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The replacement problem: A polyhedral and complexity analysis. The complete version

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Abstract

We consider an optimization model for determining optimal opportunistic maintenance (that is, component replacement) schedules when data is deterministic. This problem, which generalizes that of Dickman, Epstein, and Wilamowsky [4], is a natural starting point for the modelling of replacement schedules when component lives are non-deterministic, whence a mathematical study of the model is of large interest. We show that the convex hull of the set of feasible replacement schedules is full-dimensional, and that all the necessary inequalities are facet-inducing. We show that when maintenance occasions are fixed, the remaining problem reduces to a linear program; in some cases the latter is solvable through a greedy procedure. We further show that this basic replacement problem is NP-hard.

1 Introduction

The importance of performing maintenance operations well—and of improving the state of the art—seems to be impossible to overestimate: according to [8, Chapter 1], maintenance costs in plants in the US alone accounted for more than \$600 billion in 1981, more than \$800 billion in 1991, and were then projected to increase to become more than \$1.2 trillion by the year 2000. It is stated that these evaluations indicate that on average, one third, or \$250 billion, of all maintenance dollars are wasted through ineffective maintenance management methods. According to a recent study (made by Forum Vision Instandhaltung, Germany), maintenance costs in the manufacturing industry within the EU amount to roughly \$2 trillion per year. Studies over the last 20 years have indicated that around Europe, the direct cost of maintenance is equivalent to between 4% and 8% of total sales turnover. Also in these cases, it is quite natural to assume that not all the money spent is spent well: according to information gathered by the Swedish Center for Maintenance Management, maintenance is quite often performed too frequently, and surprisingly often equipment failure is triggered by inspections and the condition monitoring itself. One objective with constructing and studying mathematical models for the optimization of the scheduling of maintenance and inspection activities is to mitigate some of these problems, and to thereby contribute to a shift of focus from considering maintenance as mainly a cost-inducing activity to that of an investment in availability improvement.

One strategy for planning maintenance activities is so called opportunistic maintenance, in which a mathematical model is utilized to decide whether, at a (possibly already planned) maintenance occasion, more than the necessary maintenance activities should be performed. According

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to Dickman et al. [4], Jorgenson and Radner [7] introduced the original opportunistic replacement/maintenance problem. They considered a system of stochastically failing components, which incur extensive maintenance costs upon failure, that is, for shutting down and disassembling the system. When the system is down for whatever reason, components may be replaced at no additional maintenance cost. Thereby, opportunities arise to trade off remaining life of components in order to avoid maintenance costs associated with component failure, perhaps already in the near future. This is their main motive for studying the problem.

In [6, 3] a problem associated with fusion power plants is addressed. For safety reasons, components are assigned life limits at which removal is mandatory, and before which the probability of failure is effectively zero. Their problem therefore is deterministic. (Later, in [5], this problem was solved for the case of an infinite horizon and for a system of two components.)

Our original motivation for studying the replacement problem was a project concerning the optimization of jet engine maintenance schedules at Volvo Aero Corporation (VAC). An aircraft engine consists of thousands of parts. Some of the parts are safety-critical, as in fusion power plants, which means that if they fail there will be an engine breakdown, possibly with catastrophic consequences. Therefore, the safety-critical parts have fixed life limits, and must be replaced before they are reached. Hence, we consider the safety-critical parts as having deterministic lives. All other parts of the engine are considered to have stochastic lives; therefore, their life limits need to be estimated, which in turn makes it much more difficult to compute a reliable replacement schedule. For some of these parts failure distributions may be computed from historical data and monitoring observations. This information could then be discretized and be used as an input into optimization models. This was the subject of two PhD projects (see [1, 10]).

Taking into account parts that are either deterministic or stochastic in a unified model is quite a lot more complex than what has been studied in the past; even stochastic models found in the literature typically do not incorporate failure distributions but failure intensities only, and solution approaches provide simple maintenance policies for infinite horizon problems.

The purpose of the present paper is to initiate a detailed mathematical study of opportunistic maintenance optimization models through an analysis of a basic replacement model. In the near future we will consider extensions to this problem. In a recent case study at Volvo Aero Corporation the structure of the jet engine, and in particular the disassembly of its parts, has been taken better into account through detailed cost dependencies between components. Further, recent applications of opportunistic maintenance optimization to the generation of wind and nuclear power have resulted in the study of multi-stage stochastic programming models, properly incorporating stochastic information about the remaining lives of components.

2 The basic replacement model

Consider a set $\mathcal{N}$ of components and define $N = |\mathcal{N}|$. Consider also a set $\mathcal{T} = \{1, \ldots, T\}$ of times, with $T \geq 2$. Suppose that a new component $i \in \mathcal{N}$ has a (deterministic) life of is T_i time steps. (Without loss of generality, $2 \leq T_i \leq T$.) The purchase cost at time $t \in \mathcal{T}$ for component i is $c_{it} > 0$. There is a fixed cost of $d_t > 0$ associated with a maintenance occasion at time t, independent of the number of parts replaced at this occasion.

The objective is to minimize the cost of having a working system between times 1 and T.

Letting

$$z_t = \begin{cases} 1, & \text{if maintenance shall occur at time } t, \\ 0, & \text{otherwise,} \end{cases} \qquad t \in \mathcal{T},$$

$$x_{it} = \begin{cases} 1, & \text{if part } i \text{ shall be replaced at time } t, \\ 0, & \text{otherwise,} \end{cases} \qquad i \in \mathcal{N}, \quad t \in \mathcal{T},$$

the replacement problem is that to

$$\begin{aligned}
\underset{(x,z)}{\text{minimize}} \quad & \sum_{t \in \mathcal{T}} \left(\sum_{i \in \mathcal{N}} c_{it} x_{it} + d_t z_t \right), && && \text{(1a)} \\
\text{subject to} \quad & \sum_{t=\ell+1}^{\ell+T_i} x_{it} \geq 1, && \ell = 0, \dots, T - T_i, \quad i \in \mathcal{N}, && \text{(1b)} \\
& x_{it} \leq z_t, && t \in \mathcal{T}, \quad i \in \mathcal{N}, && \text{(1c)} \\
& x_{it} \geq 0, && t \in \mathcal{T}, \quad i \in \mathcal{N}, && \text{(1d)} \\
& z_t \leq 1, && t \in \mathcal{T}, && \text{(1e)} \\
& x_{it} \in \{0,1\}, && t \in \mathcal{T}, \quad i \in \mathcal{N}. && \text{(1f)} \\
& z_t \in \{0,1\}, && t \in \mathcal{T}. && \text{(1g)}
\end{aligned}$$

The constraint (1b) ensures that each part is replaced before the end of its life; the constraint (1c) enforces the payment of the fixed maintenance cost d_t whenever any part is replaced at time t, and, once this cost is paid, induces maintenance opportunities at no extra maintenance cost. The remaining constraints are definitional; the removal of (1f)–(1g) amounts to a continuous relaxation of the problem.

