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FINITE ELEMENT APPROXIMATION OF THE CAHN-HILLIARD-COOK EQUATION

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Abstract. We study the nonlinear stochastic Cahn-Hilliard equation driven by additive colored noise. We show almost sure existence and regularity of solutions. We introduce spatial approximation by a standard finite element method and prove error estimates of optimal order on sets of probability arbitrarily close to 1. We also prove strong convergence without known rate.

1. Introduction

We study the Cahn-Hilliard equation perturbed by noise, also known as the Cahn-Hilliard-Cook equation (cf. [1, 3]),

$$\begin{aligned} &\mathrm{d}u - \Delta w\,\mathrm{d}t = \mathrm{d}W && \text{in } \mathcal{D}\times[0,T],\\ &w + \Delta u + f(u) = 0 && \text{in } \mathcal{D}\times[0,T],\\ &\frac{\partial u}{\partial n} = \frac{\partial w}{\partial n} = 0 && \text{on } \partial\mathcal{D}\times[0,T],\\ &u(0) = u_0. && \text{in } \mathcal{D}. \end{aligned}$$

Here $\mathcal{D}$ is a bounded domain in $\mathbf{R}^d$, $d = 1, 2, 3$, and $f(s) = s^3 - s$. Using the framework of [9] we write this as an abstract evolution equation of the form

$$\mathrm{d}X + \big(A^2X + Af(X)\big)\,\mathrm{d}t = \mathrm{d}W, \quad t > 0; \quad X(0) = X_0, \tag{1.1}$$

where A denotes the Neumann Laplacian considered as an unbounded operator in the Hilbert space $H = L_2(\mathcal{D})$ and W is a Q-Wiener process in H with respect to a filtered probability space $(\Omega, \mathcal{F}, \mathbf{P}, \{\mathcal{F}_t\}_{t\ge 0})$. See Section 2 for details.

Our goal is to study the convergence properties of the spatially semidiscrete finite element approximation X_h of X, which is defined by an equation of the form

$$\mathrm{d}X_h + \big(A_h^2X_h + A_hP_hf(X_h)\big)\,\mathrm{d}t = P_h\mathrm{d}W, \quad t > 0; \quad X_h(0) = P_hX_0.$$

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In order to do so, we need to prove existence and regularity for solutions of (1.1). Such results were first proved in [4]. Under the assumption that the covariance operator $Q = I$ (space-time white noise, cylindrical noise) it was shown that there is a process which belongs to $C([0,T],H^{-1})$ almost surely (a.s.) and which is the unique solution of (1.1). Under the stronger assumption that A and Q commute and that $\mathrm{Tr}(A^{\delta-1}Q)<\infty$ for some $\delta>0$ (colored noise) it was shown that the solution belongs to $C([0,T],H)$ a.s. Such regularity is insufficient for proving convergence of a numerical solution. Our first aim is therefore to prove existence of a solution in $C([0,T],H^{\beta})$ a.s. for some $\beta>0$.

Following the semigroup approach of [9] we write the equation (1.1) as the integral equation (mild solution)

$$\begin{aligned} X(t) &= \mathrm{e}^{-tA^2}X_0 - \int_0^t \mathrm{e}^{-(t-s)A^2} Af(X(s))\,\mathrm{d}s + \int_0^t \mathrm{e}^{-(t-s)A^2}\,\mathrm{d}W(s) \\ &= Y(t) + W_A(t), \end{aligned}$$

where e^{-tA^2} is the analytic semigroup generated by $-A^2$. This naturally splits the solution as $X = Y + W_A$, where $W_A(t) = \int_0^t \mathrm{e}^{-(t-s)A^2} A\,\mathrm{d}W(s)$ is a stochastic convolution. This convolution, and its finite element approximation, was studied in [8]. In particular, it was shown there that if $\|A^{\frac{\beta-2}{2}}Q^{\frac12}\|_{\mathrm{HS}}^2 = \mathrm{Tr}(A^{\beta-2}Q)<\infty$ for some $\beta\ge 0$, then we have regularity of order β in a mean square sense; that is,

$$\mathbf{E}\big[\|W_A(t)\|_{H^\beta}^2\big] \le \|A^{\frac{\beta-2}{2}}Q^{\frac12}\|_{\mathrm{HS}}^2, \quad t\ge 0. \tag{1.2}$$

The other part, Y, solves a differential equation with random coefficient,

$$\dot Y + A^2 Y + Af(Y+W_A) = 0, \quad t>0; \quad Y(0)=X_0. \tag{1.3}$$

This can be solved once W_A is known. This approach was also used in [4], but while they used Galerkin's method and energy estimates to solve (1.3), we use a semigroup approach similar to that of [5]. However, published results for the deterministic Cahn-Hilliard equation do not apply directly due to the limited regularity in (1.3).

The nonlinear term is only locally Lipschitz and we need to control the Lipschitz constant. In the deterministic case studied in [5] this is achieved by the Lyapunov functional

$$J(u) = \frac12\|\nabla u\|^2 + \int_{\mathcal{D}} F(u)\,\mathrm{d}x, \quad u\in H^1, \quad F(s)=\tfrac14 s^4 - \tfrac12 s^2,$$

which is nonincreasing along paths, so that $\|X(t)\|_{H^1}\le C$ for $t\ge 0$. Due to the stochastic perturbation, this is not true for the stochastic equation (1.1). However, it is possible find a bound for the growth of the expected value of $J(X(t))$, and hence a bound

$$\mathbf{E}\big[\|X(t)\|_{H^1}^2\big] \le C(t), \quad t\ge 0. \tag{1.4}$$

This was shown in [4] under the assumption

$$\|A^{1/2}Q^{1/2}\|_{\mathrm{HS}}^2 = \mathrm{Tr}(AQ) < \infty, \tag{1.5}$$

which is consistent with $\beta = 3$ in (1.2). We repeat this in Theorem 3.1 with several improvements. First of all we reduce the growth of the bound from exponential to quadratic with respect to t. We also relax the assumptions: we do not assume that A and Q commute; that is, have a common eigenbasis, and we do not assume that the eigenbasis of Q consists of bounded functions. Moreover, we prove the same bound for the finite element solution X_h.

By means of Chebyshev's inequality we may then show that for each $T > 0$ and $\epsilon \in (0, 1)$ there are K_T and $\Omega_\epsilon \subset \Omega$ with $\mathbf{P}(\Omega_\epsilon) \geq 1 - \epsilon$ and such that

$$\|X(t)\|_{H^1}^2 + \|X_h(t)\|_{H^1}^2 \leq \epsilon^{-1} K_T \quad \text{on } \Omega_\epsilon,\ t \in [0, T].$$

This bound controls the nonlinear term and we show that $X \in C([0, T], H^3)$ for $\omega \in \Omega_\epsilon$ under the assumption (1.5) (see Theorem 4.2). We also obtain an error estimate (see Theorem 5.3)

$$\|X_h(t) - X(t)\| \leq C(\epsilon^{-1}K_T, T)h^2|\log(h)| \quad \text{on } \Omega_\epsilon,\ t \in [0, T].$$

The constant grows rapidly with $\epsilon^{-1}K_T$, but nevertheless we may use this to show strong convergence (see Theorem 5.4),

$$\max_{t\in[0,T]} \mathbf{E}\big[\|X_h(t) - X(t)\|^2\big] \to 0 \quad \text{as } h \to 0. \tag{1.6}$$

To prove strong convergence with an estimate of the rate remains a challenge for future work. In this connection we note that even for numerical methods for stochastic ordinary differential equations with local Lipschitz nonlinearity there are few results on convergence rates (cf. [6]).

