mathbf{n}}\,ds=\frac{1}{\hbar_i\bar{k}_j\bar{\ell}_n}\Big(\chi_{ijn}*\rho\Big)(x_i,y_j,z_n),\qquad(x_i,y_j,z_n)\in\Omega^h_{\mathbf{x}}.$$

Stability and convergence of this method are generalization of Süli's [15] results in two dimensions for the Dirichlet problem. $|\nabla_x\varphi|=0$ on $\partial\Omega_x$ with extended $\varphi(\infty)=0$ yield $\varphi=0$ on $\partial\Omega_x$. The first assertion in Theorem 1, may be proved repeating the arguments in [15] (we skip) for the 3d case in discrete $H^1(\Omega^h_{\mathbf{x}})$ and $L_2(\Omega^h_{\mathbf{x}})$ norms:

$$||\psi||_{1,h}=\Big(||\psi||^2+|\psi|^2_{1,h}\Big)^{1/2},\quad\text{and}\quad||\psi||=(\psi,\psi)^{1/2},\qquad\text{where}$$

$$(\phi,\psi)=\sum_{i=1}^{I-1}\sum_{j=1}^{J-1}\sum_{n=1}^{N-1}\hbar_i\bar{k}_j\bar{\ell}_n\phi_{ijn}\psi_{ijn},\qquad\text{and}$$

$$|\psi|_{1,h}=\Big(||\Delta_x^-\psi||_x^2+||\Delta_y^-\psi||_y^2+||\Delta_z^-\psi||_z^2\Big)^{1/2},\qquad\text{with}$$

divided differences $\Delta_x^-\psi_{ijn}=(\psi_{ijn}-\psi_{i-1,j,n})/\hbar_i$, $\Delta_y^-\psi_{ijn}=(\psi_{ijn}-\psi_{i,j-1,n})/\bar{k}_j$ and $\Delta_z^-\psi_{ijn}=(\psi_{ijn}-\psi_{i,j,n-1})/\bar{\ell}_n$, and the, *one-sided discrete L_2-norms*

$$||\Delta_x^-\psi||_x^2=(\psi,\psi]_x,\qquad(\phi,\psi]_x=\sum_{i=1}^{I}\sum_{j=1}^{J-1}\sum_{n=1}^{N-1}\hbar_i\bar{k}_j\bar{\ell}_n\phi_{ijn}\psi_{ijn},$$

with the similar notations corresponding to the y and z directions.

3 Streamline diffusion and Discontinuous Galerkin approaches

For a finite element scheme over the phase-space-time domain $\Omega_T := [0,T]\times\Omega$ we start with a phase-space subdivision of Ω, into the product of triangular elements τ_x and τ_v , as $\mathscr{T}_h := \{\tau = \tau_x\times\tau_v\}$ combined with a partition of the time interval $(0,T)$: $0 = t_0 < t_1 < \ldots < t_M = T$, and let $I_m := (t_m,t_{m+1})$; $m = 0,1,\ldots,M-1$. Then the corresponding partition of Ω_T is given by the *prism-type triangulation*

$$\mathscr{C}_h := \{K|K := \tau\times I_m,\ \tau\in\mathscr{T}_h\}.$$

We seek piecewise polynomial approximations for the solution of problem (1) in a finite dimensional space

$$V_h := \{f\in\mathscr{H} : f|_K\in\mathscr{P}_k(\tau)\times\mathscr{P}_k(I_m);\ \forall K = \tau\times I_m\in\mathscr{C}_h\},$$

with V_h being continuous in x and v, possibly discontinuous in t across time levels t_m and $\mathscr{H} := \prod_{m=0}^{M-1}H^1(\Omega_m)$; $\Omega_m = \Omega\times I_m$. We shall also use the standard notation

$$(f,g)_m = (f,g)_{\Omega_m} = \int_{\Omega_m} fg\,dxdvdt, \qquad ||g||_m = (g,g)_m^{1/2},$$

$$\langle f,g\rangle_m = \int_{\Omega} f(\cdot,\cdot,t_m)g(\cdot,\cdot,t_m)\,dxdv, \qquad |g|_m =< g,g >_m^{1/2},$$

$$< f^{\mp},g^{\mp} >_{\Gamma^{\pm}} = \int_{\Gamma^{\pm}} f^{\mp}g^{\mp}|G^h\cdot n|dv, \qquad \text{and the jumps}$$

$$[g] = g^+ - g^- \qquad g^{\pm} = \lim_{s\to 0\pm} g(x,v,t+s),$$

$$< f^{\mp},g^{\mp} >_{\lambda\pm} = \int_{I_m} < f^{\mp},g^{\mp} >_{\Gamma^{\pm}}\ dt.$$

