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Abstract

We consider the spherically symmetric, asymptotically flat, non-vacuum Einstein equations, using as matter model a collisionless gas as described by the Vlasov equation. We find explicit conditions on the initial data which guarantee the formation of a trapped surface in the evolution which in particular implies that weak cosmic censorship holds for these data. We also analyze the evolution of solutions after a trapped surface has formed and we show that the event horizon is future complete. Furthermore we find that the apparent horizon and the event horizon do not coincide. This behavior is analogous to what is found in certain Vaidya spacetimes. The analysis is carried out in Eddington-Finkelstein coordinates.

1 Introduction

A major open problem in the mathematical analysis of General Relativity are the geometric properties of spacetimes in the case when singularities develop out of regular initial data. Of particular interest is the validity of the cosmic censorship conjecture which in non-technical terms states that generically every spacetime singularity which evolves out of regular, asymptotically flat data is covered by an event horizon so that it cannot be observed from infinity. For a precise form (or precise forms) of a statement of cosmic censorship we refer to [30] or [12]. An important insight from the mathematical analysis of this problem is that the answer depends on the model used to describe the matter which collapses. For dust which by definition is a perfect fluid with pressure zero Christodoulou [8] showed that singularities which violate cosmic censorship—so-called naked singularities—arise for an open subset of spherically symmetric, asymptotically flat data. Christodoulou also carried out an extensive analysis of this problem with a massless scalar field as matter model [9, 10, 11]. Again, there do exist spherically symmetric data which give rise to naked singularities, but this behavior is shown to be unstable, and for generic data the spacetime has a complete future null infinity which is a more formal statement of the validity of the weak cosmic censorship conjecture, cf. [12].

In [4, 5, 6] gravitational collapse was studied using as matter model a collisionless gas, i.e., the Vlasov equation. The Einstein-Vlasov system describes an ensemble of particles which interact only through gravity and where collisions among the particles are neglected. It is a model for astrophysical systems like galaxies or globular clusters, although in astrophysics relativistic effects are most of the time neglected and the non-relativistic limit [23, 27] of the system, the Vlasov-Poisson system, is used, cf. [7] and the references therein. The main result for the Einstein-Vlasov system in the cited investigations is that a class of spherically symmetric, asymptotically flat initial data is identified which lead to gravitational collapse and to a spacetime with a complete future null infinity so that cosmic censorship holds in the sense of [12]. The analysis is carried out in Schwarzschild coordinates. It is shown that a piece of the maximal development of the data is covered by these coordinates which is sufficiently large to deduce the desired conclusions, but the analysis is hampered by the well known fact that Schwarzschild coordinates cannot cover regions of spacetime which contain trapped surfaces. Closely related to this defect is the fact that these coordinates do not seem to completely cover the radially outgoing null geodesic which generates the event horizon; at least it has not been possible to show that they do.

It is hence natural to attempt an analogous analysis of the Einstein-Vlasov system using coordinates which do not break down at trapped surfaces. In the present paper we propose to do so using Eddington-Finkelstein coordinates. For the purpose of illustration we recall that the Schwarzschild metric, when written in Schwarzschild coordinates, takes the form

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 \ + r^2(d\theta^2 + \sin^2\theta d\varphi^2).$$

Here $M > 0$ is a constant, the time coordinate $t \in \mathbb{R}$ coincides with the proper time of an observer who is at rest at spatial infinity, the area radius $r > 0$ labels the surfaces of symmetry, i.e., the orbits of the group SO(3) which have surface area $4\pi r^2$ as measured in this metric, and $\theta \in [0, \pi]$, $\varphi \in [0, 2\pi]$ parametrize these symmetry orbits. The metric is non-singular with signature $(- + + +)$ in the region $r > 2M$, but the coordinates break down at $r = 2M$. In Eddington-Finkelstein coordinates t is replaced by

$$v := t + r + 2M \ln(r - 2M).$$

The Schwarzschild metric takes the form

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dv^2 + 2dv\, dr \ + r^2(d\theta^2 + \sin^2\theta d\varphi^2)$$

which is completely regular for $r > 0$, and $r = 0$ is revealed as the true spacetime singularity where for example

$$R^{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\delta} \sim r^{-6} \text{ as } r \to 0;$$

${R_{\alpha\beta\gamma}}^{\delta}$ is the Riemann curvature tensor. The lines $v = const$ correspond to radially ingoing null geodesics, and v is referred to as an advanced null coordinate or advanced time [15, 5.5]. The line $r = 2M$ is also null and represents the event horizon, while the surfaces of constant v and r are trapped if $r < 2M$ in the sense that with increasing v also the radially outgoing null geodesics $dr/dv = 1 - 2M/r$ at $r < 2M$ move towards the center. We want to use such coordinates to study dynamic spacetimes so we generalize the above form of the Schwarzschild metric as follows:

$$ds^2 = -a(v, r)\, b^2(v, r)\, dv^2 + 2b(v, r)\, dv\, dr + r^2 \left(d\theta^2 + \sin^2\theta\, d\varphi^2\right).$$

While b is required to be strictly positive, a can change sign, and as long as $b > 0$ this metric is non-degenerate with signature $(- + + +)$. Asymptotic flatness means that the metric quantities a and b satisfy the boundary conditions

$$\lim_{r\to\infty} a(v, r) = \lim_{r\to\infty} b(v, r) = 1. \tag{1.1}$$

For a metric of this form the non-trivial components of the Einstein equations

$$G_{\alpha\beta} = 8\pi T_{\alpha\beta}$$

are found to be

$$-\frac{b}{r^2}\Big(r\partial_v a + rab\partial_r a + a^2 b - ab\Big) = 8\pi T_{00}, \tag{1.2}$$

$$\frac{b}{r^2}\Big(r\partial_r a + a - 1\Big) = 8\pi T_{01}, \tag{1.3}$$

$$\frac{2}{rb}\partial_r b = 8\pi T_{11}, \tag{1.4}$$

$$\begin{aligned}\frac{r^2}{2b^2}\Big(2\partial_{rv} b + b^2\partial_{rr} a + 2ab\partial_{rr} b - \frac{2\partial_r b\partial_v b}{b} + 3b\partial_r a\partial_r b \\ + \frac{2b^2\partial_r a}{r} + \frac{2ab\partial_r b}{r}\Big) = 8\pi T_{22};\end{aligned} \tag{1.5}$$

the also non-trivial 33 component is a multiple of the 22 component.

In a collisionless gas the world lines of the particles are timelike geodesics. The ensemble can be described by a non-negative number density function f on the tangent bundle TM or equivalently on the cotangent bundle TM^* of the spacetime. The latter choice turns out to be advantageous for our analysis. We denote by (p^0, p^1, p^2, p^3) the canonical momenta corresponding to the coordinates $(x^0, x^1, x^2, x^3) = (v, r, \theta, \varphi)$ so that $(v, r, \theta, \varphi, p_0, p_1, p_2, p_3)$ coordinatize the cotangent bundle TM^*. On TM^* the geodesic equations take the form

$$\frac{dx^\alpha}{d\tau} = p^\alpha = g^{\alpha\beta}p_\beta, \quad \frac{dp_\alpha}{d\tau} = -\frac{1}{2}\frac{\partial g^{\beta\gamma}}{\partial x^\alpha}p_\beta p_\gamma,$$

where $g^{\alpha\beta}$ is the inverse of the Lorentz metric $g_{\alpha\beta}$. Due to spherical symmetry the angular momentum

$$L := (p_2)^2 + \frac{1}{\sin^2\theta}(p_3)^2$$

is conserved along geodesics,

$$\frac{dL}{d\tau} = 0.$$

We shall as usual assume that all the particles in the ensemble have the same rest mass which we normalize to unity. Then their density f is supported on the mass shell defined by

$$-1 = g^{\alpha\beta}p_\alpha p_\beta = \frac{2}{b}p_0 p_1 + a\,(p_1)^2 + \frac{L}{r^2}.$$

