THESIS FOR THE DEGREE OF LICENTIATE OF PHILOSOPHY

Equivariant KK-theory and twists

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Abstract

Twists play an important role in the theory of quantum groups. In this thesis the notion of a twist is introduced for reduced locally compact quantum groups. It is shown that the S-equivariant KK-theory KK_S forms a triangulated category. The triangulated category KK_S is up to equivalence invariant under twist of the quantum group S. The twist equivalence is expressed explicitly on spectral triples over a certain type of quantum homogeneous spaces. In the special case of a group action, the Ext-invariant is generalized to $*$-algebras.

Keywords: Equivariant KK-theory, twists of locally compact quantum groups, triangulated categories, Baum-Connes conjecture, extension theory.

Preface

This text constitutes my thesis for a Licentiate degree at the department of Mathematical Sciences at the Chalmers University of Technology and the University of Gothenburg. The work in this thesis is based on material gathered under my time as a PhD-student in Gothenburg and frequent visits to the University of Copenhagen under the time period June 2007 to December 2008.

Acknowledgements

There are quite many people I would like to thank. I started to write down everybody, but it got out of hand. So I'll keep it short. I would especially like to thank my advisors Prof. Grigori Rozenblioum and Prof. Ryszard Nest who always have been supportive and encouraged me to go my own way. A big thanks to my examiner Prof. Lyudmila Turowska who gave some very useful remarks on the thesis. I would also like to thank my family and all of my friends and colleagues.

Notations

Before starting with the main content of the thesis, let us set some notations. Every vector space is a vector space over $\mathbb{C}$. If X is a subset of a Banach space, we use the standard notation $[X]$ for the closed linear span of X. If X is a Banach space we will let X^* denote the dual Banach space. The symbol $\otimes$ will denote tensor product in the relevant category. For C^*-algebras we choose the minimal tensor product. Algebraic tensor product will be denoted $\otimes^{alg}$. H will denote a separable Hilbert space and the von Neumann algebra of bounded operators on H will be denoted $\mathscr{B}(H)$. The pre-dual of $\mathscr{B}(H)$ is as usual denoted by $\mathscr{B}(H)_*$. The C^*-algebra of compact operators will be denoted by $\mathscr{K}(H)$. If H is of dimension $n < \infty$ we will denote $M_n := M_n(\mathbb{C}) = \mathscr{K}(H)$. The commutant of a subset $X \subseteq \mathscr{B}(H)$ will be denoted X'.

We reserve the letters S and R for reduced locally compact quantum groups which we assume to be separable. We will let W denote the left regular corepresentation of a quantum group and V the right regular corepresentation. The letters A, B, C will denote C^*-algebras which we always assume to be separable. Thus they are also σ-unital. The multiplier algebra of a C^*-algebra A will be denoted by $\mathscr{M}(A)$. For a unitary $u \in \mathscr{M}(A)$ the $*$-automorphism $a \mapsto uau*$ of A will be denoted by $Ad(u)$ and is called the adjoint action of u. We will not deal so much with groups, except for examples and in chapter 5, and in those settings G will denote a second countable locally compact group.

Given two spaces X and Y the flip mapping will be denoted by $\sigma : X \otimes Y \to Y \otimes X$. It will sometimes also denote the automorphism $\sigma \in \mathrm{Aut}(\mathscr{B}(H) \otimes \mathscr{B}(H))$ given by adjointing the flip mapping acting on $H \otimes H$. We will use leg numbering, so if $T \in A \otimes B$, or $T \in \mathscr{M}(A \otimes B)$, then $T_{12} \in \mathscr{M}(A \otimes B \otimes C)$ is defined as $T_{12} := T \otimes 1_C$ and $T_{21} \in B \otimes A$ is defined as $T_{21} := \sigma(T)$. Similarly operators such as T_{32}, T_{13} and so forth are defined.

If X is a topological space, then $C_0(X)$ will denote the C^*-algebra of continuous functions vanishing at infinity and $C_b(X)$ will denote the C^*-algebra of bounded continuous functions. The vector space of continuous functions with compact support on X will be denoted by $C_c(X)$. We denote the group $\mathbb{Z}/2\mathbb{Z}$ by $\mathbb{Z}_2$ and identify $\mathbb{Z}_2$ with $\{1, -1\} \subseteq \mathbb{Z}$. There will be no risk of confusion since we do not deal with any p-adic numbers in this thesis.

Contents

"*When I get sad, I stop being sad and be awesome instead. True story.*"

Barney Stinson

Chapter 1

Introduction

The notion of algebraic invariants have been in the toolbox of geometers since the days of Poincaré. They provide a method to loosen the constraints on how to compare different geometric objects. For instance, the winding number of a closed curve in the plane is a way to compare homotopy classes of maps $\mathbb{T} \to \mathbb{R}^2 \setminus \{0\}$.

In analysis, algebraic invariants has shown their strength in for example the Atiyah-Singer index theorem:

$$\operatorname{ind}(\not{D}) = \int_M \sigma(\not{D}) Td(M).$$

For an overview of the Atiyah-Singer index theorem, see more in [1]. This very deep index theorem expresses the analytic index of an elliptic operator $\not{D}$ on a closed manifold M as an explicit integral. The explicit integral is topological in nature and comes from a pairing between the homology and the cohomology of the manifold. The pairing in the Atiyah-Singer index theorem may also be described on the level of K-theory as a pairing between the K-theory and K-homology. The strength in the K-theoretic formulation lies in that K-theory has generalizations to operator algebras.

Another application is in representation theory. The Baum-Connes conjecture again relates topological properties with analytical properties of a group. This conjecture was presented in [5] and put in a more general setting in [6]. With a locally compact group G one can associate a topological space $\underline{E}G$ called the classifying space for proper actions of G. The space $\underline{E}G$ is a proper G-space with the universal property that for every proper G-space X there exists an equivariant mapping $f : X \to \underline{E}G$ that is unique up to homotopy. So the classifying space of a group is determined uniquely up to homotopy by it's universal property. With the universal classifying space one can associate the K-homology group with G-compact support $K_*^G(\underline{E}G)$ and a natural mapping

$$\mu : K_*^G(\underline{E}G) \to K_*(C_r^*(G))$$

called the assembly map. Here $C_r^*(G)$ denotes the reduced group C^*-algebra of G. The Baum-Connes conjecture states that the assembly map is an isomorphism for all G. The conjecture has important implications in geometry and group theory, for example does injectivity of μ imply the Novikov conjecture and surjectivity of μ imply the Kaplansky conjecture. If A is a separable C^*-algebra with a continuous action of G one can construct an assembly map from the K-homology group with G-compact support and coefficients from A

$$\mu_A : K_*^G(\underline{E}G; A) \to K_*(A \rtimes_r G).$$

The Baum-Connes conjecture with coefficients states that the assembly map μ_A is an isomorphism for all G and all $G - C^*$-algebras A. The motivation to study the Baum-Connes conjecture is that in applications it is usually easier to calculate $K_*^G(\underline{E}G; A)$ than to calculate $K_*(A \rtimes_r G)$. The Baum-Connes conjecture with coefficients has been proved to hold for a large class of groups, but Gromov [16] has stated existence of a group with properties implying that the Baum-Connes conjecture with coefficients fails to hold. We will say that if the Baum-Connes conjecture with coefficients hold for G then G has the Baum-Connes property.

The problems we address in this thesis are related to modern approaches in K-theory with equivariance properties from a quantum group, in particular equivariant KK-theory. The main part of the thesis is a study of KK-theory equivariant with respect to a reduced locally compact quantum group S. The main examples of locally compact quantum groups are Woronowicz deformations of compact Lie groups and Drinfeld-Jimbo twists of the discrete duals of compact Lie groups. As is shown in [46] these two notions are dual. With the Drinfeld-Jimbo twists as motivation we will develop a general theory for twists of reduced locally compact quantum groups, although most of the theory will not be able to deal with Drinfeld-Jimbo twists. Questions which are studied are whether regularity of a quantum group is invariant under twist? Can the Takesaki-Takai duality be refined to contain a twist? Is Meyers notion of torsion-free, discrete quantum group invariant under twist?

When it comes to equivariant KK-theory we will study this as a triangulated category. The viewpoint on equivariant KK-theory as a triangulated category was put into firm ground in [35], where the Baum-Connes assembly map was constructed as a natural transformation from a derived functor to the functor $A \mapsto K_*(A \rtimes_r G)$ which reformulated the Baum-Connes property to a property of a certain triangulated category. This reformulation of the Baum-Connes property seems to be promising for generalizing the Baum-Connes property to quantum groups, so far it has only been generalized to discrete quantum groups which are torsion-free in the sense of Meyer [34] and duals of compact, connected Lie groups [36].

The question arises how the equivariant KK-theory is affected by a twist of the quantum group? And does Baum-Connes property for a discrete, torsion-free quantum group imply the Baum-Connes property of it's twists? Is there a Pimsner-Voiculescu sequence for reduced crossed products by the dual of a compact, connected Lie groups? This last question corresponds to constructing a compactly induced "simplicial approximation" of the one-point space in the equivariant KK-category in the analogies of [35]. Is it possible to twist spectral triples on quantum homogeneous spaces if we have a twist of a quantum group? Is this twist of spectral triples induced from a functor between KK-categories?

The KK-groups is constructed via Kasparov modules but in the construction of a triangulated structure on equivariant KK-theory the Ext-invariant naturally arises. It is a monoid invariant consisting of certain equivalence classes of short exact sequences and the invertible elements form the KK-group. But since C^*-algebras in general are hard to do cohomological calculations in, it would be interesting to have a similar invariant for $*$-algebras? The generalization of KK-theory to kk-theory, which works for bornological algebras, was constructed in [12]. However, kk-theory was constructed as a homotopy theory. Would a straight forward generalization of the Ext-invariant give something that behaves similarly to the Ext-invariant on a simple example? Do the invertible elements have an analytic property similar to that of the usual Ext-invariant?

Overview of thesis

Chapter 2: Quantum groups

We recall the notion of a reduced locally compact quantum groups from [28] and set some notations for the theory of Hilbert modules. Taking reduced crossed product by a quantum group is one of the most important tool in equivariant KK-theory. After defining reduced crossed products, following [3], some standard results on reduced crossed products from [4] are presented. The definition of a regular quantum group from [4] is presented.

After that we define some different types of twists and cocycle twists of reduced locally compact quantum groups. Twists are used to twist the dual coaction on a reduced crossed product. A drawback is that the Drinfeld-Jimbo twists are not twists in our sense, it is only a cocycle twist. The construction is simple, but it is crucial in the study of twists on the level of KK-theory. We prove a regularity result for twists. The usual Takesaki-Takai duality for quantum groups is used to prove a twisted Takesaki-Takai duality (TTT-duality).

We also present the example of a Drinfeld-Jimbo twist of the dual of a semisimple, compact, connected Lie groups on the level of reduced locally compact groups. After recalling Meyers notion of torsion-free, discrete quantum groups we present a refined version of the twisted Takesaki-Takai duality to show that this notion is invariant under twists.

Chapter 3: Equivariant KK-theory

Chapter 3 is an introduction to equivariant KK-theory for C^*-algebras and triangulated categories. The chapter does not contain any unknown results, it is only a gathering of different results and some generalizations from the non-equivariant setting in KK-theory. The main goal of the chapter is, besides presenting some useful results in KK-theory, to show that the equivariant KK-category KK_S has a triangulated structure. In [42], the usual triangles in KK_S was stated to define a triangulated structure on KK_S. A proof was sketched using generalized homomorphisms from the Cuntz picture of KK-theory which reduced the proof to an argument analogous to a group which was studied in [35]. Our proof, using the Ext-invariant, lies closer to the idea of triangulated categories. The idea is to use the Ext-invariant to explicitly describe mapping cones of morphisms in KK_S.

Chapter 4: Twists in KK-theory

The notions introduced in Chapter 2 are used in this chapter to study what happens to the triangulated structure on KK_S when twisting the quantum group S by a twist $\mathscr{F}$. Using Baaj-Skandalis duality, every coaction is Morita equivalent to a dual coaction on a reduced crossed product and can be twisted. Twists of coactions are used to construct a triangulated equivalence $KK_S \cong KK_{S_{\mathscr{F}}}$ showing that equivariant KK-theory is independent of twists. The twisted Takesaki-Takai duality from Chapter 2 implies that the Baaj-Skandalis dual equivalence $KK_{\hat{S}} \cong KK_{\widehat{S_{\mathscr{F}}}}$ acts trivially on C^*-algebras with trivial $\hat{S}$-coactions.

Again using the results from Chapter 2, the twist equivalence has implications for the Baum-Connes property. The Baum-Connes property was generalized in [34] to a discrete, torsion-free quantum group S. From the results of Chapter 2 it follows that if $\mathscr{F}$ is a twist of S, the Baum-Connes property is well defined for $S_{\mathscr{F}}$. We prove in this chapter that if the Baum-Connes property holds for S, it also holds for $S_{\mathscr{F}}$. A very interesting question is whether this property also holds for cocycle twists?

Using the Pimsner-Voiculescu sequence for $\mathbb{Z}$-actions we are able to construct a generalized Pimsner-Voiculescu sequence for coactions of compact, connected Lie groups. For compact Lie groups satisfying the Hodgkin condition, which is equivalent to the dual being torsion-free, we are able to twist the generalized Pimsner-Voiculescu sequence to a twisted Pimsner-Voiculescu triangle in the twisted KK-category. This is also possible for compact Lie groups which do not satisfy the Hodgkin condition but in this case some terms from the "torsion part" of the discrete dual appears.

To end this chapter we describe how the twist equivalence $KK_S \cong KK_{S_{\mathscr{F}}}$ acts on the level of spectral triples over classical quantum homogeneous spaces. The notation of a classical quantum homogeneous spaces was introduced in [36]. Examples of such spaces are any quantum group and the Poodles sphere. We use the ideas from [41] to construct a rather general class of classical quantum homogeneous spaces and twists thereof.

Chapter 5: Extension theory for $*$-algebras

With the triangulated structure on KK_S as motivation we construct a functor on $*$-algebras which generalizes the Ext-functor. We restrict ourselves to actions of a second countable locally compact group G and $*$-algebras admitting C^*-closures. In the first part of this chapter we define suitable categories and construct a bivariant functor $\mathscr{E}xt_G$ to the category of abelian monoids. As a set, $\mathscr{E}xt_G(\mathscr{A}, \mathfrak{I})$ consists of equivalence classes of short exact sequences starting in a $*$-algebra $\mathfrak{I}$ and ending in another $*$-algebra $\mathscr{A}$.

After that we will move on to study the invertible elements. As it turns out, the invertible elements are those extensions which arise from G-equivariant algebraic $\mathscr{A} - \mathfrak{I}$-Kasparov modules. Using the similarities in the definition of $\mathscr{E}xt_G$ with that of the C^*-algebraic analogue Ext_G we can relate the elements in this monoid with the extensions of the C^*-closures of the $*$-algebras. This induces a natural transformation Θ.

Outlook and future problems

1. Show that every cocycle twist is manageable (see Definition 2.4.2) and a cocycle twist of a regular quantum group is strongly manageable (see Definition 2.4.6).

2. Develop a KK-theory equivariant with respect to quasi-coactions and deal with cocycle twists of the dual coaction on a crossed product.

3. Explore KK-theory equivariant with respect to universal quantum groups and their twists.

4. Twisting spectral triples on arbitrary quantum homogeneous spaces by cocycle twists.

5. Study the relations between the $\mathscr{E}xt_G$-functor and the kk-theory of bornological algebras from [12].

6. Generalize notions such as classifying space of proper actions, torsion and the localizing subcategory of compactly induced objects to quantum groups.

Chapter 2

Quantum groups

Quantum groups have long been studied as a natural generalization of groups. On the algebraic level they became interesting due to their applications in condensed matter theory and their relation with the algebraic Bethe ansatz. To the operator algebraists, quantum groups was first studied as Kac algebras which gave a good setting to generalize Pontryagin duals to non-abelian groups, see more in [15].

In [52], Woronowicz introduced a compact bi-C^*-algebra $SU_\mu(2)$ which was non-commutative and non-cocommutative containing many of the group-like structures that were found in Kac algebras, but without a tracial Haar weight, so it was not a Kac algebra. The discovery of these, more esoteric, quantum groups lead to the notion of multiplicative unitaries in [4]. The general notion of reduced quantum groups was introduced in [28], it is the locally compact quantum groups admitting faithful Haar weights. This setting even allows for Pontryagin duality.

The first half of this chapter is old material and proofs can be found in the references. The bulk of the second half of this chapter is new material. The section on the Drinfeld-Jimbo twist for non-simply connected Lie groups, lies very close to old material describing cocycle twists of simply connected Lie groups.

2.1 Groups from a C^*-algebraic point of view

A topological group is characterized by the property that it has a continuous associative, binary product $G \times G \to G$. In terms of the C^*-algebra $C_0(G)$ the product induces a non-degenerate $*$-homomorphism $\Delta : C_0(G) \to C_b(G \times G)$ by $\Delta(f)(g,h) := f(gh)$. The mapping Δ is called the comultiplication on $C_0(G)$.

Since G is a group the comultiplication satisfies the properties

$$(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta \tag{2.1}$$

$$C_0(G \times G) = [(1 \otimes C_0(G))\Delta(C_0(G))] = [(C_0(G) \otimes 1)\Delta(C_0(G))]. \tag{2.2}$$

Here we use the standard notation that if X is a subset of a Banach space, then $[X]$ denotes the closed linear span. The first condition states the coassociativity of Δ and is equivalent to the fact that the product on G is associative. The second condition is equivalent to the fact that G has left and right cancellation by Proposition 3.1 of [31]. If G is compact, that is $C_0(G) = C(G)$ is unital, the commutative C^*-algebra $C_0(G)$ together with the comultiplication satisfying the cancellation condition (2.2) encodes all properties of the group. This fact is shown in Proposition 3.2 of [31]. If G is not compact we need two more mappings. We need the counit $\varepsilon : C_0(G) \to \mathbb{C}$ which is a $*$-homomorphism given by $f \mapsto f(e)$ and the coinverse $\kappa : C_0(G) \to C_0(G)$ defined as the $*$-anti-homomorphism $\kappa(f)(g) := f(g^{-1})$. Since $C_0(G)$ is commutative the mapping κ is also a homomorphism, but in the general setting of quantum groups it will be an anti-homomorphism.

The three mappings Δ, κ and ε together with the multiplication m on $C_0(G)$ satisfy:

$$(\varepsilon \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \varepsilon)\Delta = \mathrm{id} \tag{2.3}$$

$$m(\kappa \otimes \mathrm{id})\Delta = m(\mathrm{id} \otimes \kappa)\Delta = \varepsilon \tag{2.4}$$

Conversely, assume we are given a commutative C^*-algebra A, and mappings Δ, κ and ε satisfying the equations (2.1)-(2.4). Let G denote the character space of A. The comultiplication Δ induces a binary product $G \times G \to G$ which is associative since Δ is coassociative. The character ε may be identified with a point $e \in G$ which forms a unit for the product on G. The coinverse κ induces a mapping $\kappa^* : G \to G$ and equation (2.4) implies that $\kappa^*(g)g = g\kappa^*(g) = e$.

So the group properties of G may be encoded in the C^*-algebra $C_0(G)$. The C^*-algebra $C_0(G)$ depends contravariantly on G in the sense that if $k : G \to H$ is a continuous homomorphism then $k^* : C_0(H) \to C_b(G)$ is a $*$-homomorphism such that $\Delta_G \circ k^* = (k^* \otimes k^*) \circ \Delta_H$.

Another important C^*-algebra one can associate with a group G is it's group C^*-algebra, for a more thorough presentation see [8]. This is defined to be the universal C^*-algebra $C^*(G)$ such that any continuous unitary representation of G on a Hilbert space H induces a non-degenerate $*$-homomorphism

$C^*(G) \to \mathscr{B}(H)$. More explicitly, let $L^1(G)$ denote the Banach space of L^1-functions with respect to a fixed left invariant Haar measure μ on G. Let δ denote the modular function on G. The Banach space $L^1(G)$ forms a Banach $*$-algebra in the product

$$f_1 * f_2(g) := \int_G f_1(h) f_2(h^{-1}g) \mathrm{d}\mu \quad \text{and the involution}$$
$$f^*(g) := \delta(g^{-1}) \overline{f(g^{-1})}.$$

Given a continuous unitary representation $\pi : G \to \mathscr{U}(H)$ we may integrate this to a continuous $*$-representation $\pi : L^1(G) \to \mathscr{B}(H)$ by

$$\pi(f)x := \int_G f(g)\pi(g)x \mathrm{d}\mu.$$

The representation π defines a C^*-semi-norm on $L^1(G)$ by $\|f\|_\pi := \|\pi(f)\|_{\mathscr{B}(H)}$. Define the C^*-semi-norm

$$\|f\|_{C^*(G)} := \sup\{\|f\|_\pi : \pi \text{ continuous unitary representation of } G\}.$$

The group C^*-algebra $C^*(G)$ is defined to be the closure of $L^1(G)$ in the norm $\|\cdot\|_{C^*(G)}$, it coincides with the C^*-envelope of $L^1(G)$. The universal property holds for $C^*(G)$, since any continuous unitary representation of G integrates to $L^1(G)$. Consider the left regular representation λ of G on $L^2(G)$ given by $\lambda(g)x(h) := x(g^{-1}h)$ for $x \in L^2(G)$. The universal property implies that λ integrates to a $*$-homomorphism $\lambda : C^*(G) \to \mathscr{B}(L^2(G))$. The image of λ is called the reduced group C^*-algebra and is denoted by $C^*_r(G)$. The reduced group C^*-algebras coincides with the closure of $\lambda(L^1(G))$ in $\mathscr{B}(L^2(G))$. It is universal in the sense that any summand of the left regular representation of G integrates to $C^*_r(G)$.

If we have two unitary representations π on H and π' on H' of G, then $H \otimes H'$ carries a unitary representation π'' of G by $\pi''(g) := \pi(g) \otimes \pi'(g)$. This induces a mapping $\hat{\Delta} : C^*(G) \to \mathscr{M}(C^*(G) \otimes C^*(G))$ satisfying

$$(\pi \otimes \pi')\hat{\Delta} = \pi''.$$

The comultiplication is defined on the level of $L^1(G)$ by

$$\hat{\Delta}(f_1)(f_2 \otimes f_3)(g_1, g_2) = \int_G f_1(h) f_2(h^{-1}g_1) f_2(h^{-1}g_2) \mathrm{d}\mu.$$

Therefore it also induces a mapping $\hat{\Delta} : C_r^*(G) \to \mathscr{M}(C_r^*(G) \otimes C_r^*(G))$. Just by looking at L^1-level we may deduce that also $C^*(G)$ has a cancellation property similar to that of $C_0(G)$ as in equation (2.2)

$$C^*(G) \otimes C^*(G) = [(1 \otimes C^*(G))\hat{\Delta}(C^*(G))] = [(C^*(G) \otimes 1)\hat{\Delta}(C^*(G))].$$

The C^*-algebra $C^*(G)$ depends covariantly on G in the sense that a group homomorphism $k : G \to H$ induces a $*$-homomorphism $k_* : C^*(G) \to \mathscr{M}(C^*(H))$ via

$$k_*(a)b(h) := \int_G a(g)b(k(g^{-1})h)\mathrm{d}\mu_G.$$

If G is abelian the Fourier-Plancherel transform induces an isomorphism $C_0(\hat{G}) \cong C^*(G)$, where $\hat{G}$ denotes the Pontryagin dual of G. This motivates the terminology that $C^*(G)$ is called the full Pontryagin dual of $C_0(G)$ and $C_r^*(G)$ it's reduced dual.

There is one more structure on $C^*(G)$ which is needed in generalizing locally compact groups to the quantum setting. It is the left invariant Haar weight which is defined by

$$\varphi(f) := f(e) \quad \text{for} \quad f \in C_c(G).$$

The left invariant Haar weight satisfy $\omega \otimes \varphi(\hat{\Delta}(f)) = \omega(1)\varphi(f)$ for $\omega \in C^*(G)^*$. If we let H_φ denote the GNS-construction of φ, then $L^2(G) \cong H_\varphi$. So φ extends to a faithful weight on $C_r^*(G)$. The Haar weight is also right invariant, $\varphi \otimes \omega(\hat{\Delta}(f)) = \omega(1)\varphi(f)$, since $C^*(G)$ is cocommutative in the sense that $\sigma \circ \hat{\Delta} = \hat{\Delta}$.

Similarly the left Haar measure induces a left invariant weight on $C_0(G)$ and the right Haar measure a right invariant Haar weight. As we will see in the next chapter, and is shown in [28], the Haar weight together with the cancellation property of the comultiplication is sufficient to reconstruct the group properties of G and it's reduced dual.

2.2 Definition of reduced locally compact quantum group

This section will be a short review of the theory of reduced locally compact quantum groups in the sense of Kustermans-Vaes. Their definition of reduced locally compact quantum group can be found in [28] and they have also studied von Neumann algebraic quantum groups in [29]. Both the C^*-algebraic and the von Neumann algebraic settings are reduced in the sense that they both require existence of faithful Haar weights. So the classical analogue is the reduced group C^*-algebra. In [27] the universal setting was studied by Kustermans. But we will only work in the reduced setting when the quantum group has a well behaved (co-)representation theory.

Definition 2.2.1 (Definition 2.1 of [28])**.** *If S is a C^*-algebra and $\Delta : S \to \mathcal{M}(S \otimes S)$ is a non-degenerate $*$-homomorphism satisfying*

$$(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta, \tag{2.5}$$

the pair (S, Δ) is called a bi-C^*-algebra and Δ a comultiplication. The mapping in equation (2.5) *will be denoted by $\Delta^{(2)}$.*

If S is a commutative C^*-algebra, that is $S = C_0(X)$ for some locally compact Hausdorff space X, Δ induces a binary operation $X \times X \to X$ which is associative thus making X into a semigroup. Precisely as in equation (2.2), cancellation properties on X can be interpreted as properties of S.

If $\alpha : S \to S'$ is a $*$-homomorphism between the bi-C^*-algebras (S, Δ) and (S', Δ') it is called a morphism of bi-C^*-algebras if

$$(\alpha \otimes \alpha)\Delta = \Delta' \alpha.$$

If we are given a $\omega \in S^*$, the topological dual of S, then we may extend ω to a strictly continuous functional on $\mathcal{M}(S)$. This fact follows from that S is strictly dense in $\mathcal{M}(S)$.

A weight φ on a C^*-algebra S is called proper if it is non-zero, densely defined and lower semicontinuous. By Definition 1.8 of [28] proper weights on S extends to proper weights on $\mathcal{M}(S)$. Using this extension allows us to define left and right invariant weights on a bi-C^*-algebra. For a weight φ on S we define the set

$$\mathcal{M}_\varphi^+ := \{s \in S_+ : \varphi(s) < \infty\}.$$

A weight φ is said to be left invariant if it satisfies

$$\varphi((\omega \otimes \mathrm{id}(\Delta(a))) = \omega(1)\varphi(a)$$

for all $a \in \mathcal{M}_\varphi^+$ and $\omega \in S_+^*$. Similarly, a weight ψ is said to be right invariant if

$$\psi((\mathrm{id} \otimes \omega(\Delta(a))) = \omega(1)\psi(a)$$

for all $a \in \mathcal{M}_\psi^+$ and $\omega \in S_+^*$. Let us recall the definition of a locally compact group in the reduced setting (Definition 4.1 from [28]).

Definition 2.2.2 (Reduced C^*-algebraic quantum group, Definition 4.1 of [28]). *Suppose that S is a bi-C^*-algebra. If S satisfies*

$$S = [(\mathrm{id} \otimes \omega(\Delta(a)) : a \in S, \omega \in S^*] = [(\omega \otimes \mathrm{id}(\Delta(a)) : a \in S, \omega \in S^*]$$

and there exists faithful approximate KMS-weights φ and ψ, which should be left invariant respectively right invariant, then we say that S is a reduced C^*-algebraic quantum group.

The weights φ and ψ are called left, respectively right, Haar weights. By Corollary 6.11 of [28] the density conditions in the definition of a reduced locally compact quantum group together with the existence of Haar weights imply the cancellation condition

$$S \otimes S = [\Delta(S)(1 \otimes S)] = [\Delta(S)(S \otimes 1)]. \tag{2.6}$$

See also the von Neumann-algebraic version of a quantum group in [29]. In the von Neumann-setting a density requirement on the quantum group is unnecessary. In [29] the conceptually remarkable result that there exists a unique reduced C^*-algebraic quantum group contained in every von Neumann-algebraic quantum group is proved.

We recall the standard terminology $\mathcal{N}_\varphi := \{a \in S : a^*a \in \mathcal{M}_\varphi^+\}$. Then $\langle a, b\rangle_\varphi := \varphi(a^*b)$ defines a non-degenerate scalar product on $\mathcal{N}_\varphi$ since φ is faithful. The Hilbert space closure of $\mathcal{N}_\varphi$ will be denoted by H_S. This Hilbert space carries a faithful representation $\lambda : S \to \mathcal{B}(H_S)$ given by the GNS-representation associated with φ. The representation λ is called the left regular representation of S. We denote the embedding $\mathcal{N}_\varphi \hookrightarrow H_S$ by Λ_φ. Define a linear mapping W on $H_S \otimes H_S$ in the dense subspace $\mathrm{im}(\Lambda_\varphi \otimes \Lambda_\varphi)$ by

$$W^*(\Lambda_\varphi(a) \otimes \Lambda_\varphi(b)) := \Lambda_\varphi \otimes \Lambda_\varphi(\Delta(b)a \otimes 1) \quad \text{for} \quad a, b \in \mathcal{N}_\varphi.$$

Proposition 2.2.3 (Proposition 3.17 and equation (4.2) of [28]). *The operator W is unitary and*

$$\lambda \otimes \lambda(\Delta(a)) = W^*(1 \otimes \lambda(a))W.$$

Furthermore

$$\lambda(S) = [(\mathrm{id} \otimes \omega)(W) : \omega \in B(H_S)_*] = [(\mathrm{id} \otimes \omega)(W^*) : \omega \in B(H_S)_*].$$

The unitary operator W satisfies the pentagonal equation

$$W_{12}W_{13}W_{23} = W_{23}W_{12},$$

so we say that W is a multiplicative unitary. The pentagonal equation implies that for $\omega, \omega' \in \mathscr{B}(H_S)_*$

$$(\mathrm{id}\otimes\omega)(W^*)\cdot(\mathrm{id}\otimes\omega')(W^*) = (\mathrm{id}\otimes\omega\otimes\omega')(W_{12}^*W_{13}^*) = (\mathrm{id}\otimes\omega'')(W^*)$$

where $\omega''(x) := (\omega\otimes\omega')(W(x\otimes 1)W^*)$. Thus the vector space

$$\{(\mathrm{id}\otimes\omega)(W^*) : \omega\in B(H_S)_*\}$$

forms a dense subalgebra of $\lambda(S)$.

We may define an antipode on S, but this can only be defined on a dense subalgebra. Define

$$S\left((\mathrm{id}\otimes\omega)(W)\right) := (\mathrm{id}\otimes\omega)(W^*) \quad \text{for} \quad \omega\in\mathscr{B}(H_S)_*.$$

The mapping S is well defined and an anti-automorphism of it's domain. By Proposition 5.22 and 5.24 of [28] there is a polar decomposition

$$S = R\tau_{-\frac{i}{2}},$$

where R is an anti-automorphism of S and $\tau_{-\frac{i}{2}}$ is the densely defined extension of the strongly continuous one-parameter group $(\tau_t)_{t\in\mathbb{R}}$ associated with the left-invariant weight φ to $-\frac{i}{2}$. As is shown in Proposition 5.26 of [28] the anti-automorphism R satisfies

$$\sigma\circ(R\otimes R)\circ\Delta = \Delta\circ R.$$

So the weight $a\mapsto\overline{\varphi(R(a)^*)}$ satisfies the conditions on a right invariant Haar weight. This allows us to assume that $\psi\equiv\bar{\varphi}\circ R$. As is shown in [28] the GNS-construction ρ, of this particular choice of right invariant Haar weight, may be expressed via the antipode R, the left regular representation λ and the modular conjugation operator J of φ via

$$\rho(a) = J\lambda(R(a)^*)J.$$

This equation shows that $\rho(S)\subseteq\lambda(S)'$ and $\lambda(S)\subseteq\rho(S)'$, that is the representations λ and ρ commutes.

Similarly to the results of Proposition 2.2.3 we define a subalgebra of $\mathscr{B}(H_S)$ by

$$\{(\omega\otimes\mathrm{id})(W) : \omega\in\mathscr{B}(H_S)_*\}. \tag{2.7}$$

Let $\hat{S}$ denote the norm closure of the algebra in equation (2.7). This forms a C^*-algebra by Proposition 1.4 and 3.5 from [4]. Another equivalent approach to this is by using the multiplicative unitary $\hat{W} := \sigma W^*\sigma$ and defining $\hat{S}$ as

$$\hat{S} := [(\mathrm{id}\otimes\omega)(\hat{W}) : \omega\in\mathscr{B}(H_S)_*]. \tag{2.8}$$

Define the mapping $\hat{\Delta}: \hat{S} \to \mathscr{B}(H_S \otimes H_S)$ as

$$\hat{\Delta}(x) := \hat{W}^*(1 \otimes x)\hat{W}$$

Theorem 2.2.4 (Theorem 8.20 and 8.29 of [28]). *There exist Haar weights $\hat{\varphi}$ and $\hat{\psi}$ making the pair $(\hat{S}, \hat{\Delta})$ into a reduced C^*-algebraic quantum group and there exist a natural Pontryagin duality $\hat{\hat{S}} \cong S$.*

Similarly to the setting for S we have a left regular and a right regular representation of $\hat{S}$ on H_S. We will denote them by $\hat{\lambda}$ and $\hat{\rho}$. Using the definition in equation (2.8) we have that $\hat{\lambda}$ coincides with the inclusion $\hat{S} \subseteq \mathscr{B}(H_S)$, because of Proposition 8.16 of [28]. As is explained in [51] the relations between the right and the left regular representations can be described via the modular conjugation operators of it's dual as follows

$$\rho(a) = J\hat{J}\lambda(a)\hat{J}J \quad \text{and} \quad \hat{\rho}(a) = \hat{J}J\hat{\lambda}(a)J\hat{J}.$$

This motivates the terminology that W is the left regular corepresentation of S. Similarly $\hat{W}$ is called the left regular corepresentation of $\hat{S}$. From the reasonings in chapter 3.4 of [28] it follows that $W \in (\lambda \otimes \hat{\lambda})(\mathscr{M}(S \otimes \hat{S}))$ and $\hat{W} \in (\hat{\lambda} \otimes \lambda)(\mathscr{M}(\hat{S} \otimes S))$. We will in this thesis abuse the notation somewhat by sometimes identifying W with it's pre-image in $\mathscr{M}(S \otimes \hat{S})$.

Similarly one can define the right regular corepresentation of the quantum group S as the unitary $V \in (\hat{\rho} \otimes \rho)(\mathscr{M}(\hat{S} \otimes S))$ such that

$$(\rho \otimes \rho)(\Delta(a)) = V(\rho(a) \otimes 1)V^*.$$

A straight forward calculation shows that

$$V = (J\hat{J} \otimes J\hat{J})\hat{W}(\hat{J}J \otimes \hat{J}J).$$

Similarly the right regular corepresentation $\hat{V}$ of $\hat{S}$ may be defined and by either Pontryagin duality or by another straight forward calculation it follows that

$$\hat{V} = (\hat{J}J \otimes \hat{J}J)W(J\hat{J} \otimes J\hat{J}).$$

Again, as an example, we consider $S = C_0(G)$ for a locally compact group G. This quantum group carries the left invariant Haar weight defined by $\varphi(f) := \int f \, d\mu$, for $f \in C_c(G)$. The right invariant Haar weight ψ can be defined similarly, with the left invariant Haar measure replaced by the right invariant Haar measure. The left regular representation of S on H_S is pointwise multiplication on $L^2(G)$. The multiplicative unitary W is given by

$$W\xi(g,h) = \xi(g, g^{-1}h). \tag{2.9}$$

Also, there is an isomorphism $\hat{S} \cong C_r^*(G)$ of locally compact quantum groups. We will prove this claim for G being a discrete group. Let $\lambda_g \in C_r^*(G)$ be defined on $f \in L^2(G)$ by

$$\lambda_g f(h) := f(gh).$$

Define $\delta_g \in C_0(G)$ by $\delta_g(h) := 1$ if $g = h$, and $\delta_g(h) = 0$ otherwise. Then by equation (2.9) we may express W by a sum, convergent in the weak operator topology, as

$$W = \sum_{g \in G} \delta_g \otimes \lambda_{g^{-1}}.$$

If $\omega_{g',g}(x) := \langle x\delta_{g'}, \delta_g \rangle$ then

$$(\mathrm{id} \otimes \omega_{g',g})(W) = \delta_{g'g^{-1}} \quad \text{and}$$
$$(\omega_{g',g} \otimes \mathrm{id})(W) = \delta_g \lambda_{g'^{-1}}.$$

Since finite rank operators are weak$_*$-dense in $\mathscr{B}(H_S)_*$ it follows that

$$C_0(G) = [(\mathrm{id} \otimes \omega)(W) : \omega \in \mathscr{B}(H_S)_*] \quad \text{and}$$
$$C_r^*(G) = [(\omega \otimes \mathrm{id})(W) : \omega \in \mathscr{B}(H_S)_*].$$

2.3 Hilbert modules, reduced crossed products and regularity

To start this section we will review some general theory of Hilbert modules. They form a good framework to construct reduced crossed products by quantum groups in. The main part of this section is based on the theory presented in [26] and it's equivariant generalizations in [3]. Then we will prove some general properties of reduced crossed products.

If $\mathscr{E}$ is a right A-module it is called an A-pre-Hilbert module if there exist a $\mathbb{C}$-sesquilinear mapping $\langle \cdot, \cdot \rangle_{\mathscr{E}} : \mathscr{E} \times \mathscr{E} \to A$ such that for $a \in A$ and $x, y \in \mathscr{E}$

i) $\langle x, ya \rangle_{\mathscr{E}} = \langle x, y \rangle_{\mathscr{E}} a$,

ii) $\langle x, y \rangle_{\mathscr{E}} = \langle y, x \rangle_{\mathscr{E}}^*$,

iii) $\langle x, x \rangle_{\mathscr{E}} \geq 0$ with equality if and only if $x = 0$.

