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VARIOUS APPROACHES TO PRODUCTS OF RESIDUE CURRENTS

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Abstract. We describe various approaches to Coleff-Herrera products of residue currents R^j (of Cauchy-Fantappiè-Leray type) associated to holomorphic mappings f_j. More precisely, we study to which extent (exterior) products of natural regularizations of the individual currents R^j yield regularizations of the corresponding Coleff-Herrera products. Our results hold globally on an arbitrary pure-dimensional complex space.

1. Introduction

Let $f = (f_1, \dots, f_p)$ be a holomorphic mapping from the unit ball $\mathbb{B} \subset \mathbb{C}^n$ to $\mathbb{C}^p$. If $p = 1$ and f is a monomial it is elementary to show, e.g., by integrations by parts or by a Taylor expansion, that the principal value current $\varphi \mapsto \lim_{\epsilon \to 0} \int_{|f|^2 > \epsilon} \varphi/f$, $\varphi \in \mathscr{D}_{n,n}(\mathbb{B})$, exists and defines a $(0,0)$-current $1/f$. From Hironaka's theorem it then follows that such limits exist in general for $p = 1$ and also that $\mathbb{B}$ may be replaced by a complex space, [19]. The $\bar\partial$-image, $\bar\partial(1/f)$, is the residue current of f. It has the useful property that its annihilator ideal is equal to the principal ideal $\langle f \rangle$ and by Stokes' theorem it is given by $\varphi \mapsto \lim_{\epsilon \to 0} \int_{|f|^2 = \epsilon} \varphi/f$, $\varphi \in \mathscr{D}_{n,n-1}(\mathbb{B})$. For $p > 1$, Coleff-Herrera, [16], proposed the following generalization. Define the residue integral

$$I_f^\varphi(\epsilon) = \int_{T(\epsilon)} \varphi/(f_1 \cdots f_p), \quad \varphi \in \mathscr{D}_{n,n-p}, \tag{1}$$

where $T(\epsilon) = \cap_1^p \{|f_j|^2 = \epsilon_j\}$ is oriented as the distinguished boundary of the corresponding polyhedron. Coleff-Herrera showed that if $\epsilon \to 0$ *along an admissible path*, which means that $\epsilon \to 0$ inside $(0,\infty)^p$ in such a way that $\epsilon_j/\epsilon_{j+1}^k \to 0$ for all $k \in \mathbb{N}$, then the limit of $I_f^\varphi(\epsilon)$ exists and defines a $(0,p)$-current. We call this current the Coleff-Herrera product associated to f.

If f defines a complete intersection, Coleff-Herrera showed that the Coleff-Herrera product associated to f depends only in an alternating fashion on the ordering of f, (see [15] and [28] for stronger results implying this). Moreover, in the complete intersection case, it has turned out that the Coleff-Herrera product is a good notion of a multivariable

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residue of f. In particular, its annihilator ideal is equal to $\langle f \rangle$, ([17], [23]). Moreover, the Coleff-Herrera product is the "minimal" extension to a current of Grothendieck's cohomological residue (see, e.g., [23] for the definition) in the sense that it is annihilated by anti-holomorphic functions vanishing on $\{f = 0\}$. This is also related to the fact that the Coleff-Herrera product has the so called *Standard Extension Property*, SEP, which means that it has no mass concentrated on the singular part of $\{f = 0\}$, (see, e.g., [14] and [16]).

The Coleff-Herrera product in the complete intersection case has also found applications, e.g., to explicit division-interpolation formulas and Briançon-Skoda type results ([2], [9]), explicit versions of the fundamental principle ([12]), the $\bar{\partial}$-equation on complex spaces ([5], [18]), explicit Green currents in arithmetic intersection theory [11], etc. However, if f does not define a complete intersection, then the Coleff-Herrera product does not depend in any simple way on the ordering of f. For example, the Coleff-Herrera product associated to (zw, z) is zero while the Coleff-Herrera product associated to (z, zw) is equal to $\bar{\partial}(1/z^2) \wedge \bar{\partial}(1/w)$, which is to be interpreted simply as a tensor product. Nevertheless, it has turned out that the Coleff-Herrera product indeed describes interesting phenomena also in the non-complete intersection case. For instance, the Stückrad-Vogel intersection algorithm in excess intersection theory can be described by the Coleff-Herrera method of multiplying currents; this is shown in a forthcoming paper by M. Andersson, the second author, E. Wulcan, and A. Yger.

In this paper, we describe various approaches to Coleff-Herrera type products, both in general and in the complete intersection case. More precisely, we study to which extent (exterior) products of natural regularizations of the individual currents $1/f_j$ and $\bar{\partial}(1/f_j)$ yield regularizations of the corresponding Coleff-Herrera products. Moreover, we do this globally on a complex space and we also consider products of Cauchy-Fantappiè-Leray type currents.

Let Z be a complex space of pure dimension n, let $E_1^*, \dots, E_p^*$ be hermitian holomorphic line bundles over Z, and let f_j be a holomorphic section of E_j^*. Then $1/f_j$ is a meromorphic section of the dual bundle E_j and we define it as a current on Z by

$$\frac{1}{f_j} := \frac{|f_j|^{2\lambda_j}}{f_j}\Big|_{\lambda_j=0}.$$

The right hand side is a well-defined and analytic current-valued function for $\mathfrak{Re}\, \lambda_j \gg 1$ and we will see in Section 2 that it has a current-valued analytic continuation to $\lambda_j = 0$; it is well-known and easy to show that this definition of the current $1/f$ indeed coincides with the principal value definition of Herrera-Lieberman described above, (see, e.g, Lemma 5 below). The residue current of f_j is then defined as the

$\bar\partial$-image of $1/f_j$, i.e.,

$$\bar\partial\frac{1}{f_j} = \frac{\bar\partial|f_j|^{2\lambda_j}}{f_j}\Big|_{\lambda_j=0}.$$

It follows that $\bar\partial(1/f_j)$ coincides with the limit of the residue integral associated to f_j. A conceptual reason for this equality is that $\bar\partial|f_j|^{2\lambda_j}/f_j$ in fact is the Mellin transform of the residue integral. The technique of using analytic continuation in residue current theory has its roots in the work of Atiyah, [7], and Bernstein-Gel'fand, [13], and has turned out to be very useful. In the context of residue currents it has been developed by several authors, e.g., Barlet-Maire, [8], Yger, [31], Berenstein-Gay-Yger,[10], Passare-Tsikh, [25], and recently by the second author in [28].

We use this technique to define products of the residue currents $\bar\partial(1/f_j)$ by defining recursively

$$\bar\partial\frac{1}{f_k}\wedge\cdots\wedge\bar\partial\frac{1}{f_1} := \frac{\bar\partial|f_k|^{2\lambda_k}}{f_k}\wedge\bar\partial\frac{1}{f_{k-1}}\wedge\cdots\wedge\bar\partial\frac{1}{f_1}\Big|_{\lambda_k=0}. \tag{2}$$

The existence of the right hand side of (2) follows from the fact that this type of products of residue currents are pseudomeromorphic, see Section 2 for details.

A natural way of regularizing the current $\bar\partial(1/f_j)$ inspired by Passare, [22], is as $\bar\partial\chi(|f_j|^2/\epsilon)/f_j$, where χ is a smooth approximation of $\mathbf{1}_{[1,\infty)}$, (the characteristic function of $[1,\infty)$). This regularization corresponds to a mild average of the residue integral $I^{\varphi}_{f_j}(\epsilon)$ and again, it is well-known and easy to show that $\lim_{\epsilon\to 0}\bar\partial\chi(|f_j|^2/\epsilon)/f_j = \bar\partial(1/f_j)$, (see, e.g., Lemma 5). We define the regularized residue integral associated to f by

$$\mathcal{I}^{\varphi}_f(\epsilon) = \int_Z \frac{\bar\partial\chi^{\epsilon}_p}{f_p}\wedge\cdots\wedge\frac{\bar\partial\chi^{\epsilon}_1}{f_1}\wedge\varphi, \tag{3}$$

where $\chi^{\epsilon}_j = \chi(|f_j|^2/\epsilon_j)$ and φ is a test form with values in $\Lambda(E^*_1\oplus\cdots\oplus E^*_p)$. Notice that if $\chi = \mathbf{1}_{[1,\infty)}$ (and the E_j are trivial), then (3) becomes (1).

Theorem 1. *With the notation of Definition 10, we have*

$$\bar\partial\frac{1}{f_p}\wedge\cdots\wedge\bar\partial\frac{1}{f_1}.\,\varphi = \lim_{\epsilon_1\ll\cdots\ll\epsilon_p\to 0}\mathcal{I}^{\varphi}_f(\epsilon).$$

Moreover, if we allow $\chi = \mathbf{1}_{[1,\infty)}$ in (3)*, then the limit of* (3) *along any admissible path also equals $\bar\partial(1/f_p)\wedge\cdots\wedge\bar\partial(1/f_1).\,\varphi$.*

Remark 2. The requirement that $\epsilon\to 0$ along an admissible path if $\chi = \mathbf{1}_{[1,\infty)}$ is not really necessary. However, since it is not completely obvious what, e.g., $(\bar\partial\chi(|f_2|^2/\epsilon_2))/f_2\wedge\bar\partial(1/f_1)$ means if $\chi = \mathbf{1}_{[1,\infty)}$ we prefer to add the requirement.

Theorem 1 thus says that the Coleff-Herrera product associated to f equals the successively defined current in (2) and also that it can be smoothly regularized by (3). It also follows that $\bar\partial(1/f_p)\wedge\cdots\wedge\bar\partial(1/f_1)=\lim_{\epsilon\to 0}(\bar\partial\chi(|f_p|^2/\epsilon)/f_p)\wedge\bar\partial(1/f_{p-1})\wedge\cdots\wedge\bar\partial(1/f_1)$.

Theorem 1 is a special case of Theorem 12 below, where we show a similar result for products of Cauchy-Fantappiè-Leray type currents, which can be thought of as analogues of the currents $1/f_j$ and $\bar\partial(1/f_j)$ in the case when the bundles E_j have ranks >1. Products of such currents were first defined in [30], but the definition of the products given there is in general not the same as our. The proof of Theorem 1 (and Theorem 12) is very similar to the proof of Proposition 1 in [22] but it needs to be modified in our case since extra technical difficulties arise when the metrics of the bundles E_j are not supposed to be trivial.