This implies that $p_1 \neq 0$ always, and since we want $dv/d\tau = p^0 = p_1/b > 0$, i.e., all particles move forward in advanced time, we require $p_1 > 0$ and can express p_0 as

$$p_0 = -\frac{b}{2}\left(a\,p_1 + \frac{1+L/r^2}{p_1}\right). \tag{1.6}$$

Due to spherical symmetry the particle density f is a function of the variables (v, r, p_1, L). Rewriting the relevant components of the geodesic equations using v as the parameter we find the characteristic system (cf. (2.7), (2.8) below) of the first order conservation law, i.e., the Vlasov equation, satisfied by f:

$$\begin{aligned}\partial_v f &+ \frac{b}{2}\left(a - \frac{1+L/r^2}{(p_1)^2}\right)\partial_r f \\ &+\frac{1}{2}\left(\frac{2bL}{r^3 p_1} - \partial_r(a\,b)\,p_1 - \partial_r b\,\frac{1+L/r^2}{p_1}\right)\partial_{p_1} f = 0.\end{aligned} \tag{1.7}$$

In order to close the system we have to define the energy momentum tensor in terms of f and the metric. In general,

$$T_{\alpha\beta} = |g|^{-1/2}\int p_\alpha p_\beta f\,\frac{dp_0 dp_1 dp_2 dp_3}{m},$$

where $|g|$ denotes the modulus of the determinant of the metric and m the rest mass of the particle with coordinates (x^α, p_β). In the above coordinates and using the restriction to the mass shell,

$$T_{00}(v,r) = \frac{\pi}{r^2}\int_0^\infty\int_0^\infty \frac{(p_0)^2}{p_1} f(v,r,p_1,L)\,dL\,dp_1, \tag{1.8}$$

$$T_{01}(v,r) = \frac{\pi}{r^2}\int_0^\infty\int_0^\infty p_0 f(v,r,p_1,L)\,dL\,dp_1, \tag{1.9}$$

$$T_{11}(v,r) = \frac{\pi}{r^2}\int_0^\infty\int_0^\infty p_1 f(v,r,p_1,L)\,dL\,dp_1, \tag{1.10}$$

$$T_{22}(v,r) = \frac{\pi}{2r^2}\int_0^\infty\int_0^\infty \frac{L}{p_1} f(v,r,p_1,L)\,dL\,dp_1, \tag{1.11}$$

where p_0 has to be expressed via (1.6).

As our main result for the spherically symmetric and asymptotically flat Einstein-Vlasov system (1.1)–(1.11) we prove that data given at $v = 0$ which satisfy certain explicit conditions and do not contain a trapped surface, i.e., $a(0,\cdot) > 0$, launch solutions where $a(v,r) < 0$ for some $v > 0, r > 0$, i.e., a trapped surface forms. A consequence of this result is that weak cosmic censorship holds for our initial data in view of the results [13, 14]. The solutions we construct have the additional property that for v sufficiently large, all the matter is strictly within $\{r < 2M\}$ and the generator of the event horizon, which coincides with $r = 2M$ for v large, is future complete. The structure of the data at $v = 0$ is essentially that a static, v independent state is surrounded by a shell of matter, and the particles in this shell move towards the center in a specified way. We also study solutions which at $v = 0$ contain a black hole of mass $0 < m_0 < M$ where M denotes the ADM mass for the complete spacetime. The data are such that the initial black hole is irradiated by Vlasov matter so that the apparent horizon is at $r = 2m_0$ for small advanced time v and then grows from $r = 2m_0$ to $r = 2M$ in finite advanced time. The event horizon coincides with $r = 2M$ for large v but lies strictly between $r = 2m_0$ and $r = 2M$ for small v so in particular, apparent horizon and event horizon do not coincide. This behavior is analogous to what is found in certain Vaidya spacetimes where matter, however, is described by the ad-hoc model of null dust, cf. [18, 5.1.8].

Our main result described above should be related to a previous study by Rendall [29] where he shows that there exist initial data for the spherically symmetric Einstein-Vlasov system such that trapped surfaces form in the evolution. However, the proof in [29] rests on a continuity argument and does not tell whether a given initial data set will evolve into a spacetime containing a trapped surface nor how this happens. As opposed to this we give explicit conditions on the data and analyze the evolution of the corresponding solutions to see how a trapped surface forms. The aim of such an analysis is to uncover general mechanisms leading to the formation of trapped surfaces which hopefully will play a role in a proof of weak cosmic censorship for the Einstein-Vlasov system. It is also natural to relate our result to the corresponding one for a scalar field [9]. The conditions we pose on the initial data imply that $2m/r$ is strictly less than but close to 1, whereas the conditions in [9] admit, in certain situations, that $2m/r$ can be small. These conditions are clearly very different, but so are the two matter models. In particular, the Einstein-Vlasov system admits a large number of steady states whereas there are none in the case of a scalar field. For the Einstein-Vlasov system steady states exist with $\sup_r 2m/r$ arbitrarily close to 8/9, cf. [2]. Hence there are data which do not result in the formation of trapped surfaces although $2m/r$ is relatively large initially. In view of this it is reasonable to expect that conditions which guarantee the formation of trapped surfaces for small values of $2m/r$ must be rather special.

The paper proceeds as follows. In the next section we establish a local existence and uniqueness result for the initial value problem to the system

(1.1)–(1.11), together with a criterion which allows the solution to be extended as long as certain quantities are controlled. In addition we collect some general information on the solutions to be used in the further analysis. In Section 3 we show that for particles in the outer shell p_1 is bounded both from above and away from zero as long as the shell stays away from the center. This result is used in Section 4 to prove the formation of a trapped surface out of regular data. An essential step here is to show that if the particles in the outer shell move towards the center initially in a specified way they continue to do so for $v > 0$. In the last section we study the solutions after a trapped surface has formed and also obtain the Vaidya type spacetimes mentioned above.

We conclude this introduction with some further references to the literature. Background on the Einstein-Vlasov system and relativistic kinetic theory can be found in [1, 28]. A distinguishing feature of the Vlasov matter model is that in the Newtonian case, i.e., for the Vlasov-Poisson system, no gravitational collapse occurs and solutions exist globally in time, cf. [16, 17]. For more background on the Vlasov-Poisson system we refer to [21].

2 Local existence, conservation laws, and a-priori-bounds

In this section we collect a number of observations, estimates, and a-priori-bounds which will play a role in what follows and which also lead to a local existence result and a continuation criterion.

To begin with, let us assume that a non-negative, compactly supported number density $f(v) \in C^1_c(]0,\infty[^2 \times [0,\infty[)$ is given at some instant $v \geq 0$. We want to show that at that instant the metric is then determined explicitly in terms of $f(v)$. Firstly, the field equation (1.4) can be integrated to yield

$$b(v,r) = \exp\left(-4\pi \int_r^\infty \eta T_{11}(v,\eta)\, d\eta\right). \tag{2.1}$$

Since the formula (1.10) for T_{11} does not contain a metric coefficient, (2.1) defines $b(v)$ in terms of $f(v)$, and

$$0 < b(v,r) \leq 1,\ \ r > 0, \tag{2.2}$$

in particular b is positive as required. The boundary condition (1.1) holds as well, indeed $b(v,r) = 1$ when r lies to the right of $\mathrm{supp}\, f(v)$. In order to express a we observe that

$$T_{01} = -\frac{b}{2}\,(a\, T_{11} + S) \tag{2.3}$$

where

$$S := \frac{\pi}{r^2} \int_0^\infty \int_0^\infty \frac{1+L/r^2}{p_1} f \, dL \, dp_1. \tag{2.4}$$

The field equation (1.3) can now be rewritten in the form

$$\partial_r (ra - r) + 4\pi r T_{11} \, (ra - r) = -4\pi r^2 (T_{11} + S). \tag{2.5}$$

Since $(ra - r)_{|r=0} = 0$ this can be integrated to give

$$a(v,r) = 1 - \frac{1}{r} \int_0^r 4\pi \eta^2 (T_{11} + S) \exp\left(- \int_\eta^r 4\pi\sigma T_{11} d\sigma \right) d\eta; \tag{2.6}$$

clearly $a(v,\infty) = 1$. Hence given $f(v)$ at some instant v the metric is explicitly determined. Notice that when in Section 5 we consider the system with a black hole at the center we replace the above boundary condition for $ra - r$ at $r = 0$ by one at $r = \infty$.