Numerical methods for the deterministic Cahn-Hilliard equation are well covered in literature. There are few studies of numerical methods for the Cahn-Hilliard-Cook equation. We are only aware of [2] in which convergence in probability was proved for a difference scheme for the nonlinear equation in multiple dimensions. For the linear equation there is [7], where strong convergence estimates were proved for the finite element method for the linear equation in 1-D, and the already mentioned work [8] on the finite element method for the stochastic convolution in multiple dimensions.

2. Preliminaries

2.1. **Norms.** Let $\mathcal{D} \subset \mathbf{R}^d$, $d = 1, 2, 3$, be a bounded convex domain with polygonal boundary $\partial\mathcal{D}$. Let $H = L_2(\mathcal{D})$ with standard inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, and

$$\dot{H} = \Big\{v \in H : \int_{\mathcal{D}} v \,\mathrm{d}x = 0\Big\}.$$

We also denote by $H^k = H^k(\mathcal{D})$ the standard Sobolev space. We define $A = -\Delta$ with domain of definition

$$D(A) = \Big\{ v \in H^2 : \frac{\partial v}{\partial n} = 0 \text{ on } \partial\mathcal{D} \Big\}.$$

Then A is a positive definite, selfadjoint, unbounded, linear operator on $\dot{H}$ with compact inverse. When extended to H it has an orthonormal eigenbasis $\{\varphi_j\}_{j=0}^{\infty}$ with corresponding eigenvalues $\{\lambda_j\}_{j=0}^{\infty}$ such that

$$0 = \lambda_0 < \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_j \leq \cdots, \quad \lambda_j \to \infty.$$

The first eigenfunction is constant, $\varphi_0 = |\mathcal{D}|^{-\frac{1}{2}}$.

Let $P\colon H \to \dot{H}$ define the orthogonal projector. Then

$$(I-P)v = \langle v, \varphi_0 \rangle \varphi_0 = |\mathcal{D}|^{-1} \int_{\mathcal{D}} v \, \mathrm{d}x,$$

is the average of v. We define seminorms and norms

$$|v|_\alpha = \Big(\sum_{j=1}^{\infty} \lambda_j^{\alpha} |\langle v, \varphi_j \rangle|^2 \Big)^{\frac{1}{2}}, \quad \alpha \geq 0,$$

$$\|v\|_\alpha = \Big(\sum_{j=0}^{\infty} \lambda_j^{\alpha} |\langle v, \varphi_j \rangle|^2 \Big)^{\frac{1}{2}} = \big(|v|_\alpha^2 + |\langle v, \varphi_0 \rangle|^2 \big)^{\frac{1}{2}}, \quad \alpha \geq 0,$$

and corresponding spaces

$$\dot{H}^\alpha = D(A^{\frac{\alpha}{2}}) = \Big\{ v \in H : |v|_\alpha < \infty \Big\}, \quad H^\alpha = \Big\{ v \in H : \|v\|_\alpha < \infty \Big\}.$$

For integer order $\alpha = k$, H^k coincides with the standard Sobolev spaces with $\|\cdot\|_k$ equivalent to the standard norm $\|\cdot\|_{H^k}$. For example,

$$\|v\|_1^2 = |v|_1^2 + |\langle v, \varphi_0 \rangle|^2 = \|\nabla v\|^2 + |\langle v, \varphi_0 \rangle|^2 \tag{2.1}$$

is equivalent to $\|v\|_{H^1}^2$ by the Poincaré inequality.

2.2. **The semigroup.** The operator $-A^2$ is the infinitesimal generator of an analytic semigroup e^{-tA^2} on H,

$$\begin{aligned} \mathrm{e}^{-tA^2} v &= \sum_{j=0}^{\infty} \mathrm{e}^{-t\lambda_j^2} \langle v, \varphi_j \rangle \varphi_j = \sum_{j=1}^{\infty} \mathrm{e}^{-t\lambda_j^2} \langle v, \varphi_j \rangle \varphi_j + \langle v, \varphi_0 \rangle \varphi_0 \\ &= \mathrm{e}^{-tA^2} P v + (I-P) v. \end{aligned}$$

The analyticity implies that

$$\|A^\alpha \mathrm{e}^{-tA^2} v\| \leq C t^{-\frac{\alpha}{2}} \mathrm{e}^{-ct} \|v\|, \quad \alpha > 0. \tag{2.2}$$

2.3. **The finite element method.** Let $\{\mathcal{T}_h\}_{h>0}$ denote a family of regular triangulations of $\mathcal{D}$ with maximal mesh size h. Let S_h be the space of continuous functions on $\mathcal{D}$, which are piecewise polynomials of degree ≤ 1 with respect to $\mathcal{T}_h$. Hence, $S_h \subset H^1$. We also define $\dot{S}_h = PS_h$; that is,

$$\dot{S}_h = \Big\{ v_h \in S_h : \int_{\mathcal{D}} v_h \, \mathrm{d}x = 0 \Big\}.$$

The space $\dot{S}_h$ is introduced only for the purpose of theory but not for computation. Now we define the "discrete Laplacian" $A_h \colon S_h \to \dot{S}_h$ by

$$\langle A_h v_h, w_h \rangle = \langle \nabla v_h, \nabla w_h \rangle, \quad \forall v_h \in S_h, \, w_h \in \dot{S}_h.$$

We note that

$$|v_h|_1 = \|A^{\frac{1}{2}} v_h\| = \|\nabla v_h\| = \|A_h^{\frac{1}{2}} v_h\|, \quad v_h \in S_h. \tag{2.3}$$

The operator A_h is selfadjoint, positive definite on $\dot{S}_h$, positive semidefinite on S_h, and A_h has an orthonormal eigenbasis $\{\varphi_{h,j}\}_{j=0}^{N_h}$ with corresponding eigenvalues $\{\lambda_{h,j}\}_{j=0}^{N_h}$. We have

$$0 = \lambda_{h,0} < \lambda_{h,1} < \cdots \le \lambda_{h,j} \le \cdots \le \lambda_{h,N_h},$$

and $\varphi_{h,0} = \varphi_0 = |\mathcal{D}|^{-\frac{1}{2}}$. Moreover, we define $\mathrm{e}^{-tA_h^2} \colon S_h \to S_h$ by

$$\mathrm{e}^{-tA_h^2} v_h = \sum_{j=0}^{N_h} \mathrm{e}^{-t\lambda_{h,j}} \langle v_h, \varphi_{h,j} \rangle \varphi_{h,j} = \sum_{j=1}^{N_h} \mathrm{e}^{-t\lambda_{h,j}} \langle v_h, \varphi_{h,j} \rangle \varphi_{h,j} + \langle v_h, \varphi_0 \rangle \varphi_0,$$

and the orthogonal projector $P_h \colon H \to S_h$ by

$$\langle P_h v, w_h \rangle = \langle v, w_h \rangle \quad \forall v \in H, \, w_h \in S_h. \tag{2.4}$$

Clearly, $P_h \colon \dot{H} \to \dot{S}_h$ and

$$\mathrm{e}^{-tA_h^2} P_h v = \mathrm{e}^{-tA_h^2} P_h P v + (I - P) v.$$

We have a discrete analog of (2.2),

$$\|A_h^{\alpha} \mathrm{e}^{-tA_h^2} v_h\| \le C t^{-\frac{\alpha}{2}} \mathrm{e}^{-ct} \|v_h\|, \quad v_h \in S_h, \ \alpha > 0. \tag{2.5}$$