This problem originates from [4]; the model in [1] replaces the original constraints $\sum_{i \in \mathcal{N}} x_{it} \leq N z_t$, $t \in \mathcal{T}$, with the equivalent but stronger constraints (1c); the model (1), in turn, generalizes the cost function in [1] to allow for time dependency.

As a numerical illustration, we consider an instance of (1) with $T = 60$, $N = 4$, $T_1 = 13$, $T_2 = 19$, $T_3 = 34$, $T_4 = 18$, $c_{1t} = 80$, $c_{2t} = 185$, $c_{3t} = 160$, and $c_{4t} = 125$ for all $t \in \mathcal{T}$. The data is chosen so that the relations between the lives and the costs are similar to those for the fan module of the RM12 engine, maintained at VAC. The model is solved for $d_t = 0$, 10, and 1000 for all t (where $d_t = 10$ represents the most reasonable value in the maintenance situation at VAC). For $d_t = 0$, the optimal total number of replacement occasions is 11 and there is no advantage with replacing a component before its life limit is reached. Increasing the value of d_t from 0 to 10 decreases the optimal total number of replacement occasions from 11 to five. It is now beneficial to replace the components in larger groups and they are often replaced before their respective life limits are reached. For $d_t = 1000$ it is very important to utilize the opportunity to replace several components at the same time. The optimal total number of replacement occasions is four (the least feasible number of replacement occasions for this instance).

Figure 1 shows optimal maintenance schedules for each of the three cases. The horizontal axis represents the 60 time steps and each maintenance occasion is represented by a vertical bar, where a dot at a certain height represents a component of the corresponding type being replaced. The figure clearly illustrates how opportunistic replacement becomes more beneficial with an increasing fixed maintenance cost.

The remainder of the paper is organized as follows. Section 3 presents some properties characterizing an optimal maintenance schedule. In particular, we show that if the variables z_t, associated with the maintenance occasions, are fixed to binary values, then the polyhedron arising from the continuous relaxation of the variables x_{it}, associated with the replacement of the parts, is integral (i.e., possesses integral extreme points). In other words, the integrality restrictions on those variables may be dropped. Moreover, we provide results, in part reached in

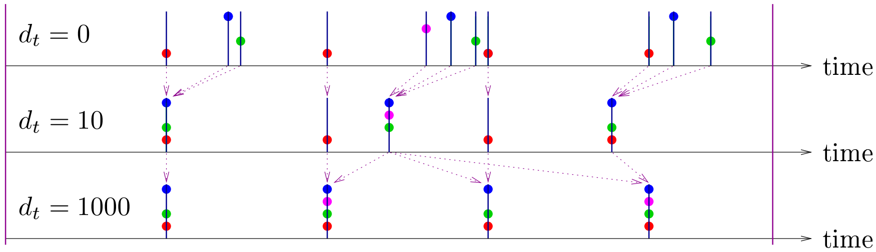

Figure 1: *Optimal maintenance schedules for $d_t = 0$, 10, and 1000 for all t. When d_t increases from 0 to 10 the replacement occasions 1–3, 5–7, and 9–11, are grouped into one occasion for each of the three groups. When d_t is increased from 10 to 1000, the last four maintenance occasions are rearranged into three occasions; the reduction from five to four occasions results in several more component replacements.*

[4], on the possibility to a priori remove some maintenance occasions from consideration. In Section 4 we perform a polyhedral study of the convex hull of the set of feasible solutions to the model (1), referred to as the *replacement polytope*. We show that the replacement polytope is full-dimensional under natural assumptions and that the necessary inequality constraints (1b)–(1e) in the original formulation (1) are facet-defining. Further, we show that they are not sufficient to completely describe the replacement polytope. By using Chvátal–Gomory rounding we construct a new class of valid inequalities and show that they are facet-defining. Finally, in Section 5, we establish that the problem (1) is NP-complete, based on a reduction from the set covering problem. We conclude with remarks on current and planned research endeavours.

3 Special properties of optimal solutions

We here present some special properties of the problem (1). First we show that the integrality constraints on the variables x_{it} can be relaxed. Then we review a result from [4] and show that for instances of the problem where costs are non-increasing with time the replacement activities will only occur at times that are sums of positive integer multiples of life limits. Finally, we show that, again for non-increasing costs and given fixed binary values of the z_t variables, the optimal x_{it} values can be obtained by a greedy algorithm.

3.1 Integrality property

The following proposition concerns integrality properties of the polyhedron in $\mathbb{R}^{N \times T}$ defined by (1b)–(1d), when the variables z_t, $t \in \mathcal{T}$, are fixed to binary values. Accordingly, we let $\tilde{z}_t \in \{0, 1\}$, $t \in \mathcal{T}$, and define $\widetilde{\mathcal{T}} = \{t \in \mathcal{T} \mid \tilde{z}_t = 1\}$; hence, $\tilde{z}_t = 0$, $t \in \mathcal{T} \setminus \widetilde{\mathcal{T}}$.

Proposition 1 (integral polyhedron) *The polyhedron defined by* (1b) *and*

$$x_{it} \leq 1, \qquad t \in \widetilde{\mathcal{T}}, \tag{2a}$$
$$x_{it} \leq 0, \qquad t \in \mathcal{T} \setminus \widetilde{\mathcal{T}}, \tag{2b}$$

for $i \in \mathcal{N}$, is integral.

