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# RESTRICTIONS OF $m$-WYTHOFF NIM AND $p$-COMPLEMENTARY BEATTY SEQUENCES

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Abstract. Fix a positive integer $m$. The game of *$m$-Wythoff Nim* (A.S. Fraenkel, 1982) is a well-known extension of *Wythoff Nim* (W.A. Wythoff, 1907). The set of $P$-positions may be represented as a pair of increasing sequences of non-negative integers. It is well-known that these sequences are so-called *complementary Beatty sequences*, that is they satisfy Beatty's theorem. For a positive integer $p$, we generalize the solution of $m$-Wythoff Nim to a pair of *$p$-complementary*—each non-negative integer is represented exactly $p$ times—Beatty sequences $a = (a_n)_{n\in\mathbb{N}_0}$ and $b = (b_n)_{n\in\mathbb{N}_0}$, which, for all $n$, satisfy $b_n - a_n = mn$. Our main result is that $\{\{a_n, b_n\} \mid n \in \mathbb{N}_0\}$ represents the solution to three new '$p$-restrictions' of $m$-Wythoff Nim—of which one has a certain blocking manoeuvre on the rook-type options. C. Kimberling has shown that the solution of Wythoff Nim satisfies the *complementary equation* $x_{x_n} = y_n - 1$. We generalize this formula to a certain '$p$-complementary equation' satisfied by our pair $a$ and $b$. Further, if $p > 1$, we prove that this pair is unique in the sense that it is the only pair of $p$-complementary Beatty sequences of which one of the sequences is strictly increasing. We also show that one may obtain our new pair of sequences by three so-called *Minimal EXclusive* algorithms.

## 1. Introduction and notation

The combinatorial game of *Wythoff Nim* ([Wyt07]) is a so-called (2-player) impartial game played on two piles of tokens. (For an introduction to impartial games see [BeCoGu82, Con76].) As an addition to the rules of the game of Nim ([Bou02]), where the players alternate in removing any finite number of tokens from precisely one of the piles (at most the whole pile), Wythoff Nim also allows removal of the same number of tokens from both piles. The player who removes the last token wins.

This game is more known as 'Corner the Queen', invented by R. P. Isaacs (1960), because the game can be played on a (large) Chess board with one single Queen. Two players move the Queen alternately but with the restriction that, for each move, the ($L^1$) distance to the lower left corner, position $(0, 0)$, must decrease. (The Queen must at all times remain on the board.) The player who moves to this *final/terminal* position wins.

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In this paper we follow the convention to denote our players with the *next player* (the player who is in turn to move) and the *previous player*. A $P$-*position* is a position from which the previous player can win (given perfect play). An $N$-*position* is a position from which the next player can win. Any position is either a $P$-position or an $N$-position. We denote *the solution*, the set of all $P$-positions, of an impartial game $G$, by $\mathcal{P} = \mathcal{P}(G)$ and the set of all $N$-positions by $\mathcal{N} = \mathcal{N}(G)$. The positive integers are denoted by $\mathbb{N}$ and the non-negative integers by $\mathbb{N}_0$.

1.1. **Restrictions of $m$-Wythoff Nim.** Let $m \in \mathbb{N}$. We next turn to a certain $m$-extension of Wythoff Nim, studied in [Fra82] by A.S. Fraenkel. In the game of *$m$-Wythoff Nim*, or just $m$WN (our notation), the Queen's 'bishop-type' options are extended so that $(x, y) \to (x + i, y + j)$ is legal if $|\, i - j \,| < m$. The 'rook-type' options are as in Nim. Hence 1-Wythoff Nim is identical to Wythoff Nim.

In this paper we define three new restrictions of $m$-Wythoff Nim—here a rough outline:

- The first has a so-called *blocking manoeuvre/Muller Twist* on the rook-type options—before the next player moves, the previous player may announce at most a predetermined number of these options as forbidden (see also [HoRe, SmSt02] and Section 1.2 of this paper);
- The second has a certain congruence restriction on the rook-type options;
- For the third, a rectangle is removed from the lower left corner of the game board (including position $(0, 0)$), so that here we get two terminal positions.

1.2. **A pair of $p$-complementary Beatty sequences.** A *Beatty sequence* is a sequence of the form $(\lfloor n\alpha + \beta \rfloor)_{n \in \mathbb{N}_0}$, where $\alpha$ is a positive irrational and $\beta$ is a real number. S. Beatty ([Bea26]) is maybe most known for a (re)[1]discovery of (the statement of) the following theorem: If $\alpha$ and $\beta$ are positive reals such that $\frac{1}{\alpha} + \frac{1}{\beta} = 1$ then $(\lfloor n\alpha \rfloor)_{n \in \mathbb{N}}$ and $(\lfloor n\beta \rfloor)_{n \in \mathbb{N}}$ split $\mathbb{N}_0$ if and only if they are Beatty sequences. This was proven by [HyOs27] (see also [Fra82]).

A pair of sequences that satisfies Beatty's theorem is *complementary* (see [Fra69, Fra73, Kim07, Kim08]).

In this paper we generalize the notion of complementarity.

**Definition 1.** Let $p \in \mathbb{N}$. Two sequences $(x_i)$ and $(y_i)$ of non-negative integers are *$p$-complementary*, if, for each $n \in \mathbb{N}_0$,

$$\#\{i \mid x_i = n\} + \#\{i \mid y_i = n\} = p.$$

As usual, a 1-complementary pair of sequences is denoted *complementary.*

We study the Beatty sequences $a = (a_n)_{n \in \mathbb{N}_0}$ and $b = (b_n)_{n \in \mathbb{N}_0}$, where for all $n \in \mathbb{N}$,

$$a_n = a_n^{m,p} = \left\lfloor \frac{n\phi_{mp}}{p} \right\rfloor \quad (1)$$

[1]This theorem was in fact discovered by J. W. Rayleigh, see [Ray94, Bry03].

and

$$b_n = b_n^{m,p} = \left\lfloor \frac{n(\phi_{mp} + mp)}{p} \right\rfloor, \tag{2}$$

and where

$$\phi_\alpha = \frac{2 - \alpha + \sqrt{\alpha^2 + 4}}{2}. \tag{3}$$

We show that $a$ and $b$ are $p$-complementary. (Notice also that, for all $n$, $b_n - a_n = mn$.)

In [Wyt07] W.A. Wythoff proved that the solution of Wythoff Nim is given by $\{\{a_n^{1,1}, b_n^{1,1}\}^2 \mid n \in \mathbb{N}_0\}$, Then in [Fra82] it was shown that the solution of $m$-Wythoff Nim is

$$\{\{a_n^{m,1}, b_n^{m,1}\} \mid n \in \mathbb{N}_0\}.$$

1.3. **Recurrence.** Let $X$ be a strict subset of the non-negative integers. Then the *Minimal EXclusive* of $X$ is defined as usual (see [Con76]):

$$\operatorname{mex} X := \min(\mathbb{N}_0 \backslash X).$$

For $n \in \mathbb{N}_0$ put

$$x_n = \operatorname{mex}\{x_i, y_i \mid i \in [0, n-1]\} \text{ and } y_n = x_n + mn. \tag{4}$$

With notation as in (4), it was proven in [Fra82] that $(x_n) = (a^{m,1})$ and $(y_n) = (b^{m,1})$. The minimal exclusive algorithm in (4) gives an exponential time solution to $m$WN whereas the Beatty-pair in (1) and (2) give a polynomial time ditto. (For interesting discussions on complexity issues for combinatorial games, see for example [Fra04, FrPe09].) We show that one may obtain $a$ and $b$ by three minimal exclusive algorithms, which in various ways generalize (4).