Finally, we define the Ritz projector $R_h \colon \dot{H}^1 \to \dot{S}_h$ by

$$\langle \nabla R_h v, \nabla w_h \rangle = \langle \nabla v, \nabla w_h \rangle, \quad \forall v \in \dot{H}^1, \, w_h \in \dot{S}_h.$$

We extend it to $R_h \colon H^1 \to S_h$ by

$$R_h v = R_h P v + (I - P) v, \quad v \in H^1. \tag{2.6}$$

We then have the error bound (cf. [10, Ch. 1])

$$\|R_h v - v\| \le C h^{\beta} |v|_{\beta}, \quad v \in H^{\beta}, \ \beta \in [1, 2]. \tag{2.7}$$

In order to simplify the presentation, we assume that P_h is bounded with respect to the H^1 and L_4 norms, and that we have an inverse bound for A_h,

$$\begin{aligned} &\|P_h v\|_1 \le C\|v\|_1, && v \in H^1, \\ &\|P_h v\|_{L_4} \le C\|v\|_{L_4}, && v \in H^1, \\ &\|A_h v_h\| \le Ch^{-2}\|v_h\|, && v_h \in S_h. \end{aligned} \tag{2.8}$$

This holds, for example, if the mesh family $\{\mathcal{T}_h\}_{h>0}$ is quasi-uniform.

2.4. **The Wiener process.** We recall the definitions of the trace and the Hilbert-Schmidt norm of a linear operator T on H:

$$\mathrm{Tr}(T) = \sum_{k=1}^{\infty} \langle Tf_k, f_k\rangle, \quad \|T\|_{\mathrm{HS}} = \Big(\sum_{k=1}^{\infty} \|Tf_k\|^2\Big)^{\frac12},$$

where $\{f_k\}_{k=1}^{\infty}$ is an arbitrary orthonormal basis of H.

Let Q be a selfadjoint, positive semidefinite, bounded, linear operator on H with $\mathrm{Tr}(Q) < \infty$. Let $\{e_k\}_{k=1}^{\infty}$ be an orthonormal eigenbasis for Q with eigenvalues $\{\gamma_k\}_{k=1}^{\infty}$. Then we define the Q-Wiener process

$$W(t) = \sum_{k=1}^{\infty} \gamma_k^{\frac12} \beta_k(t) e_k,$$

where the β_k are real-valued, independent Brownian motions. The series converges in $L_2(\Omega, H)$; that is, with respect to the norm $\|v\|_{L_2(\Omega,H)} = (\mathbf{E}[\|v\|^2])^{\frac12}$. The Q-Wiener process can be defined also when the covariance operator has infinite trace but this is not needed in the present work.

2.5. **The stochastic convolution.** We now define (cf. [9])

$$\begin{aligned} W_A(t) &= \int_0^t \mathrm{e}^{-(t-s)A^2}\,\mathrm{d}W(s) \\ &= \int_0^t \mathrm{e}^{-(t-s)A^2} P\,\mathrm{d}W(s) + \int_0^t \langle \mathrm{d}W(s), \varphi_0\rangle \varphi_0 \\ &= \int_0^t \mathrm{e}^{-(t-s)A^2} P\,\mathrm{d}W(s) + \langle W(t), \varphi_0\rangle \varphi_0 \\ &= \int_0^t \mathrm{e}^{-(t-s)A^2} P\,\mathrm{d}W(s) + (I-P)W(t). \end{aligned}$$

Similarly,

$$\begin{aligned} W_{A_h}(t) &= \int_0^t \mathrm{e}^{-(t-s)A_h^2} P_h\,\mathrm{d}W(s) \\ &= \int_0^t \mathrm{e}^{-(t-s)A_h^2} P_h P\,\mathrm{d}W(s) + \langle W(t), \varphi_0\rangle \varphi_0 \\ &= \int_0^t \mathrm{e}^{-(t-s)A_h^2} P_h P\,\mathrm{d}W(s) + (I-P)W(t). \end{aligned}$$

Hence, the constant eigenmodes cancel:

$$W_{A_h}(t) - W_A(t) = \int_0^t \left(\mathrm{e}^{-(t-s)A_h^2} P_h - \mathrm{e}^{-(t-s)A^2}\right) P \,\mathrm{d}W(s). \tag{2.9}$$

These convolutions were studied in [8]. We quote the following results from there. We use the norms

$$\|v\|_{L_2(\Omega,\dot{H}^\beta)} = \left(\mathbf{E}\left[\,|v|_\beta^2\,\right]\right)^{\frac{1}{2}}.$$

Theorem 2.1. *If* $\|A^{\frac{\beta-2}{2}} Q^{\frac{1}{2}}\|_{\mathrm{HS}} < \infty$ *for some* $\beta \geq 2$*, then*

$$\|W_A(t)\|_{L_2(\Omega,\dot{H}^\beta)} \leq C\|A^{\frac{\beta-2}{2}} Q^{\frac{1}{2}}\|_{\mathrm{HS}}, \quad t \geq 0.$$

Theorem 2.2. *If* $\|Q^{\frac{1}{2}}\|_{\mathrm{HS}} < \infty$*, then*

$$\|W_{A_h}(t) - W_A(t)\|_{L_2(\Omega,H)} \leq Ch^2|\log h|\|Q^{\frac{1}{2}}\|_{\mathrm{HS}}, \quad t \geq 0.$$

Note that $\beta = 2$ in the latter theorem. In [8] these are stated with a slightly wider range of the order β, but this is not needed in the present work.

2.6. **Gronwall's lemma.** We need the following generalization of Gronwall's lemma. A proof can found in [5].

Lemma 2.3 (Generalized Gronwall lemma)**.** *Let the function* $\varphi(t) \geq 0$ *be continuous for* $0 \leq t \leq T$*. If*

$$\varphi(t) \leq At^{-1+\alpha} + B\int_0^t (t-s)^{-1+\beta}\varphi(s)\,\mathrm{d}s, \quad t \in (0,T],$$

for some constants $A, B \geq 0$ *and* $\alpha, \beta > 0$*, then there is a constant* $C = C(B,T,\alpha,\beta)$ *such that*

$$\varphi(t) \leq CAt^{-1+\alpha}, \quad t \in (0,T].$$

We also use the standard Gronwall lemma:

Lemma 2.4 (Gronwall's lemma)**.** *Let the function* $\varphi(t)$ *be continuous on* $[0,T]$*. If, for some* $A, C \geq 0$ *and* $B > 0$*,*

$$\varphi(t) \leq A + Ct + B\int_0^t \varphi(s)\,\mathrm{d}s, \quad t \in [0,T],$$

then

$$\varphi(t) \leq \left(A + \frac{C}{B}\right)\mathrm{e}^{Bt}, \quad t \in [0,T].$$

Proof. Set $\Phi(t) = A + Ct + B\int_0^t \varphi(s)\,\mathrm{d}s$. Then

$$\Phi'(t) = C + B\varphi(t) \leq C + B\Phi(t),$$

so that $\Phi'(t) - B\Phi(t) \leq C$, which gives $\frac{\mathrm{d}}{\mathrm{d}t}(\Phi(t)\mathrm{e}^{-Bt}) \leq C\mathrm{e}^{-Bt}$. Hence

$$\Phi(t)\mathrm{e}^{-Bt} \leq \Phi(0) + C\int_0^t \mathrm{e}^{-Bs}\,\mathrm{d}s = \left(A + \frac{C}{B}\right) - \frac{C}{B}\mathrm{e}^{-Bt}.$$