If the metric coefficients a and b are given and sufficiently smooth on some interval $[0, V[$ we denote by $(R, P_1)(s, v, r, p_1, L)$ the solution of the characteristic system

$$\frac{dr}{ds} = \frac{b}{2} \left(a - \frac{1+L/r^2}{(p_1)^2} \right), \tag{2.7}$$

$$\frac{dp_1}{ds} = \frac{1}{2} \left(\frac{2bL}{r^3 p_1} - \partial_r (a\, b)\, p_1 - \partial_r b \, \frac{1+L/r^2}{p_1} \right) \tag{2.8}$$

of the Vlasov equation (1.7) with $(R, P_1)(v, v, r, p_1, L) = (r, p_1)$. Then

$$f(v, r, p_1, L) = \mathring{f}\,((R, P_1)(0, v, r, p_1, L), L) \tag{2.9}$$

is the solution of the Vlasov equation satisfying the initial condition $f_{|v=0} = \mathring{f}$. If $\mathring{f}$ is non-negative and compactly supported then these properties are inherited by $f(v)$.

The above observations allow for the following iterative scheme. If initial data $\mathring{f}$ are given we define $f_0(v, r, p_1, L) := \mathring{f}(r, p_1, L)$. If f_n is given we define $T_{11,n}$ and S_n by substituting f_n into the formulas (1.10) and (2.4). Next we define a_n and b_n through (2.1) and (2.6). Finally, we obtain the next iterate f_{n+1} via (2.9), using a_n and b_n in (2.7), (2.8). This iterative scheme leads to the following local existence result.

Theorem 2.1 *Let $\mathring{f} \in C_c^1(]0,\infty[^2 \times [0,\infty[)$ be compactly supported and non-negative. Then there exists a unique solution $f \in C^1([0,V[\times]0,\infty[^2\times[0,\infty[)$*

of the system (1.1)–(1.11) with $f_{|v=0} = \overset{\circ}{f}$ *where* $V > 0$*. Let* V *be chosen maximal. If*

$$\sup\left\{p_1 + \frac{1}{p_1} + \frac{1}{r} \mid (r, p_1, L) \in \operatorname{supp} f(v),\ 0 \leq v < V\right\} < \infty$$

then $V = \infty$*.*

Proof. In Schwarzschild coordinates the analogous proof has been carried out in detail in [20, 22]. Here we only address some key issues. Firstly, by construction the iterative scheme indicated above will converge to a solution of the subsystem where only the field equations (1.3) and (1.4) hold. The remaining field equations can then be derived exploiting the following observation. If ∇_α denotes the covariant derivative corresponding to the given metric, then the Vlasov equation implies that $\nabla_\alpha T^{\alpha\beta} = 0$. Also the Einstein tensor satisfies the relation $\nabla_\alpha G^{\alpha\beta} = 0$. The relation $\nabla_\alpha E^{\alpha\beta} = 0$ satisfied by $E^{\alpha\beta} := G^{\alpha\beta} - 8\pi T^{\alpha\beta}$ then implies that the remaining field equations hold.

Clearly, the supremum Q in the statement of the theorem together with (2.9) controls T_{11} and S, hence a and b, and via the corresponding field equations also their first order derivatives $\partial_r a, \partial_r b, \partial_v a$; notice that

$$-\frac{b}{r}\partial_v a = G_{00} + a\,b\,G_{01} = 8\pi(T_{00} + a\,b\,T_{01}). \tag{2.10}$$

But in order to extend f as a C^1 solution it is necessary to control the derivatives of R and P_1 with respect to their data r, p_1, L which seems to require second order derivatives of a and b which are not controlled by the supremum Q. However, the quantities

$$\xi = \partial_r R,\ \ \eta = \partial_r P_1 - P_1 \frac{\partial_r b}{b} \partial_r R$$

satisfy a system of ordinary differential equations where second order derivatives of the metric coefficients appear only in the combination

$$2\partial_{rv} b + b^2 \partial_{rr} a + 2ab\partial_{rr} b - \frac{2\partial_r b \partial_v b}{b}$$

which appears in the field equation (1.5) and is therefore controlled by Q as well; notice that $\partial_v b$ which appears in this combination cannot be controlled by itself in terms of Q. A similar argument also helps to obtain the bounds on the iterates required for their convergence. The differential geometric background of this maneuver is that the evolution of the derivatives of characteristics with respect to their data is governed by the geodesic deviation equation where derivatives of the metric enter only through the Riemann curvature tensor, and due to the symmetry the latter contains second order derivatives of the metric coefficients in the same combination as they appear in the Einstein tensor and hence in the field equations. □

It seems necessary that the support of the matter for the solution and hence for the initial data is bounded away from the origin. At first glance this is due to using polar coordinates (r, θ, φ), and one may hope to cure this by passing to the induced Cartesian coordinates $r(\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$, as was done in [20, 22]. However, in the case of Eddington-Finkelstein coordinates this does not seem to help; the metric, when written in Cartesian coordinates, is not C^2 at the origin, indeed, not even continuous. Since this issue does not play a role in out analysis we do not pursue it further.

In what follows below, two situations will be considered. In the first we place a static, v-independent solution in the center and surround it by a shell of non-static Vlasov matter. Since we will only need to analyze this case as long as the outer non-static matter does not reach the central static part, the considerations above are sufficient to deal with it. In the second case, studied in Section 5, we consider the system only for $r \geq r_0$ where $a(0, r_0) < 0$, so again a neighborhood of the center is avoided.

Coming back to the first case we need to see that the system indeed has static, v-independent solutions. The existence of static, spherically symmetric and compactly supported solutions has been established in [19, 24, 25] using Schwarzschild coordinates. Given such a static solution in Schwarzschild coordinates its metric takes the form

$$ds^2 = -e^{2\mu(r)}dt^2 + e^{2\lambda(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2).$$

Changing to the variable $v = v(t, r) = t + \int_0^r e^{\lambda(\eta)-\mu(\eta)}d\eta$ brings this metric into the Eddington-Finkelstein form where

$$a = e^{-2\lambda},\ b = e^{\lambda+\mu}$$

so that these transformed metric coefficients are independent of v as desired, and the same is then true for the particle density f of the steady state.

In what follows f_s denotes a fixed spherically symmetric steady state of the Einstein-Vlasov system with spatial support in $[0, r_0]$. We consider initial data of the form

$$\mathring{f} := f_s + \mathring{f}_{\text{out}} \text{ where } \mathring{f}_{\text{out}} \in C_c^1(]r_0, \infty[\times]0, \infty[\times[0, \infty[),\ \mathring{f}_{\text{out}} \geq 0. \quad (2.11)$$

For a solution launched by such data we introduce the notation

$$f_{\text{out}}(v,r,p_1,L) := \mathring{f}_{\text{out}}((R,P_1)(0,v,r,p_1,L),L), \tag{2.12}$$

i.e., f_{out} is the density of the particles which initially do not belong to the steady state. Due to spherical symmetry there are no gravitational waves, and the only way that f_{out} can influence the matter in the region $\{r < r_0\}$ is if outer particles actually reach this region. As long as this does not happen the matter in the central region stays in its equilibrium configuration. Hence we obtain the following corollary to Theorem 2.1.