It is well-known that the solution of Wythoff Nim satisfies the *complementary equation* (see for example [Kim95, Kim07, Kim08])

$$x_{x_n} = y_n - 1.$$

For arbitrary positive integers $m$ and $p$, we generalize this formula to a 'p-complementary equation'

$$x_{\varphi_n} = y_n - 1, \tag{5}$$

where $\varphi_n = \frac{x_n + (mp-1)y_n}{m}$, and show that a solution is given by $x = a$ and $y = b$.

1.4. **I.G. Connell's restriction of Wythoff Nim.** In the literature there is another generalization of Wythoff Nim that is of special interest to us. Let $p \in \mathbb{N}$. In [Con59] I.G. Connell studies the restriction of Wythoff Nim, where the the rook-type options are restricted to jumps of precise multiples of $p$. This game we call Wythoff modulo-$p$ Nim and denote with $\text{WN}^{(p)}$. Hence Wythoff modulo-1 Nim equals Wythoff Nim.

[2] As usual, $\{x, y\}$ denotes unordered pairs (of integers), that is $(x, y)$ and $(y, x)$ are considered the same.

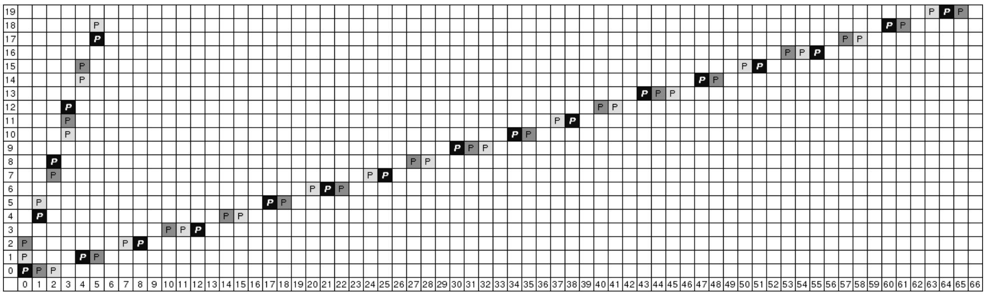

FIGURE 1. The $P$-positions of Wythoff modulo-3 Nim, $\mathrm{WN}^{(3)}$ are the positions nearest the origin such that there are precisely three positions in each row and column and one position in each NE-SW-diagonal. The black positions represent the (first few) $P$-positions of 3-Wythoff Nim, namely the positions nearest the origin such that there is precisely one position in each row and one position in every third NE-SW diagonal.

Call the $P$-positions of $\mathrm{WN}^{(p)}$ $\{\{c_n, d_n\} \mid n \in \mathbb{N}_0\}$, where $c_n = c_n^{(p)}$ and $d_n = d_n^{(p)}$ and let $\phi_\alpha$ be as in (3). The general solution of $\mathrm{WN}^{(p)}$ is given by

$$c_n = \left\lfloor \frac{n\phi_p}{p} \right\rfloor \text{ and } d_n = c_n + n,$$

a formula which can be derived from [Con59]—from which one may also deduce that $(c_i)$ and $(d_i)_{>0}$ are $p$-complementary. Notice that, for fixed $p$ and for all $n$, $a_n^{1,p} = c_n^{(p)}$ and $b_n^{1,p} = d_n^{(p)}$,.

| | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $d_n^{(3)}$ | 0 | 1 | 2 | 4 | 5 | 7 | 8 | 10 | 11 | 12 | 14 | 15 | 17 | 18 | 20 | 21 | 22 |
| $c_n^{(3)}$ | 0 | 0 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 3 | 4 | 4 | 5 | 5 | 6 | 6 | 6 |
| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |

TABLE 1. Some values of $c_n^{(3)} = \lfloor \frac{n\phi_3}{3} \rfloor$ and $d_n^{(3)} = c_n^{(3)} + n$.

**Remark 1.** In Connell's presentation, for the proof of the above formulas, he rather uses $p$ pairs of complementary sequences of integers (in analogy with the discovery of a new formulation of Beatty's theorem in [Sko57]). We have indicated this pattern of $P$-positions with different shades in Figure 1. In fact, the squares of darkest shade, starting with $(0,0)$ are $P$-positions of 3-Wythoff Nim—in general $a_n^{p,1} = c_{pn}^{(p)}$ and $b_n^{p,1} = d_{pn}^{(p)}$— and, as we will see, given a certain game constant, each lighter shade represents the solution of our third variation of this game.

**Remark 2.** In [BoFr73], Fraenkel and I. Borosh study yet another variation of both $m$-Wythoff Nim and Wythoff modulo-$p$ Nim which includes a (different from ours) Beatty-type characterization of the $P$-positions.

1.5. **Exposition.** In Section 2 we define our games, exemplify them and state our main theorem. Roughly: For each of our games, given appropriate game constants, a position is $P$ if and only if it is of the form $\{a_n, b_n\}$, with $a$ and $b$ as in (1) and (2) (so that, in terms of game complexity, the solution of each of our games is polynomial). In Section 3 we generalize Beatty's theorem to $p$-complementary sequences and prove some arithmetic properties of $a$ and $b$—most important of which is that (for fixed $m$ and $p$) $a$ and $b$ are $p$-complementary. Then, in Section 4, for arbitrary $m$ and $p > 1$, we prove that our new pair of sequences is unique in the sense that it is the only pair of $p$-complementary Beatty sequences for which one of the sequences is (strictly) increasing. Section 5 is devoted to our $p$-complementary equation (5) and minimal exclusive algorithms. In Section 6 we prove our game theory results (stated in Section 2) and finally in Section 7 a few questions are posed.

Let us, before we move on to our games, give some more background to the so-called *blocking manoeuvre* in the context of Wythoff Nim.

1.6. **A bishop-type blocking variation of $m$-Wythoff Nim.** Let $m, p \in \mathbb{N}$. In [HeLa06] we gave an exponential time solution to a variation of $m$-Wythoff Nim with a 'bishop-type' blocking manoeuvre, denoted by $p$-Blocking $m$-Wythoff Nim (and with $(m, p)$-Wythoff Nim in [Lar09]).

The rules are as in $m$-Wythoff Nim, except that before the next player moves, the previous player is allowed to block off (at most) $p - 1$ bishop-type—note, not $m$-bishop-type—options and declare that the next player must refrain from these options. When the next player has moved, any blocked options are forgotten.

The solution of this game is in a certain sense 'very close' to pairs of Beatty sequences (see also the Appendix of [Lar09]) of the form

$$\left(\left\lfloor n\frac{\sqrt{m^2+4p^2}+2p-m}{2p}\right\rfloor\right) \text{ and } \left(\left\lfloor n\frac{\sqrt{m^2+4p^2}+2p+m}{2p}\right\rfloor\right).$$

But we explain why there can be no Beatty-type solution to this game for $p > 1$. However, in [Lar09], for the cases $p \,|\, m$, we give a certain 'Beatty-type' characterisation. For these kind of questions, see also [BoFr84]. However, a recent discovery, in [Had, FrPe09], provides a polynomial time algorithm for the solution of $(m, p)$-Wythoff Nim (for any combination of $m$ and $p$).

An interesting connection to 4-Blocking 2-Wythoff Nim is presented in [DuGr08], where the authors give an explicit bijection of solutions to a variation of Wythoff's original game, where a player's bishop-type move is restricted to jumps by multiples of a predetermined positive integer.

For another variation, [Lar09] defines the rules of a so-called move-size dynamic variation of two-pile Nim, $(m, p)$-Imitation Nim, for which the $P$-positions, treated as starting positions, are identical to the $P$-positions of $(m, p)$-Wythoff Nim.

This discovery of a 'dual' game to $(m, p)$-Wythoff Nim has in its turn motivated the study of dual constructions of the 'rook-type' blocking manoeuvre in this paper.

## 2. Three games

This section is devoted to defining and exemplifying our new game rules and to state our main result. We begin by introducing some (non-standard) notation whereby we 'decompose' the Queen's moves into *rook-type* and *bishop-type* ditto.