Multiplying both sides by e^{Bt} gives

$$\Phi(t) \le \Big(A + \frac{C}{B}\Big)\mathrm{e}^{Bt} - \frac{C}{B} \le \Big(A + \frac{C}{B}\Big)\mathrm{e}^{Bt}.$$

But $\varphi(t) \le \Phi(t)$, so the desired result follows. □

2.7. **Bounds for the nonlinear term.**

Lemma 2.5. *For $u, v \in H^3$ and $f(s) = s^3 - s$ we have*

$$\|\Delta f(u)\| \le C(1 + \|u\|_1^2)\|u\|_3, \tag{2.10}$$

$$\|A_h^{-\frac12} P\big(f(u) - f(v)\big)\| \le C\left(1 + \|u\|_1^2 + \|v\|_1^2\right)\|u - v\|. \tag{2.11}$$

Proof. We have $f'(s) = 3s^2 - 2s$, $f''(s) = 6s^2$. Using Hölder's inequality, Sobolev's inequality $\|u\|_{L_6} \le C\|u\|_{H^1}$ (for $d \le 3$), and $\|u\|_{H^k} \le \|u\|_k$, we get

$$\begin{aligned}
\|\Delta f(u)\| &= \|f'(u)\Delta u + f''(u)|\nabla u|^2\| \\
&\le \|f'(u)\|_{L_3}\|\Delta u\|_{L_6} + \|f''(u)\|_{L_6}\|\nabla u\|_{L_6} \\
&\le C\left(1 + \|u\|_{L_6}^2\right)\|\Delta u\|_{L_6} + C\|u\|_{L_6}\|\nabla u\|_{L_6}^2 \\
&\le C\left(1 + \|u\|_{H^1}^2\right)\|u\|_{H^3} + C\|u\|_{H^1}\|\nabla u\|_{H^2}^2 \\
&\le C\left(1 + \|u\|_1^2\right)\|u\|_3 + C\|u\|_1\|u\|_2^2 \\
&\le C\left(1 + \|u\|_1^2\right)\|u\|_3,
\end{aligned}$$

where we used $\|u\|_2 \le C\|u\|_1^{\frac12}\|u\|_3^{\frac12}$ in the last step. This proves (2.10).

For (2.11) we apply (2.3) and Hölder and Sobolev's inequalities ($d \le 3$) to get

$$\begin{aligned}
\|A_h^{-\frac12} P\varphi\| &= \sup_{v_h \in S_h} \frac{\langle A_h^{-\frac12} P\varphi, v_h\rangle}{\|v_h\|} = \sup_{v_h \in S_h} \frac{\langle \varphi, A_h^{-\frac12} P v_h\rangle}{\|v_h\|} \\
&= \sup_{w_h \in \dot S_h} \frac{\langle \varphi, w_h\rangle}{|w_h|_1} \le \sup_{w_h \in \dot S_h} \frac{\|\varphi\|_{L_{6/5}}\|w_h\|_{L_6}}{|w_h|_1} \le C\|\varphi\|_{L_{6/5}}.
\end{aligned}$$

We use this with $\varphi = f(u) - f(v) = \int_0^1 f'(su + (1-s)v)\,\mathrm{d}s\,(u - v) = \int_0^1 f'(u_s)\,\mathrm{d}s\,(u - v)$, where $u_s = su + (1-s)v$,

$$\begin{aligned}
\|A_h^{-\frac12} P\big(f(u) - f(v)\big)\| &= \|A_h^{-\frac12} P\varphi\| \le C\|\varphi\|_{L_{6/5}} \\
&\le C\int_0^1 \|f'(u_s)\|_{L_3}\,\mathrm{d}s\,\|u - v\| \le C\int_0^1 (1 + \|u_s\|_{L_6}^2)\,\mathrm{d}s\,\|u - v\| \\
&\le C\int_0^1 (1 + \|u_s\|_1^2)\,\mathrm{d}s\,\|u - v\| \le C(1 + \|u\|_1^2 + \|v\|_1^2)\|u - v\|.
\end{aligned}$$

This is (2.11). □

3. The Cahn-Hilliard-Cook equation

3.1. **The continuous problem.** The Cahn-Hilliard-Cook equation is

$$
\begin{aligned}
&\mathrm{d}u - \Delta w \,\mathrm{d}t = \mathrm{d}W && \text{in } \mathcal{D}\times[0,T],\\
&w + \Delta u + f(u) = 0 && \text{in } \mathcal{D}\times[0,T],\\
&\frac{\partial u}{\partial n} = \frac{\partial w}{\partial n} = 0 && \text{on } \partial\mathcal{D}\times[0,T],\\
&u(0) = u_0. && \text{in } \mathcal{D}.
\end{aligned}
\tag{3.1}
$$

The finite element approximation is based on its weak form, which is

$$
\begin{aligned}
&\langle u(t), v\rangle = \langle u_0, v\rangle + \int_0^t \langle w(s), \Delta v\rangle \,\mathrm{d}s + \int_0^t \langle \mathrm{d}W(s), v\rangle, \quad t>0,\\
&\langle w, v\rangle = \langle \nabla u, \nabla v\rangle + \langle f(u), v\rangle, \quad t>0,\\
&u(0) = u_0,
\end{aligned}
\tag{3.2}
$$

for all $v \in H^2$ with $\frac{\partial v}{\partial n} = 0$ on $\partial\mathcal{D}$. With the operator A, defined in Section 2, we write (3.1) in the formal abstract form on $H = L_2(\mathcal{D})$:

$$
\mathrm{d}X + \big(A^2X + Af(X)\big)\,\mathrm{d}t = \mathrm{d}W, \quad t>0; \quad X(0) = X_0. \tag{3.3}
$$

A weak solution of (3.3) satisfies

$$
\langle X(t), v\rangle - \langle X_0, v\rangle + \int_0^t \langle X, A^2 v\rangle \,\mathrm{d}s + \int_0^t \langle f(X(s)), Av\rangle \,\mathrm{d}s = \int_0^t \langle \mathrm{d}W(s), v\rangle,
$$

for all $v \in \dot H^4 = D(A^2)$. A mild solution of (3.3) is a solution of

$$
X(t) = \mathrm{e}^{-tA^2}X_0 - \int_0^t \mathrm{e}^{-(t-s)A^2} Af(X(s))\,\mathrm{d}s + \int_0^t \mathrm{e}^{-(t-s)A^2}\,\mathrm{d}W(s). \tag{3.4}
$$