To give some intuition for Theorem 1, we recall Björk's realization of the Coleff-Herrera product, see, e.g., [14], [3], or [15] for proofs. Given a holomorphic function f_1 in $\mathbb{B}\subset\mathbb{C}^n$, there exists a holomorphic differential operator Q, a holomorphic function h, and a holomorphic $(n-1)$-form dX such that

$$(4)\quad \bar\partial\frac{1}{f_1}.\,\varphi\wedge dz=\lim_{\epsilon\to 0}\int_{\{f_1=0\}}\chi\left(\frac{|h|^2}{\epsilon}\right)\frac{Q(\varphi)\wedge dX}{h},\quad \varphi\in\mathscr{D}_{0,n-1}(\mathbb{B}),$$

where $\chi=\mathbf{1}_{[1,\infty)}$ or a smooth approximation thereof. This representation makes it possible to define the principal value of $1/f_2$ *on* the current $\bar\partial(1/f_1)$. In fact, $\lim_{\epsilon\to 0}\int_{\{f_1=0\}}\chi(|hf_2|^2/\epsilon)Q(\varphi/f_2)\wedge dX/h$ exists and defines a current $(1/f_2)\bar\partial(1/f_1)$. The $\bar\partial$-image of this current is then well-defined and, (e.g., by Theorem 1), it equals $\bar\partial(1/f_2)\wedge\bar\partial(1/f_1)$. But $\bar\partial(1/f_2)\wedge\bar\partial(1/f_1)$ has a representation similar to (4) and one can thus define the principal value of $1/f_3$ on $\bar\partial(1/f_2)\wedge\bar\partial(1/f_1)$, and so on. Intuitively, this procedure corresponds to first letting $\epsilon_1\to 0$ in (1) (or (3)), then letting $\epsilon_2\to 0$ etc.

We now turn to the case that the sections f_j define a complete intersection on Z. Then we know that the Coleff-Herrera product is anti-commutative but we have in fact the following result generalizing Theorem 1 in [15].

Theorem 3. *Assume that $f_1,\dots,f_p$ define a complete intersection. Then*

$$\left|\mathcal{I}_f^\varphi(\epsilon)-\bar\partial\frac{1}{f_p}\wedge\cdots\wedge\bar\partial\frac{1}{f_1}.\,\varphi\right|\le C\|\varphi\|_{C^M}(\epsilon_1^{\omega_1}+\cdots+\epsilon_p^{\omega_p}),$$

where the positive constants M and ω_j only depend on f, Z, and $\operatorname{supp}\varphi$ *while C also depends on the C^M-norm of the χ-functions appearing in the regularized residue integral $\mathcal{I}_f^\varphi$, (3).*

We also have a similar statement for products of Cauchy-Fantappiè-Leray currents, Theorem 13 below. Notice that it is necessary that the χ-functions are smooth; if $p \geq 2$ and $\chi = \mathbf{1}_{[1,\infty)}$ in (3), then the corresponding statement is false in view of the examples by Passare-Tsikh, [24], and Björk, [14].

We also have a generalization of Theorem 1 in [28] to products of Cauchy-Fantappiè-Leray currents, namely our Theorem 14 in Section 2. In the special case of line bundles discussed here, Theorem 14 becomes the following Theorem 4. However, Theorem 4 also follows from the results in [28]; the presence of non-trivial metrics does not cause any additional problems.

Theorem 4. *Assume that $f_1, \ldots, f_p$ define a complete intersection. If φ is a test form, then*

$$\Gamma^{\varphi}(\lambda) := \int \frac{\bar{\partial}|f_p|^{2\lambda_p}}{f_p} \wedge \cdots \wedge \frac{\bar{\partial}|f_1|^{2\lambda_1}}{f_1} \wedge \varphi$$

has an analytic continuation to a neighborhood of the half space $\{\mathfrak{Re}\,\lambda_j \geq 0\}$.

In the classical case, $\Gamma^{\varphi}(\lambda)$ is the iterated Mellin transform of the residue integral (1) and it is well known that it has a meromorphic continuation to $\mathbb{C}^p$ that is analytic in $\cap_1^p\{\mathfrak{Re}\,\lambda_j > 0\}$; (this is also true in the non-complete intersection case). The analyticity of $\Gamma^{\varphi}(\lambda)$ in a neighborhood of 0 when $p = 2$ was proved by Berenstein-Yger (see, e.g., [9]).

In Section 2, we give the necessary background and the general formulations of our results. Section 3 contains the proof of Theorems 1 and 12. The proof of Theorems 3, 13, and 14 is the content of Section 4; the crucial part is Lemma 19 which enables us to effectively use the assumption about complete intersection.

2. Formulation of the general results

Let $E_1^*, \ldots, E_q^*$ be holomorphic hermitian vector bundles over a reduced complex space Z of pure dimension n. The metrics are supposed to be smooth in the following sense. We say that φ is a smooth (p,q)-form on Z if φ is smooth on Z_{reg}, and for a neighborhood of any $p \in Z$, there is a smooth (p,q)-form $\tilde{\varphi}$ in an ambient complex manifold such that the pullback of $\tilde{\varphi}$ to Z_{reg} coincides with $\varphi|_{Z_{reg}}$ close to p. The (p,q)-test forms on Z, $\mathscr{D}_{p,q}(Z)$, are defined as the smooth compactly supported (p,q)-forms (with a suitable topology) and the (p,q)-currents on Z, $\mathscr{D}'_{p,q}(Z)$, is the dual of $\mathscr{D}_{n-p,n-q}(Z)$; see, e.g., [21] for a more thorough discussion.

We recall from [6] the definition of *pseudomeromorphic* currents, $\mathcal{PM}$. A current is pseudomeromorphic if it is a (locally finite) sum of push-forwards of elementary currents under modifications of Z. A current, T, is elementary if it is a current on $\mathbb{C}^n_x$ of the form

$$T = \frac{1}{x^\alpha} \bigwedge_{\beta_j \neq 0} \bar\partial \frac{1}{x_j^{\beta_j}} \wedge \vartheta, \tag{5}$$

where α and β are multiindices with disjoint supports and ϑ is a smooth compactly supported (possibly bundle valued) form. (We are abusing notation slightly; $\Lambda_{\beta_j \neq 0} \bar\partial(1/x_j^{\beta_j})$ is only defined up to a sign.) Elementary currents are thus merely tensor products of one-variable principal value currents $1/x_i^{\alpha_i}$ and $\bar\partial$-images of such (modulo smooth forms).

Lemma 5. *Let f be a holomorphic function, and let $T \in \mathcal{PM}(Z)$. If $\tilde f$ is a holomorphic function such that $\{\tilde f = 0\} = \{f = 0\}$ and v is a smooth non-zero function, then $(|\tilde f v|^{2\lambda}/f)T$ and $(\bar\partial |\tilde f v|^{2\lambda}/f) \wedge T$ have current-valued analytic continuations to $\lambda = 0$ and the values at $\lambda = 0$ are pseudomeromorphic and independent of the choices of $\tilde f$ and v. Moreover, if $\chi = \mathbf{1}_{[1,\infty)}$, or a smooth approximation thereof, then*

$$\left.\frac{|\tilde f v|^{2\lambda}}{f} T\right|_{\lambda=0} = \lim_{\epsilon \to 0^+} \frac{\chi^\epsilon}{f} T \quad \textit{and} \quad \left.\frac{\bar\partial |\tilde f v|^{2\lambda}}{f} \wedge T\right|_{\lambda=0} = \lim_{\epsilon \to 0^+} \frac{\bar\partial \chi^\epsilon}{f} \wedge T, \tag{6}$$

where $\chi^\epsilon = \chi(|\tilde f v|^2/\epsilon)$.

Proof. The first part is essentially Proposition 2.1 in [6], except that there, Z is a complex manifold, $\tilde f = f$ and $v \equiv 1$. However, with suitable resolutions of singularities, the proof in [6] goes through in the same way in our situation, as long as we observe that in $\mathbb{C}$

$$\frac{|x^{\alpha'} v|^{2\lambda}}{x^\alpha} \frac{1}{x^\beta} \quad \text{and} \quad \frac{|x^{\alpha'} v|^{2\lambda}}{x^\alpha} \bar\partial \frac{1}{x^\beta}$$

have analytic continuations to $\lambda = 0$, and the values at $\lambda = 0$ are $1/x^{\alpha+\beta}$ and 0 respectively, independently of α' and v, as long as $\alpha' > 0$ and $v \neq 0$ (and similarly with $\bar\partial |x^{\alpha'} v|^{2\lambda}/x^\alpha$).

By Leibniz rule, it is enough to consider the first equality in (6), since if we have proved the first equality, then

$$\begin{aligned} \lim_{\epsilon \to 0} \frac{\bar\partial \chi^\epsilon}{f} \wedge T &= \lim_{\epsilon \to 0} \bar\partial \left(\frac{\chi^\epsilon}{f} T \right) - \frac{\chi^\epsilon}{f} \bar\partial T \\ &= \left.\left(\bar\partial \left(\frac{|\tilde f v|^{2\lambda}}{f} T \right) - \frac{|\tilde f v|^{2\lambda}}{f} \bar\partial T \right)\right|_{\lambda=0} = \left.\frac{\bar\partial |\tilde f v|^{2\lambda}}{f} \wedge T\right|_{\lambda=0}. \end{aligned}$$

To prove the first equality in (6), we observe first that in the same way as in the first part, we can assume that $f = x^\gamma u$ and $\tilde f = x^{\tilde\gamma} \tilde u$, where u and $\tilde u$ are non-zero holomorphic functions. Since T is a sum of push-forwards of elementary currents, we can assume that T is of the form (5). Note that if $\operatorname{supp} \gamma \cap \operatorname{supp} \beta \neq \emptyset$, then $(|\tilde f v|^{2\lambda}/f)T = 0$ for $\mathfrak{Re}\, \lambda \gg 1$ and $(\chi(|\tilde f v|^2/\epsilon)/f)T = 0$ for $\epsilon > 0$, since $\operatorname{supp} T \subseteq \{x_i = 0, i \in$