Proof We derive the result by showing that the constraint matrix of (1b) and (2) is totally unimodular (TU) using the characterization in [9, pp. 542–543, Thm. 2.7]. The inequalities (1b) and (2) separate over $i \in \mathcal{N}$; therefore it suffices to show that the constraint matrix of the inequality system

$$\sum_{t=\ell+1}^{\ell+T_i} x_{it} \geq 1, \qquad \ell \in \{0, \ldots, T - T_i\}, \tag{3a}$$

$$-x_{it} \geq \begin{cases} -1, & t \in \widetilde{\mathcal{T}}, \\ 0, & t \in \mathcal{T} \setminus \widetilde{\mathcal{T}}, \end{cases} \tag{3b}$$

is TU for each $i \in \mathcal{N}$. Let $A^i \in \mathbb{B}^{(T-T_i+1)\times T}$ be the constraint matrix defined by the left hand sides of the inequalities (3a), that is, for each $r \in \{0, \ldots, T - T_i\}$, let $a^i_{rs} = 1$ for $s \in \{r + 1, \ldots, r + T_i\}$ and $a^i_{rs} = 0$ for $s \in \mathcal{T} \setminus \{r + 1, \ldots, r + T_i\}$. The essential property of the matrix A^i is that the ones appear consecutively in each row, that is, if $a^i_{r\ell} = a^i_{rk} = 1$ and $1 \leq \ell \leq k \leq T$, then $a^i_{rs} = 1$ for all $s \in \{\ell, \ldots, k\}$. Let $B \in \mathbb{B}^{T\times T}$ be the constraint matrix defined by the left hand sides of the inequalities (3b). Then $B = -I^T$ (minus the $T \times T$ identity matrix). Therefore, it is enough to show that property (ii) of [9, pp. 542–543, Thm. 2.7] is satisfied for $\mathcal{J} = \mathcal{T}$. Let $\mathcal{J}_1 = \{s \in \mathcal{T} \mid s \text{ odd}\}$ and $\mathcal{J}_2 = \mathcal{T} \setminus \mathcal{J}_1$. For each $\ell \in \{0, \ldots, T - T_i\}$ it holds that

$$\sum_{s\in\mathcal{J}_1} a^i_{\ell s} - \sum_{s\in\mathcal{J}_2} a^i_{\ell s} = \begin{cases} 1, & \text{if } T_i \text{ is odd and } \ell \text{ is even}, \\ -1, & \text{if } T_i \text{ is odd and } \ell \text{ is odd}, \\ 0, & \text{if } T_i \text{ is even}, \end{cases}$$

and for each $\ell \in \mathcal{T}$ it holds that

$$\sum_{s\in\mathcal{J}_1} b_{\ell s} - \sum_{s\in\mathcal{J}_2} b_{\ell s} = \begin{cases} -1, & \text{if } \ell \text{ is even}, \\ 1, & \text{if } \ell \text{ is odd}. \end{cases}$$

It follows that the property (ii) stated in [9, pp. 542–543, Thm. 2.7] holds. Hence, the constraint matrix $((A^i)^{\mathrm{T}}, B^{\mathrm{T}})^{\mathrm{T}}$ of (3) is TU. Since the right-hand sides of (3) are all integral it follows from [2, p. 221] that the corresponding polyhedron is integral. □

The result of Proposition 1 implies that the binary requirements on the variables x_{it} can be relaxed, provided that the model (1) is to be solved using an algorithm that detects extreme optimal solutions to linear programming subproblems.

3.2 Non-increasing costs

The results presented in this subsection are derived for instances of the model (1) for which the costs are non-increasing with time, that is, $c_{i,t+1} \leq c_{it}$ and $d_{t+1} \leq d_t$ for all i and t.

The following proposition extends [4, Thm. 2] from three to N components. It implies that we may a priori set $z_t = 0$ in (1) for each t which is not a non-negative sum of life limits.

Proposition 2 (a priori variable elimination) *For all instances of* (1) *with costs fulfilling* $c_{i,t+1} \leq c_{it}$ *and* $d_{t+1} \leq d_t$ *for all* i *and* t*, an optimal solution exists with* $z_t = 0$ *for every* $t \in \mathcal{T}$ *which is not a sum of non-negative integer multiples of the life limits (that is, for every* $t \in \mathcal{T}$ *such that* $\left\{\ell \in \mathbb{Z}^N_+ \mid \sum_{i\in\mathcal{N}} \ell_i T_i = t\right\} = \emptyset$*).*

Proof Consider a feasible solution to (1) with $z_t = 1$ for some t that is not a positive sum of life limits T_i, $i \in \mathcal{N}$. Let the corresponding objective value be f. Assume, without loss of generality, that t is the earliest time with such a property (that is, all previous replacements have occurred at times that are positive sums of lives T_i). This implies that all parts must have remaining lives greater than zero, since otherwise t would have been a positive sum of lives. We can therefore postpone all replacements made at t to the time $\tilde{t} > t$, when some part $i \in \mathcal{N}$ reaches a remaining life of zero. The time $\tilde{t}$ is then a positive sum of lives. The adjusted solution, with $z_t = 0$ and $z_{\tilde{t}} = 1$, is feasible and the corresponding cost $\tilde{f} \leq f$. This procedure can be applied to all times t that are not positive sums of life limits and for which $z_t = 1$. The result follows. □

The next proposition shows that given values of z_t, $t \in \mathcal{T}$, the greedy algorithm defined below produces corresponding optimal values of x_{it}, $i \in \mathcal{N}$, $t \in \mathcal{T}$.

Definition 3 (greedy rule) Let the set $\widetilde{\mathcal{T}}$ be defined by the fixed values of the variables z_t, $t \in \mathcal{T}$ (as in Section 3.1). The greedy rule for the basic replacement problem is defined as follows: Give all parts their initial lives. (*) Move to the earliest time $\tilde{t} \in \widetilde{\mathcal{T}}$. Let all parts age. Replace each part having a remaining life that is shorter than the time left to next replacement occasion. Let $\widetilde{\mathcal{T}} := \widetilde{\mathcal{T}} \setminus \{\tilde{t}\}$. If $\widetilde{\mathcal{T}} \neq \emptyset$, repeat from (*). □

Proposition 4 (greedy rule yields optimum) *Consider the problem* (1) *and assume that* $c_{i,t+1} \leq c_{it}$ *holds for all* i *and all* t. *Let the set* $\widetilde{\mathcal{T}}$ *be such that for each* $t \in \widetilde{\mathcal{T}} \cup \{0\}$ *there is an* $s \in \widetilde{\mathcal{T}} \cup \{T+1\}$ *with* $0 < s - t \leq \min_{i \in \mathcal{N}} T_i$. *Let* $\tilde{z}_t$, $t \in \mathcal{T}$, *be defined by* $\widetilde{\mathcal{T}}$. *Then the greedy rule of Definition 3 yields corresponding optimal values of the variables* x_{it}, $i \in \mathcal{N}$, $t \in \mathcal{T}$.