The vector space $\mathscr{E}$ is a normed space in the norm $\|x\|_{\mathscr{E}} := \|\sqrt{\langle x, x \rangle_{\mathscr{E}}}\|$. If $\mathscr{E}$ is complete with respect to this norm, $\mathscr{E}$ is called an A-Hilbert module. If $\langle \mathscr{E}, \mathscr{E} \rangle_{\mathscr{E}}$ is dense in A we say that $\mathscr{E}$ is essential. Let $\mathscr{E}$ and $\mathscr{E}'$ be two A-Hilbert modules.

An A-linear mapping $T : \mathcal{E} \to \mathcal{E}'$ is called adjointable if there exist an A-linear mapping $T^* : \mathcal{E}' \to \mathcal{E}$ such that

$$\langle Tx, y\rangle_{\mathcal{E}'} = \langle x, T^*y\rangle_{\mathcal{E}} \quad \text{for} \quad x \in \mathcal{E},\ y \in \mathcal{E}'.$$

If T is adjointable it is bounded, because of the Banach-Steinhaus theorem. An adjointable A-linear mapping will also be called a mapping of A-Hilbert modules. The property of being adjointable is closed under composition since $(T_1T_2)^* = T_2^*T_1^*$. We let $\mathcal{L}_A(\mathcal{E}, \mathcal{E}')$ denote the space of adjointable A-linear mappings $\mathcal{E} \to \mathcal{E}'$. The space $\mathcal{L}_A(\mathcal{E}, \mathcal{E}')$ is a Banach space in the operator norm. If $\mathcal{E} = \mathcal{E}'$ then we define $\mathcal{L}_A(\mathcal{E}) := \mathcal{L}_A(\mathcal{E}, \mathcal{E})$ which is a C^*-algebra in it's operator norm. If $\mathcal{E} = A$ we define $\mathcal{M}(\mathcal{E}') := \mathcal{L}_A(A, \mathcal{E}')$. Given elements $x \in \mathcal{E}$ and $y \in \mathcal{E}'$ we may define an operator $T_{x,y} \in \mathcal{L}_A(\mathcal{E}, \mathcal{E}')$ by

$$T_{x,y}(\xi) := y\langle x, \xi\rangle_{\mathcal{E}} \quad \text{for} \quad \xi \in \mathcal{E}.$$

This mapping has the adjoint $T_{x,y}^* = T_{y,x}$. The Banach subspace $\mathcal{K}_A(\mathcal{E}, \mathcal{E}') \subseteq \mathcal{L}_A(\mathcal{E}, \mathcal{E}')$ is defined as the closure of the linear span of the operators $T_{x,y}$ for $x \in \mathcal{E}$, $y \in \mathcal{E}'$. Clearly, for a third A-Hilbert module $\mathcal{E}''$ there are inclusions

$$\mathcal{K}_A(\mathcal{E}, \mathcal{E}')\mathcal{L}_A(\mathcal{E}', \mathcal{E}'') \subseteq \mathcal{K}_A(\mathcal{E}, \mathcal{E}'') \quad \text{and} \quad \mathcal{L}_A(\mathcal{E}'', \mathcal{E})\mathcal{K}_A(\mathcal{E}, \mathcal{E}') \subseteq \mathcal{K}_A(\mathcal{E}'', \mathcal{E}').$$

A grading on a C^*-algebra A is a $*$-automorphism $\alpha \in Aut(A)$ such that $\alpha^2 = \mathrm{id}$. The grading automorphism induces a linear decomposition $A = A^+ \oplus A^-$ satisfying that $\alpha|_{A^\pm} = \pm\mathrm{id}$. Since α is a $*$-homomorphism, it holds that A^+ is a C^*-subalgebra and A^- is a closed self-adjoint subspace satisfying

$$A^-A^- = A^+ \quad \text{and} \quad A^-A^+ = A^+A^- = A^-.$$

Elements $a \in A^\pm$ will be called homogeneous and we define $\deg(a) := \pm 1 \in \mathbb{Z}_2$. Notice that the degree mapping is multiplicative, that is $\deg(ab) = \deg(a)\deg(b)$ for homogeneous elements a and b. The elements of A^+ are called even, and the elements of A^- odd. If A is graded then we define the graded commutator for homogeneous elements $a, b \in A$ as

$$[a, b] := ab - \deg(ab)ba.$$

If A and B are graded C^*-algebras with grading automorphisms α respectively β the tensor product $A \otimes B$ can be given a grading by $\alpha \otimes \beta$. A mapping $\tau : A \to B$ is called graded if $\beta \circ \tau = \tau \circ \alpha$.

Suppose that B is a graded C^*-algebra with grading automorphism β. A grading on a B-Hilbert module $\mathcal{E}$ is a linear bijection $\gamma : \mathcal{E} \to \mathcal{E}$ such that $\gamma^2 = 1$ and for $b \in B$, $x, y \in \mathcal{E}$

$$\gamma(xb) = \gamma(x)\beta(b) \quad \text{and} \quad \langle\gamma(x), \gamma(y)\rangle = \beta(\langle x, y\rangle).$$

A grading induces a decomposition $\mathscr{E} = \mathscr{E}^+ \oplus \mathscr{E}^-$ of B^+-modules where $\gamma|_{\mathscr{E}^\pm} = \pm 1$. The B^+-linear projection onto $\mathscr{E}^\pm$ is given by $\frac{1}{2}(1 \pm \gamma)$. If $\mathscr{E}$ is a graded B-Hilbert module with grading γ the C^*-algebra $\mathscr{L}_B(\mathscr{E})$ is graded via the grading automorphism $T \mapsto \gamma T \gamma^{-1}$. This also induces a grading on the C^*-algebra $\mathscr{K}_B(\mathscr{E})$. An operator in $\mathscr{L}_B(\mathscr{E})^+$ is called even and an operator in $\mathscr{L}_B(\mathscr{E})^-$ is called odd.

If we have two graded B-Hilbert modules $\mathscr{E}$ and $\mathscr{E}'$ with gradings γ respectively γ' their direct sum $\mathscr{E} \oplus \mathscr{E}'$ is graded by the grading $\gamma \oplus \gamma'$. If $\mathscr{E}$ is graded by γ we may also define the opposite grading $-\gamma$. The B-Hilbert module $\mathscr{E}$ together with the opposite grading will be denoted by $-\mathscr{E}$.

From now on we will let A, B, C and D denote graded C^*-algebras. Suppose that $\mathscr{E}$ is a graded B-Hilbert module. If we have a graded $*$-homomorphism $\pi : A \to \mathscr{L}_B(\mathscr{E})$ we say that $\mathscr{E}$ is a graded $A-B$-Hilbert bimodule. This terminology is motivated by that the A-action commutes with the B-module structure on $\mathscr{E}$, so $\mathscr{E}$ forms a graded $A-B$-bimodule. If π is non-degenerate, we say that $\mathscr{E}$ is non-degenerate. If $\mathscr{E}$ is a graded $A-B$-Hilbert bimodule and $\mathscr{E}'$ a graded $B-C$-Hilbert bimodule, the internal tensor product $\mathscr{E} \otimes_B \mathscr{E}'$ is defined as the closure of the graded algebraic tensor product $\mathscr{E} \otimes_B^{alg} \mathscr{E}'$ in the scalar product

$$\langle x \otimes y, x' \otimes y' \rangle_{\mathscr{E} \otimes_B \mathscr{E}'} := \langle y, \pi(\langle x, x' \rangle_{\mathscr{E}}) y' \rangle_{\mathscr{E}'},$$

where $\pi : B \to \mathscr{L}_C(\mathscr{E}')$ denotes the left action of B on $\mathscr{E}'$. Since the representation of B on $\mathscr{E}'$ is graded, the grading $\gamma \otimes_B \gamma'$ is a well defined grading on $\mathscr{E} \otimes_B^{alg} \mathscr{E}'$ which clearly extends to the closure. The graded representation of A on $\mathscr{E}$ induces a graded representation on the internal tensor product $\mathscr{E} \otimes_B \mathscr{E}'$, so it forms a graded $A-C$-Hilbert bimodule in this scalar product. If $\pi : B \to \mathscr{L}(\mathscr{E}')$ denotes the left action of B on $\mathscr{E}'$ we will sometimes also denotes the internal tensor product by $\mathscr{E} \otimes_\pi \mathscr{E}'$.

We may also define an external tensor product. If $\mathscr{E}$ is a graded $A-B$-Hilbert bimodule and $\mathscr{E}'$ a graded $C-D$-Hilbert bimodule, the external tensor product $\mathscr{E} \otimes \mathscr{E}'$ is defined to be the closure of the $A \otimes B - C \otimes D$-bimodule $\mathscr{E} \otimes_{\mathbb{C}}^{alg} \mathscr{E}'$ in the scalar product

$$\langle x \otimes y, x' \otimes y' \rangle_{\mathscr{E} \otimes \mathscr{E}'} := \langle x, x' \rangle_{\mathscr{E}} \otimes \langle y, y' \rangle_{\mathscr{E}'}.$$

The $A \otimes B - C \otimes D$-Hilbert bimodule $\mathscr{E} \otimes \mathscr{E}'$ is graded by the grading $\gamma \otimes \gamma'$ so it is a graded Hilbert bimodule. If $\mathscr{E}$ is a graded B-Hilbert module and $\mathscr{E}'$ is a graded D-Hilbert module, we may set $A = C = \mathbb{C}$ and obtain an external tensor products of graded Hilbert modules.

An important concept in the theory of Hilbert modules is that of an imprimitivity module. This was introduced by Rieffel in [44]. For a brief introduction, see [32]. An essential graded $A-B$-Hilbert bimodule $\mathscr{E}$ is called an imprimitivity module from A to B if π is an isomorphism $\pi : A \xrightarrow{\sim} \mathscr{K}_B(\mathscr{E})$. Let B be a graded B-Hilbert module over itself and define the dual to be the graded vector space $\mathscr{E}^* := \mathscr{K}_B(\mathscr{E}, B)$. Since $\mathscr{E}$ is essential there exist an B-anti linear isomorphism $\mathscr{E} \cong \mathscr{E}^*$. Define a right graded action of A on $\mathscr{E}^*$ by $f.a(x) := f(ax)$ and an A-valued scalar product

$$\langle f_1, f_2 \rangle_{\mathscr{E}^*} := \pi^{-1}(T_{f_1,f_2})$$

where we identify $\mathscr{E} \cong \mathscr{E}^*$ using that $\mathscr{E}$ is essential. There is a graded representation $\pi_B : B \to \mathscr{L}_A(\mathscr{E}^*)$ given by $\pi_B(b)f(x) := bf(x)$. Clearly $\pi_B(B) = \mathscr{K}_A(\mathscr{E}^*)$ so $\mathscr{E}^*$ forms an imprimitivity bimodule from B to A. If there exists an imprimitivity bimodule from A to B then A is said to be Morita equivalent to B, this is denoted by $A \sim_M B$.

Proposition 2.3.1 ([44]). *Morita equivalence is an equivalence relation. If A and B are Morita equivalent, there is an equivalence between the category of A-Hilbert modules and the category of B-Hilbert modules.*

The theory of Hilbert modules also works well in the equivariant setting. Let $\mathscr{E}$ be a graded B-Hilbert module and D a trivially graded C^*-algebra. Define the graded $B \otimes D$-Hilbert module

$$\mathscr{M}_D(\mathscr{E} \otimes D) := \{T \in \mathscr{L}_{B\otimes D}(B \otimes D, \mathscr{E} \otimes D) : (1 \otimes d)T,\ T(1 \otimes d) \in \mathscr{E} \otimes D\ \forall d \in D\}.$$

Similarly the graded $D \otimes A$-Hilbert module $\mathscr{M}_D(D \otimes \mathscr{E})$ is defined. The notation $\tilde{M}(\mathscr{E} \otimes D)$ was used in [3], but the notation $\mathscr{M}_D(\mathscr{E} \otimes D)$ is easier understood and was used in for instance [42]. Note that if $\mathscr{E} = B$ then $\mathscr{M}_D(B \otimes D)$ is a C^*-algebra. From this point on, we let S denote a fixed reduced locally compact quantum group and it's comultiplication by Δ. We equip S with the trivial grading.

A non-degenerate, graded $*$-homomorphism $\Delta_A : A \to \mathscr{M}_S(A \otimes S)$ is called a right coaction if it is coassociative in the sense that $(\mathrm{id} \otimes \Delta)\Delta_A = (\Delta_A \otimes \mathrm{id})\Delta_A$. If Δ_A is faithful, we call Δ_A a reduced coaction. The coaction Δ_A is called continuous if $\Delta_A(A) \cdot 1 \otimes S$ is dense in $A \otimes S$. The graded C^*-algebra A equipped with a continuous right coaction of S will be called a graded $S - C^*$-algebra.

Similarly to right coactions, we may define the graded C^*-algebra $\mathscr{M}_S(S \otimes A)$ and a left coaction to be a non-degenerate, graded $*$-homomorphism $\Delta_A : A \to \mathscr{M}_S(S \otimes A)$ such that $(\mathrm{id} \otimes \Delta_A)\Delta_A = (\Delta \otimes \mathrm{id})\Delta_A$. We will only work with right coactions, so when saying coaction, we will mean a right coaction.

Definition 2.3.2 (Definition 2.2 of [3]). *If A has a right coaction of S and $\mathscr{E}$ is a graded A-Hilbert module, we say that a graded linear mapping $\delta_{\mathscr{E}} : \mathscr{E} \to \mathscr{M}_S(\mathscr{E} \otimes S)$ is a right coaction of S if*

1. *All* $a \in A$, $x, y \in \mathscr{E}$ *satisfy*

$$\delta_{\mathscr{E}}(xa) = \delta_{\mathscr{E}}(x)\Delta_A(a) \quad \text{and}$$
$$\Delta_A(\langle x, y\rangle_{\mathscr{E}}) = \langle \delta_{\mathscr{E}}(x), \delta_{\mathscr{E}}(y)\rangle_{\mathscr{E}\otimes S}.$$

2. *The subspace* $\delta_{\mathscr{E}}(\mathscr{E})(A \otimes S)$ *is dense in* $\mathscr{E} \otimes S$.

3. *The equality* $(\delta_{\mathscr{E}} \otimes \mathrm{id})\delta_{\mathscr{E}} = (\mathrm{id} \otimes \Delta)\delta_{\mathscr{E}}$ *hold as mappings* $\mathscr{E} \to \mathscr{L}(A \otimes S \otimes S, \mathscr{E} \otimes S \otimes S)$.

A graded A-Hilbert module $\mathscr{E}$ *together with a right coaction of S is called a graded S-equivariant A-Hilbert module. Similarly left coactions on* $\mathscr{E}$ *may be defined, except in that case* $\mathscr{M}_S(\mathscr{E} \otimes S)$ *is replaced by* $\mathscr{M}_S(S \otimes \mathscr{E})$.

Equivalently, a coaction may be constructed from a certain kind of unitary $V_{\mathscr{E}} \in \mathscr{L}(\mathscr{E} \otimes_{\Delta_A} (A \otimes S), \mathscr{E} \otimes S)$ by [3]. Here $\mathscr{E} \otimes_{\Delta_A} (A \otimes S)$ denotes the internal tensor product where $A \otimes S$ has the left action of A induced by Δ_A. First, for $x \in \mathscr{E}$ we define the $A \otimes S$-linear mapping

$$t_x : A \otimes S \to \mathscr{E} \otimes_{\Delta_A} (A \otimes S)$$

by $t_x(y) := x \otimes_{\Delta_A} y$ for $y \in A \otimes S$.

Definition 2.3.3 (Definition 2.3 of [3]). *An even unitary* $V_{\mathscr{E}} \in \mathscr{L}(\mathscr{E} \otimes_{\Delta_A} (A \otimes S), \mathscr{E} \otimes S)$ *is said to be admissible if it satisfies*

1. *For all* $x \in \mathscr{E}$ *then* $V_{\mathscr{E}} t_x \in \mathscr{M}_S(\mathscr{E} \otimes S)$.

2. *The identity* $(V_{\mathscr{E}} \otimes_{\mathbb{C}} \mathrm{id})(V_{\mathscr{E}} \otimes_{\Delta_A \otimes \mathrm{id}} \mathrm{id}) \equiv V_{\mathscr{E}} \otimes_{\mathrm{id}\otimes\Delta} \mathrm{id} \in \mathscr{L}(\mathscr{E} \otimes_{\Delta_A^2} (A \otimes S \otimes S), \mathscr{E} \otimes S \otimes S)$ *holds.*

Proposition 2.3.4 (Proposition 2.4 of [3]). *A coaction* $\delta_{\mathscr{E}}$ *is uniquely determined by an admissible unitary* $V_{\mathscr{E}} \in \mathscr{L}(\mathscr{E} \otimes_{\Delta_A} (A \otimes S), \mathscr{E} \otimes S)$ *via the equation* $\delta_{\mathscr{E}}(x) = V_{\mathscr{E}} t_x$. *Conversely, given an admissible unitary* $V_{\mathscr{E}}$ *one may define a coaction* $\delta_{\mathscr{E}}(x) := V_{\mathscr{E}} t_x$.

Definition 2.3.5 (Definition 2.9 of [3]). *If* A, B *are graded* $S - C^*$*-algebras and* $\mathscr{E}$ *is a graded equivariant B-Hilbert module, we say that a graded representation* $\pi : A \to \mathscr{L}_B(\mathscr{E})$ *is equivariant if*

$$\delta_{\mathscr{E}}(\pi(a)x) = (\pi \otimes \mathrm{id})(\Delta_A(a))\delta_{\mathscr{E}}(x).$$

A graded equivariant B-Hilbert module equipped with a graded equivariant representation of A is called a graded equivariant $A - B$*-Hilbert bimodule.*

If we have a graded S-equivariant $A-B$-Hilbert bimodule $\mathscr{E}$ and a graded S-equivariant $C-D$-Hilbert bimodule we can define their external tensor product which is a graded S-equivariant $A\otimes C-B\otimes D$-Hilbert bimodule denoted by $\mathscr{E}\otimes\mathscr{E}'$. As a graded $A\otimes C-B\otimes D$-Hilbert module $\mathscr{E}\otimes\mathscr{E}'$ is the external tensor product and the coaction is given by

$$\delta_{\mathscr{E}\otimes\mathscr{E}'}(x\otimes y):=\delta_{\mathscr{E}}(x)_{13}\delta_{\mathscr{E}'}(x)_{23}.$$

Similarly, the internal tensor product between a graded S-equivariant $A-B$-Hilbert bimodule $\mathscr{E}$ and a graded S-equivariant $B-C$-Hilbert bimodule is a graded S-equivariant $A-C$-Hilbert bimodule denoted by $\mathscr{E}\otimes_B\mathscr{E}'$ constructed as the internal tensor product with coaction given by

$$\delta_{\mathscr{E}\otimes_B\mathscr{E}'}(x\otimes_B y):=\delta_{\mathscr{E}}(x)_{13}\delta_{\mathscr{E}'}(x)_{23}.$$

Similarly to the non-equivariant case, if $\mathscr{E}$ has the two properties that it is essential and π is an isomorphism $\pi:A\to\mathscr{K}_B(\mathscr{E})$ then we say that $\mathscr{E}$ is a equivariant imprimitivity bimodule from A to B. If there exists an equivariant imprimitivity bimodule from A to B, then we will again denote this by $A\sim_M B$ and we say that A and B are equivariantly Morita equivalent. This is in fact an equivalence relation.

Proposition 2.3.6. *If A is a graded $S-C^*$-algebra there is an equivariant Morita equivalence $A\sim_M A\otimes\mathscr{K}$ where the right-hand side has coaction given by identifying $\mathscr{K}\cong\mathscr{K}(H_S)$ and defining*

$$\Delta_{A\otimes\mathscr{K}(H_S)}(a\otimes k):=Ad(V_{23})(\Delta_A(a)_{13}1\otimes k\otimes 1).$$

Recall that V denotes the right regular corepresentation of S on the GNS-space H_S as defined in Chapter 2.2 and that $Ad(V_{23})(x):=V_{23}xV_{23}^*$ for $x\in\mathscr{M}(A\otimes\mathscr{K}\otimes S)$.

Proof. An equivariant imprimitivity bimodule is given by the $A\otimes\mathscr{K}(H_S)-A$-Hilbert bimodule $\mathscr{E}:=A\otimes H_S$ with coaction

$$\delta(a\otimes x):=V_{23}(\Delta_A(a)_{13}1\otimes x).$$

Clearly $\mathscr{E}$ is essential and satisfies $\mathscr{K}_A(\mathscr{E})=A\otimes\mathscr{K}(H_S)$. □

Now we are ready to define the reduced crossed product of a C^*-algebra by a coaction. The definitions are taken from [4]. With a right coaction $\Delta_A:A\to\mathscr{M}_S(A\otimes S)$ we may associate a graded representation $\lambda_A:A\to\mathscr{L}_A(A\otimes H_S)$, where $A \otimes H_S$ denotes the external product between the graded A-Hilbert module A and the trivially graded Hilbert space H_S. The reduced crossed product $A \rtimes_r S$ is defined as

$$A \rtimes_r S := [\lambda_A(A) \cdot 1 \otimes \hat{\rho}(\hat{S})] \subseteq \mathscr{L}_A(A \otimes H_S).$$

In fact, as is shown in Lemma 7.2 in [4], the closed linear span of the set $\lambda_A(A) \cdot 1 \otimes \hat{\rho}(\hat{S})$ is a $*$-algebra. So $A \rtimes_r S$ forms a graded C^*-algebra, with the grading induced from that on A. By $a \rtimes \hat{s}$ we will denote the element $\lambda_A(a)1 \otimes \hat{\rho}(\hat{s})$. The $*$-homomorphism λ_A induces a graded representation $A \to \mathscr{M}(A \rtimes_r S)$ by $b(a \rtimes \hat{s}) = ba \rtimes \hat{s}$. If the coaction of S on A is reduced this mapping is faithful.

The reduced crossed product carries an $\hat{S}$-coaction given by

$$\Delta_{A \rtimes_r S}(a \rtimes \hat{s}) := \lambda_A(a)_{12} \cdot \left((\hat{\rho} \otimes \mathrm{id})\hat{\Delta}(\hat{s})\right)_{23}.$$

This coaction is called the dual coaction on $A \rtimes_r S$ and was defined in Definition 7.3 of [4]. That the coaction is well defined follows from that

$$(\hat{\rho} \otimes \hat{\rho})\hat{\Delta}(\hat{s}) = \hat{V}(\hat{\rho}(\hat{s}) \otimes 1)\hat{V}^*$$

and since $\hat{V} \in (\rho \otimes \hat{\rho})(\mathscr{M}(S \otimes \hat{S}))$ it commutes with $\lambda_A(A)$.

A graded $*$-homomorphism of graded $S - C^*$-algebras $\alpha : A \to B$ is said to be equivariant if

$$\Delta_B \circ \alpha = (\alpha \otimes \mathrm{id})\Delta_A.$$

We define $C^*_{\mathbb{Z}_2,S}$ to be the category of separable, graded $S - C^*$-algebras with morphisms being graded, equivariant $*$-homomorphisms. The index $\mathbb{Z}_2$ is to emphasize the ($\mathbb{Z}_2$-)grading on the objects. Let C^*_S denote the category of trivially graded $S - C^*$-algebras. We will, in particular for the KK-theory study the full subcategory $C^{*,r}_{\mathbb{Z}_2,S}$ of $C^*_{\mathbb{Z}_2,S}$ consisting of the $S - C^*$-algebras with reduced coactions. Similarly the full subcategory $C^{*,r}_S$ of C^*_S is defined.

The reduced crossed product does in fact produce a functor. From the construction we may also deduce certain properties of the image of this functor.

Proposition 2.3.7. *The construction $A \mapsto A \rtimes_r S$ gives a covariant functor*

$$\rtimes_r S : C^*_{\mathbb{Z}_2,S} \to C^{*,r}_{\mathbb{Z}_2,\hat{S}}.$$

Proof. We need to prove that the coaction on $A \rtimes_r S$ is continuous and reduced. It is reduced since it is implemented by the unitary V. The density condition (2.6) implies that

$$[\Delta_{A\rtimes_r S}(A \rtimes_r S) \cdot 1_{A\rtimes_r S} \otimes \hat{S}] = [\lambda_A(A)_{12} \cdot ((\hat{\rho} \otimes \mathrm{id})\hat{\Delta}(\hat{S}))_{23} \cdot 1_{A\rtimes_r S} \otimes \hat{S}] =$$

$$= [\lambda_A(A)_{12} \cdot 1 \otimes \hat{\rho}(\hat{S}) \otimes \hat{S}] = (A \rtimes_r S) \otimes \hat{S}.$$

Suppose that $\alpha : A \to B$ is a graded, equivariant $*$-homomorphism. This induces a graded, equivariant $*$-homomorphism $\alpha \rtimes_r S : A \rtimes_r S \to B \rtimes_r S$ given by

$$\alpha \rtimes_r S(a \rtimes \hat{s}) := \alpha(a) \rtimes \hat{s}.$$

This is well defined since α is equivariant and directly from it's definition it follows that $(\alpha \circ \alpha') \rtimes_r S = (\alpha \rtimes_r S) \circ (\alpha' \rtimes_r S)$. □

Since reduced crossed product is a functor on $C^*_{\mathbb{Z}_2,S}$ the natural question is what happens if one takes a double reduced crossed product? The first observation is that we obtain a functor $\rtimes_r S \rtimes_r \hat{S} : C^*_{\mathbb{Z}_2,S} \to C^{*,r}_{\mathbb{Z}_2,S}$. To study this question we need a technical requirement on the quantum group S. Just as in Chapter 2.2, we let W denote the left regular corepresentation of S. Define the following C^*-algebra as in Proposition 3.2 of [4]

$$\mathscr{C}(W) := [(\mathrm{id} \otimes \omega)(\sigma W) : \omega \in \mathscr{B}(H_S)_*].$$

Definition 2.3.8 (Definition 3.3 of [4]). *If $\mathscr{C}(W) = \mathscr{K}(H_S)$ the quantum group S is called regular.*

We recall the Takesaki-Takai duality theorem from [3] which answers the question how the double reduced crossed product behaves when S is regular and the coaction is reduced and continuous. In [3] this is only studied in the trivially graded setting, but since we assume our coactions to be graded their result generalizes to graded C^*-algebras.

Theorem 2.3.9 (Takesaki-Takai duality, Theorem 7.5 of [3]). *If S is regular and $A \in C^{*,r}_{\mathbb{Z}_2,S}$ there is a graded, equivariant $*$-isomorphism*

$$A \rtimes_r S \rtimes_r \hat{S} \xrightarrow{\sim} A \otimes \mathscr{K}(H_S)$$

where the right hand side has the coaction given by

$$\Delta_{A\otimes\mathscr{K}}(a \otimes k) := Ad(V_{23})(\Delta_A(a)_{13} \cdot (1 \otimes k \otimes 1)).$$

For the general proof of this, see [3]. We will present a proof in the easiest case $A = \mathbb{C}$. This case is in fact rather interesting since the method of proof appears in many more contexts later on in the thesis.

Lemma 2.3.10. *If S is regular there is an equivariant isomorphism*

$$\mathbb{C} \rtimes_r S \rtimes_r \hat{S} \xrightarrow{\sim} \mathscr{K}(H_S)$$

Proof. By definition $\mathbb{C} \rtimes_r S = \hat{\rho}(\hat{S})$, so

$$\mathbb{C} \rtimes_r S \rtimes_r \hat{S} = (\hat{\rho} \otimes \hat{\lambda})(\hat{\Delta}(\hat{S}))(1 \otimes \rho(S)).$$

After applying $Ad((1 \otimes \hat{J})V^*(1 \otimes J\hat{J}))$, since λ and ρ commutes, $\mathbb{C} \rtimes_r S \rtimes_r \hat{S}$ is mapped to $[1 \otimes \hat{\lambda}(\hat{S})\lambda(S)] \cong [\hat{\lambda}(\hat{S})\lambda(S)]$. Take $\xi, \xi', \eta, \eta' \in H_S$ and consider the generic elements

$$\begin{aligned} \lambda(a) &= \mathrm{id} \otimes \omega_{\xi,\eta}(W^*) \in \lambda(S) \quad \text{and} \\ \hat{\lambda}(\hat{a}) &= \omega_{\xi',\eta'} \otimes \mathrm{id}(W) \in \hat{\lambda}(\hat{S}). \end{aligned}$$

For $x, y \in H_S$ we have that

$$\begin{aligned} \langle \hat{\lambda}(\hat{a})^* \lambda(a) x, y \rangle &= \langle W_{12}^*(x \otimes \xi \otimes \eta'), W_{31}(y \otimes \eta \otimes \xi') \rangle = \\ &= \langle (\sigma W)_{12}^*(\xi \otimes x \otimes \eta'), (\sigma W)_{13}(\xi' \otimes \eta \otimes y) \rangle. \end{aligned}$$

So there exists $\omega, \omega' \in \mathscr{B}(H_S)_*$ such that for any $x, y \in H_S$

$$\langle \hat{\lambda}(\hat{a})^* \lambda(a) x, y \rangle = \langle \big(\omega \otimes \mathrm{id}(\sigma W)^*\big) x, \big(\omega' \otimes \mathrm{id}(\sigma W)\big) y \rangle.$$

This implies that $\hat{\lambda}(\hat{S})\lambda(S) \subseteq \mathscr{C}(\hat{W})$. It follows that $[\hat{\lambda}(\hat{S})\lambda(S)] = \mathscr{K}(H_S)$ since $\hat{W}$ is regular and $[\hat{\lambda}(\hat{S})\lambda(S)]$ acts irreducibly on H_S.

There is an isomorphism $Ad(\hat{J}J) : \mathscr{K}(H_S) \cong [\hat{\lambda}(\hat{S})\lambda(S)] \to [\hat{\rho}(\hat{S})\rho(S)]$. Since $\hat{J}J \in \mathscr{B}(H_S)$ it follows that $[\hat{\rho}(\hat{S})\rho(S)] = \mathscr{K}(H_S)$. The coaction induced from $\hat{S} \rtimes \hat{S} \cong [\hat{\rho}(\hat{S})\rho(S)]$ coincides with $\hat{\rho}(\hat{s})\rho(s) \mapsto Ad(V)(\hat{\rho}(\hat{s})\rho(s) \otimes 1)$. So the isomorphism $\mathbb{C} \rtimes_r S \rtimes_r \hat{S} \xrightarrow{\sim} \mathscr{K}(H_S)$ is equivariant. □

2.4 Twists of reduced quantum groups

Twists have a central role in quantization of Lie algebras. There they are seen as quantized infinitesimal cocycles of the quantized Lie algebra. But the main examples of locally compact quantum groups have been compact quantum groups. There the usual source of examples comes from deformations. Deformations are difficult to fit into a general framework since the underlying C^*-algebra and it's deformations are very hard to relate because they have, by construction, different multiplication. Since deformations are dual to twists we can choose the equivalent approach using twists.

Definition 2.4.1. *Let S be a reduced locally compact quantum group and assume that the unitary $\mathscr{F} \in \mathscr{M}(S \otimes S)$ satisfies the cocycle condition*

$$Ad\left(\mathscr{F}_{23}(\mathrm{id} \otimes \Delta)(\mathscr{F})\right)(\Delta^{(2)}(s)) = Ad\left(\mathscr{F}_{12}(\Delta \otimes \mathrm{id})(\mathscr{F})\right)(\Delta^{(2)}(s)) \quad \forall s \in S.$$

If $\mathscr{F}$ also satisfies the density condition

$$[\omega \otimes \mathrm{id}(\mathscr{F}\Delta(a)\mathscr{F}^*) : \omega \in S^*] = [\mathrm{id} \otimes \omega(\mathscr{F}\Delta(a)\mathscr{F}^*) : \omega \in S^*] = S,$$

then $\mathscr{F}$ is called a cocycle twist of S. If a cocycle twist $\mathscr{F}$ satisfies the stronger condition

$$\mathscr{F}_{23}(\mathrm{id} \otimes \Delta)(\mathscr{F}) = \mathscr{F}_{12}(\Delta \otimes \mathrm{id})(\mathscr{F}),$$

we just say that $\mathscr{F}$ is a twist.

For example, assume that $\hat{S}$ is a compact matrix quantum group. Since $\hat{S}$ is compact, there exists a dense multiplier $*$-Hopf algebra $S_{ho} \subseteq S$ such that finite-dimensional corepresentations of $\hat{S}$ corresponds to S_{ho}-modules. Suppose that a cocycle twist $\mathscr{F} \in \mathscr{M}(S \otimes S)$ leaves $S_{ho} \otimes S_{ho}$ invariant under $Ad\ \mathscr{F}$. Then $\mathscr{F}$ corresponds to a deformation of the multiplication in $\hat{S}$.

Because of the cocycle condition we may define a new, coassociative comultiplication $\Delta_{\mathscr{F}} := Ad\mathscr{F} \circ \Delta$ on S. The cocycle condition is in fact equivalent to $\Delta_{\mathscr{F}}$ being coassociative. If $\mathscr{F}$ is a cocycle twist let $S_{\mathscr{F}}$ denote the bi-C^*-algebra S with comultiplication $\Delta_{\mathscr{F}}$.

If $\mathscr{F}$ is a cocycle twist of S we define the associator as:

$$\Phi_{\mathscr{F}} := (\mathrm{id} \otimes \Delta)(\mathscr{F}^*)\mathscr{F}_{23}^*\mathscr{F}_{12}(\Delta \otimes \mathrm{id})(\mathscr{F}).$$

The cocycle condition for $\mathscr{F}$ is equivalent to $\Phi_{\mathscr{F}}$ commuting with $\Delta^{(2)}(S)$. If S admits a counit ε and $\mathrm{id} \otimes \varepsilon \otimes \mathrm{id}(\Phi_{\mathscr{F}}) = 1$ the associator $\Phi_{\mathscr{F}}$ is the associator for the monoidal category of modules of S with the tensor product induced from the comultiplication $\Delta_{\mathscr{F}}$.

Definition 2.4.2. *If the bi-C^*-algebra $S_{\mathscr{F}}$ admits Haar weights, thus making $S_{\mathscr{F}}$ into a reduced locally compact quantum group, we say that the cocycle twist $\mathscr{F}$ is manageable.*

For a general cocycle twist of reduced quantum groups manageability is hard to study. But in the von Neumann setting this has been studied for twists in [13]. So we recall the following theorem from [13]:

Theorem 2.4.3 (Corollary 6.2 and Proposition 6.4 of [13])**.** *Let M denote the von Neumann algebra generated by S in it's left regular representation and assume that the unitary $\mathscr{F} \in M \otimes M$ satisfies*

$$\mathscr{F}_{23}(\mathrm{id} \otimes \Delta)(\mathscr{F}) = \mathscr{F}_{12}(\Delta \otimes \mathrm{id})(\mathscr{F}).$$

Then there exists Haar weights with respect to $\Delta_{\mathscr{F}}$ on M and the left regular corepresentation of $S_{\mathscr{F}}$ is given by

$$W_{\mathscr{F}} = (\hat{J}_{\mathscr{F}} \otimes J)\mathscr{F} W^*(\hat{J} \otimes J)\mathscr{F}^*$$

where $\hat{J}_{\mathscr{F}}$ denote the modular conjugation operator of $\widehat{S_{\mathscr{F}}}$.

For the proof of this theorem we refer the reader to [13]. The proof is rather technical in nature and consists of using the twist $\mathscr{F}$ to define a Galois object over M.

Theorem 2.4.4. *Every twist $\mathscr{F}$ is a manageable cocycle twist.*

Proof. Let M denote the von Neumann algebra generated by S in it's left regular representation. By Corollary 6.2 of [13] formulated above, the bi-von Neumann algebra $(M, \Delta_{\mathscr{F}})$ admits faithful Haar weights. Thus there exists Haar weights on $S_{\mathscr{F}}$. By the density condition on $\mathscr{F}$ the bi-C^*-algebra $S_{\mathscr{F}}$ is a reduced locally compact quantum group. □

If $\mathscr{F}$ is a cocycle twist of S the cocycle condition for $\mathscr{F}$ implies that for $s \in S$

$$Ad\left(\mathscr{F}_{23}^*(\mathrm{id} \otimes \Delta_{\mathscr{F}})(\mathscr{F}^*)\right)\Delta_{\mathscr{F}}^{(2)}(s) = Ad\left((\mathrm{id} \otimes \Delta)(\mathscr{F}^*)\mathscr{F}_{23}^*\right)\Delta_{\mathscr{F}}^{(2)}(s) = \tag{2.10}$$
$$= \Delta^{(2)}(s) = Ad\left((\Delta \otimes \mathrm{id})(\mathscr{F}^*)\mathscr{F}_{12}^*\right)\Delta_{\mathscr{F}}^{(2)}(s) = Ad\left(\mathscr{F}_{12}^*(\Delta_{\mathscr{F}} \otimes \mathrm{id})(\mathscr{F}^*)\right)\Delta_{\mathscr{F}}^{(2)}(s).$$

Therefore if $\mathscr{F}$ is a manageable cocycle twist of S, $\mathscr{F}^*$ is a manageable cocycle twist of $S_{\mathscr{F}}$. Similarly, if $\mathscr{F}$ is a twist it follows that

$$\mathscr{F}_{23}^*(\mathrm{id} \otimes \Delta_{\mathscr{F}})(\mathscr{F}^*) = (\mathrm{id} \otimes \Delta)(\mathscr{F}^*)\mathscr{F}_{23}^* = \tag{2.11}$$
$$= (\Delta \otimes \mathrm{id})(\mathscr{F}^*)\mathscr{F}_{12}^* = \mathscr{F}_{12}^*(\Delta_{\mathscr{F}} \otimes \mathrm{id})(\mathscr{F}^*).$$

Thus, if $\mathscr{F}$ is a twist of S then $\mathscr{F}^*$ is a twist of $S_{\mathscr{F}}$ and clearly $(S_{\mathscr{F}})_{\mathscr{F}^*} = S$.

If $\mathscr{F}$ is a manageable cocycle twist of $\hat{S}$ will use the notation $S_{\mathscr{F}}$ to denote the reduced dual of $\hat{S}_{\mathscr{F}}$. Assume that $\mathscr{F}$ is a twist of $\hat{S}$. Then we can define a twisted coaction

$$\Delta_{A \rtimes_r S}^{\mathscr{F}} : A \rtimes_r S \to \mathscr{M}_{\hat{S}_{\mathscr{F}}}(A \rtimes_r S \otimes \hat{S}_{\mathscr{F}}),$$
$$\Delta_{A \rtimes_r S}^{\mathscr{F}} := Ad((\hat{\rho} \otimes \mathrm{id}(\mathscr{F}))_{23})\Delta_{A \rtimes_r S}. \tag{2.12}$$

The C^*-algebra $A \rtimes_r S$ with this coaction of $\hat{S}_{\mathscr{F}}$ is denoted by $(A \rtimes_r S)_{\mathscr{F}}$. If $B = A \rtimes_r S$ then we will also denote it by $B_{\mathscr{F}}$.