3.2. **The finite element problem.** Recalling (3.2), we define the finite element solution $u_h(t) \in S_h$ of (3.1) by

$$
\begin{aligned}
&\langle u_h(t), v_h\rangle = \langle u_0, v_h\rangle + \int_0^t \langle \nabla w_h(s), \nabla v_h\rangle \,\mathrm{d}s + \int_0^t \langle \mathrm{d}W(s), v_h\rangle,\\
&\langle w_h, v_h\rangle = \langle \nabla u_h, \nabla v_h\rangle + \langle f(u_h), v_h\rangle,\\
&u_h(0) = u_{h,0},
\end{aligned}
$$

for all $v_h \in S_h,\ t>0$. This may also be written in the abstract form in S_h:

$$
\mathrm{d}X_h + \big(A_h^2 X_h + A_h P_h f(X_h)\big)\,\mathrm{d}t = P_h\mathrm{d}W, \quad t>0; \quad X_h(0) = P_h X_0, \tag{3.5}
$$

with mild solution

$$
\begin{aligned}
X_h(t) = \mathrm{e}^{-tA_h^2}X_0 &- \int_0^t \mathrm{e}^{-(t-s)A_h^2} A_h P_h f(X(s))\,\mathrm{d}s\\
&+ \int_0^t \mathrm{e}^{-(t-s)A_h^2} P_h\,\mathrm{d}W(s).
\end{aligned}
\tag{3.6}
$$

3.3. **A Lyapunov functional.** Define the functional

$$J(u) = \frac{1}{2}\|\nabla u\|^2 + \int_{\mathcal{D}} F(u)\,\mathrm{d}x, \quad u \in H^1, \tag{3.7}$$

where $F(s) = \frac{1}{4}s^4 - \frac{1}{2}s^2$ is a primitive of $f(s) = s^3 - s$. This is a Lyapunov functional for the deterministic Cahn-Hilliard equation, which means that in the deterministic case $J(X(t))$ does not increase along solution paths. For the stochastic equation this is not true, but we have a bound for the expected value of $J(X(t))$.

Theorem 3.1. *Assume that $\|A^{\frac{1}{2}}Q^{\frac{1}{2}}\|_{\mathrm{HS}} < \infty$ and X, X_h are weak solutions of (3.3) and (3.5) with $\mathbf{E}[J(X_0)] < \infty$ and that X_0 is $\mathcal{F}_0$-measurable with values in H^1. Then, for all $t > 0$, we have*

$$\mathbf{E}[J(X(t))] \le C\Big(\mathbf{E}[J(X_0)] + 1 + tK_Q + t^2K_Q^2\Big), \tag{3.8}$$

and

$$\mathbf{E}[J(X_h(t))] \le C\Big(\mathbf{E}[J(P_hX_0)] + 1 + tK_Q + t^2K_Q^2\Big), \tag{3.9}$$

where $K_Q = \|A^{\frac{1}{2}}Q^{\frac{1}{2}}\|^2_{\mathrm{HS}} + \langle Q\varphi_0, \varphi_0\rangle$.

Proof. We prove (3.9), the proof of (3.8) is essentially obtained by removing the subscript "h" everywhere (see also [4]).

We consider (3.5) as an Itô differential equation in S_h driven by P_hW, which is a $Q_h = P_hQP_h$-Wiener process in S_h. By assumption (2.8) it follows that $\mathbf{E}[J(P_hX_0)] < \infty$, if $\mathbf{E}[J(X_0)] < \infty$.

By applying Itô's formula ([9, Theorem 4.17]) to $J(X_h(t))$, we obtain

$$\begin{aligned} J(X_h(t)) &= J(X_h(0)) + \int_0^t \langle J'(X_h(s)), \mathrm{d}X_h(s)\rangle + \frac{1}{2}\int_0^t \mathrm{Tr}(J''(X_h(s)Q_h)\,\mathrm{d}s \\ &= J(P_hX_0) + \int_0^t \langle J'(X_h(s)), -A_h^2X_h(s) - P_hA_hf(X_h(s))\rangle\,\mathrm{d}s \\ &\quad + \frac{1}{2}\int_0^t \mathrm{Tr}(J''(X_h(s)Q_h)\,\mathrm{d}s + \int_0^t \langle J'(X_h(s)), \mathrm{d}W(s)\rangle. \end{aligned}$$

But we have

$$\langle J'(u_h), v_h\rangle = \langle \nabla u_h, \nabla v_h\rangle + \langle f(u_h), v_h\rangle = \langle A_hu_h + P_hf(u_h), v_h\rangle,$$

and

$$\begin{aligned} \langle J''(u_h)v_h, w_h\rangle &= \langle \nabla v_h, \nabla w_h\rangle + \langle f'(u_h)v_h, w_h\rangle \\ &= \langle A_hv_h + P_h[f'(u_h)v_h], w_h\rangle, \end{aligned}$$

so that

$$J'(u_h) = A_hu_h + P_hf(u_h), \quad J''(u_h) = A_h + P_h[f'(u_h)\cdot].$$

Hence, by (2.3),

$$\mathbf{E}[J(X_h(t))] = \mathbf{E}[J(P_hX_0)] - \mathbf{E}\left[\int_0^t |A_hX_h(s) + P_hf(X_h(s))|_1^2 \,\mathrm{d}s\right] + \frac{1}{2}\mathbf{E}\left[\int_0^t \Big(\operatorname{Tr}(A_hQ_h) + \operatorname{Tr}(P_h[f'(X_h(s))\cdot]Q_h)\Big)\,\mathrm{d}s\right].$$

We ignore the negative term on the right hand side to get

$$\begin{aligned}\mathbf{E}[J(X_h(t))] &\le \mathbf{E}[J(P_hX_0)] \\ &\quad + \frac{1}{2}\mathbf{E}\left[\int_0^t \Big(\operatorname{Tr}(A_hQ_h) + \operatorname{Tr}(P_h[f'(X_h(s))\cdot]Q_h)\Big)\,\mathrm{d}s\right].\end{aligned} \tag{3.10}$$

Now we compute $\operatorname{Tr}(A_hQ_h)$ and $\operatorname{Tr}(P_h[f'(X_h(s))\cdot]Q_h)$. To this end let $\{\varphi_{h,j}\}_{j=0}^{N_h}$ be an orthonormal basis of eigenvectors of A_h and $\{\lambda_{h,j}\}_{j=0}^{N_h}$ the corresponding eigenvalues. Then

$$\begin{aligned}&\operatorname{Tr}(A_hQ_h) = \operatorname{Tr}(Q_hA_h) = \sum_{j=1}^{N_h}\langle P_hQP_hA_h\varphi_{h,j}, \varphi_{h,j}\rangle = \sum_{j=1}^{N_h}\lambda_{h,j}\langle Q\varphi_{h,j}, \varphi_{h,j}\rangle \\ &= \sum_{j=1}^{N_h}\langle Q^{\frac12}A_h^{\frac12}\varphi_{h,j}, Q^{\frac12}A_h^{\frac12}\varphi_{h,j}\rangle = \sum_{j=1}^{N_h}\|Q^{\frac12}A_h^{\frac12}P_h\varphi_{h,j}\|^2 = \|Q^{\frac12}A_h^{\frac12}P_h\|_{\mathrm{HS}}^2 \\ &\le \|A_h^{\frac12}P_hQ^{\frac12}\|_{\mathrm{HS}}^2 = \|A_h^{\frac12}P_hA^{-\frac12}A^{\frac12}Q^{\frac12}\|_{\mathrm{HS}}^2 \le \|A_h^{\frac12}P_hA^{-\frac12}\|_{B(\dot H)}^2\|A^{\frac12}Q^{\frac12}\|_{\mathrm{HS}}^2 \\ &\le C\|A^{\frac12}Q^{\frac12}\|_{\mathrm{HS}}^2.\end{aligned}$$