$\operatorname{supp}\beta\}$. Thus, we can assume that $\operatorname{supp}\gamma \cap \operatorname{supp}\beta = \emptyset$. By a smooth (but non-holomorphic) change of variables, as in Section 3 (equations (15)), we can assume that $|\tilde{u}v|^2 \equiv 1$. Thus, since $(|x^{\tilde{\gamma}}|^{2\lambda}/x^{\gamma})(1/x^{\alpha})$, $(\chi(|x^{\tilde{\gamma}}|^2/\epsilon)/x^{\gamma})(1/x^{\alpha})$ depend on variables disjoint from the ones that $\wedge_{\beta_i \neq 0}\bar{\partial}(1/x_i^{\beta_i})$ depends on, it is enough to prove that

$$\left.\frac{|x^{\tilde{\gamma}}|^{2\lambda}}{x^{\gamma}}\frac{1}{x^{\alpha}}\right|_{\lambda=0} = \lim_{\epsilon\to 0}\frac{\chi(|x^{\tilde{\gamma}}|^2/\epsilon)}{x^{\gamma}}\frac{1}{x^{\alpha}},$$

which is Lemma 2 in [15]. □

Let f_j be a holomorphic section of E_j^*, $j = 1, \ldots, q$, and let s_j be the section of E_j with pointwise minimal norm such that $f_j \cdot s_j = |f_j|^2$. Outside $\{f_j = 0\}$, define

$$u_k^j = \frac{s_j \wedge (\bar{\partial}s_j)^{k-1}}{|f_j|^{2k}}.$$

It is easily seen that if $f_j = f_j^0 f_j'$, where f_j^0 is a holomorphic function and f_j' is a non-vanishing section, then $u_k^j = (1/f_j^0)^k (u')_k^j$, where $(u')_k^j$ is smooth across $\{f_j = 0\}$. We let

$$U^j = \sum_{k=1}^{\infty} |\tilde{f}_j|^{2\lambda} u_k^j \Big|_{\lambda=0}, \tag{7}$$

where $\tilde{f}_j$ is any holomorphic section of E_j^* such that $\{\tilde{f}_j = 0\} = \{f_j = 0\}$. The existence of the analytic continuation is a local statement, so we can assume that $f_j = \sum f_{j,k}\mathfrak{e}_{j,k}^*$, where $\mathfrak{e}_{j,k}^*$ is a local holomorphic frame for E_j^*. After principalization we can assume that the ideal $\langle f_{j,1}, \ldots, f_{j,k_j}\rangle$ is generated by, e.g., $f_{j,0}$. By the representation $u_k^j = (1/f_{j,0})^k (u')_k^j$, the existence of the analytic continuation of U^j in (7) then follows from Lemma 5. Let U_k^j denote the term of U^j that takes values in $\Lambda^k E_j$; U_k^j is thus a $(0, k-1)$-current with values in $\Lambda^k E_j$. Let δ_{f_j} denote interior multiplication with f_j and put $\nabla_{f_j} = \delta_{f_j} - \bar{\partial}$; it is not hard to verify that $\nabla_{f_j} U = 1$ outside $f_j = 0$. We define the Cauchy-Fantappiè-Leray type residue current, R^j, of f_j by $R^j = 1 - \nabla_{f_j} U^j$. One readily checks that

$$\begin{aligned} R^j &= R_0^j + \sum_{k=1}^{\infty} R_k^j \\ &= (1 - |\tilde{f}_j|^{2\lambda})|_{\lambda=0} + \sum_{k=1}^{\infty} \bar{\partial}|\tilde{f}_j|^{2\lambda} \wedge \left.\frac{s_j \wedge (\bar{\partial}s_j)^{k-1}}{|f_j|^{2k}}\right|_{\lambda=0}, \end{aligned} \tag{8}$$

where, as above, $\tilde{f}_j$ is a holomorphic section such that $\{\tilde{f}_j = 0\} = \{f_j = 0\}$.

Remark 6. Notice that if E_j has rank 1, then U_j simply equals $1/f_j$ and $R^j = 1 - \nabla_{f_j}(1/f_j) = 1 - f_j \cdot (1/f_j) + \bar\partial(1/f_j) = \bar\partial(1/f_j)$.

We now define a non-commutative calculus for the currents U^i_k and R^j_ℓ recursively as follows.

Definition 7. If T is a product of some U^i_k:s and R^j_ℓ:s, then we define

- $$U^j_k \wedge T = |\tilde f_j|^{2\lambda} \frac{s_j \wedge (\bar\partial s_j)^{k-1}}{|f_j|^{2k}} \wedge T\Big|_{\lambda=0}$$
- $$R^j_0 \wedge T = (1 - |\tilde f_j|^{2\lambda})T\Big|_{\lambda=0}$$
- $$R^j_k \wedge T = \bar\partial|\tilde f_j|^{2\lambda} \wedge \frac{s_j \wedge (\bar\partial s_j)^{k-1}}{|f_j|^{2k}} \wedge T\Big|_{\lambda=0},$$

where $\tilde f_j$ is any holomorphic section of E^*_j with $\{\tilde f_j = 0\} = \{f_j = 0\}$.

Note first that U^j and R^j are pseudomeromorphic. Hence, in the same way as the analytic continuation in the definition of U^j and R^j exist, we see that the analytic continuations in the definition of the currents in Definition 7 exist and also are pseudomeromorphic.

Remark 8. Under assumptions about complete intersection, these products have the suggestive commutation properties, e.g., if $\operatorname{codim}\{f_i = f_j = 0\} = \operatorname{rank} E_i + \operatorname{rank} E_j$, then $R^i_k \wedge R^j_\ell = R^j_\ell \wedge R^i_k$, $R^i_k \wedge U^j_\ell = U^j_\ell \wedge R^i_k$, and $U^i_k \wedge U^j_\ell = -U^j_\ell \wedge U^i_k$, (see, e.g., [4]). In general, there are no simple relations. However, products involving only U:s are always anti-commutative.

Now, consider collections $U = \{U^q_{k_q}, \dots, U^{p+1}_{k_{p+1}}\}$ and $R = \{R^p_{k_p}, \dots, R^1_{k_1}\}$ and put $(P_q, \dots, P_1) = (U^q_{k_q}, \dots, R^p_{k_p}, \dots, R^1_{k_1})$. For a permutation ν of $\{1, \dots, q\}$ we define

$$(UR)^\nu = P_{\nu(q)} \wedge \dots \wedge P_{\nu(1)}. \tag{9}$$

We will describe various natural ways to regularize products of this kind. For $q = 1$ we see from (7) and (8) that we have a natural λ-regularization, P^λ_j, of P_j and from Definition 7 we have $(UR)^\nu = P^{\lambda_q}_{\nu(q)} \wedge \dots \wedge P^{\lambda_1}_{\nu(1)}|_{\lambda_1=0} \cdots |_{\lambda_q=0}$. We have the following result that is proved in a forthcoming paper by M. Andersson, the second author, E. Wulcan, and A. Yger.

Theorem 9. *Let $a_1 > \dots > a_q > 0$ be integers and λ a complex variable. Then we have*

$$(UR)^\nu = P^{\lambda^{a_q}}_{\nu(q)} \wedge \dots \wedge P^{\lambda^{a_1}}_{\nu(1)}\big|_{\lambda=0}.$$

We see that one does not need to put $\lambda_1 = 0$ first, then $\lambda_2 = 0$ etc., one just has to ensure that λ_1 tends to zero much faster than λ_2 and so on. The current $(UR)^\nu$ can thus be obtained as the value at zero of a one-variable ζ-type function. From an algebraic point of view, this is desirable since one can derive functional equations and use Bernstein-Sato theory to study $(UR)^\nu$.

There are also natural ϵ-regularizations of the currents U^i_k and R^j_ℓ inspired by [16] and [22]. Let $\chi = \mathbf{1}_{[1,\infty)}$, or a smooth approximation thereof that is 0 close to 0 and 1 close to ∞. It follows from [27], or after principalization from Lemma 5, that

$$U^j_k = \lim_{\epsilon\to 0^+} \chi(|\tilde{f}_j|^2/\epsilon)\frac{s_j\wedge(\bar\partial s_j)^{k-1}}{|f_j|^{2k}}. \tag{10}$$

$$R^j_k = \lim_{\epsilon\to 0^+} \bar\partial\chi(|\tilde{f}_j|^2/\epsilon)\wedge\frac{s_j\wedge(\bar\partial s_j)^{k-1}}{|f_j|^{2k}},\ k>0, \tag{11}$$

and similarly for $k = 0$; as usual, $\{\tilde{f}_j = 0\} = \{f_j = 0\}$. Of course, the limits are in the current sense and if $\chi = \mathbf{1}_{[1,\infty)}$, then ϵ is supposed to be a regular value for $|f_j|^2$ and $\bar\partial\chi(|f_j|^2/\epsilon)$ is to be interpreted as integration over the manifold $|f_j|^2 = \epsilon$. We denote the regularizations given by (10) and (11) by P^ϵ_j.

Definition 10. Let ϑ be a function defined on $(0,\infty)^q$. We let

$$\lim_{\epsilon_1\ll\cdots\ll\epsilon_q\to 0}\vartheta(\epsilon_1,\ldots,\epsilon_q)$$

denote the limit (if it exists and is well-defined) of ϑ along any path $\delta\mapsto\epsilon(\delta)$ towards the origin such that for all $k\in\mathbb{N}$ and $j = 2,\ldots,q$ there are positive constants C_{jk} such that $\epsilon_{j-1}(\delta)\le C_{jk}\,\epsilon^k_j(\delta)$. Here, we extend the domain of definition of ϑ to points $(0,\ldots,0,\epsilon_{m+1},\ldots,\epsilon_q)$, where $\epsilon_{m+1},\ldots,\epsilon_q > 0$, by defining recursively

$$\vartheta(0,\ldots,0,\epsilon_{m+1},\ldots,\epsilon_q) = \lim_{\epsilon_m\to 0}\vartheta(0,\ldots,0,\epsilon_m,\epsilon_{m+1},\ldots,\epsilon_q),$$

if the limits exist.

Remark 11. The paths considered here are very similar to the admissible paths of Coleff-Herrera, but we also allow paths where, e.g., ϵ_1 attains the value 0 before the other parameters tend to zero.

We have the following analogue of Theorem 9.