**Definition 2.** Fix $m, p \in \mathbb{N}$ and an $l \in \{0, 1, \ldots, m\}$.

(i) An $(l, p)$-*rook* moves as in Nim, but the length of a move must be $ip + j > 0$ positions for some $i \in \mathbb{N}_0$ and $j \in \{0, 1, \ldots, l-1\}$ (we denote a $(0, p)$-rook by a $p$-rook and a $(p, p)$-rook simply by a *rook*);

(ii) A $m$-*bishop* may move $0 \le i < m$ rook-type positions and then any number of, say $j \ge 0$, bishop-type positions (a *bishop* moves as in Chess), all in one and the same move, provided $i + j > 0$ and the $L^1$-distance to $(0, 0)$ decreases.

2.1. **Game definitions.** As is clear from Definition 2 the rook-type options intersect the $m$-bishop-type options precisely when $m > 1$. For example, $(2, 3) \to (1, 3)$ is both a 2-bishop-type and a rook-type move. We will make use of this fact when defining the blocking manoeuvre. Therefore, let us introduce some new terminology.

Fix an $m \in \mathbb{N}$. A rook-type option, which is not of the form of the $m$-bishop as in Definition 2 (ii), is a *roob*(-type)[3] option. Hence, for $m = 2$, $(2, 3) \to (2, 1)$ is a roob option, but $(2, 3) \to (2, 2)$ is not (both are rook options).

Let us define our games.

**Definition 3.** Fix $m, p \in \mathbb{N}$.

(1) The game of $m$-*Wythoff* $p$-*Blocking Nim*, or $m\mathrm{WN}^p$, is a restriction of $m$-Wythoff Nim with a roob-type blocking manoeuvre.

The Queen moves as in $m$-Wythoff Nim (that is, as the $m$-bishop or the rook), but with one exception: Before the next player moves, the previous player may *block off* (at most) $p-1$ of the next player's roob options. The blocked options are then excluded from the Queen's options. As usual, each blocking manoeuvre is particular to a specific move; that is, when the next player has moved, any blocked options are forgotten and has no further impact on the game. (For $p = 1$ this game equals $m$-Wythoff Nim.)

(2) Fix an integer $0 \le l < p$. In the game of $m$-*Wythoff Modulo-*$p$ $l$-*Nim*, or $m\mathrm{WN}^{(l,p)}$, the Queen moves as the $m$-bishop or the $(l, p)$-rook. For $l = 0$ we denote this game by $m$-*Wythoff Modulo-*$p$ *Nim* or $m\mathrm{WN}^{(p)}$. (In case $m = l = 0$ the game reduces to Wythoff modulo-$p$ Nim, whereas for $l = p$ the game is simply $m$-Wythoff Nim.)

(3a) Fix an integer $0 \le l < p$. In the game of $l$-*Shifted* $m{\times}p$-*Wythoff Nim*, or $m{\times}p\mathrm{WN}_l$, the Queen moves as in $(mp)$-Wythoff Nim (that is, as the $(mp)$-bishop or the rook), except that, if $l > 0$, it is not allowed to move to a position of the form $(i, j)$, where $0 \le i < ml$ and

[3] Think of 'roob' as 'ROOk minus $m$-Bishop', or maybe 'ROOk Blocking'

$0 \leq j < m(p-l)$[4]. Hence, for this case, the terminal positions are $(ml, 0)$ and $(0, m(p-l))$. On the other hand $m \times p\mathrm{WN}_0$ is identical to $(mp)$-Wythoff Nim.

(3b) The game of $m{\times}p$-Wythoff Nim, $m{\times}p$WN: Before the first player moves, the second player may decide the parameter $l$ as in (3a). Once the parameter $l$ is fixed, it remains the same until the game has terminated, so that for the remainder of the game, the rules are as in $m{\times}p\mathrm{WN}_l$.

2.2. **Examples.** Let us illustrate some of our games, where our players are *Alice* and *Bob*—Alice makes the first move (and Bob makes the first blocking manoeuvre in case the game has a Muller twist).

**Example 1.** Suppose the starting position is $(0, 2)$ and the game is $2\mathrm{WN}^2$. Then the only bishop-type move is $(0, 2) \to (0, 1)$. There is precisely one roob option, namely $(0, 0)$. Since this is a terminal position Bob will block it off from Alice's options, so that Alice has to move to $(0, 1)$. The move $(0, 1) \to (0, 0)$ cannot be blocked off for the same reason, so Bob wins. If $y \geq 3$ there is always a move $(0, y) \to (0, x)$, where $x = 0$ or 2. This is because the previous player may block off at most one option. Altogether, this gives that $\{0, y\}$ is $P$ if and only if $y = 0$ or 2.

**Example 2.** Suppose the starting position is $(0, 2)$ and the game is $2\mathrm{WN}^{(2)}$. Alice can move to $(0, 0)$, since $0 \equiv 2 \pmod 2$, so $(0, 2)$ is $N$. On the other hand, the position $(0, 3)$ is $P$ since the only options are $(0, 2)$ and $(0, 1)$. (The latter is $N$ since the 2-bishop can move $(0, 1) \to (0, 0)$.)

**Example 3.** Suppose the starting position is $(0, 2)$ and the game is $2\mathrm{WN}^{(2,4)}$. Alice cannot move to $(0, 0)$, since $2 - 0 \not\equiv 3, 4 \pmod 4$ and since $(0, 1) \to (0, 0)$ is a 2-bishop-type move $(0, 1)$ is $N$, so that $\{0, 2\}$ must be $P$. Then $(0, 3)$ is $N$ and since $(0, y) \to (0, 0)$ is legal if $y = 4$ or 5 we get, by similar reasoning, that $\{0, y\}$ is $N$ for all $y \geq 3$.

**Example 4.** Suppose the starting position is $(0, 4)$ and the game is $2\mathrm{WN}^3$. Then the only bishop-type move is $(0, 4) \to (0, 3)$, so that the roob options are $(0, 0), (0, 1), (0, 2)$. Bob may block off 2 of these positions, say $(0, 0), (0, 2)$. Then if Alice moves to $(0, 1)$ she will loose (since she may not block off (0,0)), so suppose rather that she moves to $(0, 3)$. Than she may not block off $(0, 2)$ so Bob moves $(0, 3) \to (0, 2)$ and blocks off $(0, 0)$. Hence $(0, 4)$ is a $P$-position.

**Example 5.** Suppose the starting position is $(0, 4)$ and the game is $2\mathrm{WN}^{(3)}$. Alice cannot move to $(0, 0)$ or $(0, 2)$. But $(0, 1) \to (0, 0)$ is a 2-bishop-type option and $(0, 3) \to (0, 0)$ is a 3-rook-type option. This shows that $(0, 4)$ is a $P$-position.

Notice that, in comparison to Examples 4 and 5, the $P$-positions in the Examples 1 and 2 are distinct in spite the identical game constants ($m = p = 2$). On the other hand, the $P$-positions in Examples 1 and 3 coincide.

[4]One may think of the game as if this lower left rectangle is cut out from the game board.

**Example 6.** If the starting position is $(0,4)$ and the game is $2{\times}3\mathrm{WN}_1$, then Alice cannot move so that Bob wins. If, on the other hand, the game is $2{\times}3\mathrm{WN}_2$, the position $(0,2)$ is terminal and so Alice wins (by moving $(0,4) \to (0,2)$).

Suppose now that the starting position of $2{\times}3\mathrm{WN}_2$ is $(1,8)$. Then, Alice may move to $(0,2)$. But if the starting position of $2{\times}3\mathrm{WN}_0$ is $(1,7)$ Alice may not move to $(0,0)$ and hence Bob wins.