Proof Let $\tilde{x}_{it}$, $i \in \mathcal{N}$, $t \in \mathcal{T}$, be the solution obtained by the greedy rule. Then, $(\tilde{x}, \tilde{z})$ is clearly feasible in (1). Let $(\bar{x}, \tilde{z})$, such that $\bar{x} \neq \tilde{x}$, be feasible in (1). We can then postpone any replacement corresponding to $\bar{x}_{it}$ that is possible to postpone to the next time in $\widetilde{\mathcal{T}}$. This will transform $\bar{x}$ to $\tilde{x}$ without introducing any additional replacements and at a non-increasing cost. Hence, $\sum_{i \in \mathcal{N}} \sum_{t \in \mathcal{T}} c_{it}(\tilde{x}_{it} - \bar{x}_{it}) \leq 0$ holds and the result follows. □

4 The replacement polytope

We let the set $S \subset \mathbb{R}^{N \times T} \times \{0,1\}^T$ be defined by the values of the variables (x, z) that fulfil the constraints (1b)–(1e), (1g). The convex hull of S, denoted $\operatorname{conv} S$, is called the *replacement polytope*. By studying the facial structure of $\operatorname{conv} S$ and thereby describing it by a finite set of linear inequalities, it is possible to solve the problem using linear programming techniques. Our ambition here is to take the first steps towards such a complete linear description of the replacement polytope.

We first compute the dimension of the replacement polytope and show that all the necessary inequalities in (1b)–(1e) define facets of the same. However, by an example we show that these basic inequalities do not completely define $\operatorname{conv} S$. We then derive a new class of facets by using Chvátal–Gomory rounding.

4.1 The dimension and basic facets of $\operatorname{conv} S$

In this section we derive the dimension of the replacement polytope $\operatorname{conv} S$ and investigate the inequalities used to define S. Under weak and natural assumptions we show that the replacement polytope is full-dimensional. Further, we show that all inequalities that are necessary to define the replacement polytope are facets of the same.

Proposition 5 (dimension of the replacement polytope) *If* $T_i \geq 2$ *for all* $i \in \mathcal{N}$, *then the dimension of* $\operatorname{conv} S$ *is* $(N+1)T$, *that is,* $\operatorname{conv} S$ *is full-dimensional.*

Proof First note that since $S \subseteq \mathbb{R}^{(N+1)T}$ it holds that $\dim(\operatorname{conv} S) \leq (N+1)T$. Let the vectors $(x^k, z^k) \in \mathbb{B}^{(N+1)T}$, $k \in \{0, \ldots, (N+1)T\}$, be given by the following. For $i \in \mathcal{N}$ and $t \in \mathcal{T}$, let

$x_{it}^k = 0$ if $k \in \{(N+1)(t-1)+i, (N+1)t\}$ and $x_{it}^k = 1$ otherwise. For $t \in \mathcal{T}$, let $z_t^k = 0$ if $k = (N+1)t$ and $z_t^k = 1$ otherwise. Since $T_i \geq 2$ for $i \in \mathcal{N}$ it holds that $\sum_{t=\ell+1}^{\ell+T_i} x_{it}^k \geq 1$ for all $i \in \mathcal{N}$, all $\ell \in \{0, \dots, T - T_i\}$, and all $k \in \{0, \dots, (N+1)T\}$.

Moreover, for all $t \in \mathcal{T}$ and $k \in \{0, \dots, (N+1)T\}$ such that $z_t^k = 0$ it holds that $x_{it}^k = 0$, $i \in \mathcal{N}$; it follows that $(x^k, z^k) \in S$. It can be verified that the only solution to the system

$$\sum_{k=0}^{(N+1)T} x_{it}^k \alpha_k = 0, \quad i \in \mathcal{N}, \qquad \sum_{k=0}^{(N+1)T} z_t^k \alpha_k = 0, \quad t \in \mathcal{T}, \qquad \sum_{k=0}^{(N+1)T} \alpha_k = 0,$$

is $\alpha_k = 0$, $k \in \{0, \dots, (N+1)T\}$, implying that the vectors (x^k, z^k), $k \in \{0, \dots, (N+1)T\}$, are affinely independent. Hence, it holds that $\dim(\operatorname{conv} S) \geq (N+1)T$, thus implying that $\dim(\operatorname{conv} S) = (N+1)T$. The proposition follows. $\square$

The replacement polytope is *not* full-dimensional if $T_i = 1$ for some $i \in \mathcal{N}$, since it then holds that $x_{it} = z_t = 1$, $t \in \mathcal{T}$, for all $(x, z) \in \operatorname{conv} S$. Letting $A^=$ denote the matrix corresponding to the equality subsystem of $\operatorname{conv} S$, this would yield that $\operatorname{rank} A^= \geq 2T$ and thus that $\dim(\operatorname{conv} S) \leq (N-1)T$. However, the case that $T_i = 1$ is not interesting in practice since it would mean that component i must be replaced—and thus maintenance must be performed—at *every* time step.

The following result from polyhedral combinatorics ([9, Thm. 3.6 of Ch. I.4]) is utilized to determine facets of $\operatorname{conv} S$.

THEOREM 6 (characterization of facets) *Let P be a full-dimensional polyhedron and $F = \{x \in P \mid \pi^{\mathrm{T}} x = \pi_0\}$ a proper face of P (i.e., $\emptyset \neq F \subset P$). The following two statements are equivalent:*

1. *F is a facet of P.*
2. *If $\lambda^{\mathrm{T}} x = \lambda_0$ for all $x \in F$ then $(\lambda, \lambda_0) = \alpha(\pi, \pi_0)$ for some $\alpha \in \mathbb{R}$.* $\square$

PROPOSITION 7 (the inequalities (1b) define facets) *If $T_i \geq 2$ for all $i \in \mathcal{N}$, then each of the inequalities $\sum_{t=\ell+1}^{\ell+T_r} x_{rt} \geq 1$, $\ell = 0, \dots, T - T_r$, $r \in \mathcal{N}$, defines a facet of* $\operatorname{conv} S$.

PROOF Since $T_i \geq 2$ for $i \in \mathcal{N}$, $\operatorname{conv} S$ is full-dimensional (cf. Proposition 5). Hence, we can use the uniqueness characterization of the facet description from Theorem 6 to show the proposition.

For each $r \in \mathcal{N}$ and each $\ell \in \{0, \dots, T - T_r\}$, let $\widehat{F}_{r\ell} = \{\, (x, z) \in \operatorname{conv} S \mid \sum_{t=\ell+1}^{\ell+T_r} x_{rt} = 1 \,\}$. Further, let $x_{it}^0 = z_t^0 = 1$, $i \in \mathcal{N}$, $t \in \mathcal{T}$. Since $T_i \geq 2$ it follows that $(x^0, z^0) \in S \setminus \widehat{F}_{r\ell}$. Then, defining the vector $(x^{\mathrm{A}}, z^{\mathrm{A}})$ as $x_{it}^{\mathrm{A}} = 0$ if $i = r$ and $t \in \{\ell+2, \dots, \ell+T_r\}$, $x_{it}^{\mathrm{A}} = 1$ otherwise, and $z_t^{\mathrm{A}} = 1$, $t \in \mathcal{T}$, it follows that $(x^{\mathrm{A}}, z^{\mathrm{A}}) \in \widehat{F}_{r\ell}$ and hence that $\widehat{F}_{r\ell}$ is a proper face of $\operatorname{conv} S$.