Theorem 12. *Let $U = \{U^q_{k_q},\ldots,U^{p+1}_{k_{p+1}}\}$ and $R = \{R^p_{k_p},\ldots,R^1_{k_1}\}$ be collections of currents defined in* (7) *and* (8). *Let ν be a permutation of $\{1,\ldots,q\}$ and let $(UR)^\nu$ be the product defined in* (9). *Then*

$$(UR)^\nu = \lim_{\epsilon_1\ll\cdots\ll\epsilon_q\to 0} P^{\epsilon_q}_{\nu(q)}\wedge\cdots\wedge P^{\epsilon_1}_{\nu(1)},$$

where, as above, $(P_q,\dots,P_1)=(U^q_{k_q},\dots,R^p_{k_p},\dots,R^1_{k_1})$ *and* $P^{\epsilon_j}_{\nu(j)}$ *is an* ϵ*-regularization defined in* (10) *and* (11) *of* $P_{\nu(j)}$*. If* $\chi=\mathbf{1}_{[1,\infty)}$*, we require that* $\epsilon\to 0$ *along an admissible path.*

2.1. **The complete intersection case.** Now assume that $f_1,\dots,f_q$ defines a complete intersection, i.e., that $\operatorname{codim}\{f_1=\cdots=f_q=0\}=e_1+\cdots+e_q$, where $e_j=\operatorname{rank}E_j$. Then we know that the calculus defined in Definition 7 satisfies the suggestive commutation properties, but we have in fact the following much stronger results.

Theorem 13. *Assume that* $f_1,\dots,f_q$ *defines a complete intersection on* Z*, let* $(P_1,\dots,P_q)=(R^1_{k_1},\dots,R^p_{k_p},U^{p+1}_{k_{p+1}},\dots,U^q_{k_q})$*, and let* $P^{\epsilon_j}_j$ *be an* ϵ*-regularization of* P_j *defined by* (10) *and* (11) *with smooth* χ*-functions. Then we have*

$$\left|\int_Z P_1^{\epsilon_1}\wedge\cdots\wedge P_q^{\epsilon_q}\wedge\varphi - P_1\wedge\cdots\wedge P_q.\varphi\right|\le C\|\varphi\|_M(\epsilon_1^\omega+\cdots+\epsilon_q^\omega),$$

where M *and* ω *only depend on* $f_1,\dots,f_q$*,* Z*, and* $\operatorname{supp}\varphi$ *while* C *also depends on the* C^M*-norm of the* χ*-functions.*

Theorem 14. *Assume that* $f_1,\dots,f_q$ *defines a complete intersection on* Z*, let* $(P_1,\dots,P_q)=(R^1_{k_1},\dots,R^p_{k_p},U^{p+1}_{k_{p+1}},\dots,U^q_{k_q})$*, and let* $P^{\lambda_j}_j$ *be the* λ*-regularization of* P_j *given by* (7) *and* (8)*. Then the current valued function*

$$\lambda\mapsto P_1^{\lambda_1}\wedge\cdots\wedge P_q^{\lambda_q},$$

a priori defined for $\mathfrak{Re}\,\lambda_j\gg 1$*, has an analytic continuation to a neighborhood of the half-space* $\cap_1^q\{\mathfrak{Re}\,\lambda_j\ge 0\}$*.*

Remark 15. In case the E_j:s are trivial with trivial metrics, Theorems 13 and 14 follow quite easily from, respectively, Theorem 1 in [15] and Theorem 1 in [28] by taking averages. As an illustration, let $\varepsilon_1,\dots,\varepsilon_r$ be a nonsense basis and let $f_1,\dots,f_r$ be holomorphic functions. Then we can write $s=\bar f\cdot\varepsilon$ and so $u_k=(\bar f\cdot\varepsilon)\wedge(d\bar f\cdot\varepsilon)^{k-1}/|f|^{2k}$. A standard computation shows that

$$\int_{\alpha\in\mathbb{CP}^{r-1}}\frac{|\alpha\cdot f|^{2\lambda}\alpha\cdot\varepsilon}{(\alpha\cdot f)|\alpha|^{2\lambda}}dV=A(\lambda)|f|^{2\lambda}\frac{\bar f\cdot\varepsilon}{|f|^2},$$

where dV is the (normalized) Fubini-Study volume form and A is holomorphic with $A(0)=1$. It follows that

$$\int_{\alpha_1,\dots,\alpha_k\in\mathbb{CP}^{r-1}}\bigwedge_1^k\frac{\bar\partial|\alpha_j\cdot f|^{2\lambda}}{\alpha_j\cdot f}\wedge\frac{\alpha_j\cdot\varepsilon}{|\alpha_j|^{2\lambda}}dV(\alpha_j)=A(\lambda)^k\bar\partial(|f|^{2k\lambda}u_k).$$

Elaborating this formula and using Theorem 1 in [28] one can show Theorem 14 in the case of trivial E_j:s with trivial metrics. The general case can probably also be handled in a similar manner but the computations become more involved and we prefer to give direct proofs.

3. Proof of Theorem 12

We start by making a Hironaka resolution of singularities, [20], of Z such that the pre-image of $\cup_j\{f_j=0\}$ has normal crossings. We then make further toric resolutions (e.g., as in [26]) such that, in local charts, the pullback of each f_i is a monomial, x^{α_i}, times a non-vanishing holomorphic tuple. One checks that the pullback of P_j^ϵ is of one of the following forms:

$$\frac{\chi(|x^{\tilde{\alpha}}|^2\xi/\epsilon)}{x^\alpha}\,\vartheta,\quad 1-\chi(|x^{\tilde{\alpha}}|^2\xi/\epsilon),\quad \frac{\bar\partial\chi(|x^{\tilde{\alpha}}|^2\xi/\epsilon)}{x^\alpha}\wedge\vartheta,$$

where ξ is smooth and positive, $\operatorname{supp}\tilde{\alpha}=\operatorname{supp}\alpha$, and ϑ is a smooth bundle valued form; by localizing on the blow-up we may also suppose that ϑ has as small support as we wish. If the χ-functions are smooth, the following special case of Theorem 12 now immediately follows from Lemma 5:

$$(UR)^\nu=\lim_{\epsilon_q\to0}\cdots\lim_{\epsilon_1\to0}P^{\epsilon_q}_{\nu(q)}\wedge\cdots\wedge P^{\epsilon_1}_{\nu(1)}. \tag{12}$$

For smooth χ-functions we put

$$\mathcal{I}(\epsilon)=\int\frac{\bar\partial\chi_1^\epsilon\wedge\cdots\wedge\bar\partial\chi_p^\epsilon\chi_{p+1}^\epsilon\cdots\chi_q^\epsilon}{x^{\alpha_1+\cdots+\alpha_p+\cdots+\alpha_{q'}}}\wedge\varphi,$$

where $q'\le q$, φ is a smooth $(n,n-p)$-form with support close to the origin, and $\chi_j^\epsilon=\chi(|x^{\tilde{\alpha_j}}|^2\xi_j/\epsilon_j)$ for smooth positive ξ_j. We note that we may replace the $\bar\partial$ in $\mathcal{I}(\epsilon)$ by d for bidegree reasons. In case $\chi=\mathbf{1}_{[1,\infty)}$ we denote the corresponding integral by $I(\epsilon)$. We also put $\mathcal{I}^\nu(\epsilon_1,\ldots,\epsilon_q)=\mathcal{I}(\epsilon_{\nu(1)},\ldots,\epsilon_{\nu(q)})$ and similarly for I^ν. In view of (12), the special case of Theorem 12 when the χ-functions are smooth will be proved if we can show that

$$\lim_{\epsilon_1\ll\cdots\ll\epsilon_q\to0}\mathcal{I}^\nu(\epsilon) \tag{13}$$

exists. The case with $\chi=\mathbf{1}_{[1,\infty)}$ will then follow if we can show

$$\lim_{\delta\to0}(\mathcal{I}^\nu(\epsilon(\delta))-I^\nu(\epsilon(\delta)))=0, \tag{14}$$

where $\delta\mapsto\epsilon(\delta)$ is any admissible path.

For notational convenience, we will consider $\mathcal{I}^\nu(\epsilon)$ (unless otherwise stated), but our arguments apply just as well to $I^\nu(\epsilon)$ until we arrive at the integral (18).

Denote by $\tilde{A}$ the $q\times n$-matrix with rows $\tilde{\alpha}_i$. We will first show that we can assume that $\tilde{A}$ has full rank. The idea is the same as in [16] and [22], however because of the paths along which our limits are taken, we have to modify the argument slightly. The following lemma follows from the proof of Lemma III.12.1 in [29].

Lemma 16. *Assume that α is a $q \times n$-matrix with rows α_i such that there exists $(v_1, \dots, v_q) \neq 0$ with $\sum v_i \alpha_i = 0$. Let $j = \min\{i; v_i \neq 0\}$. Then there exist constants $C, c > 0$ such that if $\epsilon_j < C(\epsilon_{j+1} \dots \epsilon_q)^c$, then $\chi(|x^{\alpha_j}|^2 \xi_j/\epsilon_j) \equiv 1$ and $\bar\partial \chi(|x^{\alpha_j}|^2 \xi_j/\epsilon_j) \equiv 0$ for all $x \in \Delta \cap \{|x^{\alpha_i}|^2 \geq C_i \epsilon_i, i = j+1, \dots, q\}$, where Δ is the unit polydisc.*

Assume that $\tilde{A}$ does not have full rank, and let v be a column vector such that $v^t \tilde{A} = 0$. Since $(\epsilon_1, \dots, \epsilon_q)$ is replaced by $(\epsilon_{\nu(1)}, \dots, \epsilon_{\nu(q)})$ in $\mathcal{I}^\nu(\epsilon)$, we choose instead j_0 such that $\nu(j_0) \leq \nu(i)$ for all i such that $v_i \neq 0$. If $j_0 \leq p$, we let $\widetilde{\mathcal{I}}^\nu(\epsilon) = 0$, and if $j_0 \geq p+1$, we let $\widetilde{\mathcal{I}}^\nu(\epsilon)$ be $\mathcal{I}^\nu(\epsilon)$ but with $\chi^\epsilon_{j_0}$ replaced by 1. If $\epsilon = \epsilon(\delta)$ is such that $\epsilon_{\nu(j_0)} > 0$, then $\mathcal{I}^\nu(\epsilon)$ is a current acting on a test form with support on a set of the form

$$\Delta \cap \{|x^{\alpha_i}|^2 \geq C_i \epsilon_{\nu(i)}; \text{for all } i \text{ such that } \nu(i) \geq \nu(j_0)\}.$$