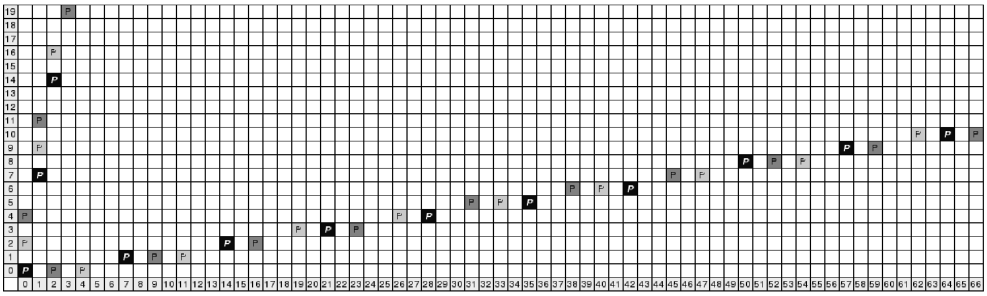

FIGURE 2. $P$-positions of $2\mathrm{WN}^{(3)}$, $2\mathrm{WN}^3$, $2\mathrm{WN}^{2,6}$ and $2\times$ 3WN—the positions nearest the origin such that there are precisely three positions in each row and column and one position in every second NE-SW-diagonal. The palest coloured squares represent $P$-positions of $2{\times}3\mathrm{WN}_1$. They are of the form $(a_{3n+1}, b_{3n+1})$ or $(b_{3n+2}, a_{3n+2})$. The darkest squares, $(\{a^{2,3}_{3i}, b^{2,3}_{3i}\})$, represent the solution of 6WN.

| | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $b^{2,3}_n$ | 0 | 2 | 4 | 7 | 9 | 11 | 14 | 16 | 19 | 21 | 23 | 26 | 28 | 31 | 33 | 35 | 38 |
| $a^{2,3}_n$ | 0 | 0 | 0 | 1 | 1 | 1 | 2 | 2 | 3 | 3 | 3 | 4 | 4 | 5 | 5 | 5 | 6 |
| $b_n - a_n$ | 0 | 2 | 4 | 6 | 8 | 10 | 12 | 14 | 16 | 18 | 20 | 22 | 24 | 26 | 28 | 30 | 32 |
| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |

TABLE 2. Some initial values of the Beatty pairs defined in (1) and (2), here $m = 2$ and $p = 3$, together with the differences of their coordinates (=$2n$).

### 2.3. Game theory results.

We may now state our main results. We prove them in Section 6, since our proofs depend on some arithmetic results presented in Section 3,4 and 5.

**Theorem 2.1.** Fix $m, p \in \mathbb{N}$ and let $a$ and $b$ be as in (1) and (2). Then

(i) $\mathcal{P}(m\mathrm{WN}^p) = \{\{a_i, b_i\} \mid i \in \mathbb{N}_0\}$;

(ii) (a) $\mathcal{P}(m\mathrm{WN}^{(p)}) = \{\{a_i, b_i\} \mid i \in \mathbb{N}_0\}$ if and only if $\gcd(m,p) = 1$;

(b) $\mathcal{P}(m\mathrm{WN}^{(m,mp)}) = \{\{a_i, b_i\} \mid i \in \mathbb{N}_0\}$;

(iii) (a) $\mathcal{P}(m{\times}p\mathrm{WN}_l) = \{(a_{ip+l}, b_{ip+l}) \mid i \in \mathbb{N}_0\} \cup \{(b_{ip-l}, a_{ip-l}) \mid i \in \mathbb{N}\}$

(b) $\mathcal{P}(m{\times}p\mathrm{WN}) = \{\{a_i, b_i\} \mid i \in \mathbb{N}_0\}$.

## 3. More on $p$-complementary Beatty sequences

As we have seen, it is customary to represent the solution of 'a removal game on two heaps' as a sequence of pairs of non-negative integers; or more precisely, as pairs of non-decreasing sequences of non-negative integers. This leads us to a certain extension of Beatty's original theorem, to (a pair of) $p$-complementary sequences.

In the literature there is a proof of this theorem in [Bry02], where K. O'Bryant uses generating functions (a method adapted from [BoBo93]). Here, we have chosen to include an elementary proof, in analogy to ideas presented in [HyOs27, Fra82].

**Theorem 3.1** (O'Bryant)**.** Let $0 < \alpha < \beta$ be real numbers such that

$$\frac{1}{\alpha} + \frac{1}{\beta} = 1.$$

Let $p \in \mathbb{N}$. Then we have that $(x_i) = (\lfloor \frac{i\alpha}{p} \rfloor)_{i \in \mathbb{N}_0}$ and $(y_i) = (\lfloor \frac{i\beta}{p} \rfloor)_{i \in \mathbb{N}}$ are $p$-complementary, that is, for each $n \in \mathbb{N}_0$,

$$p = \#\left\{ i \in \mathbb{N}_0 \mid n = \left\lfloor \frac{i\alpha}{p} \right\rfloor \right\} + \#\left\{ i \in \mathbb{N} \mid n = \left\lfloor \frac{i\beta}{p} \right\rfloor \right\}$$

if and only if $\alpha, \beta$ are irrational.

**Proof.** It suffices to establish that exactly $p$ members of the set

$$S = \{0, \alpha, \beta, 2\alpha, 2\beta, \ldots\}$$

is in the interval $[n, n+1)$ for each $n \in \mathbb{N}_0$. But

$$\begin{aligned} \#(S \cap [0, N]) &= \#(\{0, \alpha, 2\alpha, \ldots\} \cap [0, N]) + \#(\{\beta, 2\beta, \ldots\} \cap [1, N]) \\ &= \lfloor pN/\alpha \rfloor + 1 + \lfloor pN/\beta \rfloor, \end{aligned}$$

and since

$$\begin{aligned} pN/\alpha + pN/\beta - 1 &< \lfloor pN/\alpha \rfloor + 1 + \lfloor pN/\beta \rfloor \\ &< pN/\alpha + pN/\beta + 1, \end{aligned}$$

we are done. □

The following result is a special case of the generalization of Beatty's theorem to non-homogeneous sequences in [Sko57, Fra69, Bry03] (so we omit a proof).

**Proposition 3.2** (Skolem, Fraenkel)**.** With notation as in Theorem 3.1, for any integer $0 \le l < p$, the sequences

$$(x_{pi+l}) \text{ and } (y_{pi-l})$$

are complementary. □

The next result is almost immediate by definition of $a$ and $b$ and by Theorem 3.1. It is central to the rest of the paper.

**Lemma 3.3.** Fix $m, p \in \mathbb{N}$ and let $a$ and $b$ be as in (1) and (2) respectively. Then for each $n \in \mathbb{N}_0$ we have that

(i) $a$ and $b$ are $p$-complementary;

(ii) $b_n - a_n = mn$;

(iii) if $p = 1$, then

  (a) $a_{n+1} - a_n = 1$ and $b_{n+1} - b_n = m + 1$, or

  (b) $a_{n+1} - a_n = 2$ and $b_{n+1} - b_n = m + 2$;

(iv) if $p > 1$, then

  (a) $a_{n+1} - a_n = 0$ and $b_{n+1} - b_n = m$, or

  (b) $a_{n+1} - a_n = 1$ and $b_{n+1} - b_n = m + 1$.

**Proof.** Since $\phi_x$ is irrational and $\frac{1}{\phi_x} + \frac{1}{\phi_x + x} = 1$, case (i) is immediate from Theorem 3.1.

For case (ii) put $\nu = \nu_{m,p} = \frac{\phi_{mp}}{p} + \frac{m}{2}$ and observe that

$$b_n - a_n = \left\lfloor n\left(\nu + \frac{m}{2}\right)\right\rfloor - \left\lfloor n(\nu - \frac{m}{2})\right\rfloor.$$

The result follows since

$$\left\lfloor \frac{nm}{2} \right\rfloor - \left\lfloor -\frac{nm}{2} \right\rfloor = \left\lfloor \frac{nm}{2} \right\rfloor + \left\lceil \frac{nm}{2} \right\rceil = mn$$

for all $n \in \mathbb{Z}$.