Using notation $\nabla f := (\nabla_x f,\nabla_v f) = (\partial f/\partial x_1,\ldots,\partial f/\partial x_d,\partial f/\partial v_1,\ldots,\partial f/\partial v_d)$ and $G := \Big(v_1,\ldots,v_d,-\partial\phi/\partial x_1,\ldots,-\partial\phi/\partial x_d\Big)$, $\mathrm{div}G(f) = 0$. For our finite element procedure (both in the streamline diffusion and the discontinuous Galekin case) we let $\mathscr{F}$ to be a certain (linear) function space, $\tilde{f}\in\mathscr{F}$ an approximation of f and $\Pi f\in\mathscr{F}$ a projection of f into $\mathscr{F}$, then to estimate the approximation error

$$f-\tilde{f} = (f-\Pi f)+(\Pi f-\tilde{f}) \equiv \eta+\xi; \quad \xi\in\mathscr{F},$$

(i) We use interpolation theory to give sharp error bounds for a certain $|||\eta|||$-norm
(ii) Establish $|||\xi||| \le C|||\eta|||$, $\quad(|||\cdot||| := \|\cdot\|_{\Xi}$, Ξ=SD or Ξ=DG, below).

Now we consider the streamline diffusion (SD) method for (P1) with test functions of the form $u+\delta\Big(u_t+G(\tilde{f})\cdot\nabla u\Big)$ with $\delta\sim h$, the mesh size. For convenience we use the notation $\mathscr{D}w := w_t + G(f_h)\cdot\nabla w$ and formulate the SD method for problem (I) as follows: given $f_h^-(\cdot,\cdot,t_m)$ find $f_h\in V_h$ such that for $m = 0,\ldots,M-1$,

$$(P_m) \qquad B_m^{\delta}(G(f_h);f_h,u) - J_m^{\delta}(f_h,u) = L_m^{\delta}(u), \qquad \forall u\in V_h. \tag{8}$$

$$B_m^\delta := (\mathscr{D} f_h, u + \delta \mathscr{D} u)_m + \sigma(\nabla_v f_h, \nabla_v u)_m + \langle [f_h], u \rangle_m - \delta\sigma(\Delta_v f_h, \mathscr{D} u)_m, \qquad (9)$$

$$J_m^\delta := (\nabla_v \cdot (\beta v f_h), u + \delta \mathscr{D} u)_m, \qquad (10)$$

and

$$L_m^\delta := (S, u + \delta \mathscr{D} u)_m + \langle f^+, u^+ \rangle_{\lambda_m^-} + \langle f^-, u^- \rangle_{\lambda_m^+}. \qquad (11)$$

Problem P_m is a linear system of equations leading to an implicit scheme. Therefore to solve P_1 is equivalent to find $f_h \in V_h$ such that

$$B^\delta(G(f_h); f_h, u) - J^\delta(f_h, u) = L^\delta(u), \qquad \forall u \in V_h, \qquad (12)$$

$$B^\delta := \sum_{m=0}^{M-1} B_m^\delta, \quad J^\delta := \sum_{m=0}^{M-1} J_m^\delta, \quad L^\delta := \sum_{m=0}^{M-1} L_m^\delta. \qquad (13)$$

3.1 Stability and error estimates

Lemma 1. *For the SD method we have the coercivity and stability estimates*

$$\forall g \in \mathscr{H}, \quad B^\delta\Big(G(f^h); g, g\Big) \geq \frac{1}{2}||g||_{SD}^2, \qquad \textit{with}$$

$$||g||_{SD}^2 = \frac{1}{2}\Big[2\sigma||\nabla_v g||_{\Omega_T}^2 + |g|_M^2 + |g|_0^2 + \sum_{m=1}^{M-1} |[g]|_m^2 + 2||\mathscr{D} g||_{\Omega_T}^2 + \int_{\Gamma \times I} g^2 |G^h \cdot n|\Big],$$

$$||g||_{L_2(\Omega_T, SD)}^2 \leq \Big[\frac{1}{C_1}||\mathscr{D} g||^2 + \sum_{m=1}^{M-1} |[g]|_m^2 + \int_{\partial\Omega \times I} g^2 |G^h \cdot n|\Big] \delta e^{C_1 \delta}, \qquad \forall C_1 \geq 0.$$

Remark 1. In the discontinuous Galerkin case, $||g||_{DG}$ and $||g||_{L_2(\Omega_T, DG)}$ are defined by replacing the $\int$-term, in the SD case, by $\sum \int_{\partial K_-(G'')} [g]^2 |G^h \cdot n|\, ds$ where

$$\partial K_-(G'') = \{(x, v, t) \in \partial K_-(G') : n_t(x, v, t) = 0\}.$$

Theorem 3. *Assume that there is a constant C such that*

$$||\nabla f||_\infty + ||G(f)||_\infty + ||\nabla \eta||_\infty \leq C. \qquad (14)$$

Then we have the following error estimate for the SD method for (P1):

$$||f - f_{SD}||_{SD} \leq C h^{k+1/2} ||f||_{H^{k+1}(\Omega_T)},$$

where $f_{SD} \in V_h$ is the SD-approximation for f, and we have assumed $f \in H^{k+1}(\Omega_T)$.