In particular, if $\epsilon_{\nu(j_0)}(\delta)$ is sufficiently small compared to $(\epsilon_{\nu(j_0)+1}(\delta), \dots, \epsilon_q(\delta))$, then by Lemma 16, if $j_0 \leq p$, the factor $\bar\partial \chi^\epsilon_{j_0}$ is identically 0, and if $j_0 \geq p+1$, the factor $\chi^\epsilon_{j_0}$ is identically 1 and thus is equal to $\widetilde{\mathcal{I}}^\nu(\epsilon)$ for such ϵ. Similarly, if $\epsilon_{\nu(j_0)} = 0$, we have that $\mathcal{I}^\nu(\epsilon)$ is defined as a limit along $\epsilon_{\nu(j_0)} \to 0$, with $\epsilon_{\nu(j_0)+1}, \dots, \epsilon_q$ fixed and in the limit we get again that for sufficiently small $\epsilon_{\nu(j_0)}$, we can replace $\mathcal{I}^\nu(\epsilon)$ by $\widetilde{\mathcal{I}}^\nu(\epsilon)$. Thus we have

$$\lim_{\epsilon_1 \ll \dots \ll \epsilon_q \to 0} \mathcal{I}^\nu(\epsilon) = \lim_{\epsilon_1 \ll \dots \ll \epsilon_q \to 0} \widetilde{\mathcal{I}}^\nu(\epsilon),$$

and we have reduced to the case that $\tilde{A}$ is a $(q-1) \times n$-matrix of the same rank. We continue this procedure until $\tilde{A}$ has full rank.

By re-numbering the coordinates, we may suppose that the minor $A = (\tilde\alpha_{ij})_{1 \leq i,j \leq q}$ of $\tilde{A}$ is invertible and we put $A^{-1} = B = (b_{ij})$. We now use complex notation to make a non-holomorphic, but smooth change of variables:

$$\begin{aligned} (15) \qquad & y_1 = x_1 \, \xi^{b_1/2}, \dots, y_q = x_q \, \xi^{b_q/2}, y_{q+1} = x_{q+1}, \dots, y_n = x_n, \\ & \bar{y}_1 = \bar{x}_1 \, \xi^{b_1/2}, \dots, \bar{y}_q = \bar{x}_q \, \xi^{b_q/2}, \bar{y}_{q+1} = \bar{x}_{q+1}, \dots, \bar{y}_n = \bar{x}_n, \end{aligned}$$

where $\xi^{b_i/2} = \xi_1^{b_{i1}/2} \cdots \xi_q^{b_{iq}/2}$. One easily checks that $dy \wedge d\bar{y} = \xi^{b_1} \cdots \xi^{b_q}$ $dx \wedge d\bar{x} + O(|x|)$, so (15) defines a smooth change of variables between neighborhoods of the origin. A simple linear algebra computation then shows that $|x^{\tilde\alpha_i}|^2 \xi_i = |y^{\tilde\alpha_i}|^2$. Of course, this change of variables does not preserve bidegrees so $\varphi(y)$ is merely a smooth compactly supported $(2n-p)$-form. We thus have

$$(16) \qquad \mathcal{I}^\nu(\epsilon) = \int_\Delta \frac{d\chi^\epsilon_1 \wedge \dots \wedge d\chi^\epsilon_p \chi^\epsilon_{p+1} \cdots \chi^\epsilon_q}{y^{\alpha_1 + \dots + \alpha_p + \dots + \alpha_{q'}}} \wedge \varphi'(y),$$

where $\chi^\epsilon_j = \chi(|y^{\tilde\alpha_j}|^2/\epsilon_{\nu(j)})$ and $\varphi'(y) = \sum_{|I|+|J|=2n-p} \psi_{IJ} dy_I \wedge d\bar{y}_J$. By linearity we may assume that the sum only consists of one term $\varphi'(y) =$

$\psi dy_K \wedge d\bar{y}_L$, and by scaling, we may assume that $\operatorname{supp}\psi \subseteq \Delta$, Δ being the unit polydisc. By Lemma 2.4 in [16], we can write the function ψ as

$$\psi(y) = \sum_{I+J<\sum_1^{q'}\alpha_j-\mathbf{1}} \psi_{IJ}\, y^I \bar{y}^J + \sum_{I+J=\sum_1^{q'}\alpha_j-\mathbf{1}} \psi_{IJ}\, y^I \bar{y}^J, \tag{17}$$

where $a < b$ for tuples a and b means that $a_i < b_i$ for all i. In the decomposition (17) each of the smooth functions ψ_{IJ} in the first sum on the left hand side is independent of some variable. We now show that this implies that the first sum on the left hand side of (17) does not contribute to the integral (16). In case $\varphi'(y)$ has bidegree $(n, n-p)$ this is a well-known fact but we must show it for an arbitrary $(2n-p)$-form.

We change to polar coordinates:

$$dy_K \wedge d\bar{y}_L = d(r_{K_1}e^{i\theta_{K_1}}) \wedge \cdots \wedge d(r_{L_1}e^{-i\theta_{L_1}}) \wedge \cdots$$

Since χ_j^ϵ in (16) is independent of θ, it follows that we must have full degree $= n$ in $d\theta$. The only terms in the expansion of $dy_K \wedge d\bar{y}_L$ above that will contribute to (16) are therefore of the form

$$c\, r_1 \cdots r_n e^{i\theta\cdot\gamma}\, dr_M \wedge d\theta,$$

where $|M| = n-p$, c is a constant, and γ is a multiindex with entries equal to 1, -1, or 0. Substituting this and a term $\psi_{IJ}y^I\bar{y}^J = \psi_{IJ}r^{I+J}e^{i\theta\cdot(I-J)}$ from (17) into (16) gives rise to an "inner" θ-integral (by Fubini's theorem):

$$\mathscr{I}_{IJ}(r) = \int_{\theta\in[0,2\pi)^n} \psi_{IJ}(r,\theta)\; e^{i\theta\cdot(I-J-\sum_1^{q'}\alpha_j+\gamma)}\, d\theta.$$

If $I+J < \sum_1^{q'}\alpha_j - \mathbf{1}$, then $I-J-\sum_1^{q'}\alpha_j+\gamma < 0$ and ψ_{IJ} is independent of some $y_j = r_je^{i\theta_j}$. Integrating over $\theta_j \in [0,2\pi)$ thus yields $\mathscr{I}_{IJ} = 0$ if $I+J < \sum_1^{q'}\alpha_j - \mathbf{1}$. If instead $I+J = \sum_1^{q'}\alpha_j - \mathbf{1}$, then $\mathscr{I}_{IJ}(r)$ is smooth on $[0,\infty)^n$.

Summing up, we see that we can write (16) as

$$\mathcal{I}^\nu(\epsilon) = \int_{r\in(0,1)^n} d\chi_1^\epsilon \wedge \cdots \wedge d\chi_p^\epsilon \chi_{p+1}^\epsilon \cdots \chi_q^\epsilon\, \mathscr{I}(r)\, dr_M, \tag{18}$$

where $\chi_j^\epsilon = \chi(r^{2\alpha_j}/\epsilon_{\nu(j)})$, $\mathscr{I}$ is smooth, and $|M| = n-p$.

After these reductions, the integral (18) we arrive at is the same as equation (16) in [22], and we will use the fact proven there, that $\lim_{\delta\to 0}\mathcal{I}^\nu(\epsilon(\delta))$ exists along any admissible path $\epsilon(\delta)$, and is well-defined independently of the choice of admissible path. (This is not exactly what is proven there, but the fact that if $b \in \mathbb{Q}^p$, then $\lim_{\delta\to 0}\epsilon(\delta)^b$ is either 0 or ∞ independently of the admissible path chosen is the only addition we need to make for the argument to go through in our case.) Using this, if we let $\epsilon(\delta)$ be any admissible path, we will show by induction over q that

$$\lim_{\epsilon_1 \ll \cdots \ll \epsilon_q \to 0} \mathcal{I}^\nu(\epsilon) = \lim_{\delta \to 0} \mathcal{I}^\nu(\epsilon(\delta)).$$

For $q = 1$ this is trivially true, so we assume $q > 1$. Let ϵ^k be any sequence satisfying the conditions in Definition 10. Consider a fixed k, and let m be such that $\epsilon^k = (0, \dots, 0, \epsilon^k_{m+1}, \dots, \epsilon^k_q)$ with $\epsilon^k_{m+1} > 0$. Let $I_1 = \nu^{-1}(\{1, \dots, m\}) \cap \{1, \dots, p\}$ and $I_2 = \nu^{-1}(\{1, \dots, m\}) \cap \{p + 1, \dots, q\}$. We consider $\epsilon^k_{m+1}, \dots, \epsilon^k_q$ fixed in $\mathcal{I}^\nu(\epsilon)$, and define

$$\mathcal{I}_k(\epsilon_1, \dots, \epsilon_m) = \int_{[0,1]^n} \bigwedge_{i \in I_1} d\chi(r^{\alpha_i}/\epsilon_{\nu(i)}) \prod_{i \in I_2} \chi(r^{\alpha_i}/\epsilon_{\nu(i)}) \mathscr{J}_k(r) dr_M,$$

originally defined on $(0, \infty)^p$, but extended according to Definition 10, where

$$\mathscr{J}_k(r) = \pm \bigwedge_{i \in \{1,\dots,p\} \setminus I_1} d\chi(r^{\alpha_i}/\epsilon^k_{\nu(i)}) \prod_{i \in \{p+1,\dots,q\} \setminus I_2} \chi(r^{\alpha_i}/\epsilon^k_{\nu(i)}) \mathscr{J}(r)$$