Corollary 2.2 *Initial data $\mathring{f}$ as specified in (2.11) launch a unique solution $f \in C^1([0,V[\times]0,\infty[^2\times[0,\infty[)$ of the system (1.1)–(1.11) with $f_{|v=0} = \mathring{f}$ where $V > 0$. Let V be chosen maximal. If*

$$\sup\left\{p_1 + \frac{1}{p_1} \mid (r,p_1,L) \in \operatorname{supp} f_{\text{out}}(v),\ 0 \le v < V\right\} < \infty$$

and

$$\inf\{r \mid (r,p_1,L) \in \operatorname{supp} f_{\text{out}}(v),\ 0 \le v < V\} > r_0$$

then $V = \infty$.

We now collect some properties of the local solutions which are used in what follows. First we look at the ADM mass and resulting a-priori bounds.

Proposition 2.3 *Let*

$$m(v,r) := \frac{r}{2}\left(1 - a(v,r)\right),\ \textit{i.e.,}\ a(v,r) = 1 - \frac{2m(v,r)}{r}.$$

Then the quasi-local ADM mass m is given by

$$\begin{aligned} m(v,r) &= 2\pi \int_0^r \eta^2 (T_{11} + S) \exp\left(-\int_\eta^r 4\pi\sigma T_{11} d\sigma\right) d\eta \\ &= 2\pi \int_0^r \eta^2 (T_{11} + S)(v,\eta) \frac{b(v,\eta)}{b(v,r)}\, d\eta. \end{aligned} \tag{2.13}$$

For r sufficiently large,

$$m(v,r) = M = \lim_{r\to\infty} m(v,r)$$

which is a conserved quantity, the ADM mass. Moreover,

$$m(v,r) = 2\pi \int_0^r \eta^2 (aT_{11} + S)(v,\eta)\, d\eta, \tag{2.14}$$

and

$$
\begin{aligned}
a(v,r) &\leq 1, \\
b(v,r)\ m(v,r) &= 2\pi \int_0^r \eta^2 (T_{11}+S) b\, d\eta \\
&\leq 2\pi \int_0^\infty \eta^2 (T_{11}+S) b\, d\eta = M. \qquad (2.15)
\end{aligned}
$$

Proof. The formula (2.13) for $m(v,r)$ results from comparing its relation to a with (2.6) and (2.1). Recalling (2.10),

$$
\partial_v m = -\frac{r}{2}\partial_v a = 4\pi r^2 \left(\frac{1}{b} T_{00} + a T_{01} \right) \qquad (2.16)
$$

in particular, $m(\cdot, r)$ is constant for r large enough and hence to the right of the support of the matter. If for the moment we denote the right hand side of (2.14) by $\tilde{m}$ then clearly $\tilde{m}(v,0) = 0 = m(v,0)$, and the differential equation (2.5) for $ra - r$ is equivalent to the fact that $\partial_r \tilde{m} = \partial_r m$. By (2.13), $m \geq 0$ and hence $a \leq 1$, and (2.13) together with the conservation of M and the fact that $b(v,\infty) = 1$ implies (2.15). □

To conclude this section we collect some information on sign changes in a.

Proposition 2.4 *If $a(v_0, r_0) < 0$ for some $v_0 \geq 0, r_0 > 0$ then $a(v, r_0) < 0$ for all $v > 0$ for which the solution exists. If $a(v,r) < 0$ then all timelike or null geodesics at the spacetime point (v, r, θ, ϕ) move towards strictly smaller values of r.*

Proof. First we note that by (2.16),

$$
\begin{aligned}
\partial_v a &= -8\pi r \left(\frac{1}{b} T_{00} + a T_{01} \right) \\
&= -\frac{2\pi^2 b}{r} \int_0^\infty \int_0^\infty \frac{1}{(p_1)^3} \left(\left(1 + L/r^2\right)^2 - a^2 (p_1)^4 \right) f\, dL\, dp_1.
\end{aligned}
$$

Hence $\partial_v a(v,r) = 0$ if $f(v,r,\cdot) = 0$. If $f(v,r,\cdot) > 0$ then $\partial_v a(v,r) < 0$, provided $a(v,r)$ is sufficiently close to 0 so that the term in parentheses is positive; notice that $f(v,r,\cdot)$ has compact support. Hence if $a(v_0,r_0) < 0$ at some v_0, r_0 this sign must be preserved for all $v > v_0$. The remaining assertion follows from the geodesic equations. □

3 A lower and an upper bound on p_1

In this section we consider solutions launched by data as specified in (2.11), i.e., with a steady state at the center. The aim is to prove that p_1 is bounded from above and below on the support of $f_{\rm out}$ where we recall (2.12). In connection with the continuation criterion in Corollary 2.2 this means that a solution can only blow up if particles from the exterior mass shell reach the central region $\{r < r_0\}$ where the steady state part is supported. The result may be compared to [26] where it is shown that in Schwarzschild coordinates solutions which blow up at all must do so at the center first.

Theorem 3.1 *Let $V > 0$ be such that $r > r_0$ for $(r, p_1, L) \in \operatorname{supp} f_{\rm out}(v)$ and $0 \leq v \leq V$. Then there are constants $C_1, C_2 > 0$ such that $C_1 \leq p_1 \leq C_2$ for $(r, p_1, L) \in \operatorname{supp} f(v)$ and $0 \leq v \leq V$.*

In the proof of this theorem we need the following a-priori-bound.

Lemma 3.2 *Let $V > 0$ and $r_0 > 0$ be such that $f(v, r_0, \cdot, \cdot) = 0$ for $0 \leq v \leq V$. Then*

$$\int_{r_0}^{\infty} r^2 T_{11}(v, r)\, dr \leq \int_{r_0}^{\infty} r^2 T_{11}(0, r)\, dr + \frac{M}{2\pi r_0} v,\ \ 0 \leq v \leq V.$$

Proof. Using the Vlasov equation, integration by parts, and the field equations we find that

$$\frac{d}{dv} \int_{r_0}^{\infty} r^2 T_{11}\, dr = -\frac{1}{2} \int_{r_0}^{\infty} b r T_{11} (1-a) dr + \pi \int_{r_0}^{\infty} \int_0^{\infty} \int_0^{\infty} \frac{bL}{r^3 p_1} f\, dL\, dp_1\, dr.$$

Now $a \leq 1$ and $T_{11} \geq 0$ so the first term can be dropped. As to the second, we observe that

$$\frac{bL}{r^3 p_1} \leq b \frac{1 + L/r^2}{p_1} \frac{1}{r},$$

and recalling the expression (2.4) for S we obtain the estimate

$$\frac{d}{dv} \int_{r_0}^{\infty} r^2 T_{11}\, dr \leq \int_{r_0}^{\infty} r b S dr \leq \frac{1}{r_0} \int_{r_0}^{\infty} r^2 b S dr.$$

By (2.15),

$$M = 2\pi \int_0^{\infty} r^2 (T_{11} + S)\, b(v, r)\, dr \geq 2\pi \int_{r_0}^{\infty} r^2 S(v, r)\, b(v, r)\, dr,$$

hence

$$\frac{d}{dv} \int_{r_0}^{\infty} r^2 T_{11}\, dr \leq \frac{M}{2\pi r_0}$$

and the claim of the lemma follows. $\square$

Proof of Theorem 3.1. Let $[0,V] \ni s \mapsto (r(s), p_1(s), L)$ denote a characteristic in $\operatorname{supp} f_{\text{out}}$. Using (2.7) we can rewrite (2.8) in the form

$$\frac{d}{ds}p_1 = \frac{bL}{r^3 p_1} - \left(\frac{1}{2}\partial_r ab + a\partial_r b - \frac{\partial_r b}{b}\frac{dr}{ds}\right) p_1. \tag{3.1}$$