Proof. (sketch of the main ideas) Let $\tilde{f}^h$ be an interpolant of f, and split the error as

$$e = f - f_{SD} = f - \tilde{f}^h + \tilde{f}^h - f_{SD} := \eta - \xi.$$

Then, by the above coercivity estimate and Galerkin orthogonality, we may write

$$\frac{1}{2}||\xi||_{SD}^2 \le B(G(f^h);\xi,\xi) = B(G(f);f,\xi) - B(G(f^h);\tilde{f}^h,\xi) + J(f^h,\xi) - J(f,\xi)$$
$$:= \Delta B + \Delta J \le \frac{1}{8}||\xi||_{SD}^2 + C_B||\eta||_{SD}^2 + \frac{1}{8}||\xi||_{SD}^2 + C_J||\eta||_{SD}^2,$$

where to estimate J-term, we have used the inverse estimate. Further interpolation estimates give $||\eta||_{SD}^2 \le C_i h^{k+1/2}||f||_{H^{k+1}(\Omega_T)}$, which yields the desired result.

In the discontinuous Galerkin (DG) case, we assume also discontinuities in x and v over the interelement boundaries. Here, we shall use the discrete function spaces

$$W_h = \Big\{g \in L_2(Q_T) : g|_K \in P_k(K) \quad \forall K \in \mathscr{C}_h\Big\}, \quad \text{and}$$
$$W_h^d = \Big\{w \in [L_2(Q_T)]^d : w|_K \in [P_k(K)]^d \quad \forall K \in \mathscr{C}_h\Big\}.$$

Then, the corresponding final error estimate for the DG case reads as follows:

Theorem 4. *Under the assumptions (14) of Theorem 3 and regularity assumption for the exact solution as $f \in H^{k+1}(\Omega_T) \cap W^{k+1,\infty}(\Omega_T)$, we have that the discontinuous Galerkin approximation $f_{DG} \in W_h^d$ for f in (P1) satisfies the error estimate*

$$||f - f_{DG}||_{DG} \le Ch^{k+1/2}\Big(||f||_{H^{k+1}(\Omega_T)} + ||f||_{W^{k+1,\infty}(\Omega_T)}\Big).$$

Proof. (Sketchy) Here we demonstrate only the terms that are involved in estimations of the enterelement jump terms, which are additional to those in the SD-case. To this end, we introduce $R: W_h \to W^d$, see [6], defined by

$$R(g)w = -\sum_{\tau_x \times I_m} \int_{\tau_x \times I_m} \sum_{e \in E_v} \int_e [[g]] n_v \cdot (w)^0 \, dv, \qquad \forall w \in W_h^d, \tag{15}$$

E_v is the set of all interior edges of the triangulation T_h^v. Define

$$(\chi)^0 = \frac{\chi + \chi^{ext}}{2}, \quad \text{and} \quad [[\chi]] = \chi - \chi^{ext},$$

χ^{ext} is the value of χ in the element τ_v^{ext} having $e \in E_v$ common edge with τ_v. Now we let r_e be the restriction of R to the elements sharing the edge $e \in E_v$, then

$$r_e(g)w = -\sum_{\tau_x \times I_m} \int_{\tau_x \times I_m} \int_e [[g]] n_v \cdot (w)^0 \, dv, \qquad \forall w \in W_h^d. \tag{16}$$

Hence, we may easily verify that

$$\sum_{e \subset \partial\tau_v \cap E_v} r_e = R \quad \text{on } \tau_v \Longrightarrow ||R(g)||_K^2 \le \gamma \sum_{e \subset \partial\tau_v \cap E_v} ||r_e(g)||_K^2, \tag{17}$$

where τ_v corresponds to the element K and $\gamma = \gamma(d) > 0$ is a constant. Furthermore, since the support of each r_e is the union of elements sharing the edge e,

$$\sum_{e\in Ev} \|r_e(g)\|^2 = \sum_{K\in C} \sum_{e\subset\partial\tau_v\cap Ev} \|r_e(g)\|_K^2. \tag{18}$$

The corresponding discontinuous Galerkin method reads as: find $f_h \in W_h$ such that

$$B_{DG}(G(f_h); f_h, g) - K(f_h, g) = L(g), \qquad \forall g \in W_h,$$
$$\text{where,} \qquad (Kf, g) = \Big(\nabla_v(\beta v f), g + h\mathscr{D}g\Big).$$

Proving a coercivity which, compared to B_{SD}, contains also interelement jumps;

$$(B_{DG}G(f^h); g, g) \geq \alpha|||g|||^2, \qquad \forall g \in W_h,$$

and following the same procedure as in the SD case yields the DG error estimate.

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