Notice that if $\mathscr{F}$ only is a cocycle twist, the $*$-homomorphism $\Delta_{A \rtimes_r S}^{\mathscr{F}}$ is not coassociative in general. It is merely a quasi-coaction since

$$(\mathrm{id}_{A \rtimes_r S} \otimes \hat{\Delta}_{\mathscr{F}})\Delta_{A \rtimes_r S}^{\mathscr{F}} = Ad(1_A \otimes \Phi_{\mathscr{F}}^*)(\Delta_{A \rtimes_r S}^{\mathscr{F}} \otimes \mathrm{id})\Delta_{A \rtimes_r S}^{\mathscr{F}}.$$

Proposition 2.4.5. *If A has a coaction of S and $\mathscr{F}$ is a twist of $\hat{S}$ the $\hat{S}_{\mathscr{F}}$-coaction on $(A \rtimes_r S)_{\mathscr{F}}$ is continuous.*

Proof. Since the $\hat{S}$-coaction on $A \rtimes_r S$ is continuous it follows from the proof of Proposition 2.3.7 that

$$[\lambda_A(A)_{12} \cdot 1 \otimes \hat{\rho}(\hat{S}) \otimes \hat{S}] = (A \rtimes_r S) \otimes \hat{S}$$

is a C^*-algebra. Therefore

$$[\lambda_A(A)_{12} \cdot 1 \otimes \hat{\rho}(\hat{S}) \otimes \hat{S}] = [1 \otimes \hat{\rho}(\hat{S}) \otimes \hat{S} \cdot \lambda_A(A)_{12}].$$

The twist $\mathscr{F}$ is a multiplier of $\hat{S} \otimes \hat{S}$ so it follows that

$$[\Delta^{\mathscr{F}}_{A\rtimes_r S}(A \rtimes_r S) \cdot 1_{A\rtimes_r S} \otimes \hat{S}] = [\mathscr{F}_{23}\lambda_A(A)_{12}\mathscr{F}^*_{23} \cdot ((\hat{\rho} \otimes \mathrm{id})\hat{\Delta}_{\mathscr{F}}(\hat{S}))_{23} \cdot 1_{A\rtimes_r S} \otimes \hat{S}] =$$

$$= [\mathscr{F}_{23}\lambda_A(A)_{12} \cdot 1 \otimes \hat{\rho}(\hat{S}) \otimes \hat{S}] = [\mathscr{F}_{23} 1 \otimes \hat{\rho}(\hat{S}) \otimes \hat{S} \cdot \lambda_A(A)_{12}] =$$
$$= [\lambda_A(A)_{12} \cdot 1 \otimes \hat{\rho}(\hat{S}) \otimes \hat{S}] = (A \rtimes_r S) \otimes \hat{S}.$$

The second equality follows from that $\mathscr{F}$ is a twist so by Theorem 2.4.4 it is manageable which implies that $[(\hat{\rho} \otimes \mathrm{id})\hat{\Delta}_{\mathscr{F}}(\hat{S})) \cdot 1 \otimes \hat{S}] = \hat{\rho}(\hat{S}) \otimes \hat{S}$. ☐

Definition 2.4.6. *If $S_{\mathscr{F}}$ is regular we say that $\mathscr{F}$ is a strongly manageable cocycle twist.*

Theorem 2.4.7. *Every twist of a regular, reduced quantum group is strongly manageable. Conversely, if there exists a twist of S which is strongly manageable then S is regular.*

Proof. We will start by proving $\mathscr{C}(W_{\mathscr{F}}) \subseteq \mathscr{K}(H_S)$. Let $\omega_{\xi,\eta} \in \mathscr{B}(H_S)_*$ denote the functional $\omega_{\xi,\eta}(T) := \langle T\xi, \eta\rangle$ for $\xi, \eta \in H_S$. It is sufficient to prove that $(\mathrm{id} \otimes \omega_{\xi,\eta})(\sigma W_{\mathscr{F}}) \in \mathscr{K}(H_S)$ for all ξ, η. We will prove compactness using the regularity of W and a diagonal procedure.

Take a bounded sequence $(x_k)_{k\in\mathbb{N}} \subseteq H_S$ and define

$$y_k := (\mathrm{id} \otimes \omega_{\xi,\eta})(\sigma W_{\mathscr{F}})(x_k).$$

Let P_l denote the orthogonal projection onto the linear span of $x_0, x_1, \ldots, x_l$. Define $\mathscr{F}_l := (P_l \otimes P_l)\mathscr{F}(P_l \otimes P_l)$ and

$$T_l := (\hat{J}_{\mathscr{F}} \otimes J)\mathscr{F}_l W^*(\hat{J} \otimes J)\mathscr{F}^*_l.$$

We observe that for $y \in H_S$ the following equality holds:

$$\Big\langle (\mathrm{id} \otimes \omega_{\xi,\eta})(\sigma T_l)x_k, y \Big\rangle = \Big\langle W^*(\hat{J} \otimes J)\mathscr{F}^*_l(x_k \otimes \xi), \mathscr{F}^*_l(\hat{J}_{\mathscr{F}}\eta \otimes Jy) \Big\rangle.$$

Define $y_k^l := (\mathrm{id} \otimes \omega_{\xi,\eta})(\sigma T_l)(x_k)$, clearly the regularity of W implies that $y_k^l \to y_k$. Construct $(x_{kl}) \subseteq (x_k)$ inductively by taking (x_{k0}) as a subsequence such that $(\mathrm{id} \otimes \omega_{\xi,\eta}(\sigma T_0)x_k)$ converges, then take (x_{kl+1}) to be subsequence of (x_{kl}) such that $(\mathrm{id} \otimes \omega_{\xi,\eta}(\sigma T_{l+1})x_{kl})$ converges. Existence of convergent subsequences follows from compactness of T_l. The subsequence $(x_{kk}) \subseteq (x_k)$ satisfies that $(\mathrm{id} \otimes \omega_{\xi,\eta})(\sigma W_{\mathscr{F}})(x_{kk})$ converges because

$$\|(\mathrm{id} \otimes \omega_{\xi,\eta})(\sigma W_{\mathscr{F}})(x_{kk}) - (\mathrm{id} \otimes \omega_{\xi,\eta})(\sigma W_{\mathscr{F}})(x_{k'k'})\| \leq$$

$$\leq \|(\mathrm{id} \otimes \omega_{\xi,\eta})(\sigma W_{\mathscr{F}})(x_{kk}) - y_k^{k'}\| + \|y_k^{k'} - y_{k'}^{k'}\| +$$

$$+ \|y_{k'}^{k'} - (\mathrm{id} \otimes \omega_{\xi,\eta})(\sigma W_{\mathscr{F}})(x_{k'k'})\| \to 0.$$

It follows that $(\mathrm{id} \otimes \omega_{\xi,\eta})(\sigma W_{\mathscr{F}})$ is compact.

To prove that $\mathscr{C}(W_{\mathscr{F}}) = \mathscr{K}(H_S)$ we observe that Lemma 3.1 of [4] implies that

$$\mathscr{C}(W_{\mathscr{F}}) \otimes \mathscr{K}(H_S) = \sigma(\mathscr{K}(H_S) \otimes 1 \cdot W_{\mathscr{F}} \cdot 1 \otimes \mathscr{K}(H_S)).$$

Since the right-hand side acts irreducibly on $H_S \otimes H_S$, it follows that $\mathscr{C}(W_{\mathscr{F}}) = \mathscr{K}(H_S)$. □

Using the ideas from the proof of Takesaki-Takai duality from [4] we obtain a twisted Takesaki-Takai duality for regular quantum groups. The untwisted Takesaki-Takai duality from [4], formulated above as Theorem 2.3.9, states that there is a graded, equivariant $*$-isomorphism $A \rtimes_r S \rtimes_r \hat{S} \cong A \otimes \mathscr{K}(H_S)$. This isomorphism does, in fact, behave well even if we allow a twist. Thus we obtain a twisted Takesaki-Takai duality.

Theorem 2.4.8 (TTT-duality)**.** *Let S be a regular, reduced locally compact quantum group and $\mathscr{F}$ a twist of $\hat{S}$. For $A \in C^{*,r}_{\mathbb{Z}_2,S}$ there is a natural graded $*$-isomorphism*

$$(A \rtimes_r S)_{\mathscr{F}} \rtimes_r \hat{S}_{\mathscr{F}} \cong A \otimes \mathscr{K}(H_S).$$

Proof. Let $\hat{\lambda}_{\mathscr{F}} : \hat{S}_{\mathscr{F}} \to \mathscr{B}(H_S)$ denote the left regular representation and similarly for $\rho_{\mathscr{F}}$, $\lambda_{\mathscr{F}}$ and $\hat{\rho}_{\mathscr{F}}$. Since the Haar weights are faithful, there is a unitary $U \in \mathscr{B}(H_S)$ such that $\hat{\rho}_{\mathscr{F}} = Ad U \circ \hat{\rho}$. Define $\lambda_A' := Ad(1 \otimes U) \circ \lambda_A$. Consider the $*$-homomorphism $Ad(1 \otimes U \otimes 1) : (A \rtimes_r S)_{\mathscr{F}} \rtimes_r \hat{S}_{\mathscr{F}} \to \mathscr{L}(A \otimes H_S \otimes H_S)$ which acts on $(A \rtimes_r S)_{\mathscr{F}} \rtimes_r \hat{S}_{\mathscr{F}}$ as

$$(\lambda_A(a) \otimes 1) \cdot (1 \otimes (\hat{\rho} \otimes \hat{\lambda}_{\mathscr{F}})\hat{\Delta}_{\mathscr{F}}(\hat{s})) \cdot (1 \otimes 1 \otimes \rho_{\mathscr{F}}(s_{\mathscr{F}})) \mapsto$$

$$(\lambda_A'(a) \otimes 1) \cdot (1 \otimes (\hat{\rho}_{\mathscr{F}} \otimes \hat{\lambda}_{\mathscr{F}})\hat{\Delta}_{\mathscr{F}}(\hat{s})) \cdot (1 \otimes 1 \otimes \rho_{\mathscr{F}}(s_{\mathscr{F}})).$$

Since the Haar weights of S and $S_{\mathscr{F}}$ are faithful, this is well defined. The subalgebra $\lambda(S) \subseteq \mathscr{B}(H_S)$ commutes with $\rho(S)$ so it is a $*$-homomorphism. Let $\hat{W}_{\mathscr{F}}$ denote the left regular corepresentation of $\hat{S}_{\mathscr{F}}$. Composing this with $Ad(1\otimes(1\otimes J_{\mathscr{F}})\hat{W}_{\mathscr{F}}(1\otimes J_{\mathscr{F}}))$ we end up in the closed linear span of elements of the form

$$\lambda''_A(a)\cdot(1_{A\otimes H_S}\otimes\hat{\lambda}_{\mathscr{F}}(\hat{s})\lambda_{\mathscr{F}}(s_{\mathscr{F}})),$$

where $\lambda''_A(a) := Ad(1\otimes(1\otimes J_{\mathscr{F}})\hat{W}_{\mathscr{F}})(\lambda'_A(a)\otimes 1)$. Using the identification $A\otimes H_S\otimes H_S \cong (A\otimes H_S)\otimes_{\lambda'_A}(A\otimes H_S)\cong A\otimes H_S$ we obtain

$$Ad(1\otimes(1\otimes J_{\mathscr{F}})\hat{W}_{\mathscr{F}}(1\otimes J_{\mathscr{F}})1\otimes U\otimes 1)((A\rtimes_r S)_{\mathscr{F}}\rtimes_r\hat{S}_{\mathscr{F}}) = A\otimes(\hat{S}_{\mathscr{F}}\rtimes_r\hat{S}_{\mathscr{F}})\cong$$

$$\cong A\otimes\mathscr{K}(H_S)$$

where the first equality follows from the continuity of the coaction Δ_A and the second equality follows from Theorem 2.3.9, since $\mathscr{F}$ is strongly manageable by Theorem 2.4.7. □

2.5 Example: Twists induced from abelian subgroups

In [30] deformations of the quantum group $C_0(G)$ coming from twists on $C^*_r(G)$ was studied. In particular they studied twists induced from abelian subgroups. We will review that induction procedure from a dual viewpoint.

So take G to be a locally compact group and H a closed abelian subgroup. Assume that $\mathscr{F}_H \in \mathscr{M}(C^*(H)\otimes C^*(H))$ satisfies the cocycle relation

$$\mathscr{F}_{H,12}(\Delta\otimes\mathrm{id})(\mathscr{F}_H) = \mathscr{F}_{H,23}(\mathrm{id}\otimes\Delta)(\mathscr{F}_H).$$

Observe that H is abelian, thus amenable, so $C^*(H) = C^*_r(H)$. Extending the Fourier transformation $C^*(H)\cong C_0(\hat{H})$ to an isomorphism $\mathscr{M}(C^*(H)\otimes C^*(H))\cong\mathscr{M}(C_0(\hat{H})\otimes C_0(\hat{H})) = C_b(\hat{H}\times\hat{H})$ and letting χ denote the image of $\mathscr{F}_H$ under this isomorphism we obtain the identity

$$\chi(h_1,h_2)\chi(h_1h_2,h_3) = \chi(h_2,h_3)\chi(h_1,h_2h_3)\quad\forall h_1,h_2,h_3\in\hat{H}.$$

So χ is a 2-cocycle in for $\hat{H}$. By Theorem 7.1 of [24], if $h\mapsto h^2$ is an automorphism of $\hat{H}$, every 2-cocycle is similar to a bicharacter. So in general one may assume that χ is an bicharacter.

As an example, consider $H=\mathbb{R}^n$. Then every bicharacter of $\hat{H}=\mathbb{R}^n$ is of the form

$$(x_1,x_2)\mapsto \mathrm{e}^{ix_1Ax_2}\quad\text{for an}\quad A\in M_n(\mathbb{R}).$$

This follows from that bicharacters of $\hat{H}$ corresponds to homomorphisms $\hat{H} \to H$.

Returning to the twists, given a bicharacter χ on $\hat{H}$ we may associate a twist $\mathscr{F}_\chi$ of $C_r^*(G)$ via the mapping $C_b(\hat{H}\times\hat{H}) \xrightarrow{\sim} \mathscr{M}(C^*(H)\otimes C^*(H)) \hookrightarrow \mathscr{M}(C_r^*(G)\otimes C_r^*(G))$. Since the Fourier transformation is an isomorphism, the unitary $\mathscr{F}_\chi$ satisfies the cocycle identity $\mathscr{F}_{\chi,12}(\Delta\otimes\mathrm{id})(\mathscr{F}_\chi) = \mathscr{F}_{\chi,23}(\mathrm{id}\otimes\Delta)(\mathscr{F}_\chi)$. So clearly $\Delta_\chi := Ad(\mathscr{F}_\chi)\Delta$ is a comultiplication on $C_r^*(G)$.

Proposition 2.5.1. *If χ is a bicharacter on H, the unitary $\mathscr{F}_\chi$ defines a twist of $C_r^*(G)$.*

Proof. What remains to be proven is the density condition for $\mathscr{F}_\chi$. To do this we use the counit $\varepsilon : L^1(G) \to \mathbb{C}$ defined by $\varepsilon(f) := \int_G f\,\mathrm{d}g$. Then for $f, g \in L^1(G)$ we have that $\mathrm{id}\otimes\varepsilon(\mathscr{F}_\chi(f\otimes g)) = f\,\varepsilon(g)$ and similarly $\varepsilon\otimes\mathrm{id}(\mathscr{F}_\chi(f\otimes g)) = \varepsilon(f)g$. So it follows that for $f \in L^1(G)$ we have the equalities

$$\mathrm{id}\otimes\varepsilon(\Delta_\chi(f)) = \varepsilon\otimes\mathrm{id}(\Delta_\chi(f)) = f.$$

Since $L^1(G)$ is dense in $C^*(G)$ the density condition for $\mathscr{F}_\chi$ follows. □

To end this section let us construct a twist of the type above on a quotient of the Heisenberg group. This example is based on Example 3.6.2 of [30]. The Heisenberg group is a Lie group which as a manifold is $\mathbb{R}^3$ with multiplication

$$(x_1, y_1, z_3)(x_2, y_2, z_3) = (x_1 + x_2, y_1 + y_2, z_1 + z_2 + y_2 x_1).$$

Embed $\mathbb{Z}$ as the central subgroup of the Heisenberg group generated by $(0, 0, 1)$ and let G denote the quotient. The Lie group G is as a manifold diffeomorphic to $\mathbb{R}^2 \times \mathbb{T}$ and the multiplication is given by

$$(x_1, y_1, z_1)(x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 z_2 \mathrm{e}^{i y_2 x_1}).$$

Consider the closed abelian subgroup $H = \mathbb{R} \times \{0\} \times \mathbb{T}$. The Pontryagin dual of H is given by $\mathbb{R} \times \mathbb{Z}$. Define the bicharacter $\chi : \hat{H} \times \hat{H} \to \mathbb{T}$ as

$$\chi((\xi_1, n), (\xi_2, k)) := \mathrm{e}^{i(\xi_1 k - \xi_2 n)}.$$

Let us calculate how Δ_χ acts on $C_r^*(G)$. It is sufficient to study elements of the form $a = \lambda_{(x,y,z)}$, left translation by the group element (x, y, z). Although this is not an element in the group C^*-algebra, every element in $C_r^*(G)$ is an integrated form of elements of this type and since the comultiplication is strictly continuous we may deduce expressions for $a \in L^1(G)$. Since

$$(x, y, z) = (x, 0, 1)(0, y, 1)(0, 0, z)$$

and H commutes with the twist we may take $x=0$ and $z=1$. In this case we have that

$$
\begin{aligned}
&\Delta_\chi(a)f(x_1,x_2,y_1,y_2,z_1,z_2)=\\
&=\sum_{k,l,m,n}\int_{\mathbb{T}^4} w_1^k w_2^l w_3^m w_4^n f(x_1-n+l,x_2+m-k,y_1+y,y_2+y,\\
&\qquad \bar{w}_3\bar{w}_1 z_1 \mathrm{e}^{iy(x_1+k-n)},\bar{w}_4\bar{w}_2 z_1 \mathrm{e}^{iy(x_2-l+m)})\mathrm{d}w_1\mathrm{d}w_2\mathrm{d}w_3\mathrm{d}w_4=\\
&=\sum_{k,l,m,n}\int_{\mathbb{T}^2} w_3^m w_4^n f_{kl}(x_1-n+l,x_2+m-k,y_1+y,y_2+y)\\
&\qquad \bar{w}_3^k z_1^k \bar{w}_4^l z_2^l \mathrm{e}^{iyk(x_1+k-n)}\mathrm{e}^{iyl(x_2-l+m)}\mathrm{d}w_3\mathrm{d}w_4=\\
&=\sum_{k,l} f_{kl}(x_1,x_2,y_1+y,y_2+y) z_1^k z_2^l \mathrm{e}^{iyk(x_1+k-l)}\mathrm{e}^{iyl(x_2+k-l)}.
\end{aligned}
$$

So letting μ denote the left Haar measure of G then for $a\in L^1(G)$ we may express $\Delta_\chi(a)$ as

$$
\begin{aligned}
&(\Delta_\chi(a)\,f)(x_1,x_2,y_1,y_2,z_1,z_2)=\\
&\sum_{k,l}\int_G a(x,y,z)\ f_{kl}(x_1+x,x_2+x,y_1+y,y_2+y)\cdot\\
&\qquad\cdot\bar{z}^{k+l} z_1^k z_2^l \mathrm{e}^{iyk(x_1+k-l)}\mathrm{e}^{iyl(x_2+k-l)}\mathrm{d}\mu.
\end{aligned}
$$

2.6 Example: Drinfeld-Jimbo twists

The aim of this section is to give the structure of a locally compact quantum group to the Drinfeld-Jimbo quantization of a Lie algebra and describe it as a manageable cocycle twist of the group C^*-algebra. So let G be a semisimple, compact, connected Lie group of rank n and $\mathfrak{g}$ it's Lie algebra. The Drinfeld-Jimbo algebra $\mathscr{U}_q(\mathfrak{g})$ is a deformation of the universal enveloping algebra of $\mathfrak{g}$ depending on a parameter $q\in\mathbb{C}\setminus\{0,1\}$. It was independently constructed by Drinfeld and Jimbo. The Drinfeld-Jimbo twists are of interest in the theory of locally compact quantum groups since they are dual to Woronowicz deformations as was shown in [46].

Let the Cartan matrix of $\mathfrak{g}$ associated with the Cartan subalgebra $\mathfrak{h}$ be denoted by $A=(a_{ij})$. Let $d_1,\dots,d_n$ be coprime integers such that the matrix (d_ia_{ij}) is symmetric. We also take a complex number $q\in\mathbb{C}\setminus\{0,1\}$ and define

$q_i := q^{d_i}$. For an integer k, we define

$$[k]_{q_i} := \frac{q_i^k - q_i^{-k}}{q_i - q_i^{-1}}.$$

If l is a positive integer we define

$$\binom{k}{l}_{q_i} := \frac{[k]_{q_i}[k-1]_{q_i}\cdots[k-l+1]_{q_i}}{[1]_{q_i}[2]_{q_1}\cdots[l]_{q_i}}.$$

The Drinfeld-Jimbo quantization $\mathscr{U}_q(\mathfrak{g})$ is given by generators E_i, F_i, K_i, K_i^{-1}, for $1 \leq i \leq n$, satisfying the relations

$$K_iK_i^{-1} = K_i^{-1}K_i = 1, \quad K_iK_j = K_jK_i, \quad K_iE_jK_i^{-1} = q_i^{a_{ij}}E_j,$$

$$K_iF_jK_i^{-1} = q_i^{-a_{ij}}F_j, \quad [E_i, F_j] = \delta_{ij}\frac{K_i - K_i^{-1}}{q_i - q_i^{-1}},$$

$$\sum_{k=0}^{1-a_{ij}}(-1)^k\binom{1-a_{ij}}{k}_{q_i} E_i^kE_jE_i^{1-a_{ij}-k} = 0,$$

$$\sum_{k=0}^{1-a_{ij}}(-1)^k\binom{1-a_{ij}}{k}_{q_i} F_i^kF_jF_i^{1-a_{ij}-k} = 0.$$

To read more about the algebra $\mathscr{U}_q(\mathfrak{g})$ and it's representation theory, see [25]. The algebra $\mathscr{U}_q(\mathfrak{g})$ forms a multiplier Hopf algebra with comultiplication

$$\Delta_q(K_i) := K_i \otimes K_i, \quad \Delta_q(E_i) := E_i \otimes 1 + K_i \otimes E_i, \quad \Delta_q(F_i) := F_i \otimes K_i^{-1} + F_i \otimes 1,$$

and counit ε_q given by $\varepsilon_q(E_i) = \varepsilon_q(F_i) = 0$ and $\varepsilon_q(K_i) = 1$. For $q \in \mathbb{R}$ there is a $*$-structure on $\mathscr{U}_q(\mathfrak{g})$ given by

$$K_i^* = K_i, \quad E_i^* = K_iF_i \quad \text{and} \quad F_i^* = E_iK_i^{-1}.$$

Our aim is to construct a locally compact quantum group $\hat{G}_q$ such that it's finite dimensional modules are weight modules of the Drinfeld-Jimbo quantization $\mathscr{U}_q(\mathfrak{g})$, which are twists of a classical G-module. The construction is very much inspired by the notes [40] and we follow their notation.

Take $\hbar = -i\log q/\pi \in i\mathbb{R}$ and let $P \subseteq \mathfrak{h}^*$ denote the set of integral weights, here $\mathfrak{h}$ denotes a Cartan subalgebra of $\mathfrak{g}$. Given a finite dimensional $\mathscr{U}_q(\mathfrak{g})$-module V and a weight $\lambda \in P$ we let $V(\lambda)$ denote the subspace of vectors in V

of weight λ. If V splits as a direct sum of weight modules $V = \oplus_{\lambda\in P} V(\lambda)$ we call V an admissible $\mathscr{U}_q(\mathfrak{g})$-module. The category $\mathfrak{C}(\mathfrak{g},\hbar)$ of admissible $\mathscr{U}_q(\mathfrak{g})$-modules forms a semisimple category with simple generators indexed by dominant integral weights $\lambda \in P_+$. Just as in [40] we fix a simple generator V_λ^q to each $\lambda \in P_+$.

We let let P_G denote the set of weights whose representations π_λ integrates to a finite-dimensional, unitary representation of G and define the $*$-algebra

$$\widehat{\mathbb{C}[G_q]} := \bigoplus_{\lambda\in P_+\cap P_G} End(V_\lambda^q).$$

Here $End(V_\lambda^q)$ denotes the $*$-algebra of endomorphisms of the finite dimensional Hilbert space V_λ^q. This differs somewhat from the definition in [40] which produces the simply connected covering of $\widehat{\mathbb{C}[G_q]}$. The simply connected covering is defined in [40] as

$$\widehat{\mathbb{C}[G_q^u]} := \bigoplus_{\lambda\in P_+} End(V_\lambda^q).$$

We have a canonical projection $p_G : \widehat{\mathbb{C}[G_q^u]} \to \widehat{\mathbb{C}[G_q]}$. The multiplier Hopf algebra structure on $\mathscr{U}_q(\mathfrak{g})$ induces a multiplier Hopf algebra structure on $\widehat{\mathbb{C}[G_q^u]}$. Since the finite-dimensional $\widehat{\mathbb{C}[G_q^u]}$-modules correspond to the admissible $\mathscr{U}_q(\mathfrak{g})$-modules, the algebra ker p_G is a Hopf $*$-ideal and $\widehat{\mathbb{C}[G_q]}$ also forms a multiplier Hopf $*$-algebra. Let Δ_q denote the comultiplication of $\widehat{\mathbb{C}[G_q]}$.

The definition of $\widehat{\mathbb{C}[G_q]}$ is motivated by the classical limit in which $\widehat{\mathbb{C}[G]}$ is the convolution algebra of finite dimensional representations of G. This fact is a direct consequence of the Peter-Weyl theorem. In the classical limit $\widehat{\mathbb{C}[G_1^u]}$ corresponds to the convolution algebra of finite dimensional representations of the simply connected covering group G^u. The $*$-algebra $\widehat{\mathbb{C}[G]}$ is a multiplier Hopf algebra; let Δ denote it's comultiplication. The next theorem is a summary of the results in [40] relating the two bi-algebras $\widehat{\mathbb{C}[G_q^u]}$ and $\widehat{\mathbb{C}[G^u]}$.

Theorem 2.6.1 ([40]). *For $q > 0$ there is a $*$-isomorphism $\tilde{\varphi} : \widehat{\mathbb{C}[G_q^u]} \to \widehat{\mathbb{C}[G^u]}$ extending the identification of the centers of $\widehat{\mathbb{C}[G_q^u]}$ and $\widehat{\mathbb{C}[G^u]}$ and a unitary $\tilde{\mathscr{F}} \in \prod_{\lambda,\mu\in P_+} End(V_\lambda \otimes V_\mu)$ satisfying the cocycle condition such that*

$$\tilde{\varphi} \otimes \tilde{\varphi} \circ \tilde{\Delta}_q = Ad\tilde{\mathscr{F}} \circ \tilde{\Delta} \circ \tilde{\varphi},$$

where $\tilde{\Delta}_q$ denotes the comultiplication of $\widehat{\mathbb{C}[G_q^u]}$ and $\tilde{\Delta}$ the comultiplication of $\widehat{\mathbb{C}[G^u]}$. The associator of $\mathscr{F}$ coincides with the Drinfeld associator Φ_{KZ}.

Since $\widehat{\mathbb{C}[G^u]}$ is discrete it has a counit $\varepsilon : \widehat{\mathbb{C}[G^u]} \to \mathbb{C}$. It was proved in [40] that $\tilde{\mathscr{F}}$ satisfies $\mathrm{id} \otimes \varepsilon(\tilde{\mathscr{F}}) = \varepsilon \otimes \mathrm{id}(\tilde{\mathscr{F}}) = 1$. Therefore

$$\mathrm{id} \otimes \varepsilon(\Delta_{\tilde{\mathscr{F}}}(a)) = \varepsilon \otimes \mathrm{id}(\Delta_{\tilde{\mathscr{F}}}(a)) = a \quad \forall a \in \widehat{\mathbb{C}[G^u]} \tag{2.13}$$

and $\tilde{\mathscr{F}}$ is a cocycle twist.

Lemma 2.6.2. *Let G be a semisimple, compact, connected Lie group and $q > 0$. Then there exist a unitary $\mathscr{F} \in \prod_{\lambda,\mu \in P_+ \cap P_G} End(V_\lambda \otimes V_\mu)$ satisfying the cocycle condition and a $*$-isomorphism $\varphi : \widehat{\mathbb{C}[G_q]} \to \widehat{\mathbb{C}[G]}$ such that*

$$\varphi \otimes \varphi \circ \Delta_q = Ad\mathscr{F} \circ \Delta \circ \varphi.$$

Proof. The $*$-isomorphism $\tilde{\varphi}$ is an extension of the identification of the centers of $\widehat{\mathbb{C}[G_q^u]}$ and $\widehat{\mathbb{C}[G^u]}$. Therefore there exists a commutative diagram

$$\begin{array}{ccc} \widehat{\mathbb{C}[G_q^u]} & \xrightarrow{\tilde{\varphi}} & \widehat{\mathbb{C}[G^u]} \\ \downarrow & & \downarrow \\ \widehat{\mathbb{C}[G_q]} & \xrightarrow{\varphi} & \widehat{\mathbb{C}[G]} \end{array}$$

Since $\tilde{\mathscr{F}}$ maps tensor products of weight modules to themselves, clearly it restricts to a $\mathscr{F} \in \prod_{\lambda,\mu \in P_+ \cap P_G} End(V_\lambda \otimes V_\mu)$. Thus all properties of $\mathscr{F}$ and φ follow from those of $\tilde{\mathscr{F}}$ and $\tilde{\varphi}$. □

Theorem 2.6.3. *The unitary $\mathscr{F}$ extends to a manageable cocycle twist of $C^*(G)$. If G is simply connected the dual of $C^*(G)_{\mathscr{F}}$ coincides with the Woronowitz deformation of G.*

Proof. The element $\mathscr{F} \in \prod_{\lambda,\mu \in P_+ \cap P_G} End(V_\lambda \otimes V_\mu)$ is unitary, so it extends to a bounded multiplier $\mathscr{F} \in \mathscr{M}(C^*(G) \otimes C^*(G))$. From equation (2.13) it follows that $\mathscr{F}$ is a cocycle twist of $C^*(G)$. In [46] it is proved that $\widehat{\mathbb{C}[G_q^u]}$ are dual to the Woronowicz deformations which are compact quantum groups so they admit Haar states by [31], therefore there exists Haar weights on $\widehat{\mathbb{C}[G_q^u]}$. The Haar weights on $\mathbb{C}[G_q^u]$ induces Haar weights on $\widehat{\mathbb{C}[G_q]}$, therefore $\mathscr{F}$ is manageable. □

From this theorem it follows that the associator of $\mathscr{F}$ coincides with the bounded extension of the Drinfeld associator Φ_{KZ} to $C^*(G) \otimes C^*(G) \otimes C^*(G)$. Since the Drinfeld associator is non-trivial, $\mathscr{F}$ is not a twist, merely a cocycle twist.

2.7 Torsion-free, discrete quantum groups

In the paper [34], Meyer introduced the notion of torsion-free, discrete quantum groups. When Γ is a discrete group, $C_0(\Gamma)$ is torsion-free in the sense of Meyer if and only if Γ is torsion-free. For compact groups G the dual $\hat{G}$, or actually $C^*(G)$, is torsion-free precisely when G satisfies the Hodgkin condition, that is G is connected with torsion-free fundamental group. As is shown in [45], for compact Lie groups there exists Künneth formulas and UCT:s in KK^G if G satisfies the Hodgkin condition. So the notion of torsion-free, discrete quantum groups plays an important role for homological algebra of C^*-algebras.

Definition 2.7.1 ([34])**.** *A discrete quantum group S is torsion-free if every coaction of $\hat{S}$ on a finite-dimensional C^*-algebra A is equivariantly Morita equivalent to direct sums of $\mathbb{C}$ with the trivial coaction.*

Suppose that G is a compact, connected Lie group satisfying the Hodgkin condition. Let $T \subseteq G$ be a maximal torus of rank n and w the Weyl group. Then, by [48], the representation ring $R(T)$ is a free $R(G)$-module of rank $|w|$. As is shown in [36], and reviewed below in Lemma 4.3.1, this implies that there exists a natural isomorphism $KK^G(A, \mathbb{C}^{|w|}) \cong KK^T(A, \mathbb{C})$. So the fact that the quantum group $\hat{G}$ is torsion-free follows from the classical result that $\hat{T} = \mathbb{Z}^n$ is torsion-free.

Lemma 2.7.2. *Suppose that $\mathscr{F}$ is a twist of the regular, reduced quantum group $\hat{S}$ and A has a continuous, reduced coaction of S. Then there exist a C^*-algebra B with a continuous, reduced coaction of $S_{\mathscr{F}}$ such that $A \cong B$ as C^*-algebras and an equivariant Morita equivalence $(A \rtimes_r S)_{\mathscr{F}} \rtimes_r \hat{S}_{\mathscr{F}} \sim_M B$.*

Proof. Letting $B := A$ as a C^*-algebra we know that $(A \rtimes_r S)_{\mathscr{F}} \rtimes_r \hat{S}_{\mathscr{F}} \cong B \otimes \mathscr{K}(H_S)$ by TTT-duality. If B may be given a coaction of $S_{\mathscr{F}}$ such that the coaction of the left hand side in the TTT-isomorphism coincides with that of $(B \rtimes_r S_{\mathscr{F}}) \rtimes_r \hat{S}_{\mathscr{F}}$, the statement of the lemma follows. Let $B_s := B \otimes \mathscr{K}(H_S)$ with the coaction of $S_{\mathscr{F}}$ induced from the left hand side of the TTT-isomorphism from Theorem 2.4.8. Letting $\hat{V}_{\mathscr{F}}$ denote the right regular corepresentation of $\hat{S}_{\mathscr{F}}$, the coaction on B_s is induced by $\hat{V}_{\mathscr{F}}$.

On the other hand, suppose that we have a continuous, reduced coaction Δ_B of $S_{\hat{\mathscr{F}}}$ on B. Then if we define $B_s' := B \otimes \mathscr{K}(H_S)$ with coaction induced from the Takesaki-Takai isomorphism $B_s' \cong (B \rtimes_r S_{\hat{\mathscr{F}}}) \rtimes_r \hat{S}_{\mathscr{F}}$ the coaction on $a \otimes k \in B_s'$ would be given by

$$\Delta_{B_s'}(a \otimes k) = (\hat{V}_{\mathscr{F}})_{23}(\Delta_B(a)_{13} \cdot 1_B \otimes k \otimes 1)(\hat{V}^*_{\mathscr{F}})_{23}$$

Using this as motivation, we define the $*$-homomorphism $\delta_B : B \to \mathscr{M}(B_s' \otimes S_{\hat{\mathscr{F}}})$ by

$$\delta_B(a) = (\hat{V}^*_{\mathscr{F}})_{23}(\Delta_{B_s}(a \otimes 1))(\hat{V}_{\mathscr{F}})_{23}.$$

Let us prove that δ_B restricts to a coaction $\Delta_B : B \to \mathscr{M}_{S_{\hat{\mathscr{F}}}}(B \otimes S_{\hat{\mathscr{F}}})$. For $a \in B$, then for all $k \in \mathscr{K}(H_S)$

$$\delta_B(a) \cdot 1 \otimes k \otimes 1 = (\hat{V}^*_{\mathscr{F}})_{23}(\Delta_{B_s}(a \otimes 1))(\hat{V}_{\mathscr{F}})_{23} \cdot 1 \otimes k \otimes 1 =$$

$$= (\hat{V}^*_{\mathscr{F}})_{23}(\Delta_{B_s}(a \otimes k))(\hat{V}_{\mathscr{F}})_{23} = 1 \otimes k \otimes 1 \cdot (\hat{V}^*_{\mathscr{F}})_{23}(\Delta_{B_s}(a \otimes 1))(\hat{V}_{\mathscr{F}})_{23} =$$
$$= 1 \otimes k \otimes 1 \cdot \delta_B(a).$$

So $\delta_B(a) \in (1 \otimes \mathscr{K}(H_S) \otimes 1)'$, therefore it is of the form $\delta_B(a) = \Delta_B(a)_{13}$ for a unique element $\Delta_B(a) \in \mathscr{M}(B \otimes S_{\hat{\mathscr{F}}})$. Since Δ_{B_s} is a reduced coaction this implies that $\Delta_B(a) \in \mathscr{M}_{S_{\hat{\mathscr{F}}}}(B \otimes S_{\hat{\mathscr{F}}})$ and that Δ_B defines a reduced coaction on B. The coaction Δ_B is continuous since

$$[\Delta_B(B)_{13} \cdot 1 \otimes \mathscr{K} \otimes 1 \cdot 1 \otimes 1 \otimes S_{\hat{\mathscr{F}}}] = [\Delta_{B_s}(B_s) \cdot 1_{B_s} \otimes S_{\hat{\mathscr{F}}}] = B \otimes \mathscr{K} \otimes S_{\hat{\mathscr{F}}}.$$

Let B be given the coaction Δ_B, it follows directly from the definition that $\Delta_{B_s} = \Delta_{B_s'}$. Thus we have in an equivariant fashion

$$(A \rtimes_r S)_{\mathscr{F}} \rtimes_r \hat{S}_{\mathscr{F}} \cong (B \rtimes_r S_{\hat{\mathscr{F}}}) \rtimes_r \hat{S}_{\mathscr{F}} \sim_M B.$$

□

Theorem 2.7.3. *If S is discrete, torsion-free and $\mathscr{F}$ a twist of S, then $S_{\mathscr{F}}$ is also discrete, torsion-free.*

Proof. The property of being discrete is invariant under twist. Suppose that $A \in C^*_{\widehat{S_{\mathscr{F}}}}$ is finite-dimensional. Consider the object $\tilde{A} := (A \rtimes_r \widehat{S_{\mathscr{F}}})_{\mathscr{F}^*} \in C^*_S$. Since A is of finite dimension, Lemma 2.7.2 implies that there is an equivariant Morita equivalence from $\tilde{A} \rtimes_r S$ to a finite-dimensional C^*-algebra. Since S is torsion-free, $\tilde{A} \rtimes_r S$ is Morita equivalent to $\mathbb{C}^k$ for some k. So we can conclude that

$$A \sim_M (\mathbb{C}^k \rtimes_r \hat{S})_{\mathscr{F}} \rtimes_r S_{\mathscr{F}} \cong \mathbb{C}^k \otimes \mathscr{K}(H_S) \sim_M \mathbb{C}^k.$$

□

In [34] the question was posed whether duals of the Woronowicz deformations are torsion-free? Theorem 2.7.3 unfortunately does not answer this question since the Drinfeld-Jimbo twists are not twists, just cocycle twists. Though it seems as if the theory developed previously in the thesis generalize to quasi-coactions and cocycle twists.