Moreover, there exist values of $\lambda \in \mathbb{R}^{N \times T}$, $\mu \in \mathbb{R}^T$, and $\rho \in \mathbb{R}$ such that the equation

$$\sum_{t \in \mathcal{T}} \left(\sum_{i \in \mathcal{N}} \lambda_{it} x_{it} + \mu_t z_t \right) = \rho \tag{4}$$

is satisfied for all $(x, z) \in \widehat{F}_{r\ell}$. We will show that for any value of $\rho \in \mathbb{R}$, in a solution to (4) the following hold: $\lambda_{it} = \rho$ if $i = r$ and $t \in \{\ell+1, \dots, \ell+T_r\}$, $\lambda_{it} = 0$ otherwise; $\mu_t = 0$ for $t \in \mathcal{T}$.

Choose any $i \in \mathcal{N} \setminus \{r\}$ and any $t \in \mathcal{T}$. Let, for $j \in \mathcal{N}$ and $k \in \mathcal{T}$, $x_{jk}^1 = 0$ if $(j,k) = (i,t)$, $x_{jk}^1 = x_{jk}^{\mathrm{A}}$ otherwise, and let $z^1 = z^{\mathrm{A}}$. It follows that $(x^1, z^1) \in \widehat{F}_{r\ell}$. The vectors $(x^{\mathrm{A}}, z^{\mathrm{A}})$ and (x^1, z^1), respectively, inserted in (4) then yield that $\lambda_{it} = 0$. It follows that $\lambda_{it} = 0$ for all $i \in \mathcal{N} \setminus \{r\}$ and all $t \in \mathcal{T}$.

For each $t \in \mathcal{T} \setminus \{\ell+1, \ldots, \ell+T_r+1\}$, let, for $i \in \mathcal{N}$ and $k \in \mathcal{T}$, $x^2_{ik} = 0$ if $(i,k) = (r,t)$, $x^2_{ik} = x^{\mathrm{A}}_{ik}$ otherwise, and let $z^2 = z^{\mathrm{A}}$. It follows that $(x^2, z^2) \in \widehat{F}_{r\ell}$. The vectors $(x^{\mathrm{A}}, z^{\mathrm{A}})$ and (x^2, z^2), respectively, inserted in (4) then yield that $\lambda_{rt} = 0$ for all $t \in \mathcal{T} \setminus \{\ell+1, \ldots, \ell+T_r+1\}$.

Further, let, for $i \in \mathcal{N}$ and $t \in \mathcal{T}$, $x^{\mathrm{B}}_{it} = 0$ if $i = r$ and $t \in \{\ell+1, \ldots, \ell+T_r-1\}$, $x^{\mathrm{B}}_{it} = 1$ otherwise, and let $z^{\mathrm{B}}_t = 1$, $t \in \mathcal{T}$. Moreover, let, for $i \in \mathcal{N}$ and $t \in \mathcal{T}$, $x^3_{it} = 0$ if $(i,t) = (r, \ell+T_r+1)$, $x^3_{it} = x^{\mathrm{B}}_{it}$ otherwise, and let $z^3 = z^{\mathrm{B}}$. It follows that $(x^3, z^3) \in \widehat{F}_{r\ell}$. The vectors $(x^{\mathrm{B}}, z^{\mathrm{B}})$ and (x^3, z^3), respectively, inserted in (4) then yield that $\lambda_{r,\ell+T_r+1} = 0$. The equation (4) can then be rewritten as

$$\sum_{t=\ell+1}^{\ell+T_r} \lambda_{rt} x_{rt} + \sum_{t \in \mathcal{T}} \mu_t z_t = \rho. \tag{5}$$

For each $t \in \mathcal{T} \setminus \{\ell+1, \ell+T_r+1\}$, let, for $i \in \mathcal{N}$, $x^4_{ik} = z^4_k = 0$ if $k = t$, $x^4_{ik} = x^{\mathrm{A}}_{ik}$ and $z^4_k = z^{\mathrm{A}}_k$ otherwise. It follows that $(x^4, z^4) \in \widehat{F}_{r\ell}$. The vectors $(x^{\mathrm{A}}, z^{\mathrm{A}})$ and (x^4, z^4), respectively, inserted in (5) then yield that $\mu_t = 0$ for $t \in \mathcal{T} \setminus \{\ell+1, \ell+T_r+1\}$.

Further, for each $t \in \{\ell+1, \ell+T_r+1\}$, let, for $i \in \mathcal{N}$, $x^5_{ik} = z^5_k = 0$ if $k = t$, $x^5_{ik} = x^{\mathrm{B}}_{ik}$ and $z^5_k = z^{\mathrm{B}}_k$ otherwise. It follows that $(x^5, z^5) \in \widehat{F}_{r\ell}$. The vectors $(x^{\mathrm{B}}, z^{\mathrm{B}})$ and (x^5, z^5), respectively, inserted in (5) then yield that $\mu_{\ell+1} = \mu_{\ell+T_r+1} = 0$. Equation (5) can then be rewritten as

$$\sum_{t=\ell+1}^{\ell+T_r} \lambda_{rt} x_{rt} = \rho. \tag{6}$$

For each $t \in \{\ell+2, \ldots, \ell+T_r\}$, let for $i \in \mathcal{N}$ and $k \in \mathcal{T}$, $x^6_{ik} = 0$ if $(i,k) = (r, \ell+1)$, $x^6_{ik} = 1$ if $(i,k) = (r,t)$, and $x^6_{ik} = x^{\mathrm{A}}_{ik}$ otherwise, and let $z^6 = z^{\mathrm{A}}$. It follows that $(x^6, z^6) \in \widehat{F}_{r\ell}$. The vectors $(x^{\mathrm{A}}, z^{\mathrm{A}})$ and (x^6, z^6), respectively, inserted in (6) then yield that $\lambda_{r,\ell+1} = \rho = \lambda_{rt}$, $t \in \{\ell+2, \ldots, \ell+T_r\}$ Since $(x^{\mathrm{A}}, z^{\mathrm{A}}) \in \widehat{F}_{r\ell}$ it follows that $\lambda_{rt} = \rho$, $t \in \{\ell+1, \ldots, \ell+T_r\}$. The equation (6) can then be rewritten as $\sum_{t=\ell+1}^{\ell+T_r} \rho x_{rt} = \rho$. From [9, pp. 91–92] then follows that the inequality $\sum_{t=\ell+1}^{\ell+T_r} x_{rt} \geq 1$ defines a facet of $\operatorname{conv} S$. □

Proposition 8 (the inequalities (1c) define facets) *If $T_i \geq 2$ for all $i \in \mathcal{N}$, then each of the inequalities $x_{rs} \leq z_s$, $r \in \mathcal{N}$, $s \in \mathcal{T}$, defines a facet of* $\operatorname{conv} S$.