Here we used (2.3) and (2.8) to get

$$\|A_h^{\frac12}P_hA^{-\frac12}v\| = |P_hA^{-\frac12}v|_1 \le C|A^{-\frac12}v|_1 = C\|v\|, \quad v \in \dot H,$$

so that $\|A_h^{\frac12}P_hA_h^{-\frac12}\|_{B(\dot H)} \le C$. Hence, with $K_Q = \|A^{\frac12}Q^{\frac12}\|_{\mathrm{HS}}^2 + \langle Q\varphi_0, \varphi_0\rangle$,

$$\|A_h^{\frac12}Q_h^{\frac12}\|_{\mathrm{HS}}^2 = \operatorname{Tr}(A_hQ_h) \le C\|A^{\frac12}Q^{\frac12}\|_{\mathrm{HS}}^2 \le CK_Q. \tag{3.11}$$

Let $\{e_{h,j}\}_{j=0}^{N_h}$ be an orthonormal eigenbasis of Q_h and $\{\gamma_{h,j}\}_{j=0}^{N_h}$ the corresponding eigenvalues. We get

$$\begin{aligned}\operatorname{Tr}\big(P_h[f'(X_h)\cdot]Q_h\big) &= \sum_{j=0}^{N_h}\langle P_h[f'(X_h)Q_he_{h,j}], e_{h,j}\rangle \\ &= \sum_{j=0}^{N_h}\gamma_{h,j}\langle f'(X_h)e_{h,j}, e_{h,j}\rangle \\ &= \sum_{j=0}^{N_h}\langle f'(X_h)Q_h^{\frac12}e_{h,j}, Q_h^{\frac12}e_{h,j}\rangle.\end{aligned} \tag{3.12}$$

By using the bound $|f'(s)| \le C(1+s^2)$, we get by Hölder's and Sobolev's inequalities,

$$|\langle f'(u)v,v\rangle| \le C(1+\|u\|_{L_4}^2)\|v\|_{L_4}^2 \le C(1+\|u\|_{L_4}^2)\|v\|_{H^1}^2 \le C(1+\|u\|_{L_4}^2)\|v\|_1^2.$$

By (2.1) and (2.3) we have, for $v_h \in S_h$,

$$\|v_h\|_1^2 = |v_h|_1^2 + \langle v_h,\varphi_0\rangle^2 = \|A_h^{\frac12}v_h\|^2 + \langle v_h,\varphi_0\rangle^2,$$

so that, by (3.11),

$$\begin{aligned}\sum_{j=0}^{N_h}\|Q_h^{\frac12}e_{h,j}\|_1^2 &= \sum_{j=0}^{N_h}\|A_h^{\frac12}Q_h^{\frac12}e_{h,j}\|^2 + \sum_{j=0}^{N_h}\langle Q_h^{\frac12}e_{h,j},\varphi_0\rangle^2\\ &\le \|A_h^{\frac12}Q_h^{\frac12}\|_{\mathrm{HS}}^2 + \|Q_h^{\frac12}\|_{\mathrm{HS}}^2 \le \|A_h^{\frac12}Q_h^{\frac12}\|_{\mathrm{HS}}^2 + \|Q^{\frac12}\|_{\mathrm{HS}}^2\\ &\le C\|A^{\frac12}Q^{\frac12}\|_{\mathrm{HS}}^2 + \langle Q\varphi_0,\varphi_0\rangle \le CK_Q.\end{aligned}$$

Here we used the boundedness of $A^{-\frac12}$ to get

$$\begin{aligned}\|Q^{\frac12}\|_{\mathrm{HS}}^2 &= \sum_{j=0}^{\infty}\|Q^{\frac12}\varphi_j\|^2 = \sum_{j=1}^{\infty}\|A^{-\frac12}A^{\frac12}Q^{\frac12}\varphi_j\|^2 + \|Q^{\frac12}\varphi_0\|^2\\ &\le C\sum_{j=1}^{\infty}\|A^{\frac12}Q^{\frac12}\varphi_j\|^2 + \langle Q\varphi_0,\varphi_0\rangle\\ &= C\|A^{\frac12}Q^{\frac12}\|_{\mathrm{HS}} + \langle Q\varphi_0,\varphi_0\rangle \le CK_Q.\end{aligned} \tag{3.13}$$

Returning to (3.12), we now have

$$\mathrm{Tr}(P_h[f'(X_h)\cdot]Q_h) \le C(1+\|X_h\|_{L_4}^2)\sum_{j=0}^{N_h}\|Q_h^{\frac12}e_{h,j}\|_1^2 \le C(1+\|X_h\|_{L_4}^2)K_Q, \tag{3.14}$$

Putting (3.11) and (3.14) in (3.10) gives

$$\mathbf{E}[J(X_h(t))] \le \mathbf{E}[J(P_hX_0)] + CK_Q\Big(t + \int_0^t \mathbf{E}\big[\|X_h(s)\|_{L_4}^2\big]\,\mathrm{d}s\Big). \tag{3.15}$$

It remains to bound $\int_0^t \mathbf{E}[\|X_h\|_{L_4}^2]\,\mathrm{d}s$. By definition of the Lyapunov functional (3.7) and noting that $F(s) = \frac14 s^4 - \frac12 s^2 \ge c_1 s^4 - c_2$, we get

$$J(u) \ge \frac12\|\nabla u\|^2 + C_1\|u\|_{L_4}^4 - C_2,$$

which implies

$$\|u\|_{L_4}^4 \le C_3(1+J(u)).$$

Hence, by Hölder's inequality, we get, for $\epsilon > 0$,

$$\begin{aligned} CK_Q \int_0^t \mathbf{E}[\|X_h(s)\|_{L_4}^2]\,\mathrm{d}s &\le CK_Q\Big(\int_0^t \mathbf{E}[\|X_h(s)\|_{L_4}^2]\,\mathrm{d}s\Big)^{\frac12} t^{\frac12} \\ &\le \frac{\epsilon}{C_3}\int_0^t \mathbf{E}[\|X_h(s)\|_{L_4}^2]\,\mathrm{d}s + \frac{C_3}{4\epsilon} tCK_Q^2 \\ &\le \epsilon \int_0^t \mathbf{E}[1 + J(X_h(s)]\,\mathrm{d}s + C\epsilon^{-1} tK_Q^2 \\ &\le \epsilon \int_0^t \mathbf{E}[J(X_h(s))]\,\mathrm{d}s + \epsilon t + C\epsilon^{-1} tK_Q^2. \end{aligned}$$

Putting this in (3.15) gives

$$\mathbf{E}[J(X_h(t))] \le \mathbf{E}[J(P_h X_0)] + C\Big(\epsilon + K_Q + \epsilon^{-1}K_Q^2\Big)t + \epsilon\int_0^t \mathbf{E}[J(X_h(s))]\,\mathrm{d}s.$$

Now apply the Gronwall Lemma 2.4 to get, for $\epsilon > 0$,

$$\begin{aligned} \mathbf{E}[J(X_h(t))] &\le \mathrm{e}^{\epsilon t}\Big(\mathbf{E}[J(P_h X_0)] + C(1 + \epsilon^{-1}K_Q + \epsilon^{-2}K_Q^2)\Big) \\ &\le \mathrm{e}\Big(\mathbf{E}[J(P_h X_0)] + C(1 + tK_Q + t^2K_Q^2)\Big), \end{aligned}$$

where for each fixed t we have chosen $\epsilon = t^{-1}$ to get an optimal bound. □

This theorem is adapted from [4]. We have improved it in several ways. Most importantly, the growth of the bound is reduced from exponential to quadratic with respect to t. Moreover, we have removed the assumption that A and Q have a common eigenbasis and that the eigenbasis of Q satisfies $\|e_j\|_{L_\infty} \le C$. It is also important that we obtain the same bound for X_h.