Chapter 3

Equivariant KK-theory

Equivariant KK-theory was first introduced in [22] by Kasparov in the study of the Baum-Connes conjecture. The viewpoint on KK as a category was introduced by Higson in [19]. The KK-theory was generalized to KK-theory equivariant with respect to a Hopf-C^*-algebra by Baaj-Skandalis in [3]. We will review their construction and show that KK_S forms a triangulated category. We will through out this chapter assume S to be a separable, regular, reduced locally compact quantum group.

This chapter consists only of known results. However, the proofs of some results have only been published in the non-equivariant setting. So the main part of every proof consists of reducing the theory to the non-equivariant setting and use old results. The final section on the triangulated structure on KK_S is again old results, but the approach using the Ext-invariant in the setting of quantum groups is new. The idea is due to Nest who suggested this approach since the classical approach using generalized homomorphisms does not work satisfactory for KK-categories over topological spaces which is needed in the study of the Baum-Connes property (see [35]).

3.1 Kasparov modules

Definition 3.1.1 (Kasparov module, Definition 2.1.1 of [26]). *Suppose that $\mathscr{E}$ is a graded $A-B$-Hilbert bimodule and that $F \in \mathscr{L}_B(\mathscr{E})$ is an odd operator such that for all $a \in A$*

$$\pi(a)(F^* - F),\ \pi(a)(F^2 - 1),\ [F, \pi(a)] \in \mathscr{K}_B(\mathscr{E}), \tag{3.1}$$

where $\pi : A \to \mathscr{L}_B(\mathscr{E})$ defines the left action of A on $\mathscr{E}$. Then the pair $(\mathscr{E}, F)$ is called an $A-B$-Kasparov module and F a Kasparov operator.

If $\pi(a)(F^* - F) = \pi(a)(F^2 - 1) = [F, \pi(a)] = 0$ for all $a \in A$ we say that the $A - B$-Kasparov module $(\mathscr{E}, F)$ is trivial. A simple example of a Kasparov module to keep in mind is on a closed manifold M and $A = C(M)$, $B = \mathbb{C}$, $\mathscr{E}$ being L^2-sections on a vector bundle $\mathbb{E}^+ \oplus \mathbb{E}^-$, P a self-adjoint elliptic pseudo differential operator from $\mathbb{E}^+$ to $\mathbb{E}^-$ of order 0 and $F = \begin{pmatrix} 0 & Q \\ P & 0 \end{pmatrix}$ where Q is a parametrix for P.

Definition 3.1.2 (S-equivariant Kasparov module, Definition 3.1 of [3]). *If A and B are graded $S - C^*$-algebras, $\mathscr{E}$ a graded S-equivariant $A-B$-Hilbert module and $(\mathscr{E}, F)$ is an $A - B$-Kasparov module satisfying*

$$(\pi \otimes \mathrm{id})(y)\left(F \otimes_{\mathbb{C}} \mathrm{id}_S - V_{\mathscr{E}}(F \otimes_{\Delta_B} 1)V_{\mathscr{E}}^*\right) \in \mathscr{K}(E \otimes S), \tag{3.2}$$

for all $y \in A \otimes S$ the $A - B$-Kasparov module $(\mathscr{E}, F)$ is called S-equivariant.

Recall that the operator $V_{\mathscr{E}} \in \mathscr{L}(\mathscr{E} \otimes_{\Delta_A} (A \otimes S), \mathscr{E} \otimes S)$ is the admissible unitary defining the coaction of S on $\mathscr{E}$, see more in Proposition 2.3.4. If an operator $F \in \mathscr{L}_B(\mathscr{E})$ satisfies equation (3.2) we say that F is almost S-equivariant. If $T \in \mathscr{L}_B(\mathscr{E}, \mathscr{E}')$ satisfies that

$$(T \otimes_{\mathbb{C}} \mathrm{id}_S)V_{\mathscr{E}} = V_{\mathscr{E}'}(T \otimes_{\Delta_B} 1)$$

the operator T is an S-equivariant mapping of Hilbert modules. If $\mathscr{E} = \mathscr{E}'$ we will say that T is S-invariant. Suppose that $(\mathscr{E}_1, F_1)$ and $(\mathscr{E}_2, F_2)$ are two equivariant $A - B$-Kasparov bimodule and that $U \in \mathscr{L}_B(\mathscr{E}_2, \mathscr{E}_1)$ is an even S-equivariant unitary. If U intertwines the representations of A and $F_1 = UF_2U^*$ we say that U is an isomorphism of equivariant $A - B$-Kasparov modules.

Let $E_S(A, B)$ denote the set of isomorphism classes of countably generated, S-equivariant $A - B$-Kasparov modules. If $(\mathscr{E}_1, F_1), (\mathscr{E}_2, F_2) \in E_S(A, B)$ we define their direct sum by

$$(\mathscr{E}_1, F_1) \oplus (\mathscr{E}_2, F_2) := (\mathscr{E}_1 \oplus \mathscr{E}_2, F_1 \oplus F_2).$$

The direct sum of two equivariant $A - B$-Kasparov modules is again an equivariant $A - B$-Kasparov module. The set $E_S(A, B)$ forms an abelian monoid under direct sum of Kasparov modules, since $(\mathscr{E}_1 \oplus \mathscr{E}_2, F_1 \oplus F_2)$ is isomorphic to $(\mathscr{E}_2 \oplus \mathscr{E}_1, F_2 \oplus F_1)$. A trivial $A - B$-Kasparov module $(\mathscr{E}, F)$ such that F is S-invariant is called a trivial S-equivariant $A - B$-Kasparov module. Let $D_S(A, B)$ denote the set of countably generated trivial S-equivariant $A-B$-Kasparov modules. The set $D_S(A, B)$ forms a submonoid of $E_S(A, B)$.

An equivariant $*$-homomorphism $f : B \to C$ induces an additive mapping $f_* : E_S(A,B) \to E_S(A,C)$ by viewing C as a $B-C$-Hilbert bimodule via f and defining

$$f_*(\mathscr{E},F) = (\mathscr{E} \otimes_B C, F \otimes_B 1).$$

Similarly, if $f : A \to C$ is an equivariant $*$-homomorphism, f induces an additive mapping $f^* : E_S(C,B) \to E_S(A,B)$. Using associativity of tensor products it follows that E_S is functorial in it's arguments.

Consider the C^*-algebra $B[0,1] := B \otimes C[0,1]$ with trivial coaction on $C[0,1]$. For every $t \in [0,1]$, point evaluation at t induces an equivariant $*$-homomorphism $\pi_t : B[0,1] \to B$. If there exists an equivariant $A-B[0,1]$-Kasparov module $(\mathscr{E},F)$ such that $(\pi_0)_*(\mathscr{E},F) = (\mathscr{E}_0,F_0)$ and $(\pi_1)_*(\mathscr{E},F) = (\mathscr{E}_1,F_1)$ we say that $(\mathscr{E}_0,F_0)$ and $(\mathscr{E}_1,F_1)$ are directly homotopic. The notion of direct homotopy is clearly a symmetric and reflexive relation between Kasparov modules. The equivalence relation generated by direct homotopy is called homotopy equivalence. We will let $[(\mathscr{E},F)]$ denote the homotopy class of $(\mathscr{E},F)$ and

$$KK_S(A,B) := E_S(A,B)/\sim_h .$$

Lemma 3.1.3 (Lemma 2.1.20 of [26])**.** *If* $(\mathscr{E},F) \in D_S(A,B)$ *then* $[(\mathscr{E},F)] = [(0,0)]$.

Just to get a feeling for the theory we present the short proof from [26].

Proof. Define $\tilde{\mathscr{E}} := \mathscr{E} \otimes_{\mathbb{C}} C_0(0,1]$ with trivial coaction and grading on $C_0(0,1]$. Clearly $(\tilde{\mathscr{E}}, F \otimes 1)$ forms an equivariant $A-B[0,1]$-Kasparov module. Since

$$(\pi_1)_*(\tilde{\mathscr{E}}, F \otimes 1) = (\mathscr{E},F) \quad \text{and} \quad (\pi_0)_*(\tilde{\mathscr{E}}, F \otimes 1) = (0,0)$$

the proposition follows. □

Proposition 3.1.4 (Theorem 2.1.23 of [26] and Proposition 3.3 of [3])**.** *The set* $KK_S(A,B)$ *forms an abelian group induced by the monoid structure of* $E_S(A,B)$.

We refer the reader to the proof in [26]. Besides the straight forward check that direct sum is independent of representative of the homotopy class, the proof consists of proving that $[(-\mathscr{E},-F)]$ is an inverse to $[(\mathscr{E},F)]$. Recall from Chapter 2.3 that $-\mathscr{E}$ denotes the Hilbert module $\mathscr{E}$ with opposite grading. Functoriality of KK_S is a consequence of functoriality of E_S, we state this fact as a proposition and refer the reader to the proof in [26].

Proposition 3.1.5 (Lemma 2.1.26 of [26] and Proposition 3.5 of [3])**.** *Letting Ab denote the category of abelian groups, the mapping* $(A,B) \mapsto KK_S(A,B)$ *defines a functor* $C^*_S \times C^*_S \to Ab$ *contravariant in it's first argument and covariant in it's second argument.*

If $B = \mathbb{C}$, we obtain a contravariant functor $A \mapsto KK_S(A, \mathbb{C})$ which coincides with the equivariant K-homology $K_S^*(A)$. If we fix the first variable to $A = \mathbb{C}$, we obtain a covariant functor $B \mapsto KK_S(\mathbb{C}, B)$. This does in fact coincides with the equivariant K-theory $K_*^S(B)$. For a proof of this statement in the non-equivariant setting, see [7]. The equivariant version is proven in the same way.

The motivating example for the definition of K-homology by Atiyah was when $A = C(M)$, for a smooth, closed riemannian manifold M. If $\mathbb{E} \to M$ is a graded hermitian vector bundle and D an odd self-adjoint first order elliptic pseudo-differential operator on $\mathbb{E}$, then letting $F := D/(1+D^2)^{-1/2}$ the pair $(L^2(\mathbb{E}), F)$ forms a $C(M) - \mathbb{C}$-Kasparov module.

Suppose that $A \sim_M B$ via the equivariant imprimitivity $A-B$-bimodule ${}_A\mathscr{E}_B$, these notions were defined above in Chapter 2.3. Given a graded $S-C^*$-algebra C we consider the mapping $\otimes_A \mathscr{E}_B : E_S(C, A) \to E_S(C, B)$ given by

$$(\mathscr{E}_0, F) \otimes_A \mathscr{E}_B := (\mathscr{E}_0 \otimes_A \mathscr{E}_B, F \otimes_A 1).$$

Clearly $\otimes_A \mathscr{E}_B$ is an additive map, independent of homotopy class so it induces a mapping $\otimes_A \mathscr{E}_B : KK_S(C, A) \to KK_S(C, B)$. This mapping is in fact an isomorphism with inverse induced by the dual imprimitivity bimodule

$$\otimes_B \mathscr{E}_A^* : E_S(C, B) \to E_S(C, A).$$

To show this, observe that as graded $A-A$-Hilbert bimodules respectively as $B-B$-Hilbert bimodules

$${}_A\mathscr{E}_B \otimes_B \mathscr{E}_A^* \cong A \quad \text{and} \quad {}_B\mathscr{E}_A^* \otimes_A \mathscr{E}_B \cong B.$$

So it follows that the compositions

$$\begin{aligned} &(\otimes_A \mathscr{E}_B) \circ (\otimes_B \mathscr{E}_A^*) : KK_S(C, B) \to KK_S(C, B) \quad \text{and} \\ &(\otimes_B \mathscr{E}_A^*) \circ (\otimes_A \mathscr{E}_B) : KK_S(C, A) \to KK_S(C, A) \end{aligned}$$

are the identity maps. Thus, we may conclude the following proposition:

Proposition 3.1.6. *If we assume that $A \sim_M B$, there is a natural isomorphism $KK_S(C, A) \cong KK_S(C, B)$ for all $C \in C^*_{\mathbb{Z}_2, S}$.*

From Proposition 2.3.6 it follows that $KK_S(C, A) \cong KK_S(C, A \otimes \mathscr{K})$ for all $A, C \in C^*_{\mathbb{Z}_2, S}$. The same kind of stability holds for the first variable of the KK_S-functor.

Proposition 3.1.7 (Proposition 17.8.7 of [7]). *Consider the C^*-algebra $A \otimes \mathscr{K}$ with the coaction defined in Proposition 2.3.6. There is a natural isomorphism*

$$KK_S(A,B) \cong KK_S(A \otimes \mathscr{K}, B)$$

for all $B \in C^*_{\mathbb{Z}_2,S}$.

The proof in [7] is in the non-equivariant setting. We will present a proof that reduces the isomorphism $KK_S(A,B) \cong KK_S(A \otimes \mathscr{K}, B)$ to the isomorphism $KK_S(A,B) \cong KK_S(A \otimes \mathscr{K}_0, B)$ where $\mathscr{K}_0$ has trivial coaction and then use the methods from [7].

Proof. By the remark after Proposition 3.1.6 it is sufficient to show existence of a natural isomorphism

$$KK_S(A,B) \cong KK_S(A \otimes \mathscr{K}, B \otimes \mathscr{K}).$$

Define the mapping $\tau : KK_S(A,B) \to KK_S(A \otimes \mathscr{K}, B \otimes \mathscr{K})$ by $\tau[(\mathscr{E}, F)] := [(\mathscr{E} \otimes \mathscr{K}, F \otimes 1)]$, where $\mathscr{E} \otimes \mathscr{K}$ denotes the external tensor product between $\mathscr{E}$ and the equivariant $\mathscr{K} - \mathscr{K}$-Hilbert bimodule $\mathscr{K}(H_S)$.

Let $\mathscr{K}_0$ denote the compact operators on $\ell^2(\mathbb{N})$ with trivial coaction of S. Since the coaction of S on $\mathscr{K}$ is given by $\Delta_{\mathscr{K}}(k) = Ad(V)(k \otimes 1)$ by viewing $V \in \mathscr{M}(\mathscr{K} \otimes S)$, there is an isomorphism

$$t : KK_S(A \otimes \mathscr{K}, B \otimes \mathscr{K}) \xrightarrow{\sim} KK_S(A \otimes \mathscr{K}_0, B \otimes \mathscr{K}_0)$$

given by defining $t[(\mathscr{E}, F)]$ to be the Hilbert module $\mathscr{E}$ with coaction $(1_A \otimes V) \circ \delta_{\mathscr{E}}$.

Now let $p \in \mathscr{K}_0$ denote a rank-one projection and define $e : A \to A \otimes \mathscr{K}_0$ as the graded, equivariant $*$-homomorphism $e(a) := a \otimes p$. Define

$$\tau_0' : KK_S(A \otimes \mathscr{K}_0, B \otimes \mathscr{K}_0) \to KK_S(A,B)$$

by $\tau_0'[(\mathscr{E}, F)] := e^*[(\mathscr{E} \otimes_{B \otimes \mathscr{K}_0} (B \otimes \ell^2(\mathbb{N})), F \otimes 1)]$. We combine the two maps τ_0' and t to define a mapping:

$$\tau' := \tau_0' \circ t : KK_S(A \otimes \mathscr{K}, B \otimes \mathscr{K}) \to KK_S(A,B).$$

The remaining part of the proof consists of showing that τ' is an inverse to τ. Take an equivariant $A - B$-Kasparov module $(\mathscr{E}, F)$ and consider $\tau'\tau[(\mathscr{E}, F)]$. This element can be represented by

$$e^*[(\mathscr{E} \otimes \ell^2(\mathbb{N}), F \otimes 1)] = [(\mathscr{E}, F)] + e^*[(\mathscr{E} \otimes (1-p)\ell^2(\mathbb{N}), F \otimes 1)] = [(\mathscr{E}, F)].$$

It follows that τ' is a left inverse to τ. The proof that τ' is a right inverse is analogous and may be found in [7] or [26]. □

Recall the reduced crossed product functor $\rtimes_r S : C^*_{\mathbb{Z}_2,S} \to C^{*,r}_{\mathbb{Z}_2,\hat{S}}$ from Chapter 2.3. This functor will be of great importance when studying category theoretical aspects of equivariant KK-theory. Let $\mathscr{E}$ be a graded, equivariant $A-B$-Hilbert bimodule. Define the graded $A \rtimes_r S - B \rtimes_r S$-Hilbert bimodule

$$\mathscr{E} \rtimes_r S := \mathscr{E} \otimes_{\Delta_B} B \rtimes_r S.$$

This is in fact a graded, $\hat{S}$-equivariant $A \rtimes_r S - B \rtimes_r S$-Hilbert bimodule under the coaction

$$\delta_{\mathscr{E}\rtimes_r S}(x \otimes b) := (t_x \otimes 1)\Delta_{B\rtimes_r S}(b).$$

If $(\mathscr{E}, F)$ is an equivariant $A-B$-Kasparov module, we define

$$(\mathscr{E}, F) \rtimes_r S := (\mathscr{E} \rtimes_r S, F \otimes 1).$$

The pair $(\mathscr{E} \rtimes_r S, F \otimes 1)$ clearly forms an $\hat{S}$-equivariant $A \rtimes_r S - B \rtimes_r S$-Kasparov module. Furthermore, the Kasparov operator $F \otimes 1$ is $\hat{S}$-equivariant.

Theorem 3.1.8 (Baaj-Skandalis duality). *The functor $\rtimes_r S : C^{*,r}_S \to C^{*,r}_{\hat{S}}$ induces a natural isomorphism*

$$J_S : KK_S(A,B) \xrightarrow{\sim} KK_{\hat{S}}(A \rtimes_r S, B \rtimes_r S).$$

Proof. Given an element $\alpha = [(\mathscr{E}, F)] \in KK_S(A,B)$ we define $J_S(\alpha)$ to be given by the homotopy class of the Kasparov module $(\mathscr{E} \rtimes_r S, F \otimes 1)$. Since $J_S(\alpha)$ is a homotopy class, it will be independent of the homotopy class of $(\mathscr{E}, F)$.

To show that J_S is an isomorphism, we observe that by Takesaki-Takai duality (see Theorem 2.3.9), $J_{\hat{S}} J_S = \tau$, where τ is the mapping from the proof of Proposition 3.1.7. Since τ is an isomorphism, J_S is injective. By Pontryagin duality, J_S is surjective. □

This duality was proved in [3] as Theorem 6.20 in the classical setting $S = C_0(G)$. In Remark 7.7.*b* of [3] it was stated that J_S is an isomorphism for arbitrary separable, regular quantum group S. However the line of proof is standard and analogous to that of the case $S = C_0(G)$ in [3].

Let $\mathbb{C}_n := C\ell(\mathbb{C}^n)$ denote the graded Clifford algebra of $\mathbb{C}^n$. This finite dimensional C^*-algebra has a grading induced from the mapping $v \mapsto -v$ on $\mathbb{C}^n$. Given a graded C^*-algebra B we define the graded $S - C^*$-algebra $B_{(n)} := B \otimes \mathbb{C}_n$ with the tensor grading and trivial coaction on $\mathbb{C}_n$. The higher KK_S-groups are defined as

$$KK^n_S(A,B) := KK_S(A, B_{(n)}).$$

Another approach to the higher KK_S-groups is to use the suspension functor $\Sigma A := C_0(\mathbb{R}) \otimes A$ and define $KK_S^n(A,B) := KK_S(A, \Sigma^n B)$. As is shown below in Proposition 3.4.6 these two definitions are, up to a natural isomorphism, the same. But the proof of Bott periodicity becomes a bit more tricky using the suspension functor.

Proposition 3.1.9 (Bott periodicity). *For $n \in \mathbb{N}$ there is a natural isomorphism*

$$KK_S^n(A,B) \cong KK_S^{n+2}(A,B).$$

Proof. By Proposition 3.1.6 it is sufficient to prove that $\mathbb{C}_n \sim_M \mathbb{C}_{n+2}$, since this induces an equivariant Morita equivalence $B \otimes \mathbb{C}_n \sim_M B \otimes \mathbb{C}_{n+2}$. We will prove the claim by induction on n. For $n = 0$ we have that $\mathbb{C}_0 = \mathbb{C}$. Since $\mathbb{C}_2 \cong \mathscr{K}_{\mathbb{C}}(\mathbb{C}^2)$ for a suitable non-trivial grading on $\mathbb{C}^2$, the Morita equivalence $\mathbb{C} \sim_M \mathbb{C}_2$ follows. Suppose that the claim holds for all $k < n$, and let V denote an imprimitivity bimodule from $\mathbb{C}_{n-1}$ to $\mathbb{C}_{n+1}$. There is a graded isomorphism $\mathbb{C}_{m+1} \cong \mathbb{C}_1 \otimes \mathbb{C}_m$ for all m. Thus $\mathbb{C}_1 \otimes V$ may be viewed as a graded $\mathbb{C}_n - \mathbb{C}_{n+2}$-Hilbert bimodule. Since V is a $\mathbb{C}_{n-1} - \mathbb{C}_{n+1}$-imprimitivity bimodule, the bimodule $\mathbb{C}_1 \otimes V$ forms an imprimitivity bimodule from $\mathbb{C}_n$ to $\mathbb{C}_{n+2}$. □

A consequence of Bott periodicity is that we may view the group

$$KK_S^*(A,B) := \bigoplus_{n \geq 0} KK_S^n(A,B)$$

as a $\mathbb{Z}_2$-graded abelian group.

If A and B are trivially graded, the description of $KK_S^1(A,B)$ may be simplified somewhat. Then we may in fact skip the gradings totally. Suppose that $\mathscr{E}$ is a trivially graded equivariant $A - B$-Hilbert module and that $F \in \mathscr{L}_B(\mathscr{E})$ is an almost equivariant operator satisfying the equations in (3.1). Then we say that $(\mathscr{E}, F)$ is an equivariant KK^1-cycle for A, B. Notice that in this setting the graded commutators in equation (3.1) coincide with the usual commutator since everything is trivially graded. Let $\widehat{KK}_S^1(A,B)$ denote the group of homotopy classes of equivariant KK^1-cycles for A, B.

Define the B-Hilbert module $H_B := \ell^2(\mathbb{N}) \otimes B$ and the $B_{(1)}$-Hilbert module $\hat{H}_B := H_B \otimes \mathbb{C}_1$. They form graded, equivariant Hilbert modules under the action and grading induced from B respectively $B_{(1)}$. So H_B is a trivially graded B-Hilbert module and $\hat{H}_B$ decomposes as a right B-module as $\hat{H}_B = H_B \oplus H_B$. We will view H_B and $\hat{H}_B$ as graded, equivariant $A - B$-Hilbert bimodules in the trivial action of A.

Theorem 3.1.10 (Proposition 3.3.6 of [26]). *If A and B are trivially graded and $(\tilde{\mathscr{E}}, \tilde{F})$ is an equivariant $A - B_{(1)}$-Kasparov module there exists an equivariant KK^1-cycle $(\mathscr{E}, F)$ for A, B such that*

$$(\tilde{\mathscr{E}} \oplus \hat{H}_B, \tilde{F} \oplus 0) \cong (\mathscr{E} \otimes \mathbb{C}_1, F \oplus (-F)).$$

The mapping $[(\tilde{\mathscr{E}}, \tilde{F})] \mapsto [(\mathscr{E}, F)]$ is a well defined natural isomorphism

$$KK^1_S(A,B) \xrightarrow{\sim} \widehat{KK}^1_S(A,B).$$

Proof. By the Kasparov stabilization theorem (see Theorem 1.1.24 of [26]) there is a graded isomorphism $\tilde{\mathscr{E}} \oplus \hat{H}_B \cong \hat{H}_B = H_B \otimes \mathbb{C}_1$ as $B_{(1)}$-Hilbert modules. Define $\mathscr{E} := H_B$ with the coaction of S induced from this isomorphism. Since the A-action on $\tilde{\mathscr{E}} \oplus \hat{H}_B$ is graded there exists an equivariant representation $\pi : A \to \mathscr{L}_B(\mathscr{E})$ such that $\pi \oplus \pi$ induces the A-action on $\tilde{\mathscr{E}} \oplus \hat{H}_B \cong \mathscr{E} \otimes \mathbb{C}_1$. Since $\tilde{F} \oplus 0$ is odd, it induces an operator of form $F \oplus (-F)$. That $(\mathscr{E}, F)$ is an equivariant KK^1-cycle for A, B follows from that $(\tilde{\mathscr{E}} \oplus \hat{H}_B, \tilde{F} \oplus 0)$ is an equivariant $A - B$-Kasparov module. ☐

The proof in [26] is in the non-equivariant setting. The difference from the equivariant setting is that one can assume $\mathscr{E} = H_B$ in the non-equivariant setting.

3.2 Invertible extensions and KK^1_S

In this section we will present the theory of extensions for trivially graded $S - C^*$-algebras. This has been studied for C^*-algebras with a group action in [50] and is thoroughly studied in the non-equivariant case in [26]. We will assume that all C^*-algebras are trivially graded in this section.

For $B \in C^{*,r}_S$ we will let $B_s := B \otimes \mathscr{K}$ with trivial coaction on $\mathscr{K}$. Consider a short exact sequence in $C^{*,r}_S$

$$0 \to B_s \to E \xrightarrow{p} A \to 0.$$

If E fits into such a short exact sequence we will say that E is a stable equivariant extension of A by B. We say that this extension is semi-split if there exists an equivariant, completely positive mapping $s : A \to E$ such that $ps = \mathrm{id}_A$. Two stable equivariant extensions E and E' are said to be unitarily equivalent if there exist a $*$-homomorphism $\Psi : E \to E'$ and an S-invariant unitary $u \in \mathscr{M}(B_s)$ such that the following diagram commutes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B_s & \longrightarrow & E & \xrightarrow{\varphi} & A & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle Ad(u)} & & \downarrow{\scriptstyle \Psi} & & \| & & \\ 0 & \longrightarrow & B_s & \longrightarrow & E' & \xrightarrow{\varphi'} & A & \longrightarrow & 0 \end{array} \tag{3.3}$$

If $u = 1$, then E and E' are said to be weakly isomorphic. Because of the five lemma for vector spaces, if Ψ fits into a unitary equivalence then Ψ is a $*$-isomorphism. So the notion of being unitarily equivalent is an equivalence relation.

Let $Q(B_s) := \mathscr{M}(B_s)/B_s$ denote the corona algebra of B_s and define $q_B : \mathscr{M}(B_s) \to Q(B_s)$ to be the canonical projection. Since B_s is an S-invariant ideal in E there is an equivariant mapping $i_E : E \to \mathscr{M}(B_S)$ by the universal property of the multiplier algebra. Define the equivariant mapping $\beta_E : A \to Q(B_s)$ by $a \mapsto q_B(i_E(p^{-1}(a)))$. The mapping β_E is a well defined $*$-homomorphism, since $q_B(i_E(p^{-1}(a)))$ is independent of what pre-image of a one chooses. The $*$-homomorphism β_E is called the Busby mapping of E. If β_E lifts to an equivariant $*$-homomorphism $\pi : A \to \mathscr{M}(B_s)$ we say that β_E is a trivial Busby mapping. Two Busby maps β_E and $\beta_{E'}$ are said to be unitarily equivalent if there exist a $u \in \mathscr{M}(B_S)$ such that $q_B(u)$ is an S-invariant unitary and $\beta_E = Ad(q_B(u))\beta_{E'}$.

Proposition 3.2.1. *Two stable equivariant extensions E and E' are weakly isomorphic if and only if $\beta_E = \beta_{E'}$. Furthermore, E and E' are unitarily equivalent if and only if β_E and $\beta_{E'}$ are unitarily equivalent.*

For the proof of this Proposition, see Theorem 3.1.2 and Lemma 3.2.2 of [26]. Although in [26], a stronger equivalence is used, it follows by Proposition 15.6.4 of [7] that it generalizes to our notion of unitary equivalence. The reason for this correspondence between these equivalences is that all information that is contained in the weak isomorphism class of an extension can be found in the Busby map.

Theorem 3.2.2. *Given an equivariant $\beta : A \to Q(B_s)$ there is a stable equivariant extension E_β of A by B which is unique up to weak isomorphism.*

Compare this theorem to Theorem 3.1.4 of [26] and it's generalization to group equivariant extensions which is contained in Theorem 2.1 of [50]. The definition of E_β as a C^*-algebra is rather straight forward, the problem lies in showing that it has a well behaved coaction of S.

Proof. Uniqueness follows from Proposition 3.2.1. To prove existence of E_β, define

$$E_\beta := \{a \oplus x \in A \oplus \mathscr{M}(B_s) : \beta(a) = q_B(x)\}.$$

The projection $E_\beta \to A$ and the embedding $B_s \hookrightarrow E_\beta$ induces a short exact sequence of C^*-algebras

$$0 \to B_s \to E_\beta \to A \to 0. \tag{3.4}$$

What remains of the existence part of the proof is to construct a reduced, continuous S-coaction on E_β making the sequence (3.4) equivariant. From the S-coactions on A and B_s we can construct the $*$-homomorphism $\Delta_{E_\beta} := \Delta_A \oplus \overline{\Delta}_{B_s}$: $E_\beta \to \mathscr{M}(E_\beta \otimes S)$, where $\overline{\Delta}_{B_s}$ denotes the extension of Δ_{B_s} to $\mathscr{M}(B_s)$. If Δ_{E_β} defines a reduced, continuous coaction, then this clearly makes (3.4) into a stable equivariant extension. The mapping Δ_{E_β} is coassociative since Δ_A and $\overline{\Delta}_{B_s}$ are.

The Busby mapping β is equivariant by assumption, therefore we claim that $\Delta_{E_\beta}(E_\beta) \subseteq \mathscr{M}_S(E_\beta \otimes S)$. Take an $a \oplus x \in E_\beta$ and an $s \in S$. Then consider the element $\Delta_{E_\beta}(a \oplus x)1 \otimes s = \Delta_A(a)1 \otimes s + \overline{\Delta}_{B_s}(x)1 \otimes s$. Since Δ_A is a coaction the first term is in $A \otimes S$. The second term satisfies that

$$(q_{B_s} \otimes \mathrm{id})(\overline{\Delta}_{B_s}(x)1 \otimes s) = \overline{\Delta}_{B_s}(q_{B_s}(x))1 \otimes s =$$

$$= (\beta \otimes \mathrm{id})(\Delta_A(a)1 \otimes s) \in Q(B_s) \otimes S,$$

because of equivariance of β. Therefore $\overline{\Delta}_{B_s}(x)1 \otimes s \in \mathscr{M}(B_s) \otimes S$ and $\Delta_{E_\beta}(a \oplus x)1 \otimes s \in E_\beta \otimes S$.

To show continuity, consider the commutative diagram of vector spaces with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & [\Delta_{B_s}(B)1 \otimes S] & \longrightarrow & [\Delta_{E_\beta}(E_\beta)1 \otimes S] & \longrightarrow & [\Delta_A(A)1 \otimes S] & \longrightarrow & 0 \\
 & & \Vert & & \downarrow & & \Vert & & \\
0 & \longrightarrow & B_s \otimes S & \longrightarrow & E_\beta \otimes S & \longrightarrow & A \otimes S & \longrightarrow & 0
\end{array}$$

The five lemma implies that $[\Delta_{E_\beta}(E_\beta)(1 \otimes S)] = E_\beta \otimes S$. □

Let $X_S(A,B)$ denote the set of unitary equivalence classes of equivariant $*$-homomorphisms $A \to Q(B_s)$, or equivalently the set of stable equivariant extensions of A by B. Our aim now is to construct a monoid structure on $X_S(A,B)$.

Proposition 3.2.3. *There is an equivariant isomorphism $M_2 \otimes B_s \cong B_s$ given by the adjoint action of an S-invariant unitary operator $V = V_1 \oplus V_2 : B_s \oplus B_s \to B_s$ between Hilbert modules.*

Proof. It is sufficient to construct two S-invariant isometries $V_1, V_2 \in \mathscr{M}(B_s)$ such that $V_1V_1^* + V_2V_2^* = 1$. Then $V := V_1 \oplus V_2$ is an S-invariant unitary. Let K denote a separable Hilbert space with trivial S-coaction. Choose a unitary $V' : K \oplus K \to K$. Let $V_1', V_2' \in \mathscr{B}(K)$ be defined by $V'(x_1 \oplus x_2) := V_1'x_1 + V_2'x_2$. We may take the isometries V_1 and V_2 to be the image of V_1' and V_2' under the equivariant, unital embedding

$$\mathscr{B}(K) = \mathscr{M}(\mathscr{K}) \hookrightarrow \mathscr{M}(B_s).$$

□

For $\beta_1, \beta_2 \in X_S(A,B)$, we define $\beta_1 + \beta_2 \in X_S(A,B)$ as

$$\beta_1 + \beta_2 := Ad(V) \circ (\beta_1 \oplus \beta_2).$$

Proposition 3.2.4 (Lemma 3.1 of [50]). *The set $X_S(A,B)$ forms an abelian semigroup in the operation $+$ independent of V.*

We refer the reader to Lemma 3.1 of [50]. There the proof is in the classical setting $S = C_0(G)$, but generalizes word by word to the setting of quantum groups. Let $X_S^{tr}(A,B)$ denote the sub semigroup of trivial Busby maps. Define the abelian monoid $Ext_S(A,B)$ as the semigroup quotient:

$$Ext_S(A,B) := X_S(A,B)/X_S^{tr}(A,B).$$

We will denote the equivalence class of an extension by $[E]$, and the equivalence class of a Busby mapping by $[\beta]$. As an abuse of notation we will make the identification $[E_\beta] = [\beta]$.

An important remark is that in [50], the notation $Ext_G(A,B)$ is used for the subgroup $Ext^{-1}_{C_0(G)}(A,B)$, the group of invertible elements in $Ext_{C_0(G)}(A,B)$. We use the notation $Ext_S(A,B)$ for the full extension monoid. The reason for this is that we will return to these monoids in the more general setting of $*$-algebras in Chapter 5 and there the full extension monoid is needed to describe analytical properties of extensions.

Lemma 3.2.5 (Lemma 3.2 of [50]). *An extension $[E] \in Ext_S(A,B)$ is invertible if and only if there exists an operator $F_E \in \mathscr{M}(B_s)$ making (B_s, F_E) into an equivariant KK^1-cycle for A, B_s such that β_E is unitarily equivalent to*

$$a \mapsto q_B\left(\frac{1+F_E}{2}\right) q_B(a) q_B\left(\frac{1+F_E}{2}\right). \tag{3.5}$$

The inverse is given by the Busby mapping

$$a \mapsto q_B\left(\frac{1-F_E}{2}\right) q_B(a) q_B\left(\frac{1-F_E}{2}\right). \tag{3.6}$$

Since the notation of [50] is somewhat different from ours, we present a proof here. The methods are precisely the same and the generalization from group to quantum group does not pose any problem.

Proof. Assume that E is a stable equivariant extension of A by B with Busby mapping $\beta_1 : A \to Q(B_s)$ which is invertible in $Ext_S(A,B)$. By definition, there is a mapping $\beta_2 : A \to Q(B_s)$ and an operator $U \in \mathcal{M}(B_s)$ such that

$$U^*(\beta_1 \oplus \beta_2)U : \mathcal{A} \to M_2 \otimes Q(B_s)$$

can be lifted to an equivariant $*$-homomorphism $\pi : \mathcal{A} \to M_2 \otimes \mathcal{M}(B_s)$. Define the almost S-invariant operator

$$F_E := U^* \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} U \in M_2 \otimes \mathcal{M}(B_s)$$

and the Busby mappings

$$\beta_1'(a) := q_B\left(\frac{1+F_E}{2}\right) q_B(a) q_B\left(\frac{1+F_E}{2}\right),$$

$$\beta_2'(a) := q_B\left(\frac{1-F_E}{2}\right) q_B(a) q_B\left(\frac{1-F_E}{2}\right).$$

Furthermore, for $a \in A$, we have

$$\begin{aligned}\beta_1(a) = q_B\left(U\frac{1+F_E}{2}U^*\right)(\beta_1(a)\oplus\beta_2(a))q_B\left(U\frac{1+F_E}{2}U^*\right) = \\ = q_B\left(U\frac{1+F_E}{2}\right) q(\pi(a)) q_B\left(\frac{1+F_E}{2}U^*\right) = q_B(U)\beta_1'(a)q_B(U^*),\end{aligned}$$

which implies that up to unitary equivalence β_1' is a Busby mapping for E. Similarly, β_2' is unitarily equivalent to β_1.

Conversely assume that E has a Busby mapping given as in equation (3.5). We define β_2 as in equation (3.6), if this defines an inverse to β_1 we are done. To prove this, define the operator

$$U := \begin{pmatrix} \frac{1+F_E}{2} & \frac{1-F_E}{2} \\ \frac{1-F_E}{2} & \frac{1+F_E}{2} \end{pmatrix},$$

and the operator

$$P_2 := \frac{1+F_E}{2} \oplus \frac{1-F_E}{2}.$$

They satisfy that $q_B(U)$ is an S-invariant symmetry and $q_B(P_2)$ is an S-invariant projection. They are related by the equation $q_B(UP_2U) = 1 \oplus 0$. We make the observation that the Busby maps $a \mapsto q_B(P_2\pi \oplus \pi(a)P_2)$ and $a \mapsto q_B(P_2U\pi \oplus \pi(a)UP_2)$ coincides and thus defines the same extension by Proposition 3.2.1. Since

$$\pi(a) \oplus 0 - UP_2U(\pi(a) \oplus \pi(a))UP_2U \in M_2 \otimes B_s$$

it follows that

$$\begin{aligned}[E] + [E_{\beta_2}] = [q_B \circ (P_2(\pi \oplus \pi)P_2)] = [q_B \circ (UP_2U^2(\pi \oplus \pi)U^2P_2U)] = \\ = [q_B \circ (UP_2U(\pi \oplus \pi)UP_2U)] = [q_B \circ \pi \oplus 0] = 0.\end{aligned}$$

□

Theorem 3.2.6. *The mapping*

$$[E] \mapsto [(B_s \otimes \mathscr{K}(H_S), F_E)]$$

induces a well defined, natural isomorphism of groups

$$Ext_S^{-1}(A \otimes \mathscr{K}(H_S), B \otimes \mathscr{K}(H_S)) \cong \widehat{KK}_S^1(A, B_s).$$

Thus there is a natural isomorphism

$$Ext_S^{-1}(A \otimes \mathscr{K}(H_S), B \otimes \mathscr{K}(H_S)) \cong KK_S^1(A, B).$$

We will not prove this theorem here. The proof is a straight forward generalization of the results in [50]. The generalization lies in placing the word "quantum" in front of every occurrence of the word group.