(where the sign is chosen such that $\mathcal{I}_k(0) = \mathcal{I}^\nu(\epsilon^k)$). Since $m < q$ and $\mathscr{J}_k$ is smooth, we have by induction that

$$\mathcal{I}_k(0) = \lim_{\epsilon_m \to 0} \dots \lim_{\epsilon_1 \to 0} \mathcal{I}_k(\epsilon_1, \dots, \epsilon_m) = \lim_{\delta \to 0} \mathcal{I}_k(\epsilon'(\delta)),$$

where $\epsilon'(\delta)$ is any admissible path, and the first equality follows by definition of $\mathcal{I}_k(0)$. We fix an admissible path $\epsilon'(\delta)$. For each k we can choose δ_k such that if $\epsilon^{k'} = (\epsilon'_1(\delta_k), \dots, \epsilon'_m(\delta_k))$, then $\lim_{k\to\infty}(\mathcal{I}_k(\epsilon^{k'}) - \mathcal{I}_k(0)) = 0$ and if $\tilde{\epsilon}^k = (\epsilon^{k'}, \epsilon^k_{m+1}, \dots, \epsilon^k_q)$, then $\tilde{\epsilon}^k$ forms a subsequence of an admissible path. Since $\mathcal{I}_k(0) = \mathcal{I}^\nu(\epsilon^k)$, and $\mathcal{I}_k(\epsilon^{k'}) = \mathcal{I}^\nu(\tilde{\epsilon}^k)$, we thus have

$$\lim_{k\to\infty} \mathcal{I}^\nu(\epsilon^k) = \lim_{k\to\infty} \mathcal{I}^\nu(\tilde{\epsilon}^k) = \lim_{\delta\to 0} \mathcal{I}^\nu(\epsilon(\delta))$$

where the second equality follows from the existence and uniqueness of $\mathcal{I}^\nu(\epsilon(\delta))$ along any admissible path. Hence we have shown that the limit in (13) exists and is well-defined.

Finally, if we start from (18), as (23) in [22] shows, either

$$\lim_{\epsilon_1 \ll \cdots \ll \epsilon_q \to 0} \mathcal{I}^\nu(\epsilon) = \pm \int_{r_M \in (0,1)^{n-p}} \mathscr{J}(0, r_M) dr_M,$$

or the limit is 0, depending only on α. If we consider $I^\nu(\epsilon)$ instead, we get the same limit, see [29, p. 79–80], and (14) follows.

4. Proof of Theorems 13 and 14

Recall that $(P_1, \dots, P_q) = (R^1_{k_1}, \dots, R^p_{k_p}, U^{p+1}_{k_{p+1}}, \dots, U^q_{k_q})$ and that $P^{\epsilon_j}_j$ and $P^{\lambda_j}_j$ are the ϵ-regularizations with smooth χ (given by (10), (11)) and the λ-regularizations (cf. (7), (8)) respectively of P_j. We will consider the following two integrals:

$$\mathcal{I}(\epsilon) = \int_Z P_1^{\epsilon_1} \wedge \cdots \wedge P_q^{\epsilon_q} \wedge \varphi$$

$$\Gamma(\lambda) = \int_Z P_1^{\lambda_1} \wedge \cdots \wedge P_q^{\lambda_q} \wedge \varphi,$$

where φ is a test form on Z, supported close to a point in $\{f_1 = \cdots = f_q = 0\}$, of bidegree $(n, n - k_1 - \cdots - k_q + q - p)$ with values in $\Lambda(E_1^* \oplus \cdots \oplus E_q^*)$. In the arguments below, we will assume for notational convenience that $\tilde{f}_j = f_j$ (cf., e.g., (7)); the modifications to the general case are straightforward.

The crucial parts of the proofs of Theorems 13 and 14 are contained in the following propositions.

Proposition 17. *Assume that $f_1, \dots, f_q$ define a complete intersection. For $p < s \le q$ we have*

$$\left|\mathcal{I}(\epsilon) - \mathcal{I}(\epsilon_1, \dots, \epsilon_{s-1}, 0, \dots, 0)\right| \le C\|\varphi\|_M(\epsilon_s^\omega + \cdots + \epsilon_q^\omega).$$

Note that $\mathcal{I}(\epsilon_1, \dots, \epsilon_{s-1}, 0, \dots, 0)$ is well-defined; it is the action of $U_{k_s}^s \wedge \cdots \wedge U_{k_q}^q$ on a smooth form.

Proposition 18. *Assume that $f_1, \dots, f_q$ define a complete intersection. Then $\Gamma(\lambda)$ has a meromorphic continuation to all of $\mathbb{C}^q$ and its only possible poles in a neighborhood of $\cap_1^q \{\mathfrak{Re}\, \lambda_j \ge 0\}$ are along hyperplanes of the form $\sum_{j=1}^p \lambda_j \alpha_j = 0$, where $\alpha_j \in \mathbb{N}$ and at least two α_j are positive. In particular, for $p = 1$, $\Gamma(\lambda)$ is analytic in a neighborhood of $\cap_1^q \{\mathfrak{Re}\, \lambda_j \ge 0\}$.*

Using that

$$\bar\partial |f_j|^{2\lambda} \wedge u_k^j = \bar\partial(|f_j|^{2\lambda} u_k^j) - f_j \cdot (|f_j|^{2\lambda} u_{k+1}^j), \tag{19}$$

the proof of Theorem 14 follows from Proposition 18 in a similar way as Theorem 1 in [28] follows from Proposition 4 in [28].

We indicate one way Proposition 17 can be used to prove Theorem 13. To simplify notation somewhat, we let R^j denote any R_k^j and R_ϵ^j denotes a smooth ϵ-regularization of R^j; U^j and U_ϵ^j are defined similarly. The uniformity in the estimate of Proposition 17 implies that we have estimates of the form

$$\left| \bigwedge_1^m R_\epsilon^j \wedge \bigwedge_{m+1}^p R^j \wedge \bigwedge_{p+1}^q U_\epsilon^j - \bigwedge_1^m R_\epsilon^j \wedge \bigwedge_{m+1}^p R^j \wedge \bigwedge_{p+1}^q U^j \right| \lesssim (\epsilon_{p+1}^\omega + \cdots + \epsilon_q^\omega), \tag{20}$$

where, e.g., $R^{m+1} \wedge \cdots \wedge R^p$ a priori is defined as a Coleff-Herrera product. We prove (a slightly stronger result than) Theorem 13 by induction over p. Let R^* denote the Coleff-Herrera product of some R^j:s with $j > p$ and let U^* and U^*_ϵ denote the product of some U^j:s and U^j_ϵ:s respectively, also with $j > p$ but only j:s not occurring in R^*. We prove

$$\left|R^1_\epsilon \wedge \cdots \wedge R^p_\epsilon \wedge R^* \wedge U^*_\epsilon - R^1 \wedge \cdots \wedge R^p \wedge R^* \wedge U^*\right| \lesssim \epsilon^\omega,$$

i.e., we prove Theorem 13 *on* the current R^*. The induction start, $p = 0$, follows immediately from (20). If we add and subtract $R^1_\epsilon \wedge \cdots \wedge R^p_\epsilon \wedge R^* \wedge U^*$, the induction step follows easily from (19) (construed in setting of ϵ-regularizations) and estimates like (20).

Proof of Propositions 17 and 18. We may assume that φ has arbitrarily small support. Hence, we may assume that Z is an analytic subset of a domain $\Omega \subseteq \mathbb{C}^N$ and that all bundles are trivial, and thus make the identification $f_j = (f_{j1}, \dots, f_{je_j})$, where f_{ji} are holomorphic in Ω. We choose a Hironaka resolution $\hat{Z} \to Z$ such that the pulled-back ideals $\langle \hat{f}_j \rangle$ are all principal, and moreover, so that in a fixed chart with coordinates x on $\hat{Z}$ (and after a possible re-numbering), $\langle \hat{f}_j \rangle$ is generated by $\hat{f}_{j1}$ and $\hat{f}_{j1} = x^{\alpha_j} h_j$, where h_j is holomorphic and non-zero. We then have

$$|\hat{f}_j|^2 = |\hat{f}_{j1}|^2 \xi_j, \quad \hat{u}^j_{k_j} = v^j / \hat{f}^{k_j}_{j1},$$

where ξ_j is smooth and positive and v^j is a smooth (bundle valued) form. We thus get

$$\bar\partial \chi_j(|\hat{f}_j|^2/\epsilon_j) = \tilde{\chi}_j(|\hat{f}_j|^2/\epsilon_j) \left(\frac{d\bar{\hat{f}}_{j1}}{\bar{\hat{f}}_{j1}} + \frac{\bar\partial \xi_j}{\xi_j} \right),$$

where $\tilde{\chi}_j(t) = t\chi_j'(t)$, and

$$\bar\partial |\hat{f}_j|^{2\lambda_j} = \lambda_j |\hat{f}_j|^{2\lambda_j} \left(\frac{d\bar{\hat{f}}_{j1}}{\bar{\hat{f}}_{j1}} + \frac{\bar\partial \xi_j}{\xi_j} \right).$$

It follows that $\mathcal{I}(\epsilon)$ and $\Gamma(\lambda)$ are finite sums of integrals which we without loss of generality can assume to be of the form

$$\pm \int_{\mathbb{C}^n_x} \prod_1^p \tilde{\chi}^\epsilon_j \prod_{p+1}^q \chi^\epsilon_j \bigwedge_1^m \frac{d\bar{\hat{f}}_{j1}}{\bar{\hat{f}}_{j1}} \wedge \bigwedge_{m+1}^p \frac{\bar\partial \xi_j}{\xi_j} \wedge \bigwedge_1^q \frac{v^j}{\hat{f}^{k_j}_{j1}} \wedge \varphi\rho, \tag{21}$$

$$\pm \lambda_1 \cdots \lambda_p \int_{\mathbb{C}^n_x} \prod_1^q |\hat{f}_j|^{2\lambda_j} \bigwedge_1^m \frac{d\bar{\hat{f}}_{j1}}{\bar{\hat{f}}_{j1}} \wedge \bigwedge_{m+1}^p \frac{\bar\partial \xi_j}{\xi_j} \wedge \bigwedge_1^q \frac{v^j}{\hat{f}^{k_j}_{j1}} \wedge \varphi\rho, \tag{22}$$

where ρ is a cutoff function.