In order to obtain a lower bound for p_1 we observe that

$$\begin{aligned}\frac{d}{ds}\frac{1}{p_1} &= -\frac{bL}{r^3(p_1)^3} + \left(\frac{1}{2}\partial_r ab + a\partial_r b - \frac{\partial_r b}{b}\frac{dr}{ds}\right)\frac{1}{p_1}\\ &\leq \left(\frac{1}{2}\partial_r ab + a\partial_r b - \frac{\partial_r b}{b}\frac{dr}{ds}\right)\frac{1}{p_1}.\end{aligned}$$

Consider an arbitrary instant of advanced time $0 < v_0 \leq V$. Applying Gronwall's lemma we find that we need to estimate the integral

$$\int_0^{v_0}\left[\left(\frac{1}{2}\partial_r ab + a\partial_r b\right)(s, r(s)) - \left(\frac{\partial_r b}{b}\right)(s, r(s))\frac{dr}{ds}(s)\right] ds$$

which is a curve integral along the curve

$$\gamma = \{(s, r(s)) \mid 0 \leq s \leq v_0\}.$$

In order to estimate this integral we apply Green's formula in the plane. For this we define

$$\begin{aligned}C_1 &= \{(v_0, r) \mid r(v_0) \leq r \leq R\},\\ C_2 &= \{(v, R) \mid 0 \leq v \leq v_0\},\\ C_3 &= \{(0, r) \mid r(0) \leq r \leq R\}.\end{aligned}$$

Here $R > 0$ is large, and we will let $R \to \infty$. We orient the closed curve $\Gamma = \gamma + C_1 + C_2 + C_3$ clockwise. Now

$$\begin{aligned}\int_0^{v_0}\left[\frac{1}{2}\partial_r ab + a\partial_r b - \frac{\partial_r b}{b}\frac{dr}{ds}\right] ds &= \int_\gamma \left[\left(\frac{1}{2}\partial_r ab + a\partial_r b\right) dv - \left(\frac{\partial_r b}{b}\right) dr\right]\\ &= \oint_\Gamma - \int_{C_1} - \int_{C_2} - \int_{C_3}.\end{aligned}$$

We denote by Ω the domain enclosed by Γ and apply Green's formula in the plane to find that

$$\begin{aligned}\oint_\Gamma [\cdots] &= -\frac{1}{2}\iint_\Omega \left[\frac{\partial}{\partial v}\left(\frac{2\partial_r b}{b}\right) + \frac{\partial}{\partial r}\left(\partial_r ab + 2a\partial_r b\right)\right] dr\, dv\\ &= -\iint_\Omega \left[\frac{b}{r^2}G_{22} - \frac{a\,\partial_r b}{r} - \frac{\partial_r a\, b}{r}\right] dr\, dv;\end{aligned}$$

recall the field equation (1.5) for the 22-component G_{22} of the Einstein tensor. If we now use the field equations and the definitions of the energy momentum tensor it turns out that

$$\frac{b}{r^2}G_{22} - \frac{a\,\partial_r b}{r} - \frac{\partial_r a\, b}{r} = \frac{4\pi^2}{r^2}b\int_0^\infty\int_0^\infty \frac{1+2L/r^2}{p_1} f\,dL\,dp_1 - \frac{b}{r^2}\frac{2m}{r}.$$

Hence by (2.15),

$$\oint_\Gamma \cdots \leq \iint_\Omega \frac{2mb}{r^3}\,dr\,du \leq \frac{M}{r_0^2}v_0.$$

Next we use the field equation (1.4) and Lemma 3.2 to get the estimate

$$\begin{aligned}
-\int_{C_1}\cdots &= \int_{r(v_0)}^R \frac{\partial_r b(v_0,r)}{b(v_0,r)}dr \leq 4\pi\int_{r_0}^\infty rT_{11}(v_0,r)\,dr \\
&\leq \frac{1}{r_0}4\pi\int_{r_0}^\infty r^2T_{11}(v_0,r)\,dr \\
&\leq \frac{1}{r_0}\left(4\pi\int_{r_0}^\infty r^2T_{11}(0,r)\,dr + \frac{2M}{r_0}v_0\right).
\end{aligned} \tag{3.2}$$

The C_2-contribution vanishes in the limit $R\to\infty$:

$$-\int_{C_2}\cdots = \int_0^{v_0}\left(\frac{1}{2}\partial_r ab + a\partial_r b\right)(v,R)\,dv \leq \frac{M}{R^2}v_0,$$

since for r sufficiently large and in particular outside the support of f, $b=1$ and $\partial_r a = 2M/r^2$. Finally,

$$-\int_{C_3}\cdots = -\int_{r(0)}^R \frac{\partial_r b(0,r)}{b(0,r)}dr \leq 0.$$

Altogether this implies, using Gronwall's lemma, the estimate

$$\frac{1}{p_1(v)} \leq \frac{1}{p_1(0)}\exp\left(\frac{4\pi}{r_0}\int_{r_0}^\infty r^2T_{11}(0,r)\,dr + \frac{3M}{r_0^2}v\right). \tag{3.3}$$

In order to estimate p_1 from above we start from (3.1). The term $(bL)/(r^3p_1)$ is bounded by the previous estimates so after using Gronwall's lemma we have to estimate the term

$$-\int_\gamma\left[\left(\frac{1}{2}\partial_r ab + a\partial_r b\right)dv - \left(\frac{\partial_r b}{b}\right)dr\right] = -\oint_\Gamma + \int_{C_1} + \int_{C_2} + \int_{C_3}$$

from above, where the curves are defined and oriented as before. The C_1-contribution is now negative and can be dropped. Since for r sufficiently large and outside the support of f, $b = 1$ and $\partial_r a \geq 0$ the C_2-contribution can be dropped as well. The C_3-contribution is determined by the initial data, so it remains to estimate the integral over the closed curve Γ which we again turn into an integral over Ω using Green's formula. Due to the change in sign we are left with estimating the integral

$$\iint_\Omega \frac{4\pi^2}{r^2} b \int_0^\infty \int_0^\infty \frac{1+2L/r^2}{p_1} f \, dL \, dp_1 \, dr \, dv.$$

Using the already established bounds this amounts to estimating the integral $\int_{r_0}^\infty \int_0^\infty \int_0^\infty f \, dL \, dp_1 \, dr$ which represents the number of particles in the domain $\{r \geq r_0\}$ and is conserved for $v \in [0, V]$. The proof of Theorem 3.1 is complete. $\square$

Together with the continuation criterion from the local existence result Corollary 2.2 we obtain the following corollary.

Corollary 3.3 *There exists $V > 0$ such that the solution exists on the interval $[0, V[$ and $r > r_0$ for all particles in the outer matter. If V is chosen maximal, then $V = \infty$ if*

$$\inf\{r \mid (r, p_1, L) \in \operatorname{supp} f_{\text{out}}(v),\ 0 \leq v < V\} > r_0,$$

i.e., the solution can be extended as long as the outer matter stays outside $\{r \leq r_0\}$.

4 The formation of a trapped surface

In this section we want to specify conditions on the data such that a trapped surface evolves, but is not already present in the data. The data are again of the form (2.11), i.e., they have a steady state f_s at the center whose mass we denote by m_0. We fix some notation:

$$\operatorname{supp} \mathring{f}_{\text{out}} \subset [R_0, R_1] \times [p_-, p_+] \times [0, L_+] \tag{4.1}$$

where

$$0 < r_0 < R_0 < R_1,\ 0 < p_- < p_+,\ L_+ > 0,$$

and $M > m_0$ is the mass of $\mathring{f} := f_s + \mathring{f}_{\text{out}}$. If $R_0 \geq 2M$ then the data do not contain a trapped surface in the sense that $a(0, \cdot) > 0$. We define

$$P := -\max\left\{p^1 \mid (r, p_1, L) \in \operatorname{supp} \mathring{f}_{\text{out}}\right\}. \tag{4.2}$$

Since

$$p^1 = \frac{1}{b}p_0 + ap_1 = \frac{1}{2}ap_1 - \frac{1}{2}\frac{1+L/r^2}{p_1} \leq \frac{1}{2}p_1 - \frac{1}{2}\frac{1}{p_1} \tag{4.3}$$

we have $P > 0$, i.e., all particles move inward initially, if for example $p_+ < 1$. The following theorem is the main result of the present paper.