Proof Since $T_i \geq 2$ for $i \in \mathcal{N}$, $\operatorname{conv} S$ is full-dimensional (cf. Proposition 5). Hence, we can use the uniqueness characterization of the facet description from Theorem 6 to show the proposition.

For each $r \in \mathcal{N}$ and each $s \in \mathcal{T}$, let $F_{rs} = \{ (x,z) \in \operatorname{conv} S \mid x_{rs} = z_s \}$. Further, let, for $i \in \mathcal{N}$ and $t \in \mathcal{T}$, $x^0_{it} = 0$ if $(i,t) = (r,s)$, $x^0_{it} = 1$ otherwise, and let $z^0_t = 1$, $t \in \mathcal{T}$. It follows that $(x^0, z^0) \in S \setminus F_{rs}$. Then, letting $x^{\mathrm{A}}_{it} = z^{\mathrm{A}}_t = 1$, $i \in \mathcal{N}$, $t \in \mathcal{T}$, it follows that $(x^{\mathrm{A}}, z^{\mathrm{A}}) \in F_{rs}$ and hence that F_{rs} is a proper face of $\operatorname{conv} S$.

Moreover, there exists values of $\lambda \in \mathbb{R}^{N \times T}$, $\mu \in \mathbb{R}^T$, and $\rho \in \mathbb{R}$ such that the equation (4) is satisfied for all $(x,z) \in F_{rs}$. We will show that for any value of $\mu_s \in \mathbb{R}$, in a solution to (4) the following hold: $\lambda_{it} = -\mu_s$ if $(i,t) = (r,s)$, $\lambda_{it} = 0$ otherwise; $\mu_t = 0$ for $t \in \mathcal{T} \setminus \{s\}$; $\rho = 0$.

For each $\ell \in \mathcal{T} \setminus \{s\}$, let, for $j \in \mathcal{N}$ and $t \in \mathcal{T}$, $x^1_{jt} = 0$ if $(j,t) = (r,\ell)$, $x^1_{jt} = x^{\mathrm{A}}_{jt}$ otherwise, and let $z^1 = z^{\mathrm{A}}$. It follows that $(x^1, z^1) \in F_{rs}$. The vectors $(x^{\mathrm{A}}, z^{\mathrm{A}})$ and (x^1, z^1), respectively, inserted in (4) then yield that $\lambda_{r\ell} = 0$ for $\ell \in \mathcal{T} \setminus \{s\}$.

Similarly, for each $k \in \mathcal{N} \setminus \{r\}$ and each $\ell \in \mathcal{T}$, let for $j \in \mathcal{N}$ and $t \in \mathcal{T}$, $x^2_{jt} = 0$ if $(j,t) = (k,\ell)$, $x^2_{jt} = x^{\mathrm{A}}_{jt}$ otherwise, and let $z^2 = z^{\mathrm{A}}$. It follows that $(x^2, z^2) \in F_{rs}$. The vectors $(x^{\mathrm{A}}, z^{\mathrm{A}})$ and (x^2, z^2), respectively, inserted in (4) then yield that $\lambda_{k\ell} = 0$ for $k \in \mathcal{N} \setminus \{r\}$ and $\ell \in \mathcal{T}$; hence, the equation (4) can be rewritten as

$$\lambda_{rs} x_{rs} + \sum_{t \in \mathcal{T}} \mu_t z_t = \rho. \tag{7}$$

For each $\ell \in \mathcal{T} \setminus \{s\}$, let, for $j \in \mathcal{N}$ and $t \in \mathcal{T}$, $x^3_{jt} = z^3_t = 0$ if $t = \ell$, $x^3_{jt} = z^3_t = 1$ otherwise. It follows that $(x^3, z^3) \in F_{rs}$. The vectors $(x^{\mathrm{A}}, z^{\mathrm{A}})$ and (x^3, z^3), respectively, inserted in (7) then yields that $\mu_\ell = 0$ for $\ell \in \mathcal{T} \setminus \{s\}$. Equation (7) can now be rewritten as

$$\lambda_{rs} x_{rs} + \mu_s z_s = \rho. \tag{8}$$

Let, for $j \in \mathcal{N}$ and $t \in \mathcal{T}$, $x^4_{jt} = z^4_t = 0$ if $t = s$, $x^4_{jt} = z^4_t = 1$ otherwise. It follows that $(x^4, z^4) \in F_{rs}$. The vectors (x^4, z^4) and $(x^{\mathrm{A}}, z^{\mathrm{A}})$, respectively, inserted in (8) then yield that $0 = \rho = \lambda_{rs} + \mu_s$. The equation (8) can thus be rewritten as $\mu_s x_{rs} = \mu_s z_s$, and from [9, pp. 91–92] follows that the inequality $x_{rs} \leq z_s$ defines a facet of $\operatorname{conv} S$. $\square$

PROPOSITION 9 (the inequalities (1d) define facets) *If $T_i \geq 2$ for all $i \in \mathcal{N}$, then each of the inequalities $x_{rs} \geq 0$, $r \in \mathcal{N} : T_r \geq 3$, $s \in \mathcal{T}$, defines a facet of* $\operatorname{conv} S$.

PROOF Since $T_i \geq 2$ for $i \in \mathcal{N}$, $\operatorname{conv} S$ is full-dimensional (cf. Proposition 5). Hence, we can use the uniqueness characterization of the facet description from Theorem 6 to show the proposition.

For each $r \in \mathcal{N}$ such that $T_r \geq 3$ and each $s \in \mathcal{T}$, let $\widetilde{F}_{rs} = \{ (x, z) \in \operatorname{conv} S \mid x_{rs} = 0 \}$. Further, let $x^0_{it} = z^0_t = 1$, $i \in \mathcal{N}$, $t \in \mathcal{T}$. It follows that $(x^0, z^0) \in S \setminus \widetilde{F}_{rs}$. Then letting, for $j \in \mathcal{N}$ and $t \in \mathcal{T}$, $x^{\mathrm{A}}_{jt} = 0$ if $(j, t) = (r, s)$, $x^{\mathrm{A}}_{jt} = 1$ otherwise, and letting $z^{\mathrm{A}}_t = 1$, $t \in \mathcal{T}$, it follows that $(x^{\mathrm{A}}, z^{\mathrm{A}}) \in \widetilde{F}_{rs}$ and hence that $\widetilde{F}_{rs}$ is a proper face of $\operatorname{conv} S$.