Recall that $\hat{f}_{j1} = x^{\alpha_j} h_j$ and let μ be the number of vectors in a maximal linearly independent subset of $\{\alpha_1, \dots, \alpha_m\}$; say that $\alpha_1, \dots, \alpha_\mu$

are linearly independent. We then can define new holomorphic coordinates (still denoted by x) so that $\hat{f}_{j1} = x^{\alpha_j}$, $j = 1, \dots, \mu$, see [22, p. 46] for details. Then we get

$$
\begin{aligned}
\text{(23)} \qquad \bigwedge_1^m d\hat{f}_{j1} &= \bigwedge_1^{\mu} dx^{\alpha_j} \wedge \bigwedge_{\mu+1}^{m} (x^{\alpha_j} dh_j + h_j dx^{\alpha_j}) \\
&= x^{\sum_{\mu+1}^m \alpha_j} \bigwedge_1^{\mu} dx^{\alpha_j} \wedge \bigwedge_{\mu+1}^{m} dh_j,
\end{aligned}
$$

where the last equality follows because $dx^{\alpha_1} \wedge \cdots \wedge dx^{\alpha_\mu} \wedge dx^{\alpha_j} = 0$, $\mu + 1 \le j \le m$, since $\alpha_1, \dots, \alpha_\mu, \alpha_j$ are linearly dependent. From the beginning we could also have assumed that $\varphi = \varphi_1 \wedge \varphi_2$, where φ_1 is an anti-holomorphic $(n - \sum_1^q k_j + q - p)$-form and φ_2 is a (bundle valued) $(n, 0)$-test form on Z. We now define

$$
\Phi = \bigwedge_{\mu+1}^{m} \frac{d\bar{h}_j}{\bar{h}_j} \wedge \bigwedge_{m+1}^{p} \frac{\bar\partial \xi_j}{\xi_j} \wedge \bigwedge_1^q v^j \wedge \hat{\varphi}_1.
$$

Using (23) we can now write (21) and (22) as

$$
\text{(24)} \qquad \pm \int_{\mathbb{C}^n_x} \frac{\prod_1^p \tilde{\chi}_j^\epsilon \prod_{p+1}^q \chi_j^\epsilon}{\prod_1^q \hat{f}_{j1}^{k_j}} \frac{d\bar{x}^{\alpha_1}}{\bar{x}^{\alpha_1}} \wedge \cdots \wedge \frac{d\bar{x}^{\alpha_\mu}}{\bar{x}^{\alpha_\mu}} \wedge \Phi \wedge \hat{\varphi}_2 \rho,
$$

$$
\text{(25)} \qquad \pm \lambda_1 \cdots \lambda_p \int_{\mathbb{C}^n_x} \frac{\prod_1^q |\hat{f}_j|^{2\lambda_j}}{\prod_1^q \hat{f}_{j1}^{k_j}} \frac{d\bar{x}^{\alpha_1}}{\bar{x}^{\alpha_1}} \wedge \cdots \wedge \frac{d\bar{x}^{\alpha_\mu}}{\bar{x}^{\alpha_\mu}} \wedge \Phi \wedge \hat{\varphi}_2 \rho.
$$

Lemma 19. *Let $\mathcal{K} = \{i;\, x_i \,\big|\, x^{\alpha_j},$ some $p + 1 \le j \le q\}$. For any fixed $r \in \mathbb{N}$, one can replace Φ in (24) and (25) by*

$$
\Phi' := \Phi - \sum_{J \subseteq \mathcal{K}} (-1)^{|J|} \sum_{k_1, \dots, k_{|J|} = 0}^{r+1} \frac{\partial^{|k|} \Phi}{\partial x_J^k} \bigg|_{x_J = 0} \frac{x_J^k}{k!}
$$

without affecting the integrals. Moreover, for any $I \subseteq \mathcal{K}$, we have that $\Phi' \wedge \Lambda_{i \in I} (d\bar{x}_i / \bar{x}_i)$ is C^r-smooth.

We replace Φ by Φ' in (24) and (25) and we write $d = d_{\mathcal{K}} + d_{\mathcal{K}^c}$, where $d_{\mathcal{K}}$ differentiates with respect to the variables x_i, $\bar{x}_i$ for $i \in \mathcal{K}$ and $d_{\mathcal{K}^c}$ differentiates with respect to the rest. Then we can write $(d\bar{x}^{\alpha_1}/\bar{x}^{\alpha_1}) \wedge \cdots \wedge (d\bar{x}^{\alpha_\mu}/\bar{x}^{\alpha_\mu}) \wedge \Phi'$ as a sum of terms, which we without loss of generality can assume to be of the form

$$
\begin{aligned}
&\frac{d_{\mathcal{K}^c} \bar{x}^{\alpha_1}}{\bar{x}^{\alpha_1}} \wedge \cdots \wedge \frac{d_{\mathcal{K}^c} \bar{x}^{\alpha_\nu}}{\bar{x}^{\alpha_\nu}} \wedge \frac{d_{\mathcal{K}} \bar{x}^{\alpha_{\nu+1}}}{\bar{x}^{\alpha_{\nu+1}}} \wedge \cdots \wedge \frac{d_{\mathcal{K}} \bar{x}^{\alpha_\mu}}{\bar{x}^{\alpha_\mu}} \wedge \Phi' \\
&\quad = \frac{d_{\mathcal{K}^c} \bar{x}^{\alpha_1}}{\bar{x}^{\alpha_1}} \wedge \cdots \wedge \frac{d_{\mathcal{K}^c} \bar{x}^{\alpha_\nu}}{\bar{x}^{\alpha_\nu}} \wedge \Phi'' \wedge d\bar{x}_{\mathcal{K}},
\end{aligned}
$$

where Φ'' is C^r-smooth and of bidegree $(0, n-\nu-|\mathcal{K}|)$ (possibly, $\Phi'' = 0$). Thus, (24) and (25) are finite sums of of integrals of the following type

$$\int_{\mathbb{C}^n_x} \frac{\prod_1^p \tilde{\chi}_j^\epsilon \prod_{p+1}^q \chi_j^\epsilon}{\prod_1^q \hat{f}_{j1}^{k_j}} \frac{d\bar{x}^{\alpha_1}}{\bar{x}^{\alpha_1}} \wedge \cdots \wedge \frac{d\bar{x}^{\alpha_\nu}}{\bar{x}^{\alpha_\nu}} \wedge \psi \wedge d\bar{x}_{\mathcal{K}} \wedge dx, \tag{26}$$

$$\lambda_1 \cdots \lambda_p \int_{\mathbb{C}^n_x} \frac{\prod_1^q |\hat{f}_j|^{2\lambda_j}}{\prod_1^q \hat{f}_{j1}^{k_j}} \frac{d\bar{x}^{\alpha_1}}{\bar{x}^{\alpha_1}} \wedge \cdots \wedge \frac{d\bar{x}^{\alpha_\nu}}{\bar{x}^{\alpha_\nu}} \wedge \psi \wedge d\bar{x}_{\mathcal{K}} \wedge dx, \tag{27}$$

where ψ is C^r-smooth and compactly supported.

We now first finish the proof of Proposition 18. First of all, it is well known that $\Gamma(\lambda)$ has a meromorphic continuation to $\mathbb{C}^q$. We have

$$\frac{d\bar{x}^{\alpha_1}}{\bar{x}^{\alpha_1}} \wedge \cdots \wedge \frac{d\bar{x}^{\alpha_\nu}}{\bar{x}^{\alpha_\nu}} \wedge d\bar{x}_{\mathcal{K}} = \sum_{\substack{|I|=\nu \\ I \subseteq \mathcal{K}^c}} C_I \frac{d\bar{x}_I}{\bar{x}_I} \wedge d\bar{x}_{\mathcal{K}}.$$

Let us assume that $I = \{1, \ldots, \nu\} \subseteq \mathcal{K}^c$ and consider the contribution to (27) corresponding to this subset. This contribution equals

$$C_I \lambda_1 \cdots \lambda_p \int_{\mathbb{C}^n_x} \frac{|x^{\sum_1^q \lambda_j \alpha_j}|^2}{x^{\sum_1^q k_j \alpha_j}} \bigwedge_1^\nu \frac{d\bar{x}_j}{\bar{x}_j} \wedge \Psi(\lambda, x) \wedge d\bar{x}_{\mathcal{K}} \wedge dx \tag{28}$$

$$= \frac{C_I \prod_1^p \lambda_j}{\prod_{i=1}^\nu (\sum_1^q \lambda_j \alpha_{ji})} \int_{\mathbb{C}^n_x} \frac{\bigwedge_{i=1}^\nu \bar{\partial} |x_i|^{2\sum_1^q \lambda_j \alpha_{ji}} \prod_{i=\nu+1}^n |x_i|^{2\sum_1^q \lambda_j \alpha_{ji}}}{x^{\sum_1^q k_j \alpha_j}} \wedge$$
$$\wedge \Psi(\lambda, x) \wedge d\bar{x}_{\mathcal{K}} \wedge dx,$$

where $\Psi(\lambda, x) = \psi(x) \prod_1^q (\xi_j^{\lambda_j} / h_j^{k_j})$. It is well known (and not hard to prove, e.g., by integrations by parts as in [1], Lemma 2.1) that the *integral* on the right hand side of (28) has an analytic continuation in λ to a neighborhood of $\cap_1^q \{\mathfrak{Re}\, \lambda_j \geq 0\}$. (We thus choose r in Lemma 19 large enough so that we can integrate by parts.) If $p = 0$, then the coefficient in front of the integral is to be interpreted as 1 and Proposition 18 follows in this case. For $p > 0$, we see that the poles of (28), and consequently of $\Gamma(\lambda)$, in a neighborhood of $\cap_1^q \{\mathfrak{Re}\, \lambda_j \geq 0\}$ are along hyperplanes of the form $0 = \sum_1^q \lambda_j \alpha_{ji}$, $1 \leq i \leq \nu$. But if $j > p$ and $i \leq \nu$, then $\alpha_{ji} = 0$ since $\{1, \ldots, \nu\} \subseteq \mathcal{K}^c = \{i;\, x_i \nmid x^{\alpha_j}, \forall j = p+1, \ldots, q\}$. Thus, the hyperplanes are of the form $0 = \sum_1^p \lambda_j \alpha_{ji}$ and Proposition 18 is proved except for the statement that at least for two j:s, the α_{ji} are non-zero. However, we see from (28) that if for some i we have $\alpha_{ji} = 0$ for all j but one, then the appearing λ_j in the denominator will be canceled by the numerator. Moreover, we may assume that the constant $C_I = \det(\alpha_{ji})_{1 \leq i,j \leq \nu}$ is non-zero which implies that we cannot have any λ_j^2 in the denominator.