For case (iii), by [Fra82], we are done. In case $p > 1$, by the triangle inequality, we get

$$\begin{aligned} 0 &< \frac{\phi_{m,p}}{p} \\ &= \frac{1}{p} - \frac{m}{2} + \sqrt{\frac{m}{4} + \frac{1}{p^2}} \\ &< \frac{1}{p} + \frac{1}{p} \\ &\leq 1, \text{ whenever } p > 1, \end{aligned}$$

so that we may estimate

$$a_{n+1} - a_n = \left\lfloor \frac{(n+1)\phi_{mp}}{p} \right\rfloor - \left\lfloor \frac{n\phi_{mp}}{p} \right\rfloor \in \{0, 1\}.$$

Then by (ii) we have

$$\begin{aligned} b_{n+1} - b_n &= a_{n+1} + m(n+1) - a_n - mn \\ &= a_{n+1} - a_n + m, \end{aligned}$$

so that (iv) holds. □

## 4. A unique pair of $p$-complementary Beatty sequences

Suppose that, say $(y_i)$, in Theorem 3.1, is strictly increasing. In this case, we may formulate certain 'uniqueness properties' for our pairs of $p$-complementary Beatty sequences (in case $p = 1$ see also [HeLa06] for extensive generalizations).

**Theorem 4.1.** Fix an integer $p > 1$. Suppose $x = (x_i) = (x_i)_{i \in \mathbb{N}_0}$ and $y = (y_i) = (y_i)_{i \in \mathbb{N}_0}$ are non-decreasing sequences of non-negative integers such that $x_0 = y_0 = 0$ and, for all $n$, $x_n \leq y_n$. Then the following items are equivalent:

(i) $(x_i)$ and $(y_i)_{i>0}$ are $p$-complementary and there is an $m \in \mathbb{N}$ such that, for all $n$, $y_n - x_n = mn$;

(ii) $(x_i)$ and $(y_i)_{>0}$ are $p$-complementary Beatty sequences and $(y_i)$ is (strictly) increasing;

(iii) for some fixed $m \in \mathbb{N}$ and for all $n$, $x_n = a_n^{m,p}$ and $y_n = b_n^{m,p}$;

**Proof.** By Lemma 3.3 it is clear that (iii) implies (ii) and (i). Hence, it suffices to prove that (i) implies (iii) and (ii) implies (iii).

(i)⇒ (iii): Since $x$ is non-decreasing the condition $y_n - x_n = mn$ clearly implies that $y$ is increasing. Since $p > 1$, by this and by $p$-complementarity of $x$ and $y$ we get $x_1 - x_0 = 0$ and $y_1 - y_0 = m$. Suppose further that Lemma (3.3) (iv) holds for each of the $n$ first entries of the sequences $(x_i)$ (exchanged for $(a_i)$) and $(y_i)$ (exchanged for $(b_i)$) respectively. Then, since these sequences are $p$-complementary and $y$ is increasing, we get that $x_{n+1} - x_n = 0$ or $x_{n+1} - x_n = 1$ (otherwise the integer $x_n + 1$ would have at most one representation in the sequences $x$ and $y$, a contradiction). By $y_n - x_n = mn$, we get that Lemma (3.3) (iv) is satisfied for $x$ and $y$. But, by Lemma (3.3) (i) and (ii) the same inductive argument also holds for the sequences $(a_i)$ and $(b_i)$ (in the sense that $x_{n+1} - x_n = 0$ if and only if $a_{n+1} - a_n = 0$), so we are done.

(ii)⇒ (iii): for each $n \in \mathbb{N}_0$, the 'first difference' of a Beatty sequence $z = (z_i)$ is $z_{n+1} - z_n \in \{\delta(z), \Delta(z)\}$ for some non-negative integers $0 \leq \delta(z) < \Delta(z)$.

By the conditions in (ii) we get that $\delta(x) = 0$. Then if $\Delta(x) > 1$ we must have $\delta(y) = 0$ for otherwise the number of representations of 1 is strictly less then $p$, which contradicts our assumption, so we must have $\Delta(x) = 1$.

Clearly we may take $\delta(y) = m > 0$ so we must show that $\Delta(y) = m+1$. Suppose that $\Delta(y) > m+1$. Then me may estimate the number of Sturmian words of the successive differences for the sequence $x$. We already know that (iv a) or (iv b) holds for a Beatty sequence so that $S_x(p(m+1)-1) = p(m+1)$ whenever $\Delta(y) = m+1$, and where $S_x$ is the function that counts the number of words of successive first differences of $x$ of a given length. But exchanging $m+1$ for $m+r$ with $r > 1$ gives all the same words of length $p(m+1)$ and in addition it gives the word $\zeta\zeta\ldots\zeta\eta$ where $\zeta = 00\ldots01$ and $\eta = 00\ldots0$ (where the number of successive $\zeta$:s are $m$ and the number of successive 0:s are $p-1$). Then we get $S_x(p(m+1)-1) = p(m+1)+1$, which contradicts the assumption in (ii) that $x$ is a Beatty sequence.

□

## 5. Recurrence results

We will next generalize the minimal exclusive algorithm in (4). Since our game rules are three-folded we will study three different recurrences. But first we would like to reveal some more structure of our sequences $a$ and $b$.

**Theorem 5.1.** Fix $m, p \in \mathbb{N}$ and let $a$ and $b$ be as in (1) and (2). For each $n \in \mathbb{N}_0$, define

$$\varphi_n = \varphi_n^{m,p} := \frac{a_n + (mp-1)b_n}{m}.$$

Then, for each $n \in \mathbb{N}$, $\varphi_n$ is the greatest integer such that

$$b_n - 1 = a_{\varphi_n}. \tag{6}$$

**Proof.** Notice that, for all $n$,

$$\begin{aligned} \varphi_n &= \frac{a_n + (mp-1)b_n}{m} \\ &= \frac{b_n - mn + (mp-1)b_n}{m} \\ &= \frac{mpb_n - mn}{m} \\ &= pb_n - n, \end{aligned} \tag{7}$$

so that

$$\begin{aligned} \varphi_{n+1} - \varphi_n &= pb_{n+1} - (n+1) - (pb_n - n) \\ &= p(b_{n+1} - b_n) - 1. \end{aligned} \tag{8}$$

For the base case, notice that $b_1 = m$, $a_1 = 0$ and $\varphi_1 = (mp-1)$. Recall that, for each $0 \leq j < m$, there are precisely $p$ representative(s) from $a$ and $b > 0$, (the only representative from $b$ in this interval is $b_0 = 0$ which we by definition do not count). Hence, by $a_0 = 0$, we get that

$$a_{\varphi_1} = a_{mp-1} = m - 1 = b_1 - 1$$

and

$$a_{\varphi_1+1} = a_{mp} = m = b_1.$$

Suppose that (6) holds for all $i \leq n$. Then we need to show that $b_{n+1} - 1 = a_{\varphi_{n+1}}$ and $b_{n+1} = a_{\varphi_{n+1}+1}$.

In case $a_{\varphi_{n+1}} - a_{\varphi_n} = b_{n+1} - b_n$, by $b_n - 1 = a_{\varphi_n}$ and $b_n = a_{\varphi_n+1}$ we get the result, so let us investigate the remaining cases:

(A) $a_{\varphi_{n+1}} - a_{\varphi_n} < b_{n+1} - b_n$;
(B) $a_{\varphi_{n+1}} - a_{\varphi_n} > b_{n+1} - b_n$.