Moreover, there exists values of $\lambda \in \mathbb{R}^{N \times T}$, $\mu \in \mathbb{R}^T$, and $\rho \in \mathbb{R}$ such that the equation (4) is satisfied for all $(x, z) \in \widetilde{F}_{rs}$. We will show that for any value of $\lambda_{rs} \in \mathbb{R}$, in a solution to (4) the following hold: $\lambda_{it} = 0$ if $(i, t) \in \{\mathcal{N} \times \mathcal{T}\} \setminus \{(r, s)\}$; $\mu_t = 0$ for $t \in \mathcal{T}$; $\rho = 0$.

For each $i \in \mathcal{N}$ and each $t \in \mathcal{T}$, let for $j \in \mathcal{N}$ and $k \in \mathcal{T}$, $x^1_{jk} = 0$ if $(j, k) = (i, t)$, $x^1_{jk} = x^{\mathrm{A}}_{jk}$ otherwise, and let $z^1 = z^{\mathrm{A}}$. Since $T_r \geq 3$, it follows that $(x^1, z^1) \in \widetilde{F}_{rs}$. The vectors $(x^{\mathrm{A}}, z^{\mathrm{A}})$ and (x^1, z^1), respectively, inserted in (4) then yield that $\lambda_{it} = 0$ for all $(i, t) \in \{\mathcal{N} \times \mathcal{T}\} \setminus \{(r, s)\}$. The equation (4) can then be rewritten as (7).

For each $t \in \mathcal{T}$, let, for $j \in \mathcal{N}$ and $k \in \mathcal{T}$, $x^2_{jk} = z^2_k = 0$ if $k = t$, $x^2_{jk} = x^{\mathrm{A}}_{jk}$ and $z^2_k = z^{\mathrm{A}}_k$ otherwise. Since $T_r \geq 3$, it follows that $(x^2, z^2) \in \widetilde{F}_{rs}$. The vectors $(x^{\mathrm{A}}, z^{\mathrm{A}})$ and (x^2, z^2), respectively, inserted in (7) then yield that $\mu_t = 0$ for $t \in \mathcal{T}$.

Since $x_{rs} = 0$ for all $(x, z) \in \widetilde{F}_{rs}$ it follows that $\rho = 0$. Equation (7) can then be rewritten as $\lambda_{rs} x_{rs} = 0$, and from [9, pp. 91–92] follows that the inequality $x_{rs} \geq 0$ defines a facet of $\operatorname{conv} S$. $\square$

The inequalities $x_{rs} \geq 0$, $s \in \mathcal{T}$ (cf. Proposition 9) do *not* define facets for any $r \in \mathcal{N}$ such that $T_r = 2$ since the constraints (1d) are then implied by (1b)–(1c), (1e) according to $x_{r,s+1} \geq 1 - x_{rs} \geq 1 - z_s \geq 0$, $s \in \mathcal{T} \setminus \{T\}$, and $x_{r1} \geq 1 - x_{r2} \geq 1 - z_2 \geq 0$. Hence, the constraints (1d) need to be defined only for $i \in \mathcal{N}$ such that $T_i \geq 3$.

PROPOSITION 10 (the inequalities (1e) define facets) *If $T_i \geq 2$ for all $i \in \mathcal{N}$, then each of the inequalities $z_s \leq 1$, $s \in \mathcal{T}$, defines a facet of* $\operatorname{conv} S$.

PROOF Since $T_i \geq 2$ for $i \in \mathcal{N}$, $\operatorname{conv} S$ is full-dimensional (cf. Proposition 5). Hence, we can use the uniqueness characterization of the facet description from Theorem 6 to show the proposition.

For each $s \in \mathcal{T}$, let $F_s = \{ (x, z) \in \operatorname{conv} S \mid z_s = 1 \}$. Further, let, for $j \in \mathcal{N}$ and $t \in \mathcal{T}$, $x^0_{jt} = z^0_t = 0$ if $t = s$, $x^0_{jt} = z^0_t = 1$, otherwise. It follows that $(x^0, z^0) \in S \setminus F_s$. Then, letting $x^{\mathrm{A}}_{it} = z^{\mathrm{A}}_t = 1$, $i \in \mathcal{N}$, $t \in \mathcal{T}$, it follows that $(x^{\mathrm{A}}, z^{\mathrm{A}}) \in F_s$ and that F_s is a proper face of $\operatorname{conv} S$.

Moreover, there exists values of $\lambda \in \mathbb{R}^{N \times T}$, $\mu \in \mathbb{R}^T$, and $\rho \in \mathbb{R}$ such that the equation (4) is satisfied for all $(x, z) \in F_s$. We will show that for any value of $\rho \in \mathbb{R}$, in a solution to (4) the following hold: $\lambda_{it} = 0$ for $i \in \mathcal{N}$ and $t \in \mathcal{T}$; $\mu_s = \rho$, $\mu_t = 0$ for $t \in \mathcal{T} \setminus \{s\}$.

For each $r \in \mathcal{N}$ and each $\ell \in \mathcal{T}$, let, for $j \in \mathcal{N}$ and $t \in \mathcal{T}$, $x^1_{jt} = 0$ if $(j, t) = (r, \ell)$, $x^1_{jt} = 1$ otherwise, and let $z^1 = z^{\mathrm{A}}$. It follows that $(x^1, z^1) \in F_s$. The vectors $(x^{\mathrm{A}}, z^{\mathrm{A}})$ and (x^1, z^1), respectively, inserted in (4) then yield that $\lambda_{r\ell} = 0$ for $r \in \mathcal{N}$ and $\ell \in \mathcal{T}$. Equation (4) can then be rewritten as

$$\sum_{t \in \mathcal{T}} \mu_t z_t = \rho. \tag{9}$$

For each $\ell \in \mathcal{T} \setminus \{s\}$, let, for $j \in \mathcal{N}$ and $t \in \mathcal{T}$, $x_{jt}^2 = z_t^2 = 0$ if $t = \ell$, $x_{jt}^2 = z_t^2 = 1$ otherwise. It follows that $(x^2, z^2) \in F_s$. The vectors $(x^{\mathrm{A}}, z^{\mathrm{A}})$ and (x^2, z^2), respectively, inserted in (9) then yield that $\mu_\ell = 0$ for $\ell \in \mathcal{T} \setminus \{s\}$. Equation (9) can then be rewritten as $\mu_s z_s = \rho$. Since $z_s = 1$ for all $(x, z) \in F_s$ it follows that $\mu_s = \rho$, which yields the equation $\rho z_s = \rho$. From [9, pp. 91–92] then follows that the inequality $z_s \leq 1$ defines a facet of $\operatorname{conv} S$. □

The implication of Propositions 7–10 is that all of the inequalities necessary in the description of the set S define facets of its convex hull. A natural question then arises: Is $\operatorname{conv} S$ completely described by the system (1b)–(1e)? The answer to this question is "no", which becomes apparent by the following example.