3.3 Unbounded Kasparov modules and spectral triples

The notion of a Kasparov module has certain computational shortcomings. Suppose that we have an odd zeroth order elliptic pseudo-differential operator F acting on the graded hermitian vector bundle $\mathbb{E} \to M$. Since F is of order 0 it will in general be non-local and explicit calculations with F often become quite nasty. A simpler approach would be to use first order differential operators. The problem is that they do not extend to $L^2(\mathbb{E})$. It requires a somewhat more technical definition of an unbounded Kasparov module as was done in [2]. At the end of the day we may go from an unbounded Kasparov module to a Kasparov module so it is a finer invariant.

Suppose that $\mathscr{E}$ and $\mathscr{E}'$ are B-Hilbert modules. If $X \subseteq \mathscr{E}$ is a dense submodule then a B-linear mapping $T : X \to \mathscr{E}'$ is called a densely defined mapping of Hilbert modules if there exists a dense submodule $X^* \subseteq \mathscr{E}'$ and a B-linear mapping $T^* : X^* \to \mathscr{E}$ such that

$$\langle Tx, y\rangle_{\mathscr{E}'} = \langle x, T^*y\rangle_{\mathscr{E}} \quad \text{for} \quad x \in X, y \in X^*.$$

We will denote $Dom\ T := X$ and $Dom\ T^* := X^*$. If T satisfies that $1 + T^*T$ has dense image we will say that T is a regular mapping of Hilbert modules, see more in Definition 1.1 of [2].

Definition 3.3.1 (Equivariant unbounded Kasparov module)**.** *An unbounded $A-B$-Kasparov module is a pair $(\mathscr{E}, D)$ of a graded $A-B$-Hilbert module $\mathscr{E}$ and a regular, odd mapping D on the Hilbert module $\mathscr{E}$ satisfying that*

1. *D is self-adjoint, i.e. $D = D^*$.*
2. *For all $a \in A$, we have that $\pi(a)(1 + D^2)^{-1} \in \mathscr{K}_B(\mathscr{E})$.*
3. *The $*$-subalgebra $\mathscr{A} := \{a \in A : [D, \pi(a)]$ extends to a bounded operator$\}$ is dense in A.*

If $\mathscr{E}$ is a graded S-equivariant $A-B$-Hilbert module, an unbounded $A-B$-Kasparov module $(\mathscr{E}, D)$ is called equivariant if the operator

$$(\pi \otimes \mathrm{id})(a)\left(D \otimes_{\mathbb{C}} \mathrm{id}_S - V_{\mathscr{E}}(D \otimes_{\Delta_B} 1)V_{\mathscr{E}}^*\right), \tag{3.7}$$

extends to a bounded operator for all $a \in A$.

If a regular operator T satisfies equation (3.7), we say that T is almost S-invariant. If $(T \otimes_{\mathbb{C}} \mathrm{id}_S)V_{\mathscr{E}} = V_{\mathscr{E}}(T \otimes_{\Delta_B} 1)$ we say that T is S-invariant.

Compare this definition to Definition 2.1 of [2]. They study the non-equivariant setting. An unbounded equivariant $A-\mathbb{C}$-Kasparov module will be referred to as an equivariant spectral triple. What makes the $A-\mathbb{C}$-Kasparov module to a ”triple” is the extra information in $\mathscr{A}$, so technically the spectral triple is $(\mathscr{A}, \mathscr{E}, D)$. Spectral triples where first defined by Connes (see [11]) and can be viewed as a metric structure on the algebra A. In [47] spectral triples equivariant with respect to a Hopf algebra where studied, although from a very algebraic viewpoint.

Proposition 3.3.2 (Proposition 2.2 of [2])**.** *Assume that $(\mathscr{E}, D)$ is an unbounded equivariant Kasparov module. The operator $F_D := D(1 + D^2)^{-1/2}$ is bounded and $(\mathscr{E}, F_D)$ is an equivariant $A-B$-Kasparov module.*

Proof from [2]. We need to verify that $(\mathscr{E}, F_D)$ is a Kasparov module and that F_D satisfies equation (3.2). By definition $F_D^* = F_D$ and

$$\pi(a)(1 - F_D^2) = \pi(a)(1 + D^2)^{-1} \in \mathscr{K}_B(\mathscr{E}).$$

For any $x \in \mathscr{L}_B(\mathscr{E})$ we have that

$$[F_D, x] = [D, x](1 + D^2)^{-1/2} + D[(1 + D^2)^{-1/2}, x].$$

We can write $(1 + D^2)^{-1/2}$ as a strictly convergent integral

$$(1 + D^2)^{-1/2} = \frac{1}{\pi}\int_0^\infty t^{-1/2}(1 + D^2 + t)^{-1}\mathrm{d}t.$$

So it follows that $[F_D, \pi(a)]$ is compact if $[D, \pi(a)]$ is bounded. The subalgebra $\mathscr{A} \subseteq A$ is dense, therefore $[F_D, \pi(a)] \in \mathscr{K}_B(\mathscr{E})$ for all $a \in A$. That F_D satisfies equation (3.2) follows by the same reasoning from that D is almost S-invariant. □

3.4 Kasparov product and KK_S as a category

One of the things that makes KK-theory into a powerful tool is the Kasparov product. It is an associative product $KK_S(A, B) \times KK_S(B, C) \to KK_S(A, C)$ which is functorial in every possible sense. We will shortly review the definitions of this product. The construction of the product is a bit technical. The material in this chapter is based on [26] and it's equivariant generalizations in [3]. We refer the reader to the construction there.

Suppose that we are given two S-equivariant Kasparov modules $(\mathscr{E}_1, F_1) \in E_S(A, B)$, and $(\mathscr{E}_2, F_2) \in E_S(B, C)$, let $\pi : A \to \mathscr{L}(\mathscr{E}_1)$ denote the corresponding representation. Define the morphism $t : \mathscr{E}_1 \to \mathscr{L}_C(\mathscr{E}_2, \mathscr{E}_1 \otimes_B \mathscr{E}_2)$ by $t_{x_1}(x_2) := x_1 \otimes x_2$ and

$$\tilde{t}_x := \begin{pmatrix} 0 & t_x \\ t_x^* & 0 \end{pmatrix} \in \mathscr{L}_C((\mathscr{E}_1 \otimes_B \mathscr{E}_2) \oplus \mathscr{E}_2).$$

If we have an operator $F \in \mathscr{L}_C(\mathscr{E}_1 \otimes_B \mathscr{E}_2)$ such that

$$[\tilde{t}_x, F_2 \oplus F] \in \mathscr{K}_C((\mathscr{E}_1 \otimes_B \mathscr{E}_2) \oplus \mathscr{E}_2)$$

we say that F is an F_2-connection for $\mathscr{E}_1$.

Definition 3.4.1. *An equivariant A–C-Kasparov module $(\mathscr{E}_1 \otimes_B \mathscr{E}_2, F) \in E_S(A, C)$ is called a Kasparov product of $(\mathscr{E}_1, F_1)$ and $(\mathscr{E}_2, F_2)$ if*

1. *F is an F_2-connection and*

2. *for $a \in A$ we have that $(\pi(a) \otimes 1)[F_1 \otimes 1, F](\pi(a^*) \otimes 1) \geq 0$ modulo $\mathscr{K}_C(\mathscr{E}_1 \otimes_B \mathscr{E}_2)$.*

We will denote a Kasparov product by $(\mathscr{E}_1 \otimes_B \mathscr{E}_2, F_1 \# F_2)$.

Theorem 3.4.2 (Theorem 5.3 of [3]). *Given $(\mathscr{E}_1, F_1) \in E_S(A,B)$ and $(\mathscr{E}_2, F_2) \in E_S(B,C)$ there exists a Kasparov product $(\mathscr{E}_1 \otimes_B \mathscr{E}_2, F_1 \# F_2) \in E_S(A,C)$. The class $[(\mathscr{E}_1 \otimes \mathscr{E}_2, F_1 \# F_2)]$ is uniquely determined by $[(\mathscr{E}_1, F_1)] \in KK_S(A,B)$ and $[(\mathscr{E}_2, F_2)] \in KK_S(B,C)$.*

Thus the Kasparov product

$$\begin{aligned} KK_S(A,B) \times KK_S(B,C) &\to KK_S(A,C), \\ [(\mathscr{E}_1, F_1)] \times [(\mathscr{E}_2, F_2)] &\mapsto [(\mathscr{E}_1 \otimes \mathscr{E}_2, F_1 \# F_2)] \end{aligned}$$

is a well defined mapping.

Theorem 3.4.3 (Theorem 5.5 of [3]). *The Kasparov product is associative, functorial, additive and an identity is given by $1_A := [(A,0)] \in KK_S(A,A)$. Furthermore, if $f : B \to D$ is S-equivariant, $\alpha \in KK_S(A,B)$, $\beta \in KK_S(D,C)$ then*

$$f_*(\alpha) \circ \beta = \alpha \circ f^*(\beta) \quad in \quad KK_S(A,C).$$

With this as motivation, let KK_S denote the category of C^*-algebras from $C_S^{*,r}$. The morphisms in KK_S are given by

$$Mor_{KK_S}(A,B) := KK_S(A,B).$$

Because of the theorem above, this is a category. The category KK_S is an additive category, since the Hom-sets are abelian groups and the Kasparov product is additive. There exists a functor $\iota : C_S^{*,r} \to KK_S$ which is defined on objects as $\iota(A) := A$ and given a morphism $\mu : A \to B$ we define $\iota(\mu)$ as the equivariant $A-B$-Kasparov module $(B,0)$ with A-action given by μ.

Suppose that $\mathfrak{C}$ is an additive category. A functor $\mathfrak{F} : C_S^{*,r} \to \mathfrak{C}$ is called homotopy invariant if $\mathfrak{F}(\mu_0) = \mathfrak{F}(\mu_1)$ for homotopic morphisms μ_0 and μ_1. The functor $\mathfrak{F}$ is called a stable functor if there exist a natural isomorphism $\mathfrak{F}(A \otimes \mathscr{K}(H_S)) \cong \mathfrak{F}(A)$ for all $A \in C_S^{*,r}$. If $\mathfrak{C}$ is an abelian category and if every split-exact sequence $0 \to A \to B \to C \to 0$ in $C_S^{*,r}$ is mapped to a split-exact sequence $0 \to \mathfrak{F}(A) \to \mathfrak{F}(B) \to \mathfrak{F}(C) \to 0$ the functor $\mathfrak{F}$ is called split-exact.

Theorem 3.4.4 (Theorem 4.4 of [42]). *The functor ι is the universal homotopy invariant, stable, split-exact functor in the sense that if $\mathfrak{F} : C_S^{*,r} \to \mathfrak{C}$ is a homotopy, stable, split-exact functor into an abelian category $\mathfrak{C}$, there exists a functor $\mathfrak{F}_0 : KK_S \to \mathfrak{C}$ such that $\mathfrak{F} = \mathfrak{F}_0 \circ \iota$.*

The proof of this theorem can be found in [42]. It consists of an argument similar to that in [33] where the case $S = C_0(G)$, for a locally compact group G, was proved. Both arguments consists of showing that morphisms in KK_S corresponds to homotopy classes of generalized homomorphisms.

Let $Ab^{\mathbb{Z}_2}$ denote the abelian category of graded abelian groups. For every $D \in KK_S$ there are associated Hom-functors

$$KK_S^*(D,-) : KK_S \to Ab^{\mathbb{Z}_2} \quad \text{and} \quad KK_S^*(-,D) : KK_S \to Ab^{\mathbb{Z}_2}.$$

Suppose that we have three objects $M_*^i \in Ab^{\mathbb{Z}_2}$, for $i = 1,2,3$. If we have graded mappings $\alpha : M_*^1 \to M_*^2$, $\beta : M_*^2 \to M_*^3$ and an odd degree mapping $\gamma : M_*^3 \to M_{*+1}^1$ such that $M_*^1 \xrightarrow{\alpha} M_*^2 \xrightarrow{\beta} M_*^3 \xrightarrow{\gamma} M_{*+1}^1 \xrightarrow{\alpha} M_{*+1}^2$ is exact, we will denote this by a triangle:

$$\begin{array}{ccc} M_*^1 & \longrightarrow & M_*^2 \\ & {\scriptstyle [1]}\nwarrow \quad \swarrow & \\ & M_*^3 & \end{array}$$

In fact this class of triangles together with the suspension functor $M_* \mapsto M_{*+1}$ make $Ab^{\mathbb{Z}_2}$ into a triangulated category. We will return later to more theory of triangulated categories. In particular we will equip KK_S with a triangulated structure. The first step in this direction is the following theorem from [3]:

Theorem 3.4.5 (Theorem 7.2 of [3]). *Suppose that the short exact sequence $0 \to A \to B \to C \to 0$ is semi-split, then for every $D \in C_S^{*,r}$ this exact sequence induces triangles in $Ab^{\mathbb{Z}_2}$:*

$$\begin{array}{ccc} KK_S^*(A,D) & \longrightarrow & KK_S^*(B,D) \\ & {\scriptstyle [1]}\nwarrow \quad \swarrow & \\ & KK_S^*(C,D) & \end{array} \quad \textit{and}$$

$$\begin{array}{ccc} KK_S^*(D,A) & \longrightarrow & KK_S^*(D,B) \\ & {\scriptstyle [1]}\nwarrow \quad \swarrow & \\ & KK_S^*(D,C) & \end{array}$$

Furthermore, if $0 \to A \to B \to C \to 0$ splits, the degree 1 maps vanishes.

Recall from Chapter 3.2 that a short exact sequence of $S-C^*$-algebras is called semi-split if it admits a completely positive, equivariant splitting. The exact sequences of Theorem 3.4.5 contains six terms. Therefore they are often referred to as six-term exact sequences.

We define the functor $\Sigma : KK_S \to KK_S$ by $\Sigma A := C_0(\mathbb{R}) \otimes A$, where $C_0(\mathbb{R})$ has the trivial S-coaction. On morphisms $\alpha = [(\mathscr{E}, F)] \in KK_S(A, B)$ we define $\Sigma(\alpha) := [(\mathscr{E} \otimes C_0(\mathbb{R}), F \otimes 1)]$.

Proposition 3.4.6 (Reformulation of Bott periodicity)**.** *The functor* Σ *satisfies* $\Sigma^2 \cong \mathrm{id}$ *and there is a natural graded isomorphism*

$$KK_S^*(A, \Sigma B) \cong KK_S^{*+1}(A, B).$$

The proof is based on an argument from the proof of Theorem 19.2.1 of [7].

Proof. Consider the semi-split exact sequence of C^*-algebras

$$0 \to C_0(\mathbb{R}) \to C_0(0,1] \to \mathbb{C} \to 0.$$

This is semi-split since a splitting can be chosen as a cutoff around the point 1. Given $A, B \in KK_S$ we may tensor this sequence with B and apply $KK_S(A, -)$. By Theorem 3.4.5 there is an exact triangle

$$\begin{array}{ccc} KK_S^*(A, \Sigma B) & \longrightarrow & KK_S^*(A, C_0(0,1] \otimes B) \\ & {\scriptstyle [1]}\nwarrow \quad \swarrow & \\ & KK_S^*(A, B) & \end{array}$$

If $KK_S^*(A, C_0(0,1] \otimes B) = 0$ it follows that the mapping $KK_S^{*+1}(A, B) \to KK_S(A, \Sigma B)$ is an isomorphism. Take a $t \in [0,1]$ and consider the the mapping

$$\pi_t : C_0(0,1] \otimes B \to C_0(0,1] \otimes B$$

defined on a $f \in C_0((0,1], B) \cong C_0(0,1] \otimes B$ as $\pi_t(f)(s) := f(st)$. Then $\pi_1 = \mathrm{id}$ and $\pi_0 = 0$. It follows that $0 = \mathrm{id}_* : KK_S^*(A, C_0(0,1] \otimes B) \to KK_S^*(A, C_0(0,1] \otimes B)$ and $KK_S^*(A, C_0(0,1] \otimes B) = 0$.

In a similar fashion one can prove that $KK_S^*(\Sigma A, B) \cong KK_S^{*+1}(A, B)$. It follows that $KK_S(\Sigma A, B) \cong KK_S(A, \Sigma B)$, therefore Σ is it's own adjoint functor. Since Σ is it's own adjoint, it follows by Bott periodicity that $\Sigma^2 \cong \mathrm{id}$. $\square$

An interesting remark is that one may construct an explicit inverse to the isomorphism $KK_S^{*+1}(A,B) \to KK_S(A,\Sigma B)$ by identifying $C_0(\mathbb{R})$ with the subalgebra $\{a \in C(\mathbb{T}) : a(1) = 0\} \subseteq C(\mathbb{T})$ via a homeomorphism $\mathbb{R} \cong \mathbb{T} \setminus \{1\}$ and considering the C^*-algebra $\mathscr{T}_0$ of Toeplitz operators on $\mathbb{T}$ with symbols in $C_0(\mathbb{R})$. These algebras fits into a semi-split exact sequence

$$0 \to \mathscr{K} \to \mathscr{T}_0 \to C_0(\mathbb{R}) \to 0.$$

If we tensor this semi-split exact sequence with B we obtain a semi-split extension

$$0 \to B \otimes \mathscr{K} \to B \otimes \mathscr{T}_0 \to \Sigma B \to 0.$$

Applying $KK_S(A,-)$ to this semi-split exact sequence and using Theorem 3.4.5, the boundary mapping $KK_S^*(A,\Sigma B) \to KK_S^{*+1}(A,B)$ is in fact an inverse. This is the content of Theorem 19.2.1 of [7]. For an elegant proof of that the mapping $KK_S^*(A,\Sigma B) \to KK_S^{*+1}(A,B)$ is an isomorphism, see in the proof of Theorem 4.7 of [12]. There the proof consists of showing that for any B the C^*-algebra any homotopy stable functor sends $B \otimes \mathscr{T}_0$ to 0. In our setting, the Yoneda lemma implies that $B \otimes \mathscr{T}_0 \cong 0$.

3.5 Triangulated categories

The viewpoint on KK_S as a category might seem as nothing more than just a viewpoint. But the category structure together with a triangulated structure becomes a powerful tool to study homological algebra for C^*-algebras. In this section we will very shortly review the definition of triangulated categories and some of their properties. We follow the convention from [35] to place suspension to the left in triangles. This is equivalent to the definition in [38] of a triangulated category except that the suspension functor Σ is replaced by the de-suspension functor Σ^{-1}.

The material is based on Chapter 1 of Neeman's book [38]. We follow the approach of Neeman using the mapping cone axiom instead of the equivalent octahedral axiom, these results can also be found in [38] and more thoroughly explained in [39]. As previously remarked, we follow the convention from [35], placing suspensions to the left in triangles, so our triangulated categories are triangulated categories in the conventions of [38] if the suspension functor is replaced by the de-suspension functor. For an easily accessible introduction to triangulated categories and derived categories, see [23].

We will throughout this chapter assume that $\mathfrak{T}$ is an additive category and that there exist an additive automorphism Σ of $\mathfrak{T}$. So $\mathfrak{T}$ is a suspended category and we refer to Σ as the suspension functor.

A sequence of morphisms $\Sigma Z \to X \to Y \to Z$ is called a candidate triangle if any composition of morphisms in the sequence is 0. Given two candidate triangles $\Sigma Z \to X \to Y \to Z$ and $\Sigma Z' \to X' \to Y' \to Z'$ if the three morphisms $f : X \to X'$, $g : Y \to Y'$ and $h : Z \to Z'$ makes the following diagram commutative

$$\begin{array}{ccccccc} \Sigma Z & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & Z \\ {\scriptstyle \Sigma h}\downarrow & & {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & {\scriptstyle h}\downarrow \\ \Sigma Z' & \longrightarrow & X' & \longrightarrow & Y' & \longrightarrow & Z' \end{array}$$

we say that f, g, h is a morphism of candidate triangles. If f, g and h are isomorphisms we say that they form an isomorphism of candidate triangles. If Δ is a class of candidate triangles in $\mathfrak{T}$ we say that a candidate triangle in Δ is a distinguished triangle.

Definition 3.5.1 (Definition 1.1.2 of [38]). *The suspended category $\mathfrak{T}$, equipped with a class Δ of candidate triangles, is called a pre-triangulated category if it satisfies the following axioms:*

(TR0) The class Δ is closed under isomorphism of candidate triangles and the candidate triangle

$$0 \to X \xrightarrow{\text{id}} X \to 0$$

is distinguished.

(TR1) Any morphism $f : X \to Y$ fits into a distinguished triangle

$$\Sigma Z \to X \xrightarrow{f} Y \to Z.$$

(TR2) Given two candidate triangles

$$\Sigma Z \xrightarrow{\Sigma w} X \xrightarrow{u} Y \xrightarrow{v} Z,$$

$$\Sigma X \xrightarrow{-\Sigma u} \Sigma Y \xrightarrow{-v} Z \xrightarrow{-w} X,$$

and if one of them is distinguished, so is the other.

(TR3) If $g : Y \to Y'$ and $h : Z \to Z'$ are morphisms making the following diagram commutative

$$\begin{array}{ccccccc} \Sigma Z & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & Z \\ & & & & {\scriptstyle g}\downarrow & & {\scriptstyle h}\downarrow \\ \Sigma Z' & \longrightarrow & X' & \longrightarrow & Y' & \longrightarrow & Z' \end{array},$$

where the rows are distinguished triangles, there exists an $f : X \to X'$ making the diagram

$$\begin{array}{ccccccc} \Sigma Z & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & Z \\ \Sigma h \downarrow & & f \downarrow & & g \downarrow & & h \downarrow \\ \Sigma Z' & \longrightarrow & X' & \longrightarrow & Y' & \longrightarrow & Z' \end{array}$$

into a morphism of candidate triangles.

A distinguished triangle will in short be referred to as a triangle in $\mathfrak{T}$.

We will sometimes denote a triangle by

$$\begin{array}{ccc} X & \longrightarrow & Y \\ & {}_{[1]}\nwarrow \quad \swarrow & \\ & Z & \end{array} .$$

The reason for the [1] over the arrow $Z \to X$ being that it is an arrow of degree 1, that is, a morphism $\Sigma Z \to X$.

The very useful Remark 1.1.3 from [38] states that the condition on the members of Δ to be candidate triangles is redundant. If Δ is a class of sequences $\Sigma Z \to X \to Y \to Z$ satisfying the axioms $(TR0)$, $(TR2)$ and $(TR3)$ is automatically a class of candidate triangles. So when defining the class of triangles in a pre-triangulated category it is not needed that they are candidate triangles.

If $\mathfrak{T}$ is a pre-triangulated category, so is it's dual $\mathfrak{T}^{op}$ with suspension functor $-\Sigma$. The triangles in $\mathfrak{T}^{op}$ are the triangles dual to those in $\mathfrak{T}$.

Definition 3.5.2 (Definition 1.1.7 of [38]). *Let $\mathfrak{C}$ denote an abelian category. A covariant functor $H : \mathfrak{T} \to \mathfrak{C}$ is said to be homological if for every triangle $\Sigma Z \to X \to Y \to Z$ the sequence*

$$H(X) \to H(Y) \to H(Z)$$

is exact in $\mathfrak{C}$. If $H : \mathfrak{T}^{op} \to \mathfrak{C}$ is homological, we say that H is a cohomological functor on $\mathfrak{T}$.

The motivation for this terminology is that we may define $H_n(X) := H(\Sigma^n X)$ for $n \in \mathbb{Z}$. So we obtain a functor $H_* : \mathfrak{T} \to \mathfrak{C}^{\mathbb{Z}}$ and a triangle in $\mathfrak{C}^{\mathbb{Z}}$:

$$H_{*+1}(Z) \to H_*(X) \to H_*(Y) \to H_*(Z).$$

Lemma 3.5.3 (Lemma 1.1.10 of [38]). *If $X \in \mathfrak{T}$ the functors $Hom_{\mathfrak{T}}(X,-)$ and $Hom_{\mathfrak{T}}(-,X)$ are homological respectively cohomological.*

The proof of this lemma is a rather straight forward application of Axiom $(TR3)$ and can be found in [38]. The next proposition is a consequence of this lemma and the five lemma for abelian categories. Again the proof can be found in [38].

Proposition 3.5.4 (Proposition 1.1.20 of [38]). *Suppose that we have a morphism of triangles*

$$\begin{array}{ccccccc} \Sigma Z & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & Z \\ \Sigma h \downarrow & & f \downarrow & & g \downarrow & & h \downarrow \\ \Sigma Z' & \longrightarrow & X' & \longrightarrow & Y' & \longrightarrow & Z' \end{array}$$

If two of the morphisms f, g and h are isomorphisms, then so is the third.

If we are given a mapping $f : Y \to Z$ and use Axiom $(TR1)$ to complete this to a triangle $\Sigma Z \to X \to Y \to Z$ then X is determined up to an isomorphism, which in general is not canonical. This follows since if X' is another choice we may use Axiom $(TR3)$ to complete the diagram

$$\begin{array}{ccccccc} \Sigma Z & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & Z \\ & & & & \mathrm{id} \downarrow & & \mathrm{id} \downarrow \\ \Sigma Z' & \longrightarrow & X' & \longrightarrow & Y' & \longrightarrow & Z' \end{array}$$

with a morphism $f : X \to X'$ which by Proposition 3.5.4 is an isomorphism.

Corollary 3.5.5 (Proposition 1.2.1 of [38]). *Suppose that we have a set Λ and for each $\lambda \in \Lambda$ a triangle $\Sigma Z_\lambda \to X_\lambda \to Y_\lambda \to Z_\lambda$. If the direct products $\prod_{\lambda\in\Lambda} X_\lambda$, $\prod_{\lambda\in\Lambda} Y_\lambda$ and $\prod_{\lambda\in\Lambda} Z_\lambda$ exists in $\mathfrak{T}$ the candidate triangle*

$$\Sigma\left(\prod_{\lambda\in\Lambda} Z_\lambda\right) \to \prod_{\lambda\in\Lambda} X_\lambda \to \prod_{\lambda\in\Lambda} Y_\lambda \to \prod_{\lambda\in\Lambda} Z_\lambda$$

is distinguished. Similarly, ff the direct sums $\coprod_{\lambda\in\Lambda} X_\lambda$, $\coprod_{\lambda\in\Lambda} Y_\lambda$ and $\coprod_{\lambda\in\Lambda} Z_\lambda$ exists in $\mathfrak{T}$ the following candidate triangle is distinguished

$$\Sigma\left(\coprod_{\lambda\in\Lambda} Z_\lambda\right) \to \coprod_{\lambda\in\Lambda} X_\lambda \to \coprod_{\lambda\in\Lambda} Y_\lambda \to \coprod_{\lambda\in\Lambda} Z_\lambda.$$

We now turn back to the candidate triangles. Let $CT(\mathfrak{T})$ denote the category of candidate triangles, with morphisms being morphisms of candidate triangles. This is an additive category in the usual operations by Corollary 3.5.5. The suspension functor Σ induces an additive automorphism $\tilde{\Sigma}$ on $CT(\mathfrak{T})$ by defining

$$[\Sigma Z \xrightarrow{w} X \xrightarrow{u} Y \xrightarrow{v} Z] \mapsto [\Sigma X \xrightarrow{-\Sigma u} \Sigma Y \xrightarrow{-v} Z \xrightarrow{-w} X].$$

The full subcategory $T(\mathfrak{T})$ of triangles is closed under $\tilde{\Sigma}$ by Axiom $(TR2)$. We are now going to define the mapping cone of a morphism of candidate triangles

$$\begin{array}{ccccccc}
\Sigma Z & \xrightarrow{w} & X & \xrightarrow{u} & Y & \xrightarrow{v} & Z \\
\downarrow{\scriptstyle \Sigma h} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} \\
\Sigma Z' & \xrightarrow{w'} & X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z'
\end{array}. \tag{3.8}$$

The mapping cone is defined to be the candidate triangle

$$X \oplus \Sigma Z' \xrightarrow{\begin{pmatrix} -u & 0 \\ f & w' \end{pmatrix}} Y \oplus X' \xrightarrow{\begin{pmatrix} -v & 0 \\ g & u' \end{pmatrix}} Z \oplus Y' \xrightarrow{\begin{pmatrix} -\Sigma^{-1} w & 0 \\ h & v' \end{pmatrix}} \Sigma^{-1} X \oplus Z'. \tag{3.9}$$

Suppose that the morphisms f, g, h in equation (3.8) is a morphism of triangles. Then we say that f, g, h is a good morphism if the mapping cone in equation (3.9) is a triangle.

Definition 3.5.6 (Triangulated category)**.** *Suppose that $\mathfrak{T}$ is a pre-triangulated category. It is said to satisfy the mapping cone axiom if for every commutative diagram*

$$\begin{array}{ccccccc}
\Sigma Z & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & Z \\
 & & & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} \\
\Sigma Z' & \longrightarrow & X' & \longrightarrow & Y' & \longrightarrow & Z'
\end{array},$$

where the rows are triangles, there exists an $f : X \to X'$ making the diagram

$$\begin{array}{ccccccc}
\Sigma Z & \xrightarrow{w} & X & \xrightarrow{u} & Y & \xrightarrow{v} & Z \\
\downarrow{\scriptstyle \Sigma h} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} \\
\Sigma Z' & \xrightarrow{w'} & X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z'
\end{array}$$

into a good morphism of triangles. A pre-triangulated category satisfying the mapping cone axiom is called a triangulated category.

It is shown in [39] that a pre-triangulated category satisfy the mapping cone axiom if and only if it satisfies the octahedral axiom. The content of the octahedral axiom goes beyond the scope of this thesis, but it is a more commonly used axiom that one requires for a pre-triangulated category to be a triangulated category. It is necessary in the construction of Verdier quotients of a triangulated category by a triangulated subcategory.

Definition 3.5.7 ([23])**.** *Suppose that $\mathfrak{T}_1$ and $\mathfrak{T}_2$ are triangulated categories. An additive functor $F : \mathfrak{T}_1 \to \mathfrak{T}_2$ is called triangulated if there exist a natural transformation of additive functors $\tau : F\Sigma_1 \to \Sigma_2 F$ such that if $\Sigma Z \xrightarrow{w} X \xrightarrow{u} Y \xrightarrow{v} Z$ is a triangle, the candidate triangle*

$$\Sigma_1 FZ \xrightarrow{\tau_Z Fw} FX \xrightarrow{Fu} FY \xrightarrow{Fv} FZ$$

is distinguished.

Suppose that we are given two triangulated functors $F, G : \mathfrak{T}_1 \to \mathfrak{T}_2$. Then a natural transformation of additive functors $\mu : F \to G$ is called a morphism of triangulated functors if the following square of natural transformation commutes

$$\begin{array}{ccc} F\Sigma & \xrightarrow{\tau_F} & \Sigma F \\ {\scriptstyle \mu\Sigma}\downarrow & & \downarrow{\scriptstyle \Sigma\mu} \\ G\Sigma & \xrightarrow{\tau_G} & \Sigma G \end{array} .$$

If there exists a triangulated functor and $F' : \mathfrak{T}_2 \to \mathfrak{T}_2$ such that FF' and $F'F$ are isomorphic to the identity functors on the corresponding category we say that F is a triangulated equivalence. We will state a lemma concerning triangulated equivalences, it's proof can be found in [23].

Lemma 3.5.8 (Lemma 8.2 of [23])**.** *A triangulated functor is a triangulated equivalence if and only if it is an equivalence of categories.*

3.6 The triangulated structure on KK_S

In this section we will construct a triangulated structure on KK_S. This has been done previously in [42] using generalized homomorphisms. We will use the somewhat more suggestive approach of proving the mapping cone axiom via the groups $Ext_S^{-1}(A,B)$. Although all proofs in this section are new, they are very much inspired by [35].

Recall that the functor $\Sigma : KK_S \to KK_S$ is defined as $\Sigma A := C_0(\mathbb{R}) \otimes A$, where $C_0(\mathbb{R})$ has the trivial S-coaction. The suspension functor Σ satisfies $\Sigma^2 A \cong A$ naturally by Bott periodicity (Proposition 3.4.6). So we may, after doing certain tricks, consider Σ as an automorphism of KK_S. This trick involves considering the category $\widetilde{KK_S}$ consisting of pairs (A,n) where $n \in \mathbb{Z}$ and defining morphisms by

$$Mor_{\widetilde{KK_S}}((A,n),(B,m)) := KK_S(\Sigma^{|n-m|}A, B).$$

Because of Bott periodicity, we may instead work with Σ as an automorphism in the notationally simpler category KK_S.

Let $f : B \to C$ be a surjection in $C_S^{*,r}$. The mapping cone of f is defined as

$$C(f) := \{b \oplus c \in B \oplus C_0((0,1],C) : f(b) = c(1)\}.$$

The mapping cone $C(f)$ is given the S-coaction induced from those on B and C. Since B and C has continuous, reduced coaction, so does $C(f)$. The embedding $\Sigma C \hookrightarrow C(f)$ induces a semi-split exact sequence

$$0 \to \Sigma C \to C(f) \to B \to 0.$$

If we define $A := \ker f$, we obtain mapping $A \to C(f)$, $a \mapsto a \oplus 0$. This mapping makes the following diagram commutative:

$$\begin{array}{ccccc} A & \longrightarrow & B & \longrightarrow & C \\ \downarrow & & \downarrow & & \downarrow \\ C(f) & \longrightarrow & B & \longrightarrow & C \end{array}.$$

A short exact sequence $0 \to A \to B \xrightarrow{f} C \to 0$ in $C_S^{*,r}$ is said to be admissible if the mapping $A \to C(f)$ induces an isomorphism in KK_S. Notice that this requirement is necessary to obtain a triangulated structure on KK_S, because of the remarks after Proposition 3.5.4.

Lemma 3.6.1. *If the short exact sequence* $0 \to A \to B \to C \to 0$ *is semi-split, it is admissible.*

Proof. By Theorem 3.4.5, for a given $D \in KK_S$ the functors $KK_S(-,D)$ and $KK_S(D,-)$ maps semi-split exact sequences to six-terms exact sequences. By comparing the six-term exact sequences of $0 \to A \to B \to C \to 0$ with those of the semi-split exact sequence $0 \to \Sigma C \to C(f) \to B \to 0$, the Yoneda lemma implies that $C(f) \cong A$. □

Now we are ready to define the set of distinguished triangles. Consider a sequence of arrows $A' \xrightarrow{\alpha} B' \xrightarrow{\beta} C'$ in KK_S. This sequence forms a distinguished triangle if there exists an admissible short exact sequence $0 \to A \xrightarrow{g} B \xrightarrow{f} C \to 0$ in C^*_S and KK_S-isomorphisms $A \to A'$, $B \to B'$ and $C \to C'$ making the following diagram commutative:

$$\begin{array}{ccccc} A & \xrightarrow{g} & B & \xrightarrow{f} & C \\ \downarrow & & \downarrow & & \downarrow \\ A' & \xrightarrow{\alpha} & B' & \xrightarrow{\beta} & C' \end{array}$$

The sequence $A' \xrightarrow{\alpha} B' \xrightarrow{\beta} C'$ can be completed to a candidate triangle

$$\Sigma C' \xrightarrow{\gamma} A' \xrightarrow{\alpha} B' \xrightarrow{\beta} C'$$

by taking $\gamma \in KK_S(\Sigma C', A')$ as the image of the $*$-homomorphism $\Sigma C \to C(f)$ under the isomorphism

$$KK_S(\Sigma C, C(f)) \cong KK_S(\Sigma C, A) \cong KK_S(\Sigma C', A').$$

Lemma 3.6.2. *The triangle $\Sigma C \to A \to B \to C$ is exact if and only if the rotated sequence $\Sigma B \to \Sigma C \to A \to B$ is a triangle.*

Proof. We may assume that it is given by an admissible exact sequence

$$0 \to A \to B \xrightarrow{f} C \to 0$$

in $C^{*,r}_S$, because of the definition of an exact triangle. The mapping cone construction defines a short exact sequence in $C^{*,r}_S$

$$0 \to \Sigma C \to C(f) \to B \to 0.$$

Since $C(f) \to B$ allows a completely positive linear splitting, Lemma 3.6.1 implies that this exact sequence is admissible. By assumption $A \cong C(f)$, therefore the exactness of the second triangle in the statement follows from exactness of the first. The converse follows by Bott periodicity. □

Lemma 3.6.3. *Any morphism $\alpha \in KK_S(A, B)$ fits into an exact triangle. Furthermore after stabilization by $\mathcal{K}(H_S)$, the 0-degree maps in the rotated triangle*

$$\Sigma A \otimes \mathcal{K}(H_S) \xrightarrow{\Sigma\alpha} \Sigma B \otimes \mathcal{K}(H_S) \otimes \mathcal{K} \to E \to A \otimes \mathcal{K}(H_S)$$

may be choosen as equivariant $*$-homomorphisms.