By $p$-complementarity, the number of representations from $a$ and $b$ in the interval

$$\begin{aligned} I_n &:= (a_{\varphi_n}, a_{\varphi_{n+1}}] \\ &= (a_{\varphi_n}, a_{\varphi_n + p(b_{n+1} - b_n) - 1)}] \end{aligned}$$

is $R_n := p(a_{\varphi_{n+1}} - a_{\varphi_n})$, and where the equality is by (8). By assumption, $a_{\varphi_n+1} \in I_n$ so that we have at least $p(b_{n+1} - b_n) - 1$ representations from $a$ in $I_n$. But also $b_n = a_{\varphi_n} + 1 \in I_n$ so that altogether we have at least $p(b_{n+1} - b_n)$ representations in $I_n$. Hence

$$\begin{aligned} p(b_{n+1} - b_n) &\leq R_n \\ &= p(a_{\varphi_{n+1}} - a_{\varphi_n}) \end{aligned}$$

which rules out case (A).

Notice that case (B) implies that $b_{n+1}$ lies in $I_n$ so that $a_{\varphi_n+1} = b_n < b_{n+1} \le a_{\varphi_{n+1}}$. Since both $b_n$ and $b_{n+1}$ lie in $I_n$ we get

$$\begin{aligned} 2 + \varphi_{n+1} - \varphi_n &= p(b_{n+1} - b_n) + 1 \\ &\le p(a_{\varphi_{n+1}} - (a_{\varphi_n} + 1)) + 1 \\ &= p(a_{\varphi_{n+1}} - a_{\varphi_n} + 1) - 2p + 1. \end{aligned} \tag{9}$$

By Lemma 3.3 and our assumption it is obvious that $b_{n+2} > a_{\varphi_{n+1}}$. If in addition $a_{\varphi_{n+1}+1} > a_{\varphi_{n+1}}$ we are done, since $p > 0$ together with (9) and $p$-complementarity give that there is at least one representative to little in $I_n$.

If on the other hand $a_{\varphi_{n+1}+1} = a_{\varphi_{n+1}}$ this forces $m > 1$ which together with (9) implies that there are two representatives to little, unless also $a_{\varphi_{n+1}+2} = a_{\varphi_{n+1}}$. But this forces $p > 2$ which in its turn implies that there are at least three representatives missing, and so on. □

**Remark 3.** For arbitrary $m > 0$ and $p = 1$ it is well known that $a$ and $b$ solve $x_{y_n} = x_n + y_n$. This complementary equation is studied in for example [Conn59, FrKi94, Kim07]. However, we have not been able to find any references for the complementary equation $y_n - 1 = x_{y_n - n}$ (by (7), for the cases $p = 1$, a solution is given by $a = x$ and $b = y$).

For the first of our recursive characterizations, we introduce another notation. A *multiset* (or a sequence) $X$ may be represented as (another) sequence of non-negative integers $(\xi^i)_{i \in \mathbb{N}_0}$, where, for each $i \in \mathbb{N}_0$, $\xi^i = \xi^i(X)$ counts the number of occurrences of $i$ in $X$. For a positive integer $p$, let $\text{mex}^p(\xi^i)$ denote the least non-negative integer $i \in (\xi^i)$ such that $\xi^i < p$.

**Proposition 5.2.** Let $m > 0$ and $p \ge 1$ be integers. Then the recursive characterizations (i), (ii) and (iii) are equivalent. In fact, for each $n \in \mathbb{N}_0$, $x_n = a_n^{m,p}$ and $y_n = b_n^{m,p}$ with notation as in (1) and (2).

(i) For $n \ge 0$,

$$x_n = \text{mex}^p(\xi_n^i),$$

where $\xi_n$ is the multiset, where for each $i \in \mathbb{N}_0$,

$$\begin{aligned} \xi_n^i &= \#\{j \mid i = x_j \text{ or } i = y_j, 0 \le j < n\}, \\ y_n &= x_n + mn. \end{aligned}$$

(ii) For $n \ge 0$,

$$\begin{aligned} &x_n = \text{mex}\{\nu_i^n, \mu_i^n \mid 0 \le i < n\}, \text{ where} \\ &\nu_i^n = x_i \text{ if } n \equiv i \pmod p, \text{ else } \nu_i^n = \infty, \\ &\mu_i^n = y_i \text{ if } n \equiv -i \pmod p, \text{ else } \mu_i^n = \infty; \\ &y_n = x_n + mn. \end{aligned}$$

(iii) For $n \ge 0$ and for each $0 < l < p$,

$$\begin{aligned} x_{pn} &= \text{mex}\{x_{pi}, y_{pi} \mid 0 \le i < n\}, \\ y_{pn} &= x_{pn} + mpn, \\ x_{pn+l} &= \text{mex}\{x_{pi+l}, y_{p(i+1)-l} \mid 0 \le i < n\}, \\ y_{pn+l} &= x_{pn+l} + m(pn + l). \end{aligned}$$

**Proof.** For $p = 1$ each recurrence is equivalent to (4). Hence let $p > 1$ and, for $x \in \mathbb{Z}$, let $\overline{x}$ denote the congruence class of $x$ modulo $p$. For each recurrence it is straightforward to check that $(x_i, y_i) = (a_i, b_i) = (0, mi)$ if $0 \le i < p$. Otherwise, by each definition of mex, we must at least have $x_i > 0$.

For case (i), by Theorem 4.1 and by $y_n = x_n + mn$, it suffices to prove that $(x_i)$ is non-decreasing and that $(x_i)$ and $(y_i)$ are $p$-complementary. But this is immediate by the definition of $\text{mex}^p$.

For case (ii), notice that, for $n \in \mathbb{N}_0$, (see the proof of Theorem 5.1) we have

$$\varphi_n = pb_n - n \equiv -n \pmod{p}. \tag{10}$$

If the assertion does not hold then there is a least $n \ge p$, say $n'$, such that $x_{n'} \ne a_{n'}$. Hence, we have two cases to consider.

(a) $r := x_{n'} < a_{n'}$ : By Theorem 3.2 there are two cases to consider.

Case 1: There is an $i \ge 0$ such that $\varphi(i) + p - 1 < n'$ and

$$y_i = x_{\varphi(i)+1} = x_{\varphi(i)+2} = \ldots = x_{\varphi(i)+p-1} = r.$$

But then, by

$$\{\, \overline{-i},\ \overline{-i+1},\ \ldots,\ \overline{-i+p-1}\,\} = \{\, \overline{0},\ \overline{1},\ \ldots,\ \overline{p-1}\,\} \tag{11}$$

and (10), there is a $j \in \{i, \varphi(i)+1, \ldots, \varphi(i)+p-1\}$ such that either $n' \equiv j \pmod{p}$ and $j \in \{\varphi(i)+1, \ldots \varphi(i)+p-1\}$ which implies $\nu_j^{n'} = r$, or $n' \equiv -j \pmod{p}$ and $j = i$ which implies $\mu_j^{n'} = r$. In either case the choice of $x_{n'} = r$ contradicts the definition of mex.

Case 2: There is an $i \ge 0$ such that $i + p - 1 < n'$ and

$$r = x_i = x_{i+1} = x_{i+2} = \ldots = x_{i+p-1}.$$

This case is similar but simpler, since for this case we rather use that

$$\{\, \overline{i},\ \overline{i+1},\ \ldots,\ \overline{i+p-1}\,\} = \{\, \overline{0},\ \overline{1},\ \ldots,\ \overline{p-1}\,\} \tag{12}$$

(b) $r := a_{n'} < x_{n'}$ : Then our mex-algorithm has refused $r$ as the choice for $x_{n'}$. But then there must be an indice $0 \le j < n'$ such that either $\nu_j^{n'} = r$ or $\mu_j^{n'} = r$. Hence, we get to consider two cases.

Case 1: $\overline{j} = \overline{n'}$ and $r = x_j$. On the one hand, there is a $p \in \mathbb{N}$ such that $pm + j = n'$ On the other hand, there is a greatest $p' \in \mathbb{N}$ such that $a_{n'-p'} = a_{n'-p'+1} = \ldots = a_{n'}$ and by $p$-complementarity $0 \le p' < p$. But then, since $n' - p' > n' - pm = j$, we get $a_j < r = x_j$, which contradicts the minimality of $n'$.