Example 11 (continuous relaxation) Consider a system with $N = 2$, $T_1 = 3$, and $T_2 = T = 4$. Then the problem to

$$\begin{aligned} &\text{minimize} && x_{11} + x_{12} + 2x_{13} + x_{14} + x_{21} + 5x_{22} + 5x_{23} + x_{24} + 3z_1 + 3z_2 + z_3 + 3z_4, \\ &\text{subject to} && \text{(1b)–(1g)}, \end{aligned}$$

has the two optimal solutions

$$(x_{11}, x_{12}, x_{13}, x_{14}; x_{21}, x_{22}, x_{23}, x_{24}; z_1, z_2, z_3, z_4) = (0, 0, 1, 0; \gamma, 0, 0, 1-\gamma; \gamma, 0, 1, 1-\gamma) \tag{10}$$

for $\gamma \in \{0, 1\}$, with objective value 7. Relaxing the integrality requirements yields the optimal solution

$$(x_{11}, x_{12}, x_{13}, x_{14}; x_{21}, x_{22}, x_{23}, x_{24}; z_1, z_2, z_3, z_4) = (\frac{1}{2}, 0, \frac{1}{2}, \frac{1}{2}; \frac{1}{2}, 0, 0, \frac{1}{2}; \frac{1}{2}, 0, \frac{1}{2}, \frac{1}{2}) \tag{11}$$

with objective value 6.5. Hence the convex hull of the set of feasible solutions to the system (1b)–(1g) is not completely defined by the inequalities therein. □

According to the Propositions 7–10, all of the necessary inequalities define facets of $\operatorname{conv} S$. Since, by Proposition 5, $\operatorname{conv} S$ is full-dimensional (under reasonable assumptions) the minimal description of $\operatorname{conv} S$ is unique. Therefore, all of these facets are necessary to describe $\operatorname{conv} S$.

Example 11 shows that the inequalities (1b)–(1e) are not sufficient to describe $\operatorname{conv} S$. But to completely describe $\operatorname{conv} S$ we need also other facets; facet-generating procedures will be presented in forthcoming work.

5 Complexity analysis

We next show that the problem defined in (1) is NP-hard.

Theorem 12 (reduction of set covering to (1)) *The set covering problem is polynomially reducible to* (1).

Proof Let $\{\mathcal{A}_t\}_{t=1}^m$ be a given collection of nonempty subsets of the finite set $\{1, \ldots, n\}$. Letting $a_{it} = 1$ if $i \in \mathcal{A}_t$ and 0 otherwise, the *set covering* problem is defined as to

$$\text{minimize} \quad \sum_{t=1}^m y_t, \tag{12a}$$

$$\text{subject to} \quad \sum_{t=1}^{m} a_{it} y_t \geq 1, \qquad i = 1, \ldots, n, \tag{12b}$$

$$y_t \in \{0, 1\}, \quad t = 1, \ldots, m. \tag{12c}$$

Consider then an instance of the program (1) such that $N = n$, $T = m$, $d_t = 1$, $T_i = m$, and $c_{it} = 2(1 - a_{it})$ for all $i = 1, \ldots, n$ and $t = 1, \ldots, m$. Since $T_i = T = m$, each component must be replaced once between the times 1 and T, and one replacement is always enough (for feasibility). Furthermore, in every optimal solution and for each i and t such that $a_{it} = 0$, $x_{it} = 0$ holds since $c_{it} = 2 > d$ and there exists a $\tilde{t} \in \mathcal{T}$ with $a_{i\tilde{t}} = 1$, which implies that $c_{i\tilde{t}} = 0$. Hence, this specific instance of (1) can be reformulated as the problem to

$$\text{minimize} \left\{ \sum_{t=1}^{m} z_t \;\middle|\; \sum_{t=1}^{m} a_{it} x_{it} \geq 1, \; i = 1, \ldots, n, \text{ and (1c)–(1g) holds} \right\}. \tag{13}$$

An optimal solution (x^*, z^*) to (13) is given by

$$z^* \in \operatorname*{argmin}_{z \in \{0,1\}^m} \left\{ \sum_{t=1}^{m} z_t \;\middle|\; \sum_{t=1}^{m} a_{it} z_t \geq 1, \; i = 1, \ldots, n \right\} \tag{14}$$

and $x^*_{it} = a_{it} z^*_t$, $i = 1, \ldots, n$, $t = 1, \ldots, m$. The result then follows, since the program (14) is equivalent to (12). ◻

Since (12) is an NP-hard problem it follows from [9, Prop. 6.4, p. 132] that (1) is an NP-hard problem.

Finally, it should be mentioned that the complexity of the instance of (1) for which the costs c_{it} are non-increasing with time (i.e., $c_{i,t+1} \leq c_{it}$ for all i and t) is still an open question; this includes the interesting special case for which the costs are independent of time (i.e., $c_{it} = c_i$ and $d_t = d$ for all i and t), as originally studied in [4] and [1].

6 Conclusions and future research

The opportunistic maintenance model has been shown to have a nice inherent structure, in that while the problem is NP-hard, it reduces to a linear program once the maintenance occasions are fixed; the latter can in some cases even be solved through a greedy procedure. Also, all the necessary linear constraints define facets of the convex hull of the set of feasible schedules. It would be interesting to investigate whether additional facets exist which can be relatively easily generated.

Work in progress include the proper modelling of maintenance decisions over time when component lives are nondeterministic. In particular, we study models wherein one incorporates successive improvements of life distribution estimates through the addition of measurement-based information about the condition of the system. Even in the case when costs are independent of time, we have already shown that such a stochastic extension of the current model is NP-hard. In order to provide a computationally feasible model we will therefore also investigate how to best define an accurate enough scenario representation of the component lives.

The replacement problem (1) has been utilized in studies of optimal maintenance schedules at Volvo Aero (as reported in [1, 10]), as well as to maintenance scheduling problems in the nuclear and wind power industries. In the near future, experiences from the latter activities will be reported.

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