We now prove Proposition 17. Consider (26). We have that $\alpha_1, \dots, \alpha_\nu$ are linearly independent so we may assume that $A = (\alpha_{ij})_{1\le i,j\le\nu}$ is invertible with inverse $B = (b_{ij})$. We make the non-holomorphic change of variables (15), where the "q" of (15) now should be understood as ν. Then we get $x^{\alpha_j} = y^{\alpha_j}\eta_j$, where $\eta_j > 0$ and smooth and $\eta_j^2 = 1/\xi_j$, $j = 1, \dots, \nu$. Hence, $|\hat{f}_j|^2 = |y^{\alpha_j}|^2$, $j = 1, \dots, \nu$. Expressed in the y-coordinates we get that $\Lambda_1^\nu(d\bar{x}^{\alpha_j}/\bar{x}^{\alpha_j}) \wedge \psi \wedge d\bar{x}_\mathcal{K} \wedge dx$ is a finite sum of terms of the form

$$\frac{d\bar{y}^{\alpha_1}}{\bar{y}^{\alpha_1}} \wedge \cdots \wedge \frac{d\bar{y}^{\alpha_{\nu'}}}{\bar{y}^{\alpha_{\nu'}}} \wedge \bar{y}_{\mathcal{K}'}\, d\bar{y}_{\mathcal{K}''} \wedge \psi_1, \tag{29}$$

where $\nu' \le \nu$, ψ_1 is a C^r-smooth compactly supported form, and $\mathcal{K}'$ and $\mathcal{K}''$ are disjoint sets such that $\mathcal{K}' \cup \mathcal{K}'' = \mathcal{K}$. In order to give a contribution to (26) we see that ψ_1 must contain dy. In (29) we write $d = d_\mathcal{K} + d_{\mathcal{K}^c}$, and arguing as we did immediately after Lemma 19, (29) is a finite sum of terms of the form

$$\frac{d\bar{y}^{\alpha_1}}{\bar{y}^{\alpha_1}} \wedge \cdots \wedge \frac{d\bar{y}^{\alpha_{\nu''}}}{\bar{y}^{\alpha_{\nu''}}} \wedge \psi_2 \wedge d\bar{y}_\mathcal{K} \wedge dy,$$

where $\nu'' \le \nu$ and ψ_2 is C^r-smooth and compactly supported. With abuse of notation we thus have that (26) is a finite sum of integrals of the form

$$\begin{aligned}
&\int_{\mathbb{C}^n_x} \frac{\prod_1^p \tilde{\chi}_j^\epsilon \prod_{p+1}^q \chi_j^\epsilon}{\prod_1^q \hat{f}_{j1}^{k_j}} \frac{d\bar{y}^{\alpha_1}}{\bar{y}^{\alpha_1}} \wedge \cdots \wedge \frac{d\bar{y}^{\alpha_\nu}}{\bar{y}^{\alpha_\nu}} \wedge \psi \wedge d\bar{y}_\mathcal{K} \wedge dy \\
&\quad = \int_{\mathbb{C}^n_x} \frac{\bigwedge_1^\nu d\chi_j^\epsilon \prod_{\nu+1}^p \tilde{\chi}_j^\epsilon \prod_{p+1}^q \chi_j^\epsilon}{y^{\sum_1^q k_j\alpha_j}} \wedge \Psi \wedge d\bar{y}_\mathcal{K} \wedge dy,
\end{aligned} \tag{30}$$

where Ψ is a C^r-smooth compactly supported $(n - |\mathcal{K}| - \nu)$-form; the equality follows since $\chi_j^\epsilon = \chi_j(|y^{\alpha_j}|^2/\epsilon_j)$, $j = 1, \dots, \nu$. Now, (30) is essentially equal to equation (24) of [15] and the proof of Proposition 17 is concluded as in the proof of Proposition 8 in [15]. $\square$

Proof of Lemma 19. The proof is similar to the proof of Lemma 9 in [15] but some modifications have to be done. First, it is easy to check by induction over $|\mathcal{K}|$ that $\Phi' \wedge \Lambda_{i\in I}(d\bar{x}_i/\bar{x}_i)$ is C^r-smooth for any $I \subseteq \mathcal{K}$; for $|\mathcal{K}| = 1$ this is just Taylor's formula for forms. It thus suffices to show that

$$d\bar{x}^{\alpha_1} \wedge \cdots \wedge d\bar{x}^{\alpha_\mu} \wedge \left.\frac{\partial^{|k|}\Phi}{\partial x_I^k}\right|_{x_I=0} = 0, \quad \forall I \subseteq \mathcal{K},\, k = (k_{i_1}, \dots, k_{i_{|I|}}).$$

To show this, fix an $I \subseteq \mathcal{K}$ and let $L = \{j;\, x_i \nmid x^{\alpha_j}\ \forall i \in I\}$. Say for simplicity that

$$L = \{1, \dots, \mu', \mu+1, \dots, m', m+1, \dots, p', p+1, \dots, q'\},$$

where $\mu' \leq \mu$, $m' \leq m$, $p' \leq p$, and $q' < q$. The fact that $q' < q$ follows from the definitions of $\mathcal{K}$, I, and L.

Consider, on the base variety Z, the smooth form

$$F = \bigwedge_1^{\mu'} d\bar{f}_{j1} \bigwedge_{\mu+1}^{m'} d\bar{f}_{j1} \bigwedge_{m+1}^{p'} (|f_{j1}|^2 \bar{\partial}|f_j|^2 - \bar{\partial}|f_{j1}|^2 |f_j|^2) \bigwedge_{j \in L} |f_j|^{2k_j} u^j_{k_j} \wedge \varphi_1.$$

It has bidegree $(0, n - \sum_{j \in L^c} k_j + q - q')$ so F has a vanishing pullback to $\cap_{j \in L^c}\{f_j = 0\}$ since this set has dimension $n - \sum_{j \in L^c} e_j < n - \sum_{j \in L^c} k_j + q - q'$ by our assumption about complete intersection. Thus, $\hat{F}$ has a vanishing pullback to $\{x_I = 0\} \subseteq \cap_{j \in L^c}\{\hat{f}_j = 0\}$. In fact, this argument shows that

$$\hat{F} = \sum \phi_j, \tag{31}$$

where the ϕ_j are smooth linearly independent forms such that each ϕ_j is divisible by $\bar{x}_i$ or $d\bar{x}_i$ for some $i \in I$. (It is the pull-back to $\{x_I = 0\}$ of the anti-holomorphic differentials of $\hat{F}$ that vanishes.) For the rest of the proof we let $\sum \phi_j$ denote such expressions and we note that they are invariant under holomorphic differential operators. Computing $\hat{F}$ we get

$$\hat{F} = \prod_{m+1}^{p'} |\hat{f}_{j1}|^4 \prod_{j \in L} \frac{|\hat{f}_j|^{2k_j}}{\hat{f}_{j1}^{k_j}} \bigwedge_1^{\mu'} d\bar{x}^{\alpha_j} \bigwedge_{\mu+1}^{m'} d(\bar{x}^{\alpha_j} \bar{h}_j) \bigwedge_{m+1}^{p'} \bar{\partial}\xi_j \bigwedge_{j \in L} v^j \wedge \hat{\varphi}_1.$$

The "coefficient" $\prod_{m+1}^{p'} |\hat{f}_{j1}|^4 \prod_{j \in L}(|\hat{f}_j|^{2k_j}/\hat{f}_{j1}^{k_j})$ does not contain any $\bar{x}_i$ with $i \in I$ so we may divide (31) by it (recall that the ϕ_j are linearly independent) and we obtain

$$\begin{aligned} \sum \phi_j &= \bigwedge_1^{\mu'} d\bar{x}^{\alpha_j} \bigwedge_{\mu+1}^{m'} d(\bar{x}^{\alpha_j} \bar{h}_j) \bigwedge_{m+1}^{p'} \bar{\partial}\xi_j \bigwedge_{j \in L} v^j \wedge \hat{\varphi}_1 \\ &= \prod_{\mu+1}^{m'} \bar{x}^{\alpha_j} \bigwedge_1^{\mu'} d\bar{x}^{\alpha_j} \bigwedge_{\mu+1}^{m'} d\bar{h}_j \bigwedge_{m+1}^{p'} \bar{\partial}\xi_j \bigwedge_{j \in L} v^j \wedge \hat{\varphi}_1 \\ &\quad + \bigwedge_1^{\mu'} d\bar{x}^{\alpha_j} \wedge \sum_{\mu+1}^{m'} d\bar{x}^{\alpha_j} \wedge \tau_j \end{aligned}$$

for some τ_j. We multiply this equality with

$$\bigwedge_{m'+1}^{m} d\bar{h}_j \bigwedge_{p'+1}^{p} \bar{\partial}\xi_j \bigwedge_{j \in L^c} v^j \Big/ \left(\prod_{\mu+1}^{m} \bar{h}_j \prod_{m+1}^{p} \xi_j \right)$$

and get

$$\prod_{\mu+1}^{m'} \bar{x}^{\alpha_j} \bigwedge_1^{\mu'} d\bar{x}^{\alpha_j} \wedge \Phi + \bigwedge_1^{\mu'} d\bar{x}^{\alpha_j} \wedge \sum_{\mu+1}^{m'} d\bar{x}^{\alpha_j} \wedge \tau_j = \sum \phi_j$$

for some new τ_j. We apply the operator $\partial^{|k|}/\partial x_I^k$ to this equality and then we pull back to $\{x_I = 0\}$, which makes the right hand side vanish; (we construe however the result in $\mathbb{C}_x^n$). Finally, taking the exterior product with $\Lambda_{\mu'+1}^{\mu} d\bar{x}^{\alpha_j}$, which will make each term in under the summation sign on the left hand side vanish, we arrive at

$$\prod_{\mu+1}^{m'} \bar{x}^{\alpha_j} \bigwedge_1^{\mu} d\bar{x}^{\alpha_j} \wedge \frac{\partial^{|k|}\Phi}{\partial x_I^k}\Big|_{x_I=0} = 0$$

and we are done. □

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