Proof. Take an $\alpha \in KK_S(A,B)$. The isomorphism in Theorem 3.2.6 allows us to represent the mapping α by a short exact sequence

$$0 \to \Sigma B \otimes \mathscr{K}(H_S) \otimes \mathscr{K} \to E \to A \otimes \mathscr{K}(H_S) \to 0.$$

Due to the stabilization $\mathscr{K}(H_S)$, a mild generalization of Theorem 8.1 from [50] using Baaj-Skandalis duality, implies that this short exact sequence admits an equivariant completely positive splitting. So by Lemma 3.6.2 it is admissible and thus gives an exact triangle

$$\begin{array}{ccc} \Sigma B \otimes \mathscr{K}(H_S) \otimes \mathscr{K} & \longrightarrow & E \\ & {\scriptstyle [\Sigma\alpha]} \nwarrow \quad \swarrow & \\ & A \otimes \mathscr{K}(H_S) & \end{array}.$$

with the 0-degree maps being equivariant $*$-homomorphisms. Using Lemma 3.6.2 and the natural isomorphisms $B \otimes \mathscr{K}(H_S) \otimes \mathscr{K} \cong B$ and $A \otimes \mathscr{K}(H_S) \cong A$ we can rotate the triangle to obtain

$$\begin{array}{ccc} E & \longrightarrow & A \\ & {\scriptstyle [1]} \nwarrow \quad \swarrow {\scriptstyle \alpha} & \\ & B & \end{array}.$$

□

Lemma 3.6.4. *Assume that $h \in KK_S(C,C')$ and $g \in KK_S(B,B')$ makes the following diagram commutative:*

$$\begin{array}{ccccccc} \Sigma C & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C \\ & & & & {\scriptstyle g}\downarrow & & {\scriptstyle h}\downarrow \\ \Sigma C' & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' \end{array}. \tag{3.10}$$

and the rows are admissible extensions. Then there exists an $f \in KK_S(A,A')$ completing the diagram to a morphism of triangles:

$$\begin{array}{ccccccc} \Sigma C & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C \\ {\scriptstyle \Sigma h}\downarrow & & {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & {\scriptstyle h}\downarrow \\ \Sigma C' & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' \end{array}. \tag{3.11}$$

Proof. We may assume that the algebras are stable and represent the diagram (3.10) by a diagram of $S-C^*$-algebras with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Sigma A' & \longrightarrow & \Sigma B' & \longrightarrow & \Sigma C' & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & E_B & & E_C & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Since an extension is determined by it's Busby mapping, we may assume that there exists equivariant $*$-homomorphisms

$$\beta_B : B \to Q(\Sigma B') \quad \text{and} \quad \beta_C : C \to Q(\Sigma C')$$

such that

$$E_B = \{b \oplus x \in B \oplus \mathcal{M}(\Sigma B') : q_{\Sigma B'}(x) = \beta_B(b)\} \quad \text{and}$$

$$E_C = \{c \oplus y \in C \oplus \mathcal{M}(\Sigma C') : q_{\Sigma C'}(y) = \beta_C(c)\}$$

Define a mapping $E_B \to E_C$ by extending $B \to C$ and $B' \to C'$. The five lemma implies that the mapping $E_B \to E_C$ is surjective. Defining E_A to be the kernel of this mapping and extending the mappings $A \to B$ and $A' \to B'$ we obtain a new commuting diagram of $S-C^*$-algebras with admissible columns and all but the second row admissible:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Sigma A' & \longrightarrow & \Sigma B' & \longrightarrow & \Sigma C' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & E_A & \longrightarrow & E_B & \longrightarrow & E_C & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

The mapping $\Sigma A' \to E_A$ is a well defined injection since $\Sigma A' \subseteq E_B$ and is mapped to 0 in E_C, thus we have the inclusion $\Sigma A' \subseteq E_A$. The mapping $E_A \to A$ is induced by the mapping $E_B \to B$ and is well defined, since $\Sigma C' \to E_C$ is injective. Finally it follows that $E_A \to A$ is surjective from the horizontal exactness at $\Sigma B'$ in the diagram.

This implies that if we take f to be the image of $[E_A]$ under the isomorphism $Ext_S^{-1}(A, \Sigma A') \cong KK_S(A, A')$ this completes the diagram (3.10) into a morphism of triangles. ☐

Lemma 3.6.5. *The choice of the morphism f in the proof of Lemma 3.6.4 completes the diagram* (3.10) *into a good morphism of triangles.*

Proof. Consider the diagram (3.11) and it's mapping cone triangle

$$A \oplus \Sigma C' \to B \oplus A' \to C \oplus B' \to \Sigma A \oplus C'. \tag{3.12}$$

Let us prove that this sequence is isomorphic to the mapping cone triangle of the morphism $B \oplus A' \to C \oplus B'$. The morphism $\gamma : B \oplus A' \to C \oplus B'$ can be represented by the matrix

$$\gamma = \begin{pmatrix} -\beta & 0 \\ g & -\alpha' \end{pmatrix},$$

where $\beta : B \to C$ and $\alpha' : A' \to B'$ are $*$-homomorphisms. Let us represent this morphism by an admissible extension, and thus explicitly describing a mapping cone. The $*$-homomorphism $B \to C$ can in KK_S be expressed by an invertible extension

$$0 \to \Sigma C \to E_\beta \to B \to 0$$

and similarly the $*$-homomorphism $A' \to B'$ by an invertible extension

$$0 \to \Sigma B' \to E_{\alpha'} \to A' \to 0.$$

As a linear space we have the equality $E_\beta = B \oplus \Sigma C$ but the multiplication is given by

$$(b \oplus c)(b' \oplus c') = bb' \oplus (cc' + \beta(b)c' + c\beta(b'))$$

and similarly $E_{\alpha'} = A' \oplus \Sigma B'$ as a linear space. Therefore there exists operators $F_\beta \in \mathscr{M}(\Sigma C)$ and $F_{\alpha'} \in \mathscr{M}(\Sigma B')$ defining the extensions as in Lemma 3.2.5, they can be defined by

$$F_\beta := 2 \oplus 2 - 1_{E_\beta} \quad \text{respectively} \quad F_{\alpha'} := 2 \oplus 2 - 1_{E_{\alpha'}}. \tag{3.13}$$

So the mapping cone of γ is isomorphic to the admissible extension

$$0 \to (\Sigma C \oplus \Sigma B') \otimes M_3 \to E_\gamma \to B \oplus A' \to 0,$$

where the mapping cone E_γ can be expressed by a sum in $X_S(B \oplus A', (\Sigma C \oplus \Sigma B') \otimes M_3)$ represented by the $S - C^*$-algebra

$$E_\gamma = E_\beta \oplus E_{\alpha'} \oplus E_B + (\Sigma C \oplus \Sigma B') \otimes M_3.$$

To show that the sequence (3.12) is a triangle we prove that there is an isomorphism $E_\gamma \cong A \oplus \Sigma C'$. Represent the morphisms $\Sigma h : \Sigma C \to \Sigma C'$ and $g : B \to B'$ by essential Kasparov modules $(\mathscr{E}_h, F_h)$ respectively $(\mathscr{E}_g, F_g)$. We may choose the Kasparov modules $(\mathscr{E}_h, F_h)$ and $(\mathscr{E}_g, F_g)$ to be essential by an argument similar to that of Lemma 3.3 of [33]. Define the S-equivariant $E_\gamma - A \oplus \Sigma C'$-Hilbert bimodule $\mathscr{E}_1$ as

$$\begin{aligned}
\mathscr{E}_1 := & \Big(((E_\beta \otimes_B B) \otimes_A A) \oplus \Sigma C \otimes_{\Sigma C} \mathscr{E}_h \otimes_{\Sigma C'} \Sigma C' \Big) \bigoplus \\
& \bigoplus \Big(\Sigma B' \otimes_{B'} \mathscr{K}(\mathscr{E}_g, B') \otimes_A A \oplus (\Sigma B' \otimes_{\Sigma C'} \Sigma C') \Big) \bigoplus \\
& \bigoplus \big(((E_B \otimes_B B) \otimes_A A) \oplus \Sigma B' \otimes_{\Sigma C'} \Sigma C' \big) = \\
= & \big(((E_\beta \otimes_B B) \otimes_A A) \oplus \Sigma B' \otimes_{\Sigma B'} \mathscr{K}(\mathscr{E}_g, \Sigma B') \otimes_A A \oplus \\
& \oplus ((E_B \otimes_B B) \otimes_A A) \big) \bigoplus \\
& \bigoplus \big(\Sigma C \otimes_{\Sigma C} \mathscr{E}_h \otimes_{\Sigma C'} \Sigma C' \oplus \Sigma B' \otimes_{\Sigma C'} \Sigma C' \oplus \\
& \oplus \Sigma B' \otimes_{\Sigma C'} \Sigma C' \big)
\end{aligned}$$

The first linear decomposition defines the left E_γ-action on $\mathscr{E}_1$ and the second decomposition defines the right $A \oplus \Sigma C'$-action. The second decomposition defines the $A\oplus\Sigma C'$-valued scalar product on $\mathscr{E}_1$ in the obvious way. An almost S-equivariant Kasparov operator for this bimodule is given by

$$F_1 = \left(F_\beta \oplus F_g^* \oplus F_B\right) \bigoplus \left(F_h \oplus F_{\alpha'} \oplus F_{\alpha'}\right),$$

where F_B is chosen as in equation (3.13) but for the extension E_B. If $[(\mathscr{E}_1, F_1)] \in KK_S(E_\gamma, A\oplus\Sigma C')$ is an isomorphism we are done, so let us construct it's inverse.

Similarly to above we associate to the $*$-homomorphisms $\beta' : B' \to C'$ and $\alpha : A \to B$ extensions $E_{\beta'}$ respectively E_α and Kasparov operators as in equation (3.13) $F_{\beta'}$ and F_α. Define the $A\oplus\Sigma C' - E_\gamma$-Hilbert bimodule

$$\begin{aligned}\mathscr{E}_2 :=& \big((A\otimes_A(E_\alpha\otimes_B E_\beta))\oplus(A\otimes_A \mathscr{E}_g\otimes_{B'} E_{\alpha'})\oplus(A\otimes_A(E_\alpha\otimes_B E_B))\big)\bigoplus\\ &\bigoplus\big((\Sigma C'\otimes_{\Sigma C'}\mathscr{K}(\mathscr{E}_h,\Sigma C')\otimes_{\Sigma C} E_\beta)\oplus(\Sigma C'\otimes_{\Sigma C'}\Sigma B')\oplus(\Sigma C'\otimes E_B)\big) =\\ =&\left((A\otimes_A(E_\alpha\otimes_B E_\beta))\oplus(\Sigma C'\otimes_{\Sigma C'}\mathscr{K}(\mathscr{E}_h,\Sigma C')\otimes_{\Sigma C} E_\beta)\right)\bigoplus\\ &\bigoplus\left((A\otimes_A\mathscr{E}_g\otimes_{B'} E_{\alpha'})\oplus(\Sigma C'\otimes_{\Sigma C'}\Sigma B')\right)\bigoplus\\ &\bigoplus\big((A\otimes_A(E_\alpha\otimes_B E_B))\oplus(\Sigma C'\otimes E_B)\big).\end{aligned}$$

Here the first linear decomposition defines the left $A\oplus\Sigma C'$-action and the second the right E_γ-action and the scalar product. Define the Kasparov operator on $\mathscr{E}_2$ in the second decomposition as

$$F_2 := \left(F_\alpha \oplus F_h^*\right) \oplus \left(F_g \oplus F_{\beta'}\right) \oplus \left(F_\alpha \oplus F_B\right).$$

It follows directly from the definitions that

$$\mathscr{E}_1 \otimes_{A\oplus\Sigma C'} \mathscr{E}_2 = E_\gamma \quad \text{and} \quad \mathscr{E}_2 \otimes_{E_\gamma} \mathscr{E}_1 = A\oplus\Sigma C'.$$

It is straightforward to verify that $F_1 \# F_2 = 0$ and $F_2 \# F_1 = 0$ are well defined Kasparov products. So

$$\begin{aligned}&[(\mathscr{E}_1, F_1)] \circ [(\mathscr{E}_2, F_2)] = [(E_\gamma, 0)] = \mathrm{id}_{E_\gamma} \quad \text{and}\\ &[(\mathscr{E}_2, F_2)] \circ [(\mathscr{E}_1, F_1)] = [(A\oplus\Sigma C', 0)] = \mathrm{id}_{A\oplus\Sigma C'}.\end{aligned}$$

We may conclude that $E_\gamma \cong A\oplus\Sigma C'$ in KK_S. □

Theorem 3.6.6. *The category KK_S is triangulated.*

Proof. The triangles in KK_S satisfy Axiom $(TR0)$ by definition. In Lemma 3.6.2 Axiom $(TR2)$ was proved and in Lemma 3.6.3 Axiom $(TR1)$ was shown to hold. By Lemma 3.6.4 KK_S satisfies Axiom $(TR3)$, since every triangle is isomorphic to an admissible extension. So KK_S is a pre-triangulated category. By Lemma 3.6.5 the mapping cone axiom is satisfied in KK_S. □

Theorem 3.6.7 (Baaj-Skandalis duality)**.** *The functor* $\rtimes_r S : C_S^{*,r} \to C_{\hat{S}}^{*,r}$ *induces a triangulated equivalence* $J_S : KK_S \xrightarrow{\sim} KK_{\hat{S}}$.

Proof. We have already defined J_S on objects in Proposition 2.3.7 and on morphisms in Theorem 3.1.8. To prove that J_S is covariant, take $\alpha = [(\mathscr{E}^\alpha, F_\alpha)] \in KK_S(B,C)$ and $\beta = [(\mathscr{E}^\beta, F_\beta)] \in KK_S(A,B)$. There is a natural isomorphism

$$\mathscr{E}^\beta \rtimes_r S \otimes_{B \rtimes_r S} \mathscr{E}^\alpha \rtimes_r S \cong (\mathscr{E}^\beta \otimes_B \mathscr{E}^\alpha) \rtimes_r S$$

so $J_S(\alpha \circ \beta)$ can be represented by $((\mathscr{E}^\beta \otimes_B \mathscr{E}^\alpha) \rtimes_r S, F_\beta \otimes 1 \# F_\alpha \otimes 1)$ as Kasparov module. Thus

$$J_S(\alpha \circ \beta) = [((\mathscr{E}^\beta \otimes_B \mathscr{E}^\alpha) \rtimes_r S, F_\beta \otimes 1 \# F_\alpha \otimes 1)].$$

On the other hand, up to homotopy $F_\beta \otimes 1 \# F_\alpha \otimes 1 = (F_\beta \# F_\alpha) \otimes 1$. So

$$J_S(\alpha) J_S(\beta) = [(\mathscr{E}^\beta \rtimes_r S \otimes_{B \rtimes_r S} \mathscr{E}^\alpha \rtimes_r S, F_\beta \otimes 1 \# F_\alpha \otimes 1)] = J_S(\alpha \circ \beta).$$

To prove that J_S is triangulated, we use Lemma 3.6.3 and the definition of the triangles in KK_S to reduce this statement to that $C(J_S((\alpha)) = J_S C(\alpha)$ for every equivariant $*$-homomorphism α. This statement holds by a straight forward check of the definition of a mapping cone. The functor J_S is an equivalence by Theorem 3.1.8 since $J_{\hat{S}} J_S(A) \sim_M A$ by Theorem 2.3.9 and Proposition 2.3.6. □

Chapter 4

Twists in KK-theory

In this chapter we will show that the triangulated category KK_S is independent of twist. This twist equivalence is used to show that the Baum-Connes property for torsion-free discrete quantum groups is stable under twist. We will also generalize the Pimsner-Voiculescu triangle to actions of duals of connected, compact Lie groups. Using twist invariance of KK_S the generalized Pimsner-Voiculescu triangle also holds for twists of duals of connected, compact Lie groups. The motivation for studying these triangles is that in the analogue of the Baum-Connes conjecture for discrete quantum groups the generalized Pimsner-Voiculescu triangles gives compactly induced simplicial approximation for the types of compact quantum groups arising as duals of twists of coactions of compact, connected Lie groups.

In the last section we will look at a particular type of equivariant Kasparov modules coming from equivariant spectral triples over classical quantum homogeneous spaces. The situation there is similar to that of a Dirac operator on a homogeneous space. We show that in the right setting this type of equivariant spectral triple may be twisted to a new spectral triple in a way that implements the twist equivalence of KK-categories.

4.1 Equivalences of KK-categories

For locally compact groups G the category KK^G is well studied. We also have Baaj-Skandalis duality $KK_S \cong KK_{\hat{S}}$ to describe quantum duals. But here the classical results come to an end. What we will do in this section is to show that for a twist $\mathscr{F}$ there is an equivalence $KK_{S_{\mathscr{F}}} \cong KK_S$. The construction of this equivalence consists of showing that the twisting of a dual coaction induces a functor on KK_S. So if $\mathscr{F}$ only is a cocycle twist this construction does not work.

Theorem 4.1.1 (Twist equivalence)**.** *Let S be a regular, reduced locally compact quantum group and $\mathscr{F}$ a twist of S. Then there is an equivalence of triangulated categories*

$$Q_{\mathscr{F}} : KK_S \cong KK_{S_{\mathscr{F}}}.$$

Proof. Since the Baaj-Skandalis functor $KK_{\hat{S}} \to KK_S$ is an equivalence by Theorem 3.6.7, it is sufficient to define $Q_{\mathscr{F}}$ on objects of the form $A_0 \rtimes_r \hat{S}$ for $A_0 \in KK_{\hat{S}}$ and morphisms of the form $[(\mathscr{E} \rtimes_r \hat{S}, F \otimes 1)]$ for $\hat{S}$-equivariant Kasparov modules $(\mathscr{E}, F)$. Let $A_{\mathscr{F}}$ denote the C^*-algebra A with the coaction $\Delta_A^{\mathscr{F}}$ defined by equation (2.12) and define $Q_{\mathscr{F}}(A) := A_{\mathscr{F}}$. By Proposition 2.4.5 the coaction on $A_{\mathscr{F}}$ is continuous and by Proposition 2.3.7 it is reduced. Therefore $A_{\mathscr{F}}$ is a well defined object in $KK_{S_{\mathscr{F}}}$.

We take two objects $A = A_0 \rtimes_r \hat{S}$, $B = B_0 \rtimes_r \hat{S} \in KK_S$. Then $\alpha \in KK_S(A,B)$ can be represented by an S-equivariant $A-B$-Kasparov module $(\mathscr{E}_0 \rtimes_r \hat{S}, F \otimes 1)$ for an $\hat{S}$-equivariant $A_0 - B_0$-Kasparov module $(\mathscr{E}_0, F)$. We let $\mathscr{E} := \mathscr{E}_0 \rtimes_r \hat{S}$ and by $\pi : A \to \mathscr{L}(\mathscr{E})$ denote the S-equivariant representation that induces an A-module structure on $\mathscr{E}$.

The coaction $\Delta_{\mathscr{E}} : \mathscr{E} \to \mathscr{M}_S(\mathscr{E} \otimes S)$ is defined as

$$\Delta_{\mathscr{E}_0 \rtimes_r \hat{S}}(\xi \otimes b) := (t_\xi \otimes \mathrm{id})\Delta_B(b).$$

We twist this coaction to $\Delta_{\mathscr{E}}^{\mathscr{F}} : \mathscr{E} \to \mathscr{M}_S(\mathscr{E} \otimes S_{\mathscr{F}})$ as

$$\Delta_{\mathscr{E}}^{\mathscr{F}}(\xi \otimes b) := (t_\xi \otimes \mathrm{id})\Delta_B^{\mathscr{F}}(b).$$

Let $\mathscr{E}_{\mathscr{F}}$ denote the Hilbert module $\mathscr{E}$ with the coaction $\Delta_{\mathscr{E}}^{\mathscr{F}}$. By definition $\mathscr{E}_{\mathscr{F}}$ is an $S_{\mathscr{F}}$-equivariant $B_{\mathscr{F}}$-Hilbert module and the S-equivariant representation π induces an $S_{\mathscr{F}}$ -equivariant representation $\pi_{\mathscr{F}} : A_{\mathscr{F}} \to \mathscr{L}(\mathscr{E}_{\mathscr{F}})$. Let $V \in \mathscr{L}(\mathscr{E} \otimes_B (B \otimes S), \mathscr{E} \otimes S)$ denote the unitary defined by $V t_\chi = \Delta_{\mathscr{E}}(\chi)$ for $\chi \in \mathscr{E}$, see more in Proposition 2.3.4. To show that $F \otimes 1$ satisfies the conditions making $(\mathscr{E}_{\mathscr{F}}, F \otimes 1)$ into an equivariant $A_{\mathscr{F}} - B_{\mathscr{F}}$-Kasparov module we observe that the unitary $V_{\mathscr{F}} \in \mathscr{L}(\mathscr{E}_{\mathscr{F}} \otimes_{B_{\mathscr{F}}} (B_{\mathscr{F}} \otimes S_{\mathscr{F}}), \mathscr{E}_{\mathscr{F}} \otimes S_{\mathscr{F}})$ defined by $V_{\mathscr{F}} t_\chi = \Delta_{\mathscr{E}}^{\mathscr{F}}(\chi)$ for $\chi \in \mathscr{E}_{\mathscr{F}}$ is given by $V_{\mathscr{F}} = Ad(\mathrm{id}_{\mathscr{E}_0} \otimes \mathscr{F}) \circ V$. Since $F \otimes 1$ is S-invariant and invariant under $Ad(\mathrm{id}_{\mathscr{E}_0} \otimes \mathscr{F})$ it follows that $(\mathscr{E}_{\mathscr{F}}, F \otimes 1)$ is an $S_{\mathscr{F}}$-equivariant $A_{\mathscr{F}} - B_{\mathscr{F}}$-Kasparov module. Hence, the morphism $Q_{\mathscr{F}}(\alpha) := [(\mathscr{E}_{\mathscr{F}}, F \otimes 1)] \in KK_{S_{\mathscr{F}}}(A_{\mathscr{F}}, B_{\mathscr{F}})$ is well defined.

To show that $Q_{\mathscr{F}}$ is a functor, take KK_S-morphisms $\alpha : A \to B$, $\beta : B \to C$ and represent them by Kasparov modules $(\mathscr{E}^\alpha, F_\alpha)$ and $(\mathscr{E}^\beta, F_\beta)$. We may assume that they are of the form $\mathscr{E}^\alpha = \mathscr{E}_0^\alpha \rtimes_r \hat{S}$, $F_\alpha = F'_\alpha \otimes 1$ for an $A_0 - B_0$-Kasparov module $(\mathscr{E}_0^\alpha, F'_\alpha)$ and similarly for β. Then $\mathscr{E}^\alpha \otimes \mathscr{E}^\beta = (\mathscr{E}_0^\alpha \otimes_{B_0} \mathscr{E}_0^\beta) \rtimes_r \hat{S}$ so

$$\beta \circ \alpha = [((\mathscr{E}_0^\alpha \otimes_{B_0} \mathscr{E}_0^\beta) \rtimes_r \hat{S}, F'_\alpha \otimes 1 \# F'_\beta \otimes 1)].$$

Since $Q_{\mathscr{F}}(\beta \circ \alpha)$ can be represented by $((\mathscr{E}_0^\alpha \otimes_{B_0} \mathscr{E}_0^\beta) \rtimes_r \hat{S}, F'_\alpha \otimes 1 \# F'_\beta \otimes 1)$ as Kasparov module it is clear that

$$Q_{\mathscr{F}}(\beta \circ \alpha) = \left[\left((\mathscr{E}_0^\alpha \otimes_{B_0} \mathscr{E}_0^\beta) \rtimes_r \hat{S} \right)_{\mathscr{F}}, F'_\alpha \otimes 1 \# F'_\beta \otimes 1 \right].$$

But if we construct $F'_\alpha \otimes 1 \# F'_\beta \otimes 1$ as a Kasparov product for $\mathscr{E}^\alpha \otimes_B \mathscr{E}^\beta$ it will also be a Kasparov product for $\mathscr{E}^\alpha_{\mathscr{F}} \otimes_{B_{\mathscr{F}}} \mathscr{E}^\beta_{\mathscr{F}}$ because of $Ad(\mathrm{id}_{\mathscr{E}_0} \otimes \mathscr{F})$-invariance of $F'_\alpha \otimes 1$ and $F'_\beta \otimes 1$. So

$$Q_{\mathscr{F}}(\beta) Q_{\mathscr{F}}(\alpha) = [(\mathscr{E}^\alpha_{\mathscr{F}} \otimes_{B_{\mathscr{F}}} \mathscr{E}^\beta_{\mathscr{F}}), F'_\alpha \otimes 1 \# F'_\beta \otimes 1)] = Q_{\mathscr{F}}(\beta \circ \alpha).$$

As a result, we have constructed a covariant functor $Q_{\mathscr{F}} : KK_S \to KK_{S_{\mathscr{F}}}$ so what remains to be proven is that $Q_{\mathscr{F}}$ is a triangulated equivalence. We consider the functor $Q_{\mathscr{F}^*} : KK_{S_{\mathscr{F}}} \to KK_S$ constructed in the same way but viewing $\mathscr{F}^*$ as a twist of $S_{\mathscr{F}}$ as in equation (2.11). The functors $Q_{\mathscr{F}}$ and $Q_{\mathscr{F}^*}$ satisfies that $Q_{\mathscr{F}^*} Q_{\mathscr{F}}$ and $Q_{\mathscr{F}} Q_{\mathscr{F}^*}$ are naturally isomorphic to the identity functors. That the isomorphisms are natural follows from that the Baaj-Skandalis functor is an equivalence. Since $Q_{\mathscr{F}}$ commutes with mapping cones of $*$-homomorphisms $A_0 \rtimes_r \hat{S} \to B_0 \rtimes_r \hat{S}$ we can conclude that $Q_{\mathscr{F}}$ is triangulated by Lemma 3.6.3. □

Corollary 4.1.2. *There exists a commutative diagram of triangulated categories with arrows being triangulated equivalences*

$$\begin{array}{ccc} KK_S & \xrightarrow{Q_{\mathscr{F}}} & KK_{S_{\mathscr{F}}} \\ \downarrow{\scriptstyle J_S} & & \downarrow{\scriptstyle J_{S_{\mathscr{F}}}} \\ KK_{\hat{S}} & \xrightarrow{\hat{Q}_{\mathscr{F}}} & KK_{\widehat{S_{\mathscr{F}}}} \end{array} \tag{4.1}$$

Furthermore, there is a natural isomorphism $\hat{Q}_{\mathscr{F}}(A) \cong A$ in $KK_{\widehat{S_{\mathscr{F}}}}$ if A has trivial $\hat{S}$-coaction.

Proof. The commutative diagram determines $\hat{Q}_{\mathscr{F}}$ so what needs to be proven is that $\hat{Q}_{\mathscr{F}}(A) = A$ in $KK_{\widehat{S_{\mathscr{F}}}}$ if A has trivial coaction. This is a consequence of TTT-duality since there are natural Morita equivalences

$$\hat{Q}_{\mathscr{F}}(A) \sim_M J_{S_{\mathscr{F}}} Q_{\mathscr{F}} J_{\hat{S}}(A) = A \otimes (S_{\mathscr{F}} \rtimes_r \widehat{S_{\mathscr{F}}}) \sim_M A.$$

□

4.2 Twists and the Baum-Connes property

In this section we study the generalization of the Baum-Connes property to torsion-free, discrete quantum groups defined in [34]. Using that the property of being torsion-free is twist-invariant, together with twist equivalence of KK_S, we are able to show that the Baum-Connes property is independent of taking twists.

Since the Baum-Connes property is a property of triangulated categories and $KK_S \cong KK_{\hat{S}}$ by Baaj-Skandalis duality the Baum-Connes property for S may be formulated as a property in $KK_{\hat{S}}$. This is done just as in [34]. Let $\tau : KK \to KK_{\hat{S}}$ denote the functor that map a C^*-algebra to a $\hat{S} - C^*$-algebra with trivial coaction and $\mathscr{C}\mathscr{I}_S$ it's image. We define $\langle \mathscr{C}\mathscr{I}_S \rangle$ to be the localizing subcategory generated by $\mathscr{C}\mathscr{I}_S$. An object in KK_S which is Baaj-Skandalis dual to an object in $\langle \mathscr{C}\mathscr{I}_S \rangle$ is called compactly induced. Define the localizing subcategory

$$\mathscr{C}\mathscr{C}_S := \{A \in KK_{\hat{S}} : A \rtimes_r \hat{S} \cong 0 \text{ in } KK\}.$$

Theorem 4.2.1 (Theorem 5.4 of [34]). *For any discrete quantum group S the pair $(\langle \mathscr{C}\mathscr{I}_S \rangle, \mathscr{C}\mathscr{C}_S)$ is complementary.*

Now, we let $F_S : KK_{\hat{S}} \to Ab^{\mathbb{Z}/2\mathbb{Z}}$ denote the K-theory functor $A \mapsto K_*(A)$. Let $\mathbb{L}F_S$ denote the derived functor of F_S with respect to $\mathscr{C}\mathscr{C}_S$. Existence of the derived functor $\mathbb{L}F_S$ is proven in [34]. The natural transformation $\mu_S : \mathbb{L}F_S \to F_S$ is called the assembly map.

Definition 4.2.2. *If S is a torsion-free discrete quantum group, then S is said to satisfy the Baum-Connes property if the assembly map is an isomorphism.*

Theorem 4.2.3. *Let S be a torsion-free, discrete quantum group and $\mathscr{F}$ a twist. If S satisfies the Baum-Connes property, then so does $S_{\mathscr{F}}$.*

The Baum-Connes property for the twisted quantum group makes sense, since by Theorem 2.7.3 twists of torsion-free, discrete quantum groups are torsion-free.

Proof. Consider the functor $\tilde{F}(A) := \mathbb{L}F_S(\hat{Q}_{\mathscr{F}^*}(A))$ defined for $A \in KK_{\widehat{S_{\mathscr{F}}}}$. The assembly map μ_S induces a natural transformation $\tilde{\mu} : \tilde{F} \to F_{S_{\mathscr{F}}}$ by the composition

$$\mathbb{L}F_S(\hat{Q}_{\mathscr{F}^*}(A)) \to F_S(\hat{Q}_{\mathscr{F}^*}(A)) \xrightarrow{\sim} F_{S_{\mathscr{F}}}(A),$$

where the last isomorphism comes from the isomorphism $\hat{Q}_{\mathscr{F}^*}(A) \cong A$ in KK. Clearly $\tilde{F}$ satisfies the following properties:

1. $\tilde{F}$ is a homological functor commuting with direct sums.
2. The natural transformation $\tilde{\mu}$ is an isomorphism on objects from $\langle \mathscr{C}\mathscr{I}_{S_{\mathscr{F}}} \rangle$.
3. $\tilde{F}$ vanishes on $\mathscr{C}\mathscr{C}_{S_{\mathscr{F}}}$.

The first property follows from that $\mathbb{L}F$ is homological and commutes with direct sums. The second and the third property is a consequence of that

$$\hat{Q}_{\mathscr{F}^*}(\langle \mathscr{C}\mathscr{I}_{S_{\mathscr{F}}} \rangle) = \langle \mathscr{C}\mathscr{I}_S \rangle \quad \text{and} \quad \hat{Q}_{\mathscr{F}^*}(\mathscr{C}\mathscr{C}_{S_{\mathscr{F}}}) = \mathscr{C}\mathscr{C}_S.$$

By Proposition 5.6 of [34] these three properties for $\tilde{F}$ and $\tilde{\mu}$ implies that $\tilde{F}$ is a left derived functor of $F_{S_{\mathscr{F}}}$ and that $\tilde{\mu}$ coincides with the assembly map for $S_{\mathscr{F}}$. By construction, $\tilde{\mu}$ is an isomorphism if μ_S is. □

4.3 The generalized PV-triangle for duals of compact Lie groups and their twists

We now move on to studying the situation of actions of the dual of a compact, connected Lie group G. In this chapter we will denote $KK^G := KK_{C_0(G)}$ and $KK^{\hat{G}} := KK_{C_r^*(G)}$. First we begin by some motivation coming from the easiest case of G being a torus so $\hat{G} \cong \mathbb{Z}^n$. Let $C_0(\mathbb{R}^n)$ be given a $\mathbb{Z}^n$-action from translations and consider the equivariant evaluation mapping $l : C_0(\mathbb{R}^n) \to C_0(\mathbb{Z}^n)$. It gives an $\mathbb{Z}^n$-equivariant short exact sequence

$$0 \to \Sigma^n C_0(\mathbb{Z}^n) \to C_0(\mathbb{R}^n) \xrightarrow{l} C_0(\mathbb{Z}^n) \to 0.$$

The strong Baum-Connes conjecture for $\mathbb{Z}$ implies that $C_0(\mathbb{R}) \cong \Sigma\mathbb{C}$ in $KK^{\mathbb{Z}}$ via the Dirac morphism. This exact sequence induces an exact triangle in $KK^{\mathbb{Z}^n}$ which after using the isomorphism $C_0(\mathbb{R}) \cong \Sigma\mathbb{C}$ and rotation $3n+1$ steps to the left becomes

$$\begin{array}{ccc} \Sigma^{n-1} C_0(\mathbb{Z}^n) & \longrightarrow & C_0(\mathbb{Z}^n) \\ & {}_{[1]}\nwarrow \quad \swarrow & \\ & \mathbb{C}. & \end{array}$$

Applying the Baaj-Skandalis duality functor we obtain the exact triangle in $KK^{\mathbb{T}^n}$

$$\begin{array}{ccc} \Sigma^{n-1} \mathbb{C} & \longrightarrow & \mathbb{C} \\ & {}_{[1]}\nwarrow \quad \swarrow & \\ & C(\mathbb{T}^n). & \end{array} \tag{4.2}$$

This triangle can be used to construct a Pimsner-Voiculescu triangle for $\mathbb{Z}^n$-actions. We will use this triangle to generalize the Pimsner-Voiculescu triangle to actions of duals of compact, connected Lie groups and twists of them. Let G be a compact, connected Lie group with maximal torus T of rank n. The embedding $T \hookrightarrow G$ induces a triangulated functor $Ind_T^G : KK^T \to KK^G$. Applying the triangulated functor Ind_T^G to the triangle (4.2) we obtain the exact triangle in KK^G

$$\Sigma C(G) \to \Sigma^{n-1} C(T\backslash G) \to C(T\backslash G) \to C(G).$$

To be able to see the importance of this triangle we need a lemma to describe $C(T\backslash G)$ in KK^G. Recall that a compact group G is said to satisfy the Hodgkin condition if G is connected and $\pi_1(G)$ is torsion-free.

Lemma 4.3.1. *Assume that G is a compact Lie group satisfying the Hodgkin condition. We let w denote the Weyl group of G, then*

$$C(T\backslash G) \cong \mathbb{C}^{|w|} \quad in \quad KK^G.$$

Observe that the condition on G is precisely that $\hat{G}$ is torsion-free! The less precise statement $C(T\backslash G) \cong \mathbb{C}^k$ for some k is stated and proved in [35]. We will review a conceptually important part of the proof of Proposition 2.1 in [35] and use a result from [48] showing that $k = |w|$ which proves Lemma 4.3.1.

Proof. Let R_T denote the representation ring of T and R_G that of G. Observe that $R_T \equiv KK^T(\mathbb{C},\mathbb{C})$ and $R_G \equiv KK^G(\mathbb{C},\mathbb{C})$. By [48] it holds that R_T is free of rank $|w|$ over R_G, as soon as $\pi_1(G)$ is torsion-free. If $\mathscr{S}$ denotes the localizing subcategory of KK^G generated by $\mathbb{C}$ and $C(T\backslash G)$ then Lemma 11 of [35] states that given any $A \in \mathscr{S}$ the natural homomorphism

$$R_T \otimes_{R_G} KK^G(A,\mathbb{C}) \to KK^T(Res_T^G(A),\mathbb{C})$$

is an isomorphism. Here $Res_T^G : KK^G \to KK^T$ is the restriction functor. Thus the representable functor on $\mathscr{S}$

$$A \to KK^G(A,\mathbb{C}^{|w|}) \cong R_T \otimes_{R_G} KK^G(A,\mathbb{C})$$

coincides with the representable functor

$$A \to KK^T(Res_T^G(A),\mathbb{C}) \cong KK^G(A, C(T\backslash G)).$$

The last isomorphism is a consequence of the fact that the induction functor Ind_T^G is the right adjoint of the restriction functor Res_T^G. So the Yoneda lemma implies that $C(T\backslash G) \cong \mathbb{C}^{|w|}$ in $\mathscr{S}$ and, since $\mathscr{S}$ is full, also in KK^G. ☐

Corollary 4.3.2. *If G is a compact Lie group of rank n satisfying the Hodgkin condition, there is an exact triangle in KK^G of the form*

$$\begin{array}{ccc} \Sigma^{n-1}\mathbb{C}^{|w|} & \longrightarrow & \mathbb{C}^{|w|} \\ {\scriptstyle [1]}\nwarrow & & \swarrow \\ & C(G). & \end{array} \tag{4.3}$$

We now assume that A has a coaction of $C^*(G)$ and let $A_0 := A \rtimes_r \hat{G}$. Let us consider the tensor product of the exact triangle (4.3) with A_0 and we equip every term with the diagonal G-action. This procedure gives an exact triangle in KK^G

$$\Sigma C_0(G) \otimes A_0 \to \Sigma^{n-1}\mathbb{C}^{|w|} \otimes A_0 \to \mathbb{C}^{|w|} \otimes A_0 \to C(G) \otimes A_0.$$

Again applying Baaj-Skandalis duality functor (see Theorem 3.6.7) we obtain in $KK^{\hat{G}}$

$$\begin{array}{ccc} \Sigma^{n-1}\mathbb{C}^{|w|} \otimes A & \longrightarrow & \mathbb{C}^{|w|} \otimes A \\ {\scriptstyle [1]}\nwarrow & & \swarrow \\ & (C(G) \otimes A_0) \rtimes_r G. & \end{array} \tag{4.4}$$

Given $B \in KK_{\hat{S}}$ we will by $t_S(B) \in KK_S$ denote the C^*-algebra with trivial S-coaction.