Note that the assumption $\|A^{\frac12}Q^{\frac12}\|_{\mathrm{HS}} < \infty$ is the same as the condition for regularity of order $\beta = 3$ for $W_A(t)$ in Theorem 2.1.

We now use the previous theorem to obtain norm bounds uniformly on subsets of Ω with probability arbitrarily close to 1.

Corollary 3.2. *Assume that* $\|A^{\frac12}Q^{\frac12}\|_{\mathrm{HS}} < \infty$ *and* X, X_h *are weak solutions of* (3.3) *and* (3.5) *with* X_0 $\mathcal{F}_0$*-measurable with values in* H^1 *and* $\|X_0\|_{L_2(\Omega,H^1)}^2 + \|X_0\|_{L_4(\Omega,L_4)}^4 \le \rho$. *Then, for every* $\epsilon \in (0,1)$, *there is* $\Omega_\epsilon \subset \Omega$ *with* $\mathbf{P}(\Omega_\epsilon) \ge 1 - \epsilon$ *and*

$$\|\nabla X(t)\|^2 + \|X(t)\|_{L_4}^4 \le \epsilon^{-1}K_T \quad \text{on } \Omega_\epsilon,\ t \in [0,T], \tag{3.16}$$

$$\|\nabla X_h(t)\|^2 + \|X_h(t)\|_{L_4}^4 \le \epsilon^{-1}K_T \quad \text{on } \Omega_\epsilon,\ t \in [0,T], \tag{3.17}$$

$$\|X(t)\|_1^2 + \|X_h(t)\|_1^2 \le \epsilon^{-1}K_T \quad \text{on } \Omega_\epsilon,\ t \in [0,T], \tag{3.18}$$

$$\|W_A(t)\|_3^2 \le \epsilon^{-1}K_T \quad \text{on } \Omega_\epsilon,\ t \in [0,T], \tag{3.19}$$

where $K_T = C(1 + \rho + K_QT + K_Q^2T^2)$.

Proof. Since $\mathbf{E}[J(X_0)] \le C(1+\rho)$, we obtain from Theorem 3.1,

$$\mathbf{E}[J(X(t))] \le C(1+\rho+K_Q T + K_Q^2 T^2) \le K_T \quad t\in[0,T].$$

We apply Chebyshev's inequality to get, for every $\alpha>0$ and $t\in[0,T]$,

$$\begin{aligned}\mathbf{P}\Big(\{\omega\in\Omega : \|\nabla X(t)\|^2 + \|X(t)\|_{L_4}^4 > \alpha\Big) &\le \frac{1}{\alpha}\mathbf{E}\left[\|\nabla X(t)\|^2 + \|X(t)\|_{L_4}^4\right] \\ \le \frac{1}{\alpha}C(1+\mathbf{E}\left[J(X(t))\right] &\le \frac{1}{\alpha}C(1+K_T) = \frac{K_T}{\alpha},\end{aligned}$$

where the C in K_T was adjusted. We choose $\alpha = \epsilon^{-1}K_T$ and set

$$\Omega_\epsilon = \{\omega\in\Omega : \|\nabla X(t)\|^2 + \|X(t)\|_{L_4}^4 \le \epsilon^{-1}K_T\}.$$

So (3.16) holds and

$$\mathbf{P}(\Omega_\epsilon) = 1 - \mathbf{P}\Big(\{\omega\in\Omega : \|\nabla X(t)\|^2 + \|X(t)\|_{L_4}^4 > \alpha\Big) \ge 1-\epsilon.$$

For (3.17) we replace X_h by X and note that we have $\mathbf{E}[J(P_h X_0)] \le C(1+\rho)$, by (2.8). For (3.18) we note that $\epsilon^{-1}K_T \ge 1$, and so

$$\|X(t)\|_1^2 \le \|\nabla X(t)\|^2 + \|X(t)\|^2 \le \|\nabla X(t)\|^2 + C\|X(t)\|_{L_4}^2 \le \epsilon^{-1}K_T$$

after an adjustment of the C in K_T. Finally, (3.19) follows in a similar way from Theorem 2.1 with $\beta=3$ with a constant which can be absorbed in K_T. □

4. Regularity of the solution

We quote the following from [4].

Theorem 4.1. *Let $T>0$ and assume that* $\mathrm{Tr}(A^{\delta-1}Q)<\infty$ *for some $\delta>0$ and that X_0 is $\mathcal{F}_0$-measurable with values in H. Then there is a process X, which is in $C([0,T],H)$ a.s. and which is a mild solution of* (1.1).

We now show that, under the assumption $\|A^{\frac12}Q^{\frac12}\|_{\mathrm{HS}}<\infty$, the solution is actually in H^3. In order to do this we write $X(t)=Y(t)+W_A(t)$, where we already know that W_A is in H^3 from Theorem 2.1. The regularity of Y is studied in the next theorem. Since

$$Y(t) = X(t) - W_A(t) = \mathrm{e}^{-tA^2}X_0 - \int_0^t \mathrm{e}^{-(t-s)A^2} A f(X(s))\,\mathrm{d}s,$$

it is a mild solution of

$$\dot{Y} + A^2 Y + Af(X) = 0, \quad t>0; \quad Y(0)=X_0. \tag{4.1}$$

Theorem 4.2. *Assume that $\|A^{\frac12}Q^{\frac12}\|_{\mathrm{HS}}<\infty$ and that X_0 is $\mathcal{F}_0$-measurable with values in H^3 and $\|X_0\|_{L_2(\Omega,H^1)}^2 + \|X_0\|_{L_4(\Omega,L_4)}^4 < \infty$. Let $T>0$ and $\epsilon\in(0,1)$ and let Ω_ϵ and K_T be as in Corollary 3.2. Let X be the solution* *from Theorem 4.1. Then, for each $\omega \in \Omega_\epsilon$ the mild solution Y of* (4.1) *belongs to $C([0,T],H^3)$. Moreover,*

$$\|Y(t)\|_3 \leq C(\|X_0\|_3, \epsilon^{-1}K_T, T) \quad \text{on } \Omega_\epsilon,\ t\in[0,T],$$
$$\|X(t)\|_3 \leq C(\|X_0\|_3, \epsilon^{-1}K_T, T) \quad \text{on } \Omega_\epsilon,\ t\in[0,T].$$

Proof. Let $T>0$ and $\omega \in \Omega_\epsilon$. From Corollary 3.2 we have

$$\|X(t)\|_1^2 \leq \epsilon^{-1}K_T, \quad \|W_A(t)\|_3 \leq \epsilon^{-1}K_T. \tag{4.2}$$

We take norms in

$$Y(t) = \mathrm{e}^{-tA^2}X_0 - \int_0^t \mathrm{e}^{-(t-s)A^2} Af(X(s))\,\mathrm{d}s, \tag{4.3}$$

to get

$$\begin{aligned}
|Y(t)|_3 &\leq |\mathrm{e}^{-tA^2}X_0|_3 + \int_0^t |\mathrm{e}^{-(t-s)A^2}Af(X(s))|_3\,\mathrm{d}s \\
&= \|\mathrm{e}^{-tA^2}A^{\frac32}X_0\| + \int_0^t \|A^{\frac32}\mathrm{e}^{-(t-s)A^2}Af(X(s))\|\,\mathrm{d}s \\
&\leq |X_0|_3 + C\int_0^t (t-s)^{-\frac34}\|Af(X(s))\|\,\mathrm{d}s.
\end{aligned}$$