Theorem 4.1 *Let data* $f_s + \mathring{f}_{\mathrm{out}}$ *be given such that*

$$R_0 \geq 2M > r_0,\ L_+ := 12m_0^2,\ 2P \geq 1 + \frac{L_+}{r_0^2}, \tag{4.4}$$

and such that there exists $V > 0$ *with the property that*

$$\frac{2P^2}{1+L_+/r_0^2}\exp\left(-\frac{2M}{r_0}\right)V > R_1 - 2M, \tag{4.5}$$

$$\frac{1}{2}\left(1+\frac{L_+}{r_0^2}\right)\frac{1}{p_-^2}\exp\left(\frac{2M}{r_0^2}V + \frac{4M}{r_0}\right)V < R_0 - r_0. \tag{4.6}$$

Then the solution launched by $f_s + \mathring{f}$ *forms a trapped surface at some advanced time* $v < V$.

Remark. Note that this result implies that weak cosmic censorship holds for these data in view of the results [13, 14]. In Section 5 we will see that for data as considered above an event horizon evolves which is future complete, cf. Theorem 5.2 and Corollary 5.3.

Remark. It is easy to see that data which satisfy the conditions above do exist: First we arbitrarily fix the central steady state f_s and hence also r_0 and m_0. Next we fix R_0, M, p_+ in such a way that (4.4) holds; notice that P becomes large if we chose $p_+ > 0$ small, cf. (4.3). Then we choose $0 < p_- < p_+$. If we replace the inequality in (4.6) by an equality this uniquely determines V which we then choose slightly smaller to preserve the inequality. Substituting this V into (4.5) we see that that relation is satisfied provided R_1 is sufficiently close to $2M$, and all the support parameters and the mass of $\mathring{f}_{\mathrm{out}}$ are fixed. Now we fix any non-negative, non-vanishing function g which satisfies the support conditions, consider data Ag with some amplitude $A \geq 0$ and denote the induced ADM mass and metric coefficient by M_A and a_A respectively. From (2.13) we see that M_A depends continuously on A with $M_0 = m_0$. If a_A remains positive as A increases then (2.14) shows that M_A becomes as large as we wish with increasing A. So assume that $a_{A^*}(r^*) = 0$ for some $r^* > R_0$ and some value of amplitude A^*, and a_A is positive for all $A < A^*$. Then $2m_{A^*}(r^*) = r^* > R_0 \geq 2M$ and a_{A^*}

is still non-negative, hence $M_{A^*} > M$, and $M_A = M$ for some smaller value of A. Hence $\overset{\circ}{f}_{\text{out}}$ satisfying all support conditions and having the proper mass exist. Since $R_0 \geq 2M$, $a(0,\cdot) > 0$ for any such $\overset{\circ}{f}_{\text{out}}$.

A major step in the proof of Theorem 4.1 is to show that, for suitable data, the particles move inward as long as they stay outside $r = r_0$. Since we will recycle this result in Section 5, we formulate it in a more general setting than needed here.

Lemma 4.2 *Let V be the length of the maximal existence interval of a solution f as in Corollary 3.3, and assume that*

$$L_+ \leq 12 m_0^2 \text{ and } P > 0.$$

Then

$$\max \left\{ p^1 \mid (r, p_1, L) \in \operatorname{supp} f_{\text{out}}(v),\ 0 \leq v < V \right\} \leq -P < 0.$$

Proof. By continuity and the assumption on the data there is some interval $[0, V^*[\subset [0, V[$ such that $p^1 < 0$ for all $(r, p_1, L) \in \operatorname{supp} f_{\text{out}}(v)$ with $0 \leq v < V^*$. We choose V^* maximal and have to show that $V^* = V$. To this end we consider a characteristic in $\operatorname{supp} f_{\text{out}}$ and parametrize it by proper time. After some computation we find that

$$\begin{aligned}
\dot{p}^1 &= \frac{d}{d\tau}\left(\frac{1}{b} p_0 + a p_1\right) \\
&= -\frac{\partial_r b}{b^3}(p_0)^2 + \frac{\partial_v a}{2b}(p_1)^2 + \frac{a \partial_r a}{2}(p_1)^2 + \frac{\partial_r a}{b} p_0 p_1 + a \frac{L}{r^3} \\
&= -\frac{4\pi r}{b^2} T_{11} (p_0)^2 + \left(\frac{8\pi r}{b^2} T_{01} + \frac{1}{rb}(1-a)\right) p_0 p_1 \\
&\quad + \left(-\frac{4\pi r}{b^2} T_{00} + \frac{a}{2r}(1-a)\right)(p_1)^2 + a \frac{L}{r^3} \\
&= \frac{1}{rb}(1-a) p_0 p_1 + \frac{a}{2r}(1-a)(p_1)^2 + a \frac{L}{r^3} \\
&\quad - \frac{4\pi^2}{rb^2} \int_0^\infty \int_0^\infty \left[\tilde{p}_1 (p_0)^2 - 2 \tilde{p}_0 p_0 p_1 + \frac{(\tilde{p}_0)^2}{\tilde{p}_1}(p_1)^2\right] f \, d\tilde{L} \, d\tilde{p}_1,
\end{aligned}$$

where $\tilde{p}_0$ is defined as in (1.6) but in terms of the integration variables $\tilde{p}_1$ and $\tilde{L}$. Now

$$[\ldots] = \left[\sqrt{\tilde{p}_1} p_0 - \frac{\tilde{p}_0 p_1}{\sqrt{\tilde{p}_1}}\right]^2 \geq 0,$$

so after substituting the definition (1.6) for p_0,

$$\dot{p}^1 \le -\frac{1}{2r}(1-a)\left(1+\frac{L}{r^2}\right) - (1-a)\frac{L}{r^3} + \frac{L}{r^3}. \tag{4.7}$$

If $a(v,r) < 0$, then

$$\dot{p}^1 \le -\frac{1}{2r}\left(1+\frac{L}{r^2}\right) - \frac{L}{r^3} + \frac{L}{r^3} < 0.$$

If $a(v,r) \ge 0$, then we observe the relation $a(v,r) = 1 - 2m(v,r)/r$ and investigate the behavior of $m(v,r)$ with respect to v. By Proposition 2.4, $a(\cdot,r) \ge 0$ on $[0,v]$. By (2.16) and (1.8), (1.9),

$$\partial_v m = 4\pi^2 \int_0^\infty \int_0^\infty \left(\frac{1}{b}\frac{(p_0)^2}{p_1} + ap_0\right) f\, dL\, dp_1,$$

and

$$\frac{1}{b}\frac{(p_0)^2}{p_1} + ap_0 = \frac{p_0}{p_1}p^1.$$

By assumption, $p^1 < 0$ on $[0,v] \subset [0,V^*[$ for all particles, also $p_1 > 0$ for all particles, and

$$p_0 = -\frac{b}{2}\left(ap_1 + \frac{1+L/r^2}{p_1}\right) < 0$$

because of the sign of $a(\cdot,r)$ on $[0,v]$. Hence $\partial_v m(\cdot,r) \ge 0$ on $[0,v]$, and $m(v,r) \ge m(0,r) \ge m_0$. Hence (4.7) implies that

$$\begin{aligned}
\dot{p}^1 &\le -\frac{1}{2r}\frac{2m}{r}\left(1+\frac{L}{r^2}\right) - \frac{2m}{r}\frac{L}{r^3} + \frac{L}{r^3} \\
&\le \frac{1}{r^4}\left(Lr - 3Lm_0 - r^2 m_0\right) = \frac{1}{r^4}\left(\frac{L^2}{4m_0} - \left(\sqrt{m_0}r - \frac{L}{2\sqrt{m_0}}\right)^2 - 3Lm_0\right) \\
&\le \frac{L}{4r^4 m_0}\left(L - 12m_0^2\right) \le 0.
\end{aligned}$$

Together with the case $a(v,r) < 0$ this shows that $\dot{p}^1 \le 0$ for any characteristic in $\operatorname{supp} f_{\text{out}}(v)$ with $v \in]0,V^*[$, hence $V^* = V$, and the proof is complete. □

Proof of Theorem 4.1. Assume that $a(v,r) \ge 0$ for all $r > 0$ and $0 < v \le V$ and as long as the solution exists. We will show that the solution must then exist on the interval $[0,V]$ and $m(V,r) = M$ for some $r < 2M$ so that $a(V,r) < 0$ which is the desired contradiction. The basic idea is that

(4.6) guarantees that the continuation criterion from Corollary 3.3 applies to the interval $[0,V]$ so that the solution exists there, while (4.5) guarantees that all the matter arrives inside $\{r<2M\}$ within that interval of advanced time.