Lemma 4.3.3. *Let B be a G-C^*-algebra. Then there exists a $\hat{G}$-equivariant $*$-isomorphism*

$$(C_0(G) \otimes B) \rtimes_r G \cong t_{\hat{G}}(B) \otimes (C_0(G) \rtimes_r G).$$

Proof. We will use the classical definition of a crossed product as a closure of $(C_0(G) \otimes^{alg} B) \otimes^{alg} C_c(G)$ when we construct the isomorphism in the lemma. Let $\alpha : G \to Aut(B)$ denote the G-action on B. We define

$$\varphi : (C_0(G) \otimes B) \rtimes_r G \to B \otimes (C_0(G) \rtimes_r G)$$

on the dense subspace $(C_0(G) \otimes^{alg} B) \otimes^{alg} C_c(G)$ as

$$\varphi((f_1 \otimes b) \otimes f_2)(g_1, g_2) := \alpha_{g_1^{-1}}(b) \otimes (f_1 \otimes f_2)(g_1, g_2).$$

This is easily seen to be a bounded $*$-homomorphism so it extends to a mapping $(C_0(G) \otimes B) \rtimes_r G \to B \otimes (C_0(G) \rtimes_r G)$. It is an isomorphism since an inverse can be defined on the dense subspace $B \otimes^{alg} ((C_0(G) \otimes^{alg} C_c(G))$ by

$$\varphi^{-1}(b \otimes (f_1 \otimes f_2))(g_1, g_2) := f_1(g_1)\alpha_{g_1}(b) f_2(g_2).$$

The equivariance is clear. □

Theorem 4.3.4. *Assume that G is compact Lie group of rank n satisfying the Hodgkin condition with Weyl group w and that $A \in KK^{\hat{G}}$. Then there is an exact triangle in $KK^{\hat{G}}$*

$$\begin{array}{ccc} \Sigma^{n-1}\mathbb{C}^{|w|} \otimes A & \longrightarrow & \mathbb{C}^{|w|} \otimes A \\ & {}_{[1]}\nwarrow \quad \swarrow & \\ & t_{\hat{G}}(A \rtimes_r \hat{G}). & \end{array}$$

Proof. Apply Lemma 4.3.3 to the G-algebra $A \rtimes_r \hat{G}$ the final term in equation (4.4) becomes

$$t_{\hat{G}}(A \rtimes_r \hat{G}) \otimes (C_0(G) \rtimes_r G) = t_{\hat{G}}(A \rtimes_r \hat{G}) \otimes (C_0(G) \rtimes_r G).$$

After an application of Takesaki-Takai duality

$$C_0(G) \rtimes_r G = \mathbb{C} \rtimes_r \hat{G} \rtimes_r G \cong \mathbb{C}$$

the theorem follows. □

The restriction that $\pi_1(G)$ is required to be torsion-free can be loosened. If $\pi_1(G)$ has torsion, some terms from the torsion group of $\pi_1(G)$ arise in the isomorphism in Proposition 4.3.1. We let G^u denote the universal covering Lie group of G. Since $\pi_1(G)$ is central in G^u there is an exact sequence of Lie groups

$$1 \to \pi_1(G) \to G^u \to G \to 1.$$

We define the Lie group $G^f := G^u/\pi_1(G)_{free}$ which will be connected since G is. So if $Z := \pi_1(G)_{tor}$ there is an exact sequence of Lie groups

$$1 \to Z \to G^f \to G \to 1. \tag{4.5}$$

Since Z is discrete and central in G^f, G^f will have the same rank as G. If we let $\tilde{T}$ denote the maximal torus of G^f we will have an isomorphism $C(\tilde{T}\backslash G^f) \cong \mathbb{C}^{|w|}$ in KK^{G^f}.

The quotient mapping $G^f \to G$ induces a triangulated functor $Ind_{\hat{G}}^{\hat{G}^f} : KK^{\hat{G}} \to KK^{\hat{G}^f}$ by

$$Ind_{\hat{G}}^{\hat{G}^f}(A) := (A \rtimes_r \hat{G}) \rtimes_r G^f.$$

The notation $Ind_{\hat{G}}^{\hat{G}^f}$ is inspired by the imprimitivity results of [51].

Theorem 4.3.5. *If G is a compact, connected Lie group of rank n and $A \in KK^{\hat{G}}$ there is an exact triangle in $KK^{\hat{G}^f}$*

$$\begin{array}{ccc} \Sigma^{n-1}\mathbb{C}^{|w|} \otimes Ind_{\hat{G}}^{\hat{G}^f}(A) & \longrightarrow & \mathbb{C}^{|w|} \otimes Ind_{\hat{G}}^{\hat{G}^f}(A) \\ {\scriptstyle [1]}\nwarrow & & \swarrow \\ & t_{\hat{G}^f}(A \rtimes_r \hat{G}). & \end{array}$$

Proof. Applying Theorem 4.3.4 to the $\hat{G}^f$-algebra $Ind_{\hat{G}}^{\hat{G}^f}(A)$ we obtain the exact triangle

$$\begin{array}{ccc} \Sigma^{n-1}\mathbb{C}^{|w|} \otimes Ind_{\hat{G}}^{\hat{G}^f}(A) & \longrightarrow & \mathbb{C}^{|w|} \otimes Ind_{\hat{G}}^{\hat{G}^f}(A) \\ {\scriptstyle [1]}\nwarrow & & \swarrow \\ & t_{\hat{G}^f}(Ind_{\hat{G}}^{\hat{G}^f}(A) \rtimes_r \hat{G}^f). & \end{array}$$

Baaj-Skandalis duality implies that $t_{\hat{G}^f}(Ind_{\hat{G}}^{\hat{G}^f}(A) \rtimes_r \hat{G}^f) \cong t_{\hat{G}^f}(A \rtimes_r \hat{G})$ in $KK^{\hat{G}^f}$. □

All these concepts behave well when twisting. Although we need to assume that the twist comes from a twist of the covering G^f. If $\tilde{\mathscr{F}}$ is a twist of $C^*(G^f)$ the universal map $C^*(G^f) \to C^*(G)$ induces a twist $\mathscr{F}$ of $C^*(G)$. The subgroup $Z \subseteq G^f$ is central, therefore the mapping $C^*(G^f)_{\tilde{\mathscr{F}}} \to C^*(G)_{\mathscr{F}}$ is a morphism of quantum groups. Similarly to the classical setting we can define the triangulated functor

$$Ind_{\hat{G}_{\mathscr{F}}}^{\hat{G}^f_{\tilde{\mathscr{F}}}} : KK^{\hat{G}_{\mathscr{F}}} \to KK^{\hat{G}^f_{\tilde{\mathscr{F}}}}, \quad \text{as} \quad Ind_{\hat{G}_{\mathscr{F}}}^{\hat{G}^f_{\tilde{\mathscr{F}}}}(A) := (A \rtimes_r \hat{G}_{\mathscr{F}}) \rtimes_r G^f_{\tilde{\mathscr{F}}}.$$

Corollary 4.3.6. *Assume that G is a compact, connected Lie group of rank n and that $\tilde{\mathscr{F}}$ is a twist of $C^*(G^f)$. For $A \in KK^{\hat{G}_{\mathscr{F}}}$ there is an exact triangle in $KK^{\hat{G}^f_{\tilde{\mathscr{F}}}}$*

$$\begin{array}{ccc} \Sigma^{n-1}\mathbb{C}^{|w|} \otimes Ind_{\hat{G}_{\mathscr{F}}}^{\hat{G}^f_{\tilde{\mathscr{F}}}}(A) & \longrightarrow & \mathbb{C}^{|w|} \otimes Ind_{\hat{G}_{\mathscr{F}}}^{\hat{G}^f_{\tilde{\mathscr{F}}}}(A) \\ {\scriptstyle [1]}\nwarrow & & \swarrow \\ & Q_{\tilde{\mathscr{F}}}(\mathbb{C}) \otimes t_{\hat{G}^f_{\tilde{\mathscr{F}}}}(A \rtimes_r \hat{G}_{\mathscr{F}}). & \end{array} \tag{4.6}$$

Proof. By a similar reasoning as in the proof of Theorem 4.3.5 the functor $Ind_{\hat{G}_{\mathscr{F}}}^{\hat{G}^f_{\tilde{\mathscr{F}}}}$ allows us to assume that $G = G^f$. By Theorem 4.1.1 there is a triangulated equivalence $Q_{\mathscr{F}} : KK^{\hat{G}} \to KK^{\hat{G}_{\mathscr{F}}}$, since $C^*(G)$ is regular. Thus it is sufficient to prove the statement for $A = Q_{\mathscr{F}}(B)$ for $B \in KK^{\hat{G}}$. Applying the functor $Q_{\mathscr{F}}$ to the exact triangle for B given by Theorem 4.3.4 we obtain the exact triangle in $KK^{\hat{G}_{\mathscr{F}}}$

$$\begin{array}{ccc} \Sigma^{n-1}\mathbb{C}^{|w|} \otimes A & \longrightarrow & \mathbb{C}^{|w|} \otimes A \\ & {\scriptstyle [1]}\nwarrow \quad \swarrow & \\ & Q_{\mathscr{F}} t_{\hat{G}}(B \rtimes_r \hat{G}). & \end{array}$$

View $\mathscr{F}^*$ as a twist of $C^*(G_{\mathscr{F}})$ which is possible by the remarks after Theorem 2.4.4. By Corollary 4.1.2 we have the identity $\hat{Q}_{\mathscr{F}^*} J_{\hat{G}_{\mathscr{F}}} = J_{\hat{G}} Q_{\mathscr{F}^*}$. So

$$B \rtimes_r \hat{G} = J_{\hat{G}} Q_{\mathscr{F}^*}(A) = \hat{Q}_{\mathscr{F}^*}(A \rtimes_r \hat{G}_{\mathscr{F}}).$$

Therefore

$$Q_{\mathscr{F}}(t_{\hat{G}}(B \rtimes_r \hat{G})) \cong Q_{\mathscr{F}}(\mathbb{C}) \otimes t_{\hat{G}_{\mathscr{F}}}(A \rtimes_r \hat{G}_{\mathscr{F}}).$$

□

Observe that $Q_{\mathscr{F}}(\mathbb{C}) \cong \mathbb{C}$ in KK. So a direct corollary of Corollary 4.3.6 in the simplest case that $\hat{G}$ is torsion-free follows from an application of the forgetful functor $t_{G_{\mathscr{F}}} : KK^{\hat{G}_{\mathscr{F}}} \to KK$ to the triangle (4.6) giving a triangle

$$\begin{array}{ccc} \Sigma^{n-1}\mathbb{C}^{|w|} \otimes A & \longrightarrow & \mathbb{C}^{|w|} \otimes A \qquad \text{in} \quad KK. \\ & {\scriptstyle [1]}\nwarrow \quad \swarrow & \\ & A \rtimes_r \hat{G}_{\mathscr{F}}. & \end{array}$$

Corollary 4.3.7. *If $\hat{G}$ is torsion-free and $\mathscr{F}$ is a twist of $\hat{G}$ then for every $A \in KK^{\hat{G}_{\mathscr{F}}}$ and $D \in KK$ there are six-term exact sequences in $Ab^{\mathbb{Z}_2}$:*

$$\begin{array}{ccc} KK^{*+n-1}(\mathbb{C}^{|w|} \otimes A, D) & \longrightarrow & KK^*(\mathbb{C}^{|w|} \otimes A, D) \qquad \textit{and} \\ & {\scriptstyle [1]}\nwarrow \quad \swarrow & \\ & KK^*(A \rtimes_r \hat{G}_{\mathscr{F}}, D) & \end{array}$$

$$\begin{array}{ccc} KK^{*+n-1}(D, \mathbb{C}^{|w|} \otimes A) & \longrightarrow & KK^*(D, \mathbb{C}^{|w|} \otimes A) \\ & {\scriptstyle [1]}\nwarrow \quad \swarrow & \\ & KK^*(D, A \rtimes_r \hat{G}_{\mathscr{F}}) & \end{array}$$

4.4 Spectral triples on quantum homogeneous spaces

Our aim in this section is to describe how the twist equivalence of Theorem 4.1.1 acts on the finer level of spectral triples for a special class of quantum homogenous spaces. Our methods will however not work for cocycle twists, since we can not describe the left regular representation of a cocycle twist as in Theorem 2.4.3 for twists. Twists of the Dirac operator on a compact, simply connected Lie group has previously been studied in [41]. They worked on a very algebraic level in the algebraic part of the discrete dual.

We will study the special class of classical quantum homogeneous spaces satisfying some technical conditions which is always satisfied for amenable and coamenable quantum groups. The motivation to study that special class of homogeneous spaces and their spectral triples is their resemblance with Dirac operators on a homogeneous space of the form $K\backslash G$ for K a maximal compact subgroup of the connected Lie group G. Since $K\backslash G$ is a classifying space for proper actions of G, this is of great importance in the study of the Baum-Connes conjecture, see more in [6].

Definition 4.4.1. *A quantum homogeneous S-space is a closed right $*$-coideal $\mathscr{Q} \subseteq \mathscr{M}(S)$.*

That is $\mathscr{Q}$ is a closed $*$-subalgebra such that $\Delta(\mathscr{Q}) \subseteq \mathscr{M}_S(\mathscr{Q} \otimes S)$. If $\mathscr{Q}$ satisfies that $\mathscr{Q} \subseteq S$ we say that $\mathscr{Q}$ is a proper quantum homogeneous S-space. The classical setting for this is a subgroup $H \subseteq G$ and $\mathscr{Q} = C_0(H\backslash G) \subseteq C_b(G)$. Here the embedding comes from the quotient mapping $G \to H\backslash G$. Another way of looking at this is by considering the mapping $\pi : C_0(G) \to C_0(H)$ and defining

$$C_0(H\backslash G) := \{x \in C_b(G) : \pi \otimes \mathrm{id}(\Delta(x)) = 1 \otimes x\}.$$

Suppose that R is a quantum group and $\pi : S \to \mathscr{M}(R)$ is a morphism of bi-C^*-algebras. Define

$$\mathscr{Q} := \{x \in \mathscr{M}(S) : \pi \otimes \mathrm{id}(\Delta(x)) = 1 \otimes x\}.$$

The C^*-algebra $\mathscr{Q}$ is a right $*$-coideal in $\mathscr{M}(S)$ so it is a quantum homogeneous S-space. In [36] a quantum homogeneous S-space of this type is called a classical quantum homogeneous S-space with stabilizer R.

When constructing Dirac operators on homogeneous spaces, the natural space for the Dirac operator to act on is the L^2-space of sections on a suitable vector bundle. Let G be a connected Lie group and H a compact Lie subgroup. Given a finite-dimensional, unitary representation $H \to U(V)$ this bundle is defined as $V \times_H G$ over $H\backslash G$. The space $L^2(H\backslash G, V)$, of L^2-sections, has a dense subspace of compactly supported, smooth functions $f : G \to V$ such that

$$f(gh) = h^{-1}f(g) \quad \text{for} \quad h \in H, g \in G.$$

We will model our study of spectral triples on homogeneous spaces on this setting. It will turn out to be useful to have the representation V since it provides the missing piece when we are twisting the quantum space. Returning to the general setting, let $\mathscr{Q}$ be a classical quantum homogeneous S-space with stabilizer R. We will construct an analogue of $L^2(H\backslash G, V)$ in the case that V is a graded Hilbert space with a graded representation $\hat{\lambda}_V : \hat{S} \to \mathscr{B}(V)$ which we assume to be a summand of the left regular representation, so it has a left S-coaction. Note that the graded Hilbert space $H_S \otimes V$ has a left R-coaction induced by $\pi : S \to R$ and the diagonal coaction of S. Let Δ_V^R denote this coaction. Define

$$\mathscr{Q}^V := \{z \in H_S \otimes V : \Delta_V^R(z) = z \otimes 1\}$$

The Hilbert space $\mathscr{Q}^V$ is given the grading induced from that on V. There is a graded representation $\mathscr{Q} \to \mathscr{B}(\mathscr{Q}^V)$ given by $q \mapsto \lambda(q) \otimes 1$. We equip $\mathscr{Q}^V$ with the right coaction of S given by $\Delta_{\mathscr{Q}^V} := \Delta_{H_S} \otimes \mathrm{id}_V$.

Proposition 4.4.2. *If $\mathscr{Q}$ is a classical quantum homogeneous S space with stabilizer R and V a summand of the left regular representation of $\hat{S}$, the above construction gives the structure on $\mathscr{Q}^V$ of a graded S-equivariant $\mathscr{Q} - \mathbb{C}$-Hilbert module. Furthermore, the left regular representation of $\hat{S}$ induces graded representations*

$$\tilde{\lambda}^V := (\hat{\lambda} \otimes \hat{\lambda}_V) \circ \hat{\Delta} : \hat{S} \to \mathscr{B}(\mathscr{Q}^V) \quad \textit{and}$$
$$\tilde{\lambda} := \hat{\lambda} \otimes 1 : \hat{S} \to \mathscr{B}(\mathscr{Q}^V).$$

Observe that the two representations $\tilde{\lambda}^V$ and $\tilde{\lambda}$ are unitarily equivalent. This fact follows from that $(\hat{\lambda} \otimes \hat{\lambda}) \circ \Delta(a) = \hat{W}^*(1 \otimes \hat{\lambda}(a))\hat{W}$ so let $P_V : H_S \to V$ denote the orthogonal projection the operator

$$T^V := (1 \otimes P_V)\hat{W}\sigma(1 \otimes P_V)$$

satisfies $\tilde{\lambda} = Ad(T^V) \circ \tilde{\lambda}^V$ so T^V is a unitary intertwiner of the two representations.

In the example of $S = C_0(G)$ and $R = C_0(K)$, where $K \subseteq G$ is a compact Lie subgroup, this corresponds to V having a unitary G-action. In this case $\mathscr{Q}^V \cong L^2(K\backslash G, V)$ as a G-equivariant $C_0(K\backslash G) - \mathbb{C}$-Hilbert module. So it seems somewhat strange to require an $\hat{S}$-action on V, but the reason for this condition is to be able to twist this homogeneous Hilbert bundle. The twist needs two representations to act on, so V needs to have an $\hat{S}$-action. Before we start to twist, we recall a proposition from [27]:

Proposition 4.4.3 (Proposition 5.2 from [27]). *Let C be a C^*-algebra and $u \in \mathcal{M}(S \otimes C)$ be a unitary satisfying $(\Delta \otimes \mathrm{id})(u) = u_{13}u_{23}$. Then there exists a unique $\mu : \hat{S}^u \to \mathcal{M}(C)$ such that $(\mathrm{id} \otimes \mu)(\hat{\mathcal{W}}) = u$ where $\hat{\mathcal{W}}$ is the universal left regular corepresentation of S.*

Here $\hat{S}^u$ denotes the universal dual of S. Letting $u = (\pi \otimes \mathrm{id})(W) \in \mathcal{M}(R \otimes \hat{S})$ we obtain a mapping

$$\hat{\pi}^u : \hat{R}^u \to \mathcal{M}(\hat{S}).$$

The mapping $\hat{\pi}^u$ is a morphism of bi-C^*-algebras, since $(\mathrm{id} \otimes \mu)(\hat{\mathcal{W}}_R) = u$.

Definition 4.4.4. *A classical quantum homogeneous S-space with stabilizer R is called a reduced S/R-space if $\hat{\pi}^u$ factors through a mapping $\hat{\pi} : \hat{R} \to \mathcal{M}(\hat{S})$. A cocycle twist $\mathcal{F}$ of $\hat{S}$ is said to be $\hat{R}$-invariant if*

$$[\mathcal{F}, \hat{\pi}(\hat{R}) \otimes \hat{\pi}(\hat{R})] = 0.$$

If $\mathcal{F}$ is a $\hat{R}$-invariant, it holds that $\Delta_{\mathcal{F}} \circ \hat{\pi} = (\hat{\pi} \otimes \hat{\pi}) \circ \Delta$, therefore the mapping $\hat{\pi}_{\mathcal{F}} : \hat{R} \to \mathcal{M}(\hat{S}_{\mathcal{F}}) \equiv \mathcal{M}(\hat{S})$ is a morphism of bi-C^*-algebras. Consider the unitary $u' := (\hat{\pi}_{\mathcal{F}} \otimes \mathrm{id})(\hat{W}_R) \in \mathcal{M}(\hat{S}_{\mathcal{F}} \otimes R)$. If $\mathcal{F}$ is a twist, Proposition 4.4.3 implies that there exists a unique morphism of bi-C^*-algebras

$$\tilde{\pi}_{\mathcal{F}} : S^u_{\hat{\mathcal{F}}} \to \mathcal{M}(R).$$

We say that a reduced S/R-space $\mathcal{Q}$ is $\mathcal{F}$-admissible if $\tilde{\pi}_{\mathcal{F}}$ factors through a mapping $\pi : S_{\hat{\mathcal{F}}} \to \mathcal{M}(R)$. So if $\mathcal{Q}$ is $\mathcal{F}$-admissible we may define the reduced $S_{\hat{\mathcal{F}}}/R$-space

$$\mathcal{Q}_{\hat{\mathcal{F}}} := \{x \in \mathcal{M}(S_{\hat{\mathcal{F}}}) : \mathrm{id} \otimes \pi_{\mathcal{F}}(\Delta_{\hat{\mathcal{F}}}(x)) = x \otimes 1\}.$$

Proposition 4.4.5. *The even operator $U := \hat{\lambda} \otimes \hat{\lambda}_V(\mathcal{F}) \in \mathcal{B}(\mathcal{Q}^V, \mathcal{Q}^V_{\hat{\mathcal{F}}})$ is a well defined unitary intertwining the representations $\tilde{\lambda}^V$ and $\tilde{\lambda}^V_{\mathcal{F}}$. Furthermore the unitary $\tilde{U} := T^V_{\mathcal{F}} U T^{V*}$ intertwines the two representations $\tilde{\lambda}$ and $\tilde{\lambda}_{\mathcal{F}}$ of $\hat{S}$.*

Proof. That U is well defined follows from that $\mathcal{F}$ is $\hat{R}$-invariant and it is unitary since $\mathcal{F}$ is. The representation $\tilde{\lambda}^V$ is given by $\hat{\lambda} \otimes \hat{\lambda}_V \circ \hat{\Delta}$ and the representation $\tilde{\lambda}^V_{\mathcal{F}}$ is given by $\hat{\lambda} \otimes \hat{\lambda}_V \circ Ad\mathcal{F} \circ \hat{\Delta}$. It follows that U intertwines the two representations $\tilde{\lambda}^V$ and $\tilde{\lambda}^V_{\mathcal{F}}$.

That $\tilde{U}$ intertwines $\tilde{\lambda}$ and $\tilde{\lambda}_{\mathcal{F}}$ follows from that

$$\tilde{\lambda}_{\mathcal{F}} = Ad(T^V_{\mathcal{F}}) \circ \tilde{\lambda}^V_{\mathcal{F}} = Ad(T^V_{\mathcal{F}} U) \circ \tilde{\lambda}^V = Ad(T^V_{\mathcal{F}} U T^{V*}) \circ \tilde{\lambda}.$$

□

Proposition 4.4.6. *Let $\mathscr{F}$ be a twist of $\hat{S}$ and $\mathscr{Q}$ an $\mathscr{F}$-admissible S/R-space. If R is a closed quantum subgroup of S there is a natural isomorphism $\hat{Q}_{\mathscr{F}}(\mathscr{Q}) \cong \mathscr{Q}_{\hat{\mathscr{F}}}$ in $KK_{S_{\hat{\mathscr{F}}}}$.*

Proof. Using the imprimitivity results Theorem 6.2 and Corollary 6.4 of [51] we have that

$$\hat{Q}_{\mathscr{F}}(\mathscr{Q}) = (\mathscr{Q} \rtimes_r S)_{\mathscr{F}} \rtimes_r \hat{S}_{\mathscr{F}} \sim_M \hat{R} \rtimes_r \hat{S}_{\mathscr{F}} \sim_M \mathscr{Q}_{\hat{\mathscr{F}}}.$$

□

Corollary 4.4.7. *Let $\mathscr{F}$ be a twist of the dual of a compact, connected Lie group G which we assume to satisfy the Hodgkin condition. Assume that $\mathscr{F}$ is $C^*(T)$-invariant and let $C(T\backslash G_{\mathscr{F}})$ denote the corresponding twisted homogeneous space and w the Weyl group of G. Then there is an isomorphism*

$$C(T\backslash G_{\mathscr{F}}) \cong \mathbb{C}^{|w|} \quad in \quad KK^{G_{\mathscr{F}}}.$$

See also Proposition 6.8 of [42] where this was proved for Drinfeld-Jimbo twists of $SU(2)$. A similar result would be expected to hold also for cocycle twists of duals of Hodgkin groups. However, the fact that the twist equivalence on KK-level produces *quasi*-coactions makes cocycle twists hard to deal with.

Proof. By Proposition 4.3.1 there is an isomorphism $C(T\backslash G) \cong \mathbb{C}^{|w|}$. Applying the functor $\hat{Q}_{\mathscr{F}}$ to both sides and using the natural isomorphism $\hat{Q}_{\mathscr{F}}(C(T\backslash G)) \cong C(T\backslash G)_{\hat{\mathscr{F}}} \equiv C(T\backslash G_{\mathscr{F}})$ from Proposition 4.4.6 the corollary follows. □

We will look at spectral triples over a particular type of dense subalgebra $\mathscr{Q}^{\mathfrak{U}} \subseteq \mathscr{Q}$. Suppose that $\mathfrak{U} \subseteq \mathscr{B}(H_S)_*$ is a weak* dense subset and that $\mathscr{Q}^{\mathfrak{U}}$ is a dense $*$-subalgebra of $\mathscr{Q}$ generated by

$$\{\mathrm{id} \otimes \omega(W) : \omega \in \mathfrak{U}\} \cap \mathscr{Q}.$$

Then we may define the twisted dense $*$-subalgebra $\mathscr{Q}^{\mathfrak{U}}_{\hat{\mathscr{F}}} \subseteq \mathscr{Q}_{\hat{\mathscr{F}}}$ generated by

$$\{\mathrm{id} \otimes \omega(W_{\hat{\mathscr{F}}}) : \omega \in \mathfrak{U}\} \cap \mathscr{Q}_{\hat{\mathscr{F}}},$$

where $W_{\hat{\mathscr{F}}}$ is the left regular corepresentation of $S_{\hat{\mathscr{F}}}$.

Theorem 4.4.8. *Let $\mathscr{F}$ be a twist, $\mathscr{Q}$ an $\mathscr{F}$-admissible reduced S/R-space and $(\mathscr{Q}^{\mathfrak{U}}, \mathscr{Q}^V, D)$ an S-equivariant spectral triple. If we define $D_{\mathscr{F}} := UDU^*$, where U is defined as in Proposition 4.4.5, the triple $(\mathscr{Q}^{\mathfrak{U}}_{\hat{\mathscr{F}}}, \mathscr{Q}^V_{\hat{\mathscr{F}}}, D_{\mathscr{F}})$ is an $S_{\hat{\mathscr{F}}}$-equivariant spectral triple of the same parity and dimension as $(\mathscr{Q}^{\mathfrak{U}}, \mathscr{Q}^V, D)$ if*

$$[D, \hat{J} \otimes 1] = 0 \quad and \quad [D, U] \quad is\ bounded.$$

Proof. Using Theorem 2.4.3 it follows that

$$\hat{W}_{\mathscr{F}} = (J_{\mathscr{F}} \otimes \hat{J})\mathscr{F}\hat{W}^*(J \otimes \hat{J})\mathscr{F}^*.$$

Define $\Lambda := \hat{\lambda} \otimes \hat{\lambda}_V$. We have that

$$D_{\mathscr{F},23}\hat{W}_{\mathscr{F},12} = D_{\mathscr{F},23}((J_{\mathscr{F}} \otimes \hat{J})\mathscr{F}\hat{W}^*(J \otimes \hat{J})\mathscr{F}^*)_{12} =$$

$$= (J_{\mathscr{F}} \otimes \hat{J})_{12}\left((\hat{\rho} \otimes \hat{\lambda}_V)(\mathscr{F})D(\hat{\rho} \otimes \hat{\lambda}_V)(\mathscr{F}^*)\right)_{23}\left(\mathscr{F}\hat{W}^*(J \otimes \hat{J})\mathscr{F}^*\right)_{12} =$$

$$= (J_{\mathscr{F}} \otimes \hat{J})_{12}(\hat{\rho} \otimes \hat{\lambda}_V)(\mathscr{F})_{23}[\mathscr{F}_{12}, D_{23}](\hat{\rho} \otimes \hat{\lambda}_V)(\mathscr{F}^*)_{23}\hat{W}^*_{12}\left((J \otimes \hat{J})\mathscr{F}^*\right)_{12} -$$

$$- \left((J_{\mathscr{F}} \otimes \hat{J})\mathscr{F}\right)_{12}\left((\hat{\rho} \otimes \hat{\lambda}_V)(\mathscr{F})D(\hat{\rho} \otimes \hat{\lambda}_V)(\mathscr{F}^*)\right)_{23}\hat{W}^*_{12}\left((J \otimes \hat{J})\mathscr{F}^*\right)_{12} =$$

$$= (J_{\mathscr{F}} \otimes \hat{J})_{12}(\hat{\rho} \otimes \hat{\lambda}_V)(\mathscr{F})_{23}[\mathscr{F}_{12}, D_{23}](\hat{\rho} \otimes \hat{\lambda}_V)(\mathscr{F}^*)_{23}\hat{W}^*_{12}\left((J \otimes \hat{J})\mathscr{F}^*\right)_{12} -$$

$$- \left((J_{\mathscr{F}} \otimes \hat{J})\mathscr{F}\right)_{12}\left((\hat{\rho} \otimes \hat{\lambda}_V)(\mathscr{F})D\right)_{23}\hat{W}^*_{12}(J \otimes \hat{J})_{12}(\mathrm{id} \otimes \Lambda)\left(\hat{\Delta} \otimes \mathrm{id}(\mathscr{F}^*)\mathscr{F}^*_{12}\right) =$$

$$= (J_{\mathscr{F}} \otimes \hat{J})_{12}(\hat{\rho} \otimes \hat{\lambda}_V)(\mathscr{F})_{23}[\mathscr{F}_{12}, D_{23}](\hat{\rho} \otimes \hat{\lambda}_V)(\mathscr{F}^*)_{23}\hat{W}^*_{12}\left((J \otimes \hat{J})\mathscr{F}^*\right)_{12} -$$

$$- \left((J_{\mathscr{F}} \otimes \hat{J})\mathscr{F}\right)_{12}(\hat{\rho} \otimes \hat{\lambda}_V)(\mathscr{F})_{23}[D_{23}, \hat{W}^*_{12}](J \otimes \hat{J})_{12}(\mathrm{id} \otimes \Lambda)\left(\hat{\Delta} \otimes \mathrm{id}(\mathscr{F}^*)\mathscr{F}^*_{12}\right) +$$

$$+ \left((J_{\mathscr{F}} \otimes \hat{J})\mathscr{F}\right)_{12}(\hat{\rho} \otimes \hat{\lambda}_V)(\mathscr{F})_{23}\hat{W}^*_{12}(J \otimes \hat{J})_{12}D_{23}\left((\mathrm{id} \otimes \Lambda)(\mathrm{id} \otimes \hat{\Delta})(\mathscr{F}^*)\right)U^*_{23} =$$

$$= (J_{\mathscr{F}} \otimes \hat{J})_{12}(\hat{\rho} \otimes \hat{\lambda}_V)(\mathscr{F})_{23}[\mathscr{F}_{12}, D_{23}](\hat{\rho} \otimes \hat{\lambda}_V)(\mathscr{F}^*)_{23}\hat{W}^*_{12}\left((J \otimes \hat{J})\mathscr{F}^*\right)_{12} -$$

$$- \left((J_{\mathscr{F}} \otimes \hat{J})\mathscr{F}\right)_{12}(\hat{\rho} \otimes \hat{\lambda}_V)(\mathscr{F})_{23}[D_{23}, \hat{W}^*_{12}](J \otimes \hat{J})_{12}(\mathrm{id} \otimes \Lambda)\left(\hat{\Delta} \otimes \mathrm{id}(\mathscr{F}^*)\mathscr{F}^*_{12}\right) +$$

$$+ \left((J_{\mathscr{F}} \otimes \hat{J})\mathscr{F}\right)_{12}(\hat{\rho} \otimes \hat{\lambda}_V)(\mathscr{F})_{23}\hat{W}^*_{12}(J \otimes \hat{J})_{12}\left[D_{23}, \left((\mathrm{id} \otimes \Lambda)(\mathrm{id} \otimes \hat{\Delta})(\mathscr{F}^*)\right)\right]U^*_{23} -$$

$$- \left((J_{\mathscr{F}} \otimes \hat{J})\mathscr{F}\right)_{12}\hat{W}^*_{12}(J \otimes \hat{J})_{12}\left((\mathrm{id} \otimes \Lambda)\left((\hat{\Delta} \otimes \mathrm{id})(\mathscr{F})(\mathrm{id} \otimes \hat{\Delta})(\mathscr{F}^*)\right)\right)D_{23}U^*_{23} =$$

$$= (J_{\mathscr{F}} \otimes \hat{J})_{12}(\hat{\rho} \otimes \hat{\lambda}_V)(\mathscr{F})_{23}[\mathscr{F}_{12}, D_{23}](\hat{\rho} \otimes \hat{\lambda}_V)(\mathscr{F}^*)_{23}\hat{W}^*_{12}\left((J \otimes \hat{J})\mathscr{F}^*\right)_{12} -$$

$$- \left((J_{\mathscr{F}} \otimes \hat{J})\mathscr{F}\right)_{12}(\hat{\rho} \otimes \hat{\lambda}_V)(\mathscr{F})_{23}[D_{23}, \hat{W}^*_{12}](J \otimes \hat{J})_{12}(\mathrm{id} \otimes \Lambda)\left(\hat{\Delta} \otimes \mathrm{id}(\mathscr{F}^*)\mathscr{F}^*_{12}\right) +$$

$$+ \left((J_{\mathscr{F}} \otimes \hat{J})\mathscr{F}\right)_{12}(\hat{\rho} \otimes \hat{\lambda}_V)(\mathscr{F})_{23}\hat{W}^*_{12}(J \otimes \hat{J})_{12}\left[D_{23}, \left((\mathrm{id} \otimes \Lambda)(\mathrm{id} \otimes \hat{\Delta})(\mathscr{F}^*)\right)\right]U^*_{23} -$$

$$-W_{\mathscr{F},12}D_{\mathscr{F},23}.$$

Choose an $\omega \in \mathfrak{U}$ and consider the generic element

$$a_\omega = (\omega \otimes \mathrm{id})(\hat{W}_{\mathscr{F}}) \in \mathscr{Q}^{\mathfrak{U}}_{\mathscr{F}}.$$

The commutator $[D_{\mathscr{F}}, \lambda_{\hat{\mathscr{F}}}(a_\omega) \otimes 1]$ may be written as

$$\omega \otimes \mathrm{id} \otimes \mathrm{id}([D_{\mathscr{F},23}, \hat{W}_{\mathscr{F},12}]) =$$

$$= \omega \otimes \mathrm{id} \otimes \mathrm{id}\Big((J_{\mathscr{F}} \otimes \hat{J})_{12}(\hat{\rho} \otimes \hat{\lambda}_V)(\mathscr{F})_{23}[\mathscr{F}_{12}, D_{23}](\hat{\rho} \otimes \hat{\lambda}_V)(\mathscr{F}^*)_{23}\hat{W}^*_{12}\left((J \otimes \hat{J})\mathscr{F}^*\right)_{12} -$$

$$- \big((J_{\mathscr{F}} \otimes \hat{J})\mathscr{F}\big)_{12}(\hat{\rho} \otimes \hat{\lambda}_V)(\mathscr{F})_{23}[D_{23}, \hat{W}^*_{12}](J \otimes \hat{J})_{12}(\mathrm{id} \otimes \Lambda)\left(\hat{\Delta} \otimes \mathrm{id}(\mathscr{F}^*)\mathscr{F}^*_{12}\right) +$$

$$+ \big((J_{\mathscr{F}} \otimes \hat{J})\mathscr{F}\big)_{12}(\hat{\rho} \otimes \hat{\lambda}_V)(\mathscr{F})_{23}\hat{W}^*_{12}(J \otimes \hat{J})_{12}\left[D_{23}, \big((\mathrm{id} \otimes \Lambda)(\mathrm{id} \otimes \hat{\Delta})(\mathscr{F}^*)\big)\right] U^*_{23}\Big).$$

Since $[D_{23}, \hat{W}^*_{12}]$ and $[D_{23}, W_{12}]$ are bounded, because $(\mathcal{Q}^{\mathfrak{U}}, \mathcal{Q}^V, D)$ is an equivariant spectral triple, it follows that $[D_{\mathscr{F}}, \lambda_{\hat{\mathscr{F}}}(a_\omega) \otimes 1]$ is bounded. The spectral properties of $D_{\mathscr{F}}$ follows from those of D, thus $(\mathcal{Q}^{\mathfrak{U}}_{\hat{\mathscr{F}}}, \mathcal{Q}^V_{\hat{\mathscr{F}}}, D_{\mathscr{F}})$ is a spectral triple.

To show $S_{\hat{\mathscr{F}}}$-equivariance of this spectral triple, it is sufficient to show that $[D_{\mathscr{F},12}, \hat{W}_{\mathscr{F},13}]$ extends to a bounded operator. Thus the equivariance property follows from an analogous calculation to that above showing that $[D_{\mathscr{F},12}, \hat{W}_{\mathscr{F},13}]$ is bounded because of S-equivariance of $(\mathcal{Q}^V, D)$. The statements about dimension and parity is clear since U is an even unitary. □

Corollary 4.4.9. *Under the assumption of Theorem 4.4.8 and if R is a closed quantum subgroup of S we have the equality:*

$$\hat{Q}_{\mathscr{F}}[(\mathcal{Q}^V, D)] = [(\mathcal{Q}^V_{\hat{\mathscr{F}}}, D_{\mathscr{F}})] \quad in \quad K^0_{S_{\hat{\mathscr{F}}}}(\mathcal{Q}_{\hat{\mathscr{F}}}).$$

Proof. Since $[D, U]$ is a bounded operator, it follows that $F_D - F_{D_{\mathscr{F}}}$ is compact by a reasoning similar to that in the proof of Proposition 3.3.2. Therefore the homotopy classes of the Kasparov modules $((\mathcal{Q}^V \rtimes_r S)_{\mathscr{F}}, F_D \otimes 1)$ and $(\mathcal{Q}^V_{\hat{\mathscr{F}}} \rtimes_r S_{\hat{\mathscr{F}}}, F_{D_{\mathscr{F}}} \otimes 1)$ coincide. Now the corollary follows from Baaj-Skandalis duality. □

Chapter 5

Extension theory for $*$-algebras

Extensions of C^*-algebras by stable C^*-algebras have been thoroughly studied (see [7], [9], [26], [50]) due to their close relation to Toeplitz operators and KK-theory (see [26], [50]). The starting point was the article [9] where the abelian monoid $Ext(A) \equiv Ext(A, \mathscr{K})$ was associated with a C^*-algebra A. In [50] this construction was put into the equivariant setting although only the invertible elements of $Ext_G(A, B)$ were studied.

In this chapter we will focus on extension theory for $*$-algebras. The reason for leaving the category of C^*-algebras is that most cohomology theories behave badly on C^*-algebras and one needs to look at dense subalgebras, see more in [21]. For example, if we use cohomology and Atiyah-Singers index theorem to calculate the index of a Toeplitz operator this is easily done via an explicit integral in terms of the symbol and it's derivatives if the symbol is smooth, see more in [17]. The theory developed in this chapter is a straight forward generalization of the Ext-invariant for C^*-algebras so most of the techniques are very similar to standard methods from KK-theory.