Case 2: $\overline{-j} = \overline{n'}$ and $r = y_j$. Then by Theorem 3.2, $\varphi_j + 1$ is the least indice such that $a_{\varphi_j+1} = a_{n'}$. Then, since (by minimality of $n'$) Theorem 3.2 gives $a_{n'} = b_j$, by $p$-complementarity we get $n' - (\varphi_j + 1) + 1 \le p - 1$. Then $0 < p' := n' - \varphi_j < p$ and so

$$\overline{-j + p'} = \overline{\varphi(j) + p'} = \overline{n'} = \overline{-j},$$

which is nonsense.

For case (iii), suppose that there is a least indice $n' \geq p$ such that $a_{n'} \neq x_{n'}$. Clearly, there exist unique integers, say $t$ and $0 \leq l < p$, such that $tp + l = n'$.

Suppose that $r := a_{n'} > x_{n'}$. Then, since the mex-algorithm did not choose $x_{n'} = r$, there must be an indice $0 \leq t' < t$ such that either $x_{t'p+l} = r$ or $y_{(t'+1)p-l} = r$. But then, by assumption, either $a_{t'p+l} = x_{t'p+l} = a_{tp+l}$ or $b_{(t'+1)p-l} = y_{(t'+1)p-l} = b_{(t+1)p-l} =$. By Proposition 3.2 both cases are ridiculous so we may assume $a_{n'} \leq x_{n'}$.

If $a_{n'} < x_{n'}$, by Proposition 3.2, there is an indice $0 \leq t' < t$ such that either $a_{t'p+l} = x_{n'}$ or $b_{(t'+1)p-l} = x_{n'}$. But this contradicts the mex-algorithm's choice of $x_{n'} < a_{n'}$. Hence, we get $a_{n'} = x_{n'}$. □

## 6. Solving our games

**Proof of Theorem 2.1.** For $p = 1$, the games have identical rules. This case has been established in [Fra82]. The case $m = 1$ has been studied in [Con59] for games of form (ii). (and implicitly for $1{\times}p\mathrm{WN}_l$).

For the rest of the proof assume that $p > 1$. Let us first explain the 'only if' direction of (ii)(a). Denote with $\gamma = \gcd(m, p)$, $p' = p/\gamma$ and $m' = m/\gamma$. Then the positions of the form $(0, mi)$, where $0 \leq i < p'$, are $P$-positions of $m\mathrm{WN}^{(p)}$. Now, $(0, mp')$ is an $N$-position because $m'p = mp'$ implies $(0, p'm) \to (0, 0)$. But, by definition, $b_{p'} = mp'$ if $p' < p$ which holds if and only if $\gamma > 1$.

For each game (we need another notation for Case (iii)), we need to prove that, if $(x, y)$

(A) is of the form $\{a_i, b_i\}$, then none of its options is;

(B) is not of the form $\{a_i, b_i\}$, then there is an option of this form.

By symmetry, we may assume that $0 \leq x \leq y$. Clearly, for our games in (i) and (ii), the final position $(x, y) = (0, 0)$ satisfies (A) but not (B). Hence for these games assume $y > 0$ (and so $i > 0$ for case (A)).

Case (i): Suppose $(x, y) = (a_i, b_i)$ for some $i \in \mathbb{N}_0$. By Lemma 3.3 (i) and (ii), $a$ and $b$ are $p$-complementary and $b_i - b_j \geq m$ for all $j < i$. Then any roob-type option may be blocked off, unless perhaps $a_j < a_i$ and $b_j = b_i$ for some $j < i$. But this is ridiculous since $b$ is strictly increasing. By Lemma 3.3 (ii) we get that, for $i > j$, $b_i - a_i \pm (b_j - a_j) \geq m$. Then an $m$-bishop cannot move $(a_i, b_i) \to \{a_j, b_j\}$, This proves (A).

For (B), since $p \geq 2$ and $b$ is strictly increasing, we may assume $x = a_i$ for some $i$. Then, by Lemma 3.3 (iv): (*) There exists a $j < i$ such that an $m$-bishop can move $(x, y) \to (a_j, b_j)$ (and this move is not a roob-type move) unless $y - x - (b_j - a_j) \geq m$ for all $j$ such that $a_j \leq x$. But then, since $y \geq x + (m + 1)j > b_j$ for all $j$ such that $a_j = x$, by Lemma 3.3 (i), the previous player cannot block off all $p$ roob-type options of the form $\{a_i, b_i\}$.

Case (iia): For this game, the options of the $m$-bishop are identical to those in (i). Let us analyze the $p$-rook.

Hence, suppose $(x, y) = (a_i, b_i)$ for some $i \in \mathbb{N}_0$ and that a $p$-rook can move to $\{a_j, b_j\}$. Then, since $b$ is strictly increasing, there is a $0 \leq j < i$, such that either $b_i \equiv b_j \pmod p$ and $a_i = a_j$, or $b_i \equiv a_j \pmod p$ and $a_i = b_j$. But then, for the first case, since

$\overline{mj} = \overline{b_j - a_j} = \overline{b_i - a_i} = \overline{mi}$ and $\gcd(m,p) = 1$ we must have $\bar{j} = \bar{i}$. this is ridiculous, since by $p$-complementarity we have $0 < i - j < p$. For the second case, by Theorem 5.1, we have that

$$\overline{-mj} = \overline{a_j - b_j} = \overline{b_i - a_i} = \overline{mi} = \overline{m(\varphi(j) + t)} = \overline{m(-j + t)},$$

for some $t \in \{1, \ldots, p-1\}$. This implies $\overline{0} = \overline{mt}$ but then again $\gcd(m,p) = 1$ gives a contradiction.

For (B), we follow the ideas in the second part of Case (i) up until (*). Then, for this game, we rather need to show that there is a $j$ such that $y \equiv b_j \pmod{p}$ and $a_j = x$ or $y \equiv a_j \pmod{p}$ and $b_j = x$. But this follows directly from the proof of Proposition 5.2 (ii)(a).

Case (iib): Suppose $(x,y) = (a_i, b_i)$ for some $i \in \mathbb{N}_0$ but the $(m, mp)$-rook can move to some $\{a_j, b_j\}$ (where $j < i$). Then, we have two cases:

Case 1: $b_i \equiv b_j - r \pmod{mp}$ and $a_i = a_j$, for some $r \in \{0, 1, \ldots, m-1\}$. Then $b_i - a_i \equiv b_j - a_j - r \pmod{mp}$ so that $mi \equiv mj - r \pmod{mp}$ and so $m(i-j) \equiv -r \pmod{mp}$. But this forces $r = 0$ and $i - j \equiv 0 \pmod{p}$ which is impossible since Lemma 3.3 (i) and (iv) implies $i - j \in \{1, 2, \ldots, p-1\}$.

Case 2: $b_i \equiv a_j - r \pmod{mp}$ and $a_i = b_j$, for some $r \in \{0, 1, \ldots, m-1\}$. Then $b_i - a_i \equiv a_j - b_j - r \pmod{mp}$ so that $mi \equiv -mj - r \pmod{mp}$ and so $m(i+j) \equiv -r \pmod{mp}$. By Theorem 5.1 we have that $i = \varphi(j) + s$ for some $s \in \{1, 2, \ldots, p-1\}$. Further, by (10), we have $\varphi(j) \equiv -j \pmod{p}$, so that $m(\varphi(j) + s + j) = ms \equiv -r \pmod{mp}$. Once again we have reached a contradiction.