We first need to improve the a-priori-bound from Lemma 3.2 under the assumption that $a\geq 0$. By Lemma 4.2 and (4.3),

$$\frac{1}{2}ap_1-\frac{1}{2}\frac{1+L/r^2}{p_1}=p^1\leq -P$$

on supp $f_{\mathrm{out}}(v)$ for all $v>0$; $P>0$ by (4.4). The fact that $a\geq 0$ and the assumptions on the support of the initial data imply that

$$p_1\leq\frac{1+L_+/r_0^2}{2P}\leq 1\leq\frac{1+L/r^2}{p_1}\leq\frac{1+L/r^2}{p_1}+ap_1 \tag{4.8}$$

on supp $f_{\mathrm{out}}(v)$. Since $a\geq 0$, (4.8), (2.14), and (2.15) imply that

$$\int_{r_0}^{\infty}r^2T_{11}(v,r)\,dr\leq\int_{r_0}^{\infty}r^2\left(S+aT_{11}\right)(v,r)\,dr\leq\frac{1}{2\pi}M. \tag{4.9}$$

Along any characteristic in supp f_{out} and for $0<v<V$, since $a\geq 0$,

$$\left|\frac{dr}{dv}\right|=\frac{b}{2}\left(\frac{1+L/r^2}{(p_1)^2}-a\right)\leq\frac{1}{2}\left(1+\frac{L_+}{r_0^2}\right)\frac{1}{(p_1)^2}.$$

Now we use the estimate (3.3), but we control the T_{11} integral in (3.2) by (4.9) instead of Lemma 3.2. Hence

$$\left|\frac{dr}{dv}\right|\leq\frac{1}{2}\left(1+\frac{L_+}{r_0^2}\right)\frac{1}{(p_-)^2}\exp\left(\frac{2M}{r_0^2}V+\frac{4M}{r_0}\right).$$

The assumption (4.6) now implies that $r>r_0$ for all particles in supp $f_{\mathrm{out}}(v)$ and $0<v\leq V$. Hence by Corollary 3.3 the solution exists on the interval $[0,V]$.

We now show that at $v=V$ all the matter must be strictly within $r<2M$ so that $a(V,r)<0$ for some $r<2M$, which is a contradiction. First we note that by (4.9),

$$\begin{aligned}b(v,r)&=\exp\left(-4\pi\int_r^{\infty}\eta T_{11}(v,\eta)\,d\eta\right)\\&\geq\exp\left(-\frac{4\pi}{r_0}\int_r^{\infty}\eta^2T_{11}(v,\eta)\,d\eta\right)\geq\exp\left(-\frac{2M}{r_0}\right).\end{aligned}$$

Hence along any characteristic in supp f_{out} by (4.8),

$$\left|\frac{dr}{dv}\right| = -\frac{p^1}{p^0} = b\frac{-p^1}{p_1} \geq \exp\left(-\frac{2M}{r_0}\right)\frac{P}{p_1} \geq \exp\left(-\frac{2M}{r_0}\right)\frac{2P^2}{1+L_+/r_0^2}.$$

Assumption (4.5) now implies that by $v = V$ all characteristics starting in supp $\mathring{f}_{\text{out}}$ are strictly within $r < 2M$, and the proof is complete. □

5 After the formation of a trapped surface

In this section we investigate the Einstein-Vlasov system (1.1)–(1.11) for data which are such that $a(0, r_0) < 0$ for some $r_0 > 0$, i.e., the data already contain a trapped surface. Besides being of interest in itself this problem is relevant in connection with the result of the previous section, where it was shown that such data evolve out of data not containing a trapped surface, but where the argument stopped after a becomes negative somewhere; notice that Theorem 4.1 was proven by contradiction. If we for example want to show that eventually all matter ends up in the region $\{r < 2M\}$ and that $r = 2M$ is complete and the event horizon of the evolving black hole we have to be able to continue the analysis after a has become negative somewhere.

First we check that the system is well posed on the domain $\{r \geq r_0\}$. If f and therefore T_{11} and S are given for $r \geq r_0$, we define b by (2.1) as before. Next we recall (2.5). The desired boundary condition for $ra - r$ at infinity is $\lim_{r\to\infty}(ra - r) = -2M$, and hence

$$a(v, r) = 1 - \frac{2m(v, r)}{r}, \tag{5.1}$$

where

$$m(v, r) = M - \frac{1}{2}\int_r^\infty 4\pi\eta^2(T_{11} + S)\exp\left(-\int_\eta^r 4\pi\sigma T_{11}d\sigma\right)d\eta \tag{5.2}$$

and $M > 0$ denotes the ADM mass of the spacetime.

If $a(0, r_0) < 0$ for some $r_0 > 0$ then by Proposition 2.4, $a(v, r_0) < 0$ for all $v > 0$ for which the solution exists. Since characteristics can only leave but never enter the region $\{r \geq r_0\}$ when followed forward in v,

$$f(v, r, p_1, L) = \mathring{f}((R, P_1, L)(0, v, r, p_1, L))$$

defines the solution of the Vlasov equation (1.7) on $\{r \geq r_0\}$ with initial data $\mathring{f}$.

We now specify the data which we consider in the present section: $\mathring{f} \in C^1([r_0,\infty[\times]0,\infty[\times[0,\infty[)$ is non-negative and compactly supported in $[r_0,\infty[\times]0,\infty[\times[0,\infty[$. Defining

$$\mathring{m}_{\rm out} := 2\pi \int_{r_0}^{\infty} \eta^2 (\mathring{T}_{11} + \mathring{S}) \exp\left(-\int_{\eta}^{r_0} 4\pi\sigma \mathring{T}_{11} d\sigma \right) d\eta$$

we choose some constant

$$M > \mathring{m}_{\rm out} + \frac{r_0}{2}$$

which plays the role of the ADM mass in (5.2). Then

$$a(0,r_0) = 1 - \frac{2m(0,r_0)}{r_0} < 0$$

as desired. Notice that the system can be studied on $\{r \geq r_0\}$ regardless of what represents the mass in the central region $\{r < r_0\}$ as long as that mass is there initially. In particular, this has the advantage that we need not take care that the outer matter does not interfere with a central steady state as was necessary in Theorem 4.1. First, we obtain the following global existence result.