5.1 Definitions and basic properties

To begin with we will define the suitable categories. From here on, let G be a second countable locally compact group. We will say that the group action $\alpha : G \to Aut(A)$ acts continuously on the C^*-algebra A if $g \mapsto \alpha_g(a)$ is continuous for all $a \in A$.

Definition 5.1.1. *Let C^*A_G denote the category with objects consisting of pairs $(\mathscr{A}, A)$ where A is a separable C^*-algebra with a continuous G-action and $\mathscr{A}$ is a G-invariant dense $*$-subalgebra. A morphism in C^*A_G between $(\mathscr{A}, A)$ to $(\mathscr{A}', A')$ is a G-equivariant $*$-homomorphism $\varphi : \mathscr{A} \to \mathscr{A}'$ bounded in C^*-norm.*

As an abuse of notation we will denote an object $(\mathscr{A},A)$ in C^*A_G by $\mathscr{A}$ and it's latin character A will denote the ambient C^*-algebra. Observe that a morphism in C^*A_G is the restriction of an equivariant $*$-homomorphism $\bar{\varphi}:A\to A'$ uniquely determined by φ. This follows from that if $\varphi:\mathscr{A}\to\mathscr{A}'$ is bounded in C^*-norm it extends to $\bar{\varphi}:A\to A'$ and since φ is equivariant $\bar{\varphi}$ will also be equivariant. Conversely, an equivariant $*$-homomorphism of C^*-algebras is always C^*-bounded. When a linear mapping $T:\mathscr{A}\to\mathscr{A}'$, not necessarily equivariant, between two objects is induced by a bounded mapping $\bar{T}:A\to A'$ we will say that T is C^*-bounded.

Definition 5.1.2. *If $\mathfrak{I}\in C^*A_G$ satisfies that the C^*-algebra I is equivariantly stable, that is $I\otimes\mathscr{K}\cong I$ where $\mathscr{K}$ has trivial G-action, and $\mathfrak{I}$ is an ideal in $\mathscr{M}(I)$ it is called a C^*-stable G-ideal. Let C^*SI_G denote the full subcategory of C^*A_G consisting of C^*-stable G-ideals.*

We will call a morphism $\psi:\mathfrak{I}\to\mathfrak{I}'$ of C^*-stable G-ideals an embedding of C^*-stable G-ideals if $\psi:I\to I'$ is an isomorphism.

Proposition 5.1.3. *For any C^*-stable G-ideal $\mathfrak{I}$ there is an equivariant isomorphism $M_2\otimes I\cong I$ inducing an isomorphism $M_2\otimes\mathfrak{I}\cong\mathfrak{I}$. The isomorphism is given by the adjoint action of a G-invariant unitary operator $V=V_1\oplus V_2:I\oplus I\to I$ between Hilbert modules.*

Proof. In Proposition 3.2.3 two G-invariant isometries $V_1,V_2\in\mathscr{M}(I)$ such that $V_1V_1^*+V_2V_2^*=1$ were constructed. Then $V:=V_1\oplus V_2:I\oplus I\to I$ is a G-invariant unitary mapping of Hilbert modules. Thus V will be an isomorphism of Hilbert modules so $Ad(V):M_2\otimes I\to I$ is an isomorphism and since $\mathfrak{I}$ is an ideal $Ad(V)$ induces a isomorphism $M_2\otimes\mathfrak{I}\cong\mathfrak{I}$. □

One important class of C^*-stable G-ideals is the class of symmetrically normed operator ideals such as the Schatten class ideals and the Dixmier ideals (see more in [10]) over a separable Hilbert space H with a G-action. In order to get equivariant stability we need to stabilize the Hilbert space with another Hilbert space with trivial G-action. Let H' denote a separable Hilbert space and define

$$\mathscr{L}^p_H:=(\mathscr{L}^p(H\otimes H'),\mathscr{K}(H\otimes H'))$$

and analogously for the Dixmier ideal $\mathscr{L}^{n+}_H$. The G-action on the algebras are the one induced from the G-action on H.

The main study of this chapter are equivariant extensions $0\to\mathfrak{I}\to\mathscr{E}\xrightarrow{\varphi}\mathscr{A}\to 0$ where $\mathfrak{I}$ is a C^*-stable G-ideal and $\mathscr{A}\in C^*A_G$. In particular we are interested in when such extensions admit C^*-bounded splittings of Toeplitz type.

Consider for example the 0:th order pseudodifferential extension $\Psi^0(M)$ on a closed Riemannian manifold M. This extension is an extension of the smooth functions on the cotangent sphere S^*M by the classical pseudodifferential operators of order -1 given by the short exact sequence

$$0 \to \Psi^{-1}(M) \to \Psi^0(M) \to C^\infty(S^*M) \to 0.$$

It admits an explicit splitting $T : C^\infty(S^*M) \to \Psi^0(M)$ in terms of Fourier integral operators which is not C^*-bounded if $\dim M > 1$. Read more about this in Chapter 18.6 in [20]. In this setting however, the problem can be mended. In [18] a C^*-bounded splitting is constructed for real analytic manifolds M in terms of Grauert tubes and Toeplitz operators.

We will abuse the notation somewhat by referring both to the object $\mathscr{E}$ and the extension by $\mathscr{E}$. Observe that the definition implies that there exists a commutative diagram with equivariant, exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{I} & \longrightarrow & \mathscr{E} & \xrightarrow{\varphi} & \mathscr{A} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & I & \longrightarrow & E & \xrightarrow{\bar{\varphi}} & A & \longrightarrow & 0 \end{array}$$

The $*$-homomorphism $\bar{\varphi} : E \to A$ is the extension of φ to E.

Definition 5.1.4. *Two G-equivariant extensions $\mathscr{E}$ and $\mathscr{E}'$ of $\mathscr{A}$ by $\mathfrak{I}$ are said to be isomorphic if there exists a morphism $\psi : \mathscr{E} \to \mathscr{E}'$ in C^*A_G that fits into a commutative diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{I} & \longrightarrow & \mathscr{E} & \xrightarrow{\varphi} & \mathscr{A} & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle\psi} & & \| & & \\ 0 & \longrightarrow & \mathfrak{I} & \longrightarrow & \mathscr{E}' & \xrightarrow{\varphi'} & \mathscr{A} & \longrightarrow & 0 \end{array} \tag{5.1}$$

Because of the five lemma, ψ is an isomorphism.

Choose a linear splitting $\tau : \mathscr{A} \to \mathscr{E}$ and identify $\mathfrak{I}$ with an ideal in $\mathscr{E}$. The mapping τ being a splitting of an equivariant mapping $\mathscr{E} \to \mathscr{A}$ implies that

$$\tau(ab) - \tau(a)\tau(b), \quad \tau(a^*) - \tau(a)^* \in \mathfrak{I} \quad \text{and} \tag{5.2}$$

$$\tau(g.a) - g.\tau(a) \in \mathfrak{I} \;\forall g \in G. \tag{5.3}$$

Given a C^*-stable G-ideal $\mathfrak{I}$ we define the G-$*$-algebra $\mathscr{C}_\mathfrak{I} := \mathscr{M}(I)/\mathfrak{I}$ and denote by $q_\mathfrak{I} : \mathscr{M}(I) \to \mathscr{C}_\mathfrak{I}$ the canonical surjection. By the equations (5.2) and

(5.3) the mapping $q_{\mathfrak{J}}\tau : \mathscr{A} \to \mathscr{C}_{\mathfrak{J}}$ is an equivariant $*$-homomorphism. We will call the mapping $\beta_{\mathscr{A}} := q_{\mathfrak{J}}\tau$ the Busby mapping for the extensions $\mathscr{E}$. A Busby mapping which can be lifted to a C^*-bounded G-equivariant $*$-homomorphism of $\mathscr{A}$ is called trivial.

For an equivariant $*$-homomorphism $\beta : \mathscr{A} \to \mathscr{C}_{\mathfrak{J}}$ we can define the $*$-algebra

$$\mathscr{E}_\beta := \{a \oplus x \in \mathscr{A} \oplus \mathscr{M}(I) : \beta(a) = q_{\mathfrak{J}}(x)\}.$$

The $*$-algebra $\mathscr{E}_\beta$ is closed under the G-action on $\mathscr{A} \oplus \mathscr{M}(I)$ so it is a G-$*$-algebra. Denote the norm closure of $\mathscr{E}_\beta$ in $A \oplus \mathscr{M}(I)$ by E_β. We have an injection $\mathfrak{J} \to \mathscr{E}_\beta$ and a surjection $\mathscr{E}_\beta \to \mathscr{A}$. The kernel of $\mathscr{E}_\beta \to \mathscr{A}$ is $\mathfrak{J}$, so the sequence $0 \to \mathfrak{J} \to \mathscr{E}_\beta \to \mathscr{A} \to 0$ is exact and the arrows are equivariant. The $*$-algebra $\mathscr{E}_\beta$ is a well defined object in C^*A_G, because Theorem 2.1 of [50] states that the induced G-action on E_β is continuous provided it is continuous on I and on A.

Proposition 5.1.5. *The equivariant $*$-homomorphism $\beta : \mathscr{A} \to \mathscr{C}_{\mathfrak{J}}$ determines the extension up to a isomorphism, i.e if $\mathscr{E}$ has Busby mapping β then it is isomorphic to $\mathscr{E}_\beta$.*

Proof. Suppose that β is Busby mapping for $\mathscr{E}$. Define $\psi : \mathscr{E} \to \mathscr{E}_\beta$ as

$$\psi(x) := \varphi(x) \oplus x.$$

Since φ is equivariant, so is ψ. This makes the diagram (5.1) commutative, thus ψ is an isomorphism of G-equivariant extensions. ☐

The most useful class of G-equivariant extensions are the ones arising from algebraic $\mathscr{A} - \mathfrak{J}$-Kasparov modules. This is defined as an algebraic generalization of Kasparov modules for C^*-algebras, see more in [26].

Definition 5.1.6. *A G-equivariant algebraic $\mathscr{A} - \mathfrak{J}$-Kasparov module is a C^*-bounded G-equivariant representation $\pi : \mathscr{A} \to \mathscr{M}(I)$ and a selfadjoint operator*

$$F \in \mathscr{M}(I) \quad \textit{such that} \quad F^2 = 1, \quad [F, \pi(a)] \in \mathfrak{J} \ \ \forall\, a \in \mathscr{A} \quad \textit{and}$$

$$g.F - F \in \mathfrak{J}, \ \ g \in G.$$

Since F is a grading we can define the projection $P := (F + 1)/2$. The pair (π, F) induces a $*$-homomorphism

$$\beta : \mathscr{A} \to \mathscr{C}_{\mathfrak{J}}, \ a \mapsto q_{\mathfrak{J}}(P\pi(a)P). \tag{5.4}$$

The requirement $[F, \pi(a)] \in \mathfrak{J}$ together with $g.F - F \in \mathfrak{J}$ implies that β is an equivariant $*$-homomorphism.

Let $B_G(\mathscr{A}, \mathfrak{J})$ denote the set of G-equivariant Busby mappings on $\mathscr{A}$. This is the correct set to study extensions in. By Proposition 5.1.5 it is the same set as the set of isomorphism classes of G-equivariant extensions. But we need some useful notion of equivalence of extensions, or by the previous reasoning an equivalence relation on $B_G(\mathscr{A}, \mathfrak{J})$. For an object $\mathfrak{J} \in C^*SI_G$ we define the almost invariant weakly unitaries

$$U^{aw}(\mathfrak{J}) := q_{\mathfrak{J}}^{-1}(\{v \in \mathscr{C}_{\mathfrak{J}} : g.v = v,\ v^*v = vv^* = 1\}).$$

Let the almost invariant unitaries be defined as $U^a(\mathfrak{J}) := U^{aw}(\mathfrak{J}) \cap U(\mathscr{M}(\mathfrak{J}))$.

Definition 5.1.7. *Strong equivalence on $B_G(\mathscr{A}, \mathfrak{J})$ is the equivalence of Busby mappings by the adjoint $U^a(\mathfrak{J})$-action on $\mathscr{C}_{\mathfrak{J}}$. Weak equivalence on $B_G(\mathscr{A}, \mathfrak{J})$ is that of the adjoint $U^{aw}(\mathfrak{J})$-action on $\mathscr{C}_{\mathfrak{J}}$.*

Let $E_G(\mathscr{A}, \mathfrak{J})$ denote the set of strong equivalence classes of $B_G(\mathscr{A}, \mathfrak{J})$ and let $E_G^w(\mathscr{A}, \mathfrak{J})$ denote the set of weak equivalence classes. Similarly let $D_G(\mathscr{A}, \mathfrak{J})$ denote the set of strong equivalence classes of trivial Busby mappings and let $D_G^w(\mathscr{A}, \mathfrak{J})$ denote the set of weak equivalence classes of trivial Busby maps.

The isomorphism $\lambda : M_2 \otimes \mathscr{C}_{\mathfrak{J}} \to \mathscr{C}_{\mathfrak{J}}$ induced by $Ad\ V$ from Proposition 5.1.3 can be used to define the sum of two G-equivariant Busby mappings $\beta_1, \beta_2 \in B_G(\mathscr{A}, \mathfrak{J})$ as

$$\beta_1 + \beta_2 := \lambda \circ (\beta_1 \oplus \beta_2) : \mathscr{A} \to \mathscr{C}_{\mathfrak{J}}.$$

Proposition 5.1.8. *The binary operation $+$ on $B_G(\mathscr{A}, \mathfrak{J})$ induces a well defined abelian semigroup structure on $E_G(\mathscr{A}, \mathfrak{J})$ independent of the choice of V. The set $D_G(\mathscr{A}, \mathfrak{J})$ is a subsemigroup.*

The proof of the above proposition is the same as the proof of Lemma 3.1 in [50] where the semigroup of equivariant extensions of a C^*-algebra is constructed. Two G-equivariant Busby mappings $\beta_1, \beta_2 \in B_G(\mathscr{A}, \mathfrak{J})$ are said to be stably equivalent if they differ by trivial Busby mappings. That is, if there exist C^*-bounded, G-equivariant $*$-homomorphisms $\pi_1, \pi_2 : \mathscr{A} \to \mathscr{M}(I)$ such that

$$\beta_1 \oplus q_{\mathfrak{J}}\pi_1 \equiv \beta_2 \oplus q_{\mathfrak{J}}\pi_2 : \mathscr{A} \to M_2 \otimes \mathscr{C}_{\mathfrak{J}}.$$

Stable equivalence induces a well defined equivalence relation on $E_G(\mathscr{A}, \mathfrak{J})$ and $E_G^w(\mathscr{A}, \mathfrak{J})$.

Definition 5.1.9. *We define $\mathscr{E}xt_G(\mathscr{A},\mathfrak{I})$ as the monoid of stable equivalence classes of $E_G(\mathscr{A},\mathfrak{I})$ and $\mathscr{E}xt_G^w(\mathscr{A},\mathfrak{I})$ as the monoid of stable equivalence classes of $E_G^w(\mathscr{A},\mathfrak{I})$. For $G=\{1\}$ we denote it by $\mathscr{E}xt(\mathscr{A},\mathfrak{I})$ and $\mathscr{E}xt^w(\mathscr{A},\mathfrak{I})$.*

If $\mathscr{A}=A$ and $\mathfrak{I}=I$ we use the notation $Ext_G(A,I):=\mathscr{E}xt_G(A,I)$ and $Ext_G^w(A,I):=\mathscr{E}xt_G^w(A,I)$.

The monoids $\mathscr{E}xt_G(\mathscr{A},\mathfrak{I})$ and $\mathscr{E}xt_G^w(\mathscr{A},\mathfrak{I})$ coincide with the semigroup quotients $E_G(\mathscr{A},\mathfrak{I})/D_G(\mathscr{A},\mathfrak{I})$, respectively $E_G^w(\mathscr{A},\mathfrak{I})/D_G^w(\mathscr{A},\mathfrak{I})$. It has a zero-element since the class of an element in $D_G(\mathscr{A},\mathfrak{I})$ is zero.

If we are given a G-equivariant extension $\mathscr{E}$ of $\mathscr{A}$ then we will denote the class in $\mathscr{E}xt_G(\mathscr{A},\mathfrak{I})$ of it's G-equivariant Busby mapping β by $[\mathscr{E}]$ or by $[\beta]$.

Proposition 5.1.10. *If $\mathfrak{I}=I$ then*

$$\mathscr{E}xt_G^w(\mathscr{A},I)\cong\mathscr{E}xt_G(\mathscr{A},I)\cong\mathscr{E}xt_G(A,I)\equiv Ext_G(A,I)\cong Ext_G^w(A,I).$$

Proof. We will prove the existence of the first and the second isomorphism. The proof of the last isomorphism is a special case of the first isomorphism for $\mathscr{A}=A$.

To prove the existence of the first isomorphism it is sufficient to show that weakly equivalent G-equivariant Busby mappings are strongly equivalent up to stable equivalence. Assume that $\beta_1,\beta_2\in B_G(\mathscr{A},\mathfrak{I})$ are weakly equivalent via the almost invariant weakly unitary $U\in U^{aw}(\mathfrak{I})$. Then $\beta_1\oplus 0$ and $\beta_2\oplus 0$ are weakly equivalent via the almost invariant weakly unitary $U\oplus U^*$. But the operator $U\oplus U^*$ lifts to a unitary $\tilde{U}\in\mathscr{M}(M_2\otimes I)$ since $\mathscr{C}_\mathfrak{I}$ is a C^*-algebra. In fact $\tilde{U}\in U^a(M_2\otimes\mathfrak{I})$ since U is almost invariant. Thus $\beta_1\oplus 0$ and $\beta_2\oplus 0$ are strongly equivalent. For the proof that $U\oplus U^*$ lifts to a unitary, see Proposition 3.4.1 in [7].

The second isomorphism is given by the mapping $\mathscr{E}xt_G(\mathscr{A},I)\to\mathscr{E}xt_G(A,I)$, $[\mathscr{E}]\mapsto[E]$. In terms of the G-equivariant Busby mapping β it is given by $[\beta]\mapsto[\bar{\beta}]$ and since $\mathscr{A}$ is dense this is a surjection and $\bar{\beta}$ determines β uniquely. □

The constructions of Ext_G and Ext_G^w are the same as $\mathscr{E}xt_G$ and $\mathscr{E}xt_G^w$ but with C^*-algebras. These constructions can be found in [9], [26] and [50]. Proposition 5.1.10 is a mild generalization of Proposition 15.6.4 in [7]. The proof is the same although $\mathscr{A}$ does not need to be a C^*-algebra.

Since the two theories are very similar we will focus on $\mathscr{E}xt_G$. All results stated in this chapter are easily verified to also hold for $\mathscr{E}xt_G^w$.

5.2 Functoriality of $\mathscr{E}xt_G$

In this section we will prove that $\mathscr{E}xt_G$ is a functor to the category Mo^{ab} of abelian monoids. We define this category to have objects of abelian monoids and a morphism is an additive mapping $k : M_1 \to M_2$ such that $k(0) = 0$. We know how $\mathscr{E}xt_G$ acts on the objects of C^*A_G and C^*SI_G. What needs to be defined is it's action on the morphisms. We begin by showing that $\mathscr{E}xt_G$ depends covariantly on $\mathfrak{I}$.

Let $\psi : \mathfrak{I} \to \mathfrak{I}'$ be a morphism of C^*-stable G-ideals. By definition ψ can be extended to an equivariant mapping $\mathscr{M}(I) \to \mathscr{M}(I')$ which induces an equivariant mapping $q_\psi : \mathscr{C}_{\mathfrak{I}} \to \mathscr{C}_{\mathfrak{I}'}$. Define $\psi_* : E_G(\mathscr{A}, \mathfrak{I}) \to E_G(\mathscr{A}, \mathfrak{I}')$ by $\psi_*[\beta] := [q_\psi \circ \beta]$. Clearly, $\psi_*[\beta]$ is independent of the stable equivalence class of $[\beta]$. Hence it induces a well defined mapping

$$\psi_* : \mathscr{E}xt_G(\mathscr{A}, \mathfrak{I}) \to \mathscr{E}xt_G(\mathscr{A}, \mathfrak{I}').$$

Since ψ_* acting on a trivial extension gives a trivial extension we have a homomorphism of monoids.

Let us move on to proving that $\mathscr{E}xt_G$ depends contravariantly on $\mathscr{A}$. Let $\varphi : \mathscr{A} \to \mathscr{A}'$ be a morphism in C^*A_G. Take a G-equivariant Busby mapping β of $\mathscr{A}'$. Then we can define a G-equivariant Busby mapping $\varphi^*\beta := \beta \circ \varphi$ of $\mathscr{A}$. This clearly does depends on neither strong equivalence class nor stable equivalence class of the G-equivariant Busby mapping. If β is trivial it follows that $\varphi^*\beta$ is trivial so we have a morphism of monoids

$$\varphi^* : \mathscr{E}xt_G(\mathscr{A}', \mathfrak{I}) \to \mathscr{E}xt_G(\mathscr{A}, \mathfrak{I}).$$

We have now proved the following proposition.

Proposition 5.2.1. *The functor $\mathscr{E}xt_G : C^*A_G \times C^*SI_G \to Mo^{ab}$ is a well defined functor. It is covariant in $\mathfrak{I}$ and contravariant in $\mathscr{A}$.*

As noted above, an extension $\mathscr{E}$ of the algebra $\mathscr{A}$ by $\mathfrak{I}$ gives rise to an extension E of A by I. This procedure defines a mapping $E_G(\mathscr{A}, \mathfrak{I}) \to E_G(A, I)$ which respects stable equivalences.

Let C^*_G denote the category of separable C^*-algebras with a continuous G-action and SC^*_G the full subcategory of equivariantly stable objects in C^*_G. We can define an essentially surjective functor

$$\Gamma_1 : C^*A_G \times C^*SI_G \to C^*_G \times SC^*_G,$$

$$((\mathscr{A}, A), (\mathfrak{I}, I)) \mapsto (A, I).$$

It's right adjoint is the full and faithful functor

$$\Gamma_2 : C^*_G \times SC^*_G \to C^*A_G \times C^*SI_G$$

$$(A, I) \mapsto ((A, A), (I, I)).$$

Notice that $\Gamma_1\Gamma_2$ is the identity functor on $C^*_G \times SC^*_G$. Define the functor

$$Ext_G : C^*_G \times SC^*_G \to Mo^{ab} \quad \text{by} \quad Ext_G := \mathcal{E}xt_G \circ \Gamma_2.$$

As noted above this definition coincides with the definition of the Ext_G-functor in [9] and [26].

Proposition 5.2.2. *The mapping Θ defines a natural transformation*

$$\Theta : \mathcal{E}xt_G \to Ext_G \circ \Gamma_1.$$

Proof. What the mapping $\Theta^{\mathscr{A}}_{\mathfrak{I}}$ merely does is that it extends Busby mappings to the object's C^*-closure. So $\Theta^{\mathscr{A}}_{\mathfrak{I}}$ commutes with composition of morphisms in $C^*A_G \times C^*SI_G$ since they are just equivariant C^*-bounded $*$-homomorphisms. Thus Θ is a natural transformation. □

5.3 Invertible extensions

Just as in the case of a C^*-algebra one can relate invertibility in the $\mathcal{E}xt_G$-monoid and properties of the splitting. In this section we will study invertibility in $\mathcal{E}xt_G$-monoid in terms of Toeplitz operators.

The main result to be obtained in this section tells us that there is a direct link between algebraic properties in the $\mathcal{E}xt_G$-monoid and analytical properties of the extension. But it tells us nothing about how to construct the inverse or give explicit expressions. We will study this in the case of G being the trivial group and for extensions admitting a C^*-bounded, completely positive splitting. Then these explicit constructions are possible in an ideal $\mathscr{J}_{\mathfrak{I}} \supseteq \mathfrak{I}$ such that $\mathfrak{I}$ is the linear span of $\{a^*a : a \in \mathscr{J}_{\mathfrak{I}}\}$. In this setting an explicit inverse can be given in $\mathcal{E}xt(\mathscr{A}, \mathscr{J}_{\mathfrak{I}})$.

Definition 5.3.1. *Let $\pi : \mathscr{A} \to \mathscr{M}(I)$ be an equivariant $*$-homomorphism bounded in C^*-norm and P a projection in $\mathscr{M}(I)$. Assume the following*

1. *For every $a \in \mathscr{A}$ it holds that $[P, \pi(a)] \in \mathfrak{I}$.*
2. *The projection $q_{\mathfrak{I}}(P)$ is invariant under the G-action.*

If P and π satisfy the first condition we will say that (π, P) are $\mathfrak{I}$-almost commuting and if P satisfies the second condition P is said to be $\mathfrak{I}$-almost G-invariant. Under these assumptions the linear mapping

$$\beta(a) := q_{\mathfrak{I}}(P\pi(a)P)$$

is an equivariant $*$-homomorphism. We define a G-equivariant Toeplitz quantization of $\mathscr{A}$ by $\mathfrak{I}$ as a pair (π, P) of the type above. A G-equivariant extension which admits a splitting which is a G-equivariant Toeplitz quantization is called a G-equivariant Toeplitz extension.

By the correspondence $P = (F+1)/2$ the G-equivariant Toeplitz quantizations (π, P) of $\mathscr{A}$ by $\mathfrak{I}$ stand in an one-to-one correspondence to the G-equivariant algebraic $\mathscr{A} - \mathfrak{I}$-Kasparov modules (π, F).

Theorem 5.3.2. *An extension $[\mathscr{E}] \in \mathscr{E}xt_G(\mathscr{A}, \mathfrak{I})$ is invertible if and only if $[\mathscr{E}]$ can be represented by a G-equivariant Toeplitz extension.*

For equivariant extensions of C^*-algebras this statement is proved in [50] (Lemma 3.2) and the case G trivial is well studied in [26] and [7]. Our proof of Theorem 5.3.2 is based upon the same ideas adjusted to our setting.

Lemma 5.3.3. *Every strong equivalence class of an invertible G-equivariant extension is stably equivalent to a G-equivariant Toeplitz extension.*

Proof. Assume that $\mathscr{E}$ is a G-equivariant extension of $\mathscr{A}$ by $\mathfrak{I}$ with equivariant Busby mapping $\beta_1 : \mathscr{A} \to \mathscr{C}_{\mathfrak{I}}$ which is invertible in $\mathscr{E}xt_G(\mathscr{A}, \mathfrak{I})$. By definition there is a mapping $\beta_2 : \mathscr{A} \to \mathscr{C}_{\mathfrak{I}}$ and a $U \in U^a(M_2 \otimes \mathfrak{I})$ such that

$$U^*(\beta_1 \oplus \beta_2)U : \mathscr{A} \to M_2 \otimes \mathscr{C}_{\mathfrak{I}}$$

can be lifted to an equivariant C^*-bounded representation $\pi : \mathscr{A} \to M_2 \otimes \mathscr{M}(I)$. Let $P \in M_2 \otimes \mathscr{M}(I)$ denote the almost G-invariant projection $U^* \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} U$. Define

$$\beta'(a) := q_{\mathfrak{I}}(P\pi(a)P), \quad \beta''(a) := q_{\mathfrak{I}}((1-P)\pi(a)(1-P)).$$

For $a \in \mathscr{A}$, we have

$$\beta_1(a) = q_{\mathfrak{I}}(UPU^*)(\beta_1(a) \oplus \beta_2(a))q_{\mathfrak{I}}(UPU^*) =$$

$$= q_{\mathfrak{I}}(U)q(P\pi(a)P)q_{\mathfrak{I}}(U^*) = q_{\mathfrak{I}}(U)\beta'(a)q_{\mathfrak{I}}(U^*),$$

which implies that up to strong equivalence β is the Busby mapping of the extension. By the same reasoning β'' is strongly equivalent β_2.

Define $\tau'(a) := P\pi(a)P$ and $\tau''(a) := (1-P)\pi(a)(1-P)$. We express the representation $\pi' := Ad\ U^* \circ \pi$ as follows

$$\pi'(a) = \begin{pmatrix} U\tau'(a)U^* & \pi_{12}(a) \\ \pi_{21}(a) & U\tau''(a)U^* \end{pmatrix},$$

Since $q_{\mathfrak{J}}\pi' = \beta_1 \oplus \beta_2$, it follows that $\pi_{12}(a), \pi_{21}(a) \in \mathfrak{J}$. The calculation

$$[P, \pi(a)] = U^* \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \pi'(a) \right] U = U^* \begin{pmatrix} 0 & \pi_{12}(a) \\ -\pi_{21}(a) & 0 \end{pmatrix} U \in M_2 \otimes \mathfrak{J},$$

is a consequence of that $M_2 \otimes \mathfrak{J}$ is an ideal in $M_2 \otimes I$ and implies that τ is a G-equivariant Toeplitz quantization. □

Proof of Theorem 5.3.2. If $[\mathscr{E}]$ is invertible it is given by a Toeplitz extension by Lemma 5.3.3. Conversely assume that $\mathscr{E}$ is a G-equivariant Toeplitz extension (π, P) of $\mathscr{A}$. We define $P' := 1 - P$, $P_2 := P \oplus P'$, $\tau(a) := P\pi(a)P$ and $\tau'(a) := P'\pi(a)P'$. Then the claim from which the theorem will follow is that the Busby mapping $q_{\mathfrak{J}} \circ \tau'$ defines an inverse to $\mathscr{E}$. To prove this, we define the almost G-invariant symmetry

$$U := \begin{pmatrix} P & P' \\ P' & P \end{pmatrix}.$$

This symmetry satisfies $UP_2U = 1 \oplus 0$. We make the observation that $(\pi \oplus \pi, P_2)$ and $(U\pi \oplus \pi U, P_2)$ defines the same extension because of Proposition 5.1.5 and that the pair (π, P) are $\mathfrak{J}$-almost commuting. Since

$$\pi(a) \oplus 0 = UP_2U(\pi(a) \oplus \pi(a))UP_2U$$

it follows that

$$[q_{\mathfrak{J}} \circ \tau] + [q_{\mathfrak{J}} \circ \tau'] = [q_{\mathfrak{J}} \circ (P_2(\pi \oplus \pi)P_2)] = [q_{\mathfrak{J}} \circ (UP_2U^2(\pi \oplus \pi)U^2P_2U)] =$$

$$= [q_{\mathfrak{J}} \circ (UP_2U(\pi \oplus \pi)UP_2U)] = [q_{\mathfrak{J}} \circ \pi \oplus 0] = 0.$$

□

Suppose that we are in the situation $G = \{e\}$. In this case we are able to calculate an inverse to extensions admitting positive splitting if we enlarge the ideal somewhat. This should be thought of as passing from $\mathscr{L}^n(H)$ to $\mathscr{L}^{2n}(H)$. First we need an abstract notion of this procedure.

Proposition 5.3.4. *Suppose that $\mathfrak{I}$ is a C^*-stable G-ideal. The $*$-algebra*

$$\mathscr{J}_{\mathfrak{I}} := l.s.\{x \in I : x^*x \in \mathfrak{I} \quad and \quad xx^* \in \mathfrak{I}\}.$$

defines a C^*-stable G-ideal $(\mathscr{J}_{\mathfrak{I}}, I) \in C^*SI_G$. We will call $\mathscr{J}_{\mathfrak{I}}$ the square root of $\mathfrak{I}$.

Proof. Define the two $*$-invariant subsets $\mathscr{J}_{\mathfrak{I}}^+ := \{x \in I : x^*x \in \mathfrak{I}\}$ and $\mathscr{J}_{\mathfrak{I}}^- := \{x \in I : xx^* \in \mathfrak{I}\}$. For $x \in \mathscr{J}_{\mathfrak{I}}^+$ and $a \in \mathscr{M}(I)$, $(xa)^*xa \in \mathfrak{I}$ so $xa \in \mathscr{J}_{\mathfrak{I}}^+$. Since $\mathscr{J}_{\mathfrak{I}}^+$ is $*$-invariant, $ax \in \mathscr{J}_{\mathfrak{I}}^+$. Similarly, if $x \in \mathscr{J}_{\mathfrak{I}}^+$ and $a \in \mathscr{M}(I)$ then $ax(ax)^* \in \mathfrak{I}$ so $ax \in \mathscr{J}_{\mathfrak{I}}^-$ and $xa \in \mathscr{J}_{\mathfrak{I}}^-$. The $*$-algebra $\mathscr{J}_{\mathfrak{I}} \equiv l.s.(\mathscr{J}_{\mathfrak{I}}^+ \cap \mathscr{J}_{\mathfrak{I}}^-)$ so $\mathscr{J}_{\mathfrak{I}}$ is an ideal in $\mathscr{M}(I)$. There is an embedding $\mathfrak{I} \subseteq \mathscr{J}_{\mathfrak{I}}$ because $\mathfrak{I}$ is a $*$-algebra, so $\mathscr{J}_{\mathfrak{I}}$ is dense in I. □

Theorem 5.3.5. *Let $\mathscr{E}$ be an extension of $\mathscr{A}$ by $\mathfrak{I}$ admitting a C^*-bounded splitting κ extending to a completely positive contraction $\kappa : A \to \mathscr{M}(I)$. If $i : \mathfrak{I} \to \mathscr{J}_{\mathfrak{I}}$ is the embedding of $\mathfrak{I}$ into it's square root, $i_*[q_{\mathfrak{I}} \circ \kappa]$ is invertible in $\mathscr{E}xt(\mathscr{A}, \mathscr{J}_{\mathfrak{I}})$.*

Before proving this we need to review the useful construction of the Stinespring representation. This is a standard method for operator algebras and was first introduced by Stinespring in [49].

Theorem 5.3.6 (Stinespring Representation Theorem)**.** *Assume that A is a separable C^*-algebra, I is a stable C^*-algebra and that $\kappa : A \to \mathscr{M}(I)$ is a completely positive mapping such that $\|\kappa\| \leq 1$. Then there exists a $*$-homomorphism $\pi_\kappa : A \to M_2 \otimes \mathscr{M}(I)$ of A such that*

$$\begin{pmatrix} \kappa(a) & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \pi_\kappa(a) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

The $*$-homomorphism π_κ is called a Stinespring representation of κ. For proof see [26].

Lemma 5.3.7. *Assume that $\kappa : A \to \mathscr{M}(I)$ is a completely positive contraction. In the notation above*

$$\{a \in A : \kappa(a^2) - \kappa(a)^2 \in \mathfrak{I}\} = \{a \in A : [P, \pi_\kappa(a)] \in \mathscr{J}_{\mathfrak{I}}\},$$

where $P := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

Proof. We express the representation as follows

$$\pi(a) = \begin{pmatrix} \kappa(a) & \pi_{12}(a) \\ \pi_{21}(a) & \pi_{22}(a) \end{pmatrix},$$

where $\pi_{12}(a) = P\pi(a)(1-P)$ and so on. This implies that $\pi_{12}(a)^* = \pi_{21}(a^*)$. Since π is a representation

$$\begin{pmatrix} \kappa(ab) & * \\ * & * \end{pmatrix} = \pi(ab) = \pi(a)\pi(b) = \begin{pmatrix} \kappa(a)\kappa(b) + \pi_{12}(a)\pi_{21}(b) & * \\ * & * \end{pmatrix}. \quad (5.5)$$

So

$$\kappa(ab) - \kappa(a)\kappa(b) = \pi_{12}(a)\pi_{21}(b).$$

Thus $\kappa(a^2) - \kappa(a)^2 \in \mathfrak{I}$ if and only if $\pi_{12}(a)\pi_{21}(a) \in \mathfrak{I}$. After polarization we only need to show that this is equivalent to the statement $[P, \pi_\kappa(a)] \in \mathscr{J}_{\mathfrak{I}}$ for self adjoint a. But

$$[P, \pi(a)] = \begin{pmatrix} 0 & \pi_{12}(a) \\ -\pi_{21}(a) & 0 \end{pmatrix}$$

implies

$$|[P, \pi(a)]|^2 = -[P, \pi(a)]^2 = \begin{pmatrix} \pi_{12}(a)\pi_{21}(a) & 0 \\ 0 & \pi_{21}(a)\pi_{12}(a) \end{pmatrix} \in M_2 \otimes \mathfrak{I} \quad (5.6)$$

It follows from (5.6) that $\pi_{12}(a)\pi_{21}(a) \in \mathfrak{I}$ if and only if $|[P, \pi_\kappa(a)]|^2 \in \mathfrak{I}$ if and only if $[P, \pi_\kappa(a)] \in \mathscr{J}_{\mathfrak{I}}$. □

This proves Theorem 5.3.5 since this implies that κ induces a Toeplitz quantization of $\mathscr{A}$ by $\mathscr{J}_{\mathfrak{I}}$ and by Theorem 5.3.2 the element $i_*[q_{\mathfrak{I}} \circ \kappa]$ is invertible in $\mathscr{E}xt(\mathscr{A}, \mathscr{J}_{\mathfrak{I}})$.

To see that the square root of a C^*-stable ideal is needed sometimes, consider the example of the Besov space $\mathscr{A} = \mathscr{B}_p^{1/p}$. This carries a representation $\pi : \mathscr{A} \to \mathscr{B}(L^2(\mathbb{T}))$ by multiplication as functions. Let P be the Hardy projection. By [43], if $a \in L^\infty(\mathbb{T})$ then $[P, \pi(a)] \in \mathscr{L}^p(L^2(\mathbb{T}))$ if and only if $a \in \mathscr{A}$. Making a similar decomposition of π as in the proof of Lemma 5.3.7 one can show that the completely positive mapping $\tau(a) := P\pi(a)P$ is a splitting of an extension of $\mathscr{A}$ by $\mathscr{L}^{p/2}$. Since $\mathscr{A} \equiv \{a \in L^\infty(\mathbb{T}) : [P, \pi(a)] \in \mathscr{L}^p(L^2(\mathbb{T})\}$ it follows that $[q_{\mathscr{L}^{p/2}} \circ \tau] \in \mathscr{E}xt(\mathscr{A}, \mathscr{L}^{p/2})$ is not invertible by Theorem 5.3.2. But if $i : \mathscr{L}^{p/2} \to \mathscr{L}^p$ is the inclusion mapping (which coincides with the mapping constructed in Proposition 5.3.4) then $i_*[q_{\mathscr{L}^{p/2}} \circ \tau] \in \mathscr{E}xt(\mathscr{A}, \mathscr{L}^p)$ is invertible by Theorem 5.3.2.

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