For (B), in analogy with (*), it suffices to study the $(m, mp)$-rook's options where $y$ is such that $y - x - (b_j - a_j) \geq m$ for all $j$ such that $a_j \leq x = a_i$. Hence, we need to show that there are a $j$ and an $r \in \{0, 1, \ldots m-1\}$ such that

$$y \equiv b_j - r \pmod{mp} \quad \text{and} \quad a_j = x,$$

or

$$y \equiv a_j - r \pmod{mp} \quad \text{and} \quad b_j = x.$$

Clearly, we may choose $r$ such that $y - x + r \equiv 0 \pmod{m}$. Then, for all $j$, we get $ms := y - x + r \equiv b_j \pm a_j \pmod{m}$. Hence, it suffices to find a specific $j$ such that

$$j = \frac{b_j - a_j}{m} \equiv s \pmod{p} \quad \text{and} \quad a_j = x,$$

or

$$-j = \frac{a_j - b_j}{m} \equiv s \pmod{p} \quad \text{and} \quad b_j = x.$$

But then, by (11) and (12), we are done.

Case (iiia): We may assume that $l > 0$. We have already seen that $(a'_i) := (a_{pi+l})_{i \geq 0}$ and $(b'_i) := (b_{p(i+1)-l})_{i \geq 0}$ are complementary. Our proof will be a straightforward extension of those in [Fra82] (which deals with the case $l = 0$) and [Con59] (which implicitly deals with the case $m = 0$). Observe that $a'_0 = a_l = 0$ and $b'_0 = b_{p-l} = m(p-l)$.

For (A), let $(x,y) = (a_i, b_i)$. In case $i = 0$ (by Definition 3 (3a)), the Queen has no options at all, so assume $i > 0$. Proposition 5.2

(iii) gives that $b'_i - a'_i \pm (b'_j - a'_j) \geq mp$ for all $0 \leq j < i$. Then the $mp$-bishop cannot move $(x, y) \to (a'_j, b'_j)$ for any $0 \leq j < i$. Since $a'$ and $b'$ are complementary there is no rook-type option $(a'_i, b'_i) \to \{a'_j, b'_j\}$.

For (B), we adjust the statement (*) accordingly: Suppose $x = a'_i$ (and $y \geq b'_0$). By Proposition 5.2 (iii): If the $mp$-bishop cannot move to $(a'_j, b'_j)$ for any $j < i$ we get that either $i = 0$ or $y - x - (b'_j - a'_j) \geq mp$ for all $j < i$.

But, if $i = 0$ there is a rook-type option to $(a'_0, b'_0)$ (recall here $y > b'_0$), so suppose $i > 0$. But then, since, by Proposition 5.2 (iii), both $a'$ and $b'$ are increasing we get $y \geq b'_j + mp + x - a'_j \geq b'_i + a'_i - a'_j > b'_i$. Hence, for this case, the rook-type move $(x, y) \to (a'_i, b'_i)$ suffices. Suppose on the other hand that $x = b'_i$ with $i \geq 0$. Then, since $y \geq x = b'_i > a'_i$, by complementarity, the Queen may move $(x, y) \to (b'_i, a'_i)$.

Case (iiib): Suppose that the starting position is $(a_i, b_i)$. Then $i = pj + l'$ for some (unique) pair $j \in \mathbb{N}_0$ and $0 \leq l' < p$. The second player should choose $l = l'$. If, on the other hand, the starting position is $(b_i, a_i)$. Then $i = pj - l'$ for some (unique) pair $j \in \mathbb{N}$ and $0 < l' \leq p$. The second player should choose $l = p - l'$. In either case, by Case (iiia), there is no option of the form $(a'_i, b'_i)$.

If the $(x, y)$ is not of this form, again, by Case (iiia), for any (choice of) $0 \leq l < p$, there is a move $(x, y) \to \{a'_i, b'_i\}$ for some $i \geq 0$.

□

## 7. Questions

Can one find a polynomial time solution of $m\mathrm{WN}^{(l,p)}$ for some integers $l \geq 0$, $m > 0$ and $p > 0$ whenever

- $\gcd(m, p) \neq 1$ and $l = 0$, or
- $0 < l \neq m$ or $m \nmid p$?

If this turns out to be complicated, can one at least say something about its asymptotic behaviour?

Denote the solution of $m\mathrm{WN}^{(l,p)}$ with $\{\{c_i^{(l,m,p)}, d_i^{(l,m,p)}\}\}_{i \in \mathbb{N}_0}$. Let us finish off with two tables of the initial $P$-positions of such games.

| | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $d_n^{(0,2,2)}$ | 0 | 3 | 6 | 9 | 12 | 15 | 19 | 22 | 25 | 28 | 31 | 34 | 37 | 40 | 43 | 46 | 49 |
| $c_n^{(0,2,2)}$ | 0 | 0 | 1 | 1 | 2 | 2 | 3 | 4 | 4 | 5 | 5 | 6 | 7 | 7 | 8 | 8 | 9 |
| $d_n - c_n$ | 0 | 3 | 5 | 8 | 10 | 13 | 16 | 18 | 21 | 23 | 26 | 28 | 30 | 33 | 35 | 38 | 40 |
| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |

Table 3. The first few $P$-positions of $2\mathrm{WN}^2$ together with the respective differences of their coordinates.

From these tables one may conclude that: The infinite arithmetic progressions of the sequences

$$(b_i^{m,p} - a_i^{m,p})_{i \in \mathbb{N}_0} = (mi)_{i \in \mathbb{N}_0}$$

| | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $d_n^{(1,2,3)}$ | 0 | 2 | 5 | 7 | 11 | 14 | 16 | 19 | 21 | 26 | 29 | 31 | 36 | 39 | 41 | 44 | 46 |
| $c_n^{(1,2,3)}$ | 0 | 0 | 1 | 1 | 2 | 3 | 3 | 4 | 4 | 5 | 6 | 6 | 7 | 8 | 8 | 9 | 9 |
| $d_n - c_n$ | 0 | 2 | 4 | 6 | 9 | 11 | 13 | 15 | 17 | 21 | 23 | 25 | 29 | 31 | 33 | 35 | 37 |
| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |

TABLE 4. The first few $P$-positions of $2\mathrm{WN}^{(1,3)}$. Notice that (as in Table 3) the successive differences of their coordinates are not in arithmetic progression.

(see also Table 2) are not in general seen among the sequences

$$(d_i^{(l,m,p)} - c_i^{(l,m,p)})_{i\in\mathbb{N}_0}.$$

We believe that the latter sequence is an arithmetic progression if and only if none of the items in our above question is satisfied. We also believe that, for arbitrary constants, $(c_i^{(l,m,p)})$ and $(d_i^{(l,m,p)})_{>0}$ are $p$-complementary. But the solution of these questions are left for some future work.

**Remark 4.** We may also define generalizations of $m\mathrm{WN}^p$ and $m \times p\mathrm{WN}_l$:

Fix $l \in \mathbb{N}$. Let $m\mathrm{WN}_l^p$ be as $m\mathrm{WN}^p$ but where the player may only block off *l-roob-type options* (recall, non-$l$-bishop options). Otherwise, the Queen moves as the $m$-bishop or the rook. Then obviously $m\mathrm{WN}_m^p = m\mathrm{WN}^p$.

Let $u, v \in \mathbb{N}$ and let $m \times p\mathrm{WN}_{u,v}$ be as $m \times p\mathrm{WN}_l$, but the removed (lower left) rectangle has base $u$ and hight $v$. Then for this game the final positions are $(u, 0)$ and $(0, v)$. If $l > 0$, $u = ml$ and $v = m(p - l)$ we get $m \times p\mathrm{WN}_{lm,m(p-l)} = m \times p\mathrm{WN}_l$.

We may ask questions in analogy to the above for these variations. For example, we have found a minimal exclusive algorithm satisfying $\mathcal{P}(m\mathrm{WN}_1^p)$ which is related to a polynomial time construction in [Fra98]. Is there an analog polynomial time construction for $\mathcal{P}(m\mathrm{WN}_1^p)$? Another question is if any of these further generalized games conincide via identical set of $P$-positions? But all this is left for future investigations.

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