Theorem 5.1 *Initial data $\mathring{f}$ as specified above launch a unique solution $f \in C^1([0,\infty[\times[r_0,\infty[\times]0,\infty[\times[0,\infty[)$ of the Einstein-Vlasov system (1.1)–(1.11).*

Proof. As a first step one can establish a local existence result corresponding to Theorem 2.1, using the arguments indicated there and in the well-posedness discussion above. If $V > 0$ is the length of the maximal existence interval and

$$\sup\left\{ p_1 + \frac{1}{p_1} \mid (r,p_1,L) \in \operatorname{supp} f(v),\ 0 \leq v < V \right\} < \infty$$

then $V = \infty$. But the required lower and upper bounds on p_1 along characteristics follow by the estimates in the proof of Theorem 3.1. $\square$

Let us now define $m_0 := M - \mathring{m}_{\rm out} > 0$ as the mass initially in the region $]0,r_0]$, and assume that

$$\operatorname{supp} \mathring{f} \subset [r_0,R_1] \times [p_-,p_+] \times [0,L_+]$$

where

$$R_1 > r_0,\ 0 < p_- < p_+,\ L_+ := 12 m_0^2,$$

and we require that all particles move inward initially, more precisely,

$$2P > 1 + \frac{L_+}{(2M)^2}, \tag{5.3}$$

where as before,

$$P := -\max\left\{p^1 \mid (r, p_1, L) \in \operatorname{supp} \mathring{f}\right\}.$$

We obtain the following asymptotic behavior for large advanced time.

Theorem 5.2 *For data $\mathring{f}$ as specified above all particles in the region $\{r \geq r_0\}$ are moving inwards, $p^1 < -P$. Moreover, there exist $V^* > 0$ and $r_0 < R^* < 2M$ such that for $v \geq V^*$ all the matter is in the region $\{r < R^*\}$, and $a(v, 2M) = 0$. The line $v \geq V^*$, $r = 2M$ is a radially outgoing null geodesic which is future complete and is the generator of the event horizon of the spacetime for $v \geq V^*$.*

Proof. First we observe that we can apply Lemma 4.2 also in the present situation on the region $\{r \geq r_0\}$; notice that in the proof of that lemma no assumption on the sign of a was made. Hence $p^1 < -P$ for all the particles in that region. This implies that

$$\left|\frac{dr}{dv}\right| = -b\frac{p^1}{p_1} \geq b\frac{P}{p_1}.$$

for these particles. As long as $r \geq 2M$ and hence $a \geq 0$ it follows that $b \geq \exp(-2M/2M) = 1/e$; for this estimate we again rely on (4.9) so we need the estimate corresponding to (4.8), but now only for $r \geq 2M$ which accounts for the relaxed condition (5.3). By (4.3) and since $a \geq 0$ for $r \geq 2M$,

$$\frac{1 + L/r^2}{p_1} = -2p^1 + ap_1 \geq 2P$$

so that

$$\left|\frac{dr}{dv}\right| \geq \frac{2P^2}{e(1 + L/r^2)} \geq \frac{2P^2}{e(1 + L_+/(2M)^2)}$$

as long as $r \geq 2M$. This shows that there exists $V^1 > 0$ such that for $v \geq V_1$ all particles are in the region $\{r \leq 2M\}$. If we choose some $r_0 < R < 2M$ and $V^* := V_1 + 1$, then

$$c := \inf\left\{\frac{b(v, r)}{p_1} \mid V_1 \leq v \leq V^*,\ R \leq r \leq 2M,\ (r, p_1, L) \in \operatorname{supp} f(v)\right\} > 0.$$

This implies that within the interval $[V_1, V^*]$ all particles must have moved by a uniform distance to the left which proves the existence of R^*.

For $v \geq V^*$ and $r \geq R^*$,

$$a(v,r) = 1 - 2M/r,\ b(v,r) = 1.$$

The line $r = 2M,\ v \geq V^*$ is null since $a(v, 2M) = 0$, and examining the geodesic equation for this null geodesic shows that it exists on an interval of affine parameter which is unbounded to the right. The proof is complete. $\square$

If we go back to Theorem 4.1 we see that for a solution as considered in that theorem there exists $v > 0$ such that $f_{\mathrm{out}}(v)$ defines data which satisfy the assumptions of Theorem 5.2. In particular, there exists $0 < r_0 < 2M$ such that $a(v, r_0) < 0$, and we obtain the following corollary.

Corollary 5.3 *The solutions obtained in Theorem 4.1 exhibit the same asymptotic properties as obtained in Theorem 5.2.*

To conclude this paper we exploit our estimates to construct a class of solutions where the initial data represent a black hole surrounded by a shell of Vlasov matter. These solutions will illustrate the fact that event horizons and apparent horizons do in general not coincide.

Remark. The fact that in general the apparent horizon and the event horizon need not coincide is usually illustrated by sending a shell of so-called null dust, i.e., a pressure-less fluid of photons, into a black hole, and the corresponding spacetimes are known as Vaidya spacetimes. The theorem below shows that the corresponding behavior of the horizons can also be achieved with a less artificial matter model.

We consider non-negative and compactly supported data $\mathring{f} \in C^1(]0,\infty[^2 \times [0,\infty[)$ with mass

$$0 < \mathring{m}_{\mathrm{out}} = 2\pi \int_0^\infty \eta^2 (\mathring{T}_{11} + \mathring{S}) \exp\left(- \int_\eta^\infty 4\pi\sigma \mathring{T}_{11}\, d\sigma \right) d\eta < M$$

and

$$\operatorname{supp} \mathring{f} \subset [R_0, R_1] \times [p_-, p_+] \times [0, L_+]$$

where

$$0 < 2M < R_0 < R_1,\ 0 < p_- < p_+,\ L_+ = 12 m_0^2 = 12(M - m_{\mathrm{out}})^2.$$

As before, we require that all particles are initially moving inward sufficiently fast in the sense that (5.3) holds. For $r^* := 2m_0 < 2M$ we have $a(0, r^*) = 0$

while $a(0, r^*) > 0/ < 0$ for $r > r^*/r < r^*$. In other words we have a trapped region $0 < r < r^*$, a black hole of mass m_0, surrounded by a shell of Vlasov matter which is moving inwards, and we obtain the following result.

Theorem 5.4 *Data as specified above launch a unique solution f on the domain $[0, \infty[\times]0, \infty[$. All the particles in $\operatorname{supp} f$ are moving towards the center which they all reach within a finite interval $[0, \tau^*]$ of proper time. There exist $V^* > 0$ and $0 < R^* < 2M$ such that $m(v, R^*) = M$ for $v \geq V^*$. Hence if*

$$R(v) := \sup\{r > 0 \mid a(v, r) < 0\}$$

then $R(v) = r^*$ on some interval of advanced time $[0, v^*]$ with $0 < v^* \leq V^*$, and $R(v) = 2M$ for $v \geq V^*$. On the other hand, the generator of the event horizon is a radially outgoing null geodesic which coincides with $R(v) = 2M$ for $v \geq V^*$, but lies strictly to the right of $R(v) = r^*$ for $v \in [0, v^*]$.

Proof. If we take any $0 < r_0 < r^*$ we can apply Theorem 5.1 to obtain a solution on $[0, \infty[\times]r_0, \infty[$. If we decrease r_0 we get an extension of this solution, and since $0 < r_0 < r^*$ can be arbitrary the have the solution on the asserted domain $[0, \infty[\times]0, \infty[$.

Next we apply Lemma 4.2 which shows that all particles continue to move inwards with $\sup\{p^1 \mid (v, r, p_1, L) \in \operatorname{supp} f\} < 0$ which proves the assertion on their behavior in proper time. That all particles end up strictly inside $\{r < 2M\}$ within a finite interval $[0, V^*]$ of advanced time can be shown exactly as above, and the assertions on the event horizon and the apparent horizon follow. $\square$

Concluding Remark. The analysis in the present paper leaves open the question of what happens at the center $r = 0$. Numerical evidence suggests that in the situation of Theorem 4.1 respectively Theorem 5.2 a spacetime singularity arises where the Kretschmann scalar $R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta}$ blows up at $r = 0$. At the same time the region where $a < 0$ extends all the way to $r = 0$ so that no causal curve can enter $\{r > 0\}$ out of the singularity and strong cosmic censorship holds. It remains to prove these assertions.

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