We apply (2.10) to $\|Af(X(s))\| = \|\Delta f(X(s))\|$ to get

$$\begin{aligned}
|Y(t)|_3 &\leq |X_0|_3 + C\int_0^t (t-s)^{-\frac34}(1+\|X(s)\|_1^2)\|X(s)\|_3\,\mathrm{d}s \\
&\leq |X_0|_3 + C\int_0^t (t-s)^{-\frac34}(1+\|X(s)\|_1^2)(\|Y(s)\|_3 + \|W_A(s)\|_3)\,\mathrm{d}s.
\end{aligned}$$

Since $(I-P)Y(t) = (I-P)X_0$ is constant, we get the same bound for the norm $\|Y(t)\|_3$. Using also (4.2) gives

$$\begin{aligned}
\|Y(t)\|_3 &\leq \|X_0\|_3 + C\int_0^t (t-s)^{-\frac34}(1+\epsilon^{-1}K_T)(\|Y(s)\|_3 + \epsilon^{-1}K_T)\,\mathrm{d}s \\
&\leq \|X_0\|_3 + C\epsilon^{-1}K_T(1+\epsilon^{-1}K_T)T^{\frac14} \\
&\quad + C(1+\epsilon^{-1}K_T)\int_0^t (t-s)^{-\frac34}\|Y(s)\|_3\,\mathrm{d}s.
\end{aligned}$$

Applying Gronwall's Lemma 2.3 with $\alpha = 1$, $\beta = \frac14$ and

$$A = \|X_0\|_3 + C\epsilon^{-1}K_T\left(1+\epsilon^{-1}K_T\right),\ B = C\left(1+\epsilon^{-1}K_T\right), \tag{4.4}$$

gives

$$\|Y(t)\|_3 \leq AC(B,T) = C(\|X_0\|_3, \epsilon^{-1}K_T, T), \quad t\in[0,T].$$

The bound for $\|X(t)\|_3$ then follows in view of (4.2). □

The constant $C(\|X_0\|_3, \epsilon^{-1}K_T, T)$ grows rapidly with $\epsilon^{-1}K_T$ and T. Hence, it is important that K_T grows only quadratically with T.

5. Error estimates

5.1. **Error estimate for deterministic Cahn-Hilliard equation.**

Consider the linear Cahn-Hilliard equation

$$\begin{aligned} &\dot{u} + Av = 0, && t > 0, \\ &v - Au - f = 0, && t > 0 \\ &u(0) = u_0, \end{aligned} \tag{5.1}$$

where f is a function of x, t, and the corresponding finite element problem

$$\begin{aligned} &\dot{u}_h + A_h v_h = 0, && t > 0, \\ &v_h - A_h u_h - P_h f = 0, && t > 0, \\ &u_h(0) = P_h u_0. \end{aligned} \tag{5.2}$$

We have the following error estimate. We will later use this for fixed $\omega \in \Omega_\epsilon$ with f replaced by $f(X)$ and u by the solution Y of (1.3).

Theorem 5.1. *Assume that u, v and u_h, v_h are the solutions of* (5.1) *and* (5.2)*, respectively. Then, for $t \geq 0$,*

$$\|u_h(t) - u(t)\| \leq Ch^2 \Big(|\log(h)| \max_{0 \leq s \leq t} |u(s)|_2 + \Big(\int_0^t |v(s)|_2^2 \, \mathrm{d}s \Big)^{\frac{1}{2}} \Big). \tag{5.3}$$

Proof. The weak forms of (5.1) and (5.2) are

$$\begin{aligned} &\langle \dot{u}, \varphi_1 \rangle + \langle \nabla v, \nabla \varphi_1 \rangle = 0 && \forall \varphi_1 \in H^1, \\ &\langle v, \varphi_2 \rangle - \langle \nabla u, \nabla \varphi_2 \rangle - \langle f, \varphi_2 \rangle = 0 && \forall \varphi_2 \in H^1, \\ &u(0) = u_0, \end{aligned} \tag{5.4}$$

and

$$\begin{aligned} &\langle \dot{u}_h, \varphi_{h,1} \rangle + \langle \nabla v_h, \nabla \varphi_{h,1} \rangle = 0 && \forall \varphi_{h,1} \in S_h, \\ &\langle v_h, \varphi_{h,2} \rangle - \langle \nabla u_h, \nabla \varphi_{h,2} \rangle - \langle f, \varphi_{h,2} \rangle = 0 && \forall \varphi_{h,2} \in S_h, \\ &u_h(0) = P_h u_0. \end{aligned} \tag{5.5}$$

Let P_h and R_h be as in (2.4) and (2.6) and set

$$e_u = u_h - u = (u_h - P_h u) + (P_h u - u) = \theta_u + \rho_u, \tag{5.6}$$

$$e_v = v_h - v = (v_h - R_h v) + (R_h v - v) = \theta_v + \rho_v. \tag{5.7}$$

We want to compute

$$\|e_u\| \leq \|\theta_u\| + \|\rho_u\|. \tag{5.8}$$

In (5.4) choose $\varphi_1 = \varphi_{h,1}$ and $\varphi_2 = \varphi_{h,2}$ and subtract the first two equations of (5.4) from the corresponding equations in (5.5) to get

$$\begin{aligned} &\langle \dot{e}_u, \varphi_{h,1} \rangle + \langle \nabla e_v, \nabla \varphi_{h,1} \rangle = 0 && \forall \varphi_{h,1} \in S_h, \\ &\langle e_v, \varphi_{h,2} \rangle - \langle \nabla e_u, \nabla \varphi_{h,2} \rangle = 0 && \forall \varphi_{h,2} \in S_h. \end{aligned}$$

Hence, by (5.6) and (5.7),

$$\begin{aligned}
&\langle \dot\theta_u, \varphi_{h,1}\rangle + \langle \nabla\theta_v, \nabla\varphi_{h,1}\rangle = -\langle \dot\rho_u, \varphi_{h,1}\rangle - \langle \nabla\rho_v, \nabla\varphi_{h,1}\rangle \quad \forall \varphi_{h,1} \in S_h,\\
&\langle \theta_v, \varphi_{h,2}\rangle - \langle \nabla\theta_u, \nabla\varphi_{h,2}\rangle = -\langle \rho_v, \varphi_{h,2}\rangle + \langle \nabla\rho_u, \nabla\varphi_{h,2}\rangle \quad \forall \varphi_{h,2} \in S_h.
\end{aligned}$$

By the definitions of P_h and R_h we have

$$\begin{aligned}
\langle \dot\rho_u, \varphi_{h,1}\rangle &= \langle P_h\dot u - \dot u, \varphi_{h,1}\rangle = 0 && \forall \varphi_{h,1} \in S_h,\\
\langle \nabla\rho_v, \nabla\varphi_{h,1}\rangle &= \langle \nabla R_h v - v, \nabla\varphi_{h,1}\rangle = 0 && \forall \varphi_{h,2} \in S_h,
\end{aligned}$$

so that

$$\begin{aligned}
&\langle \dot\theta_u, \varphi_{h,1}\rangle + \langle \nabla\theta_v, \nabla\varphi_{h,1}\rangle = 0 && \forall \varphi_{h,1} \in S_h\\
&\langle \theta_v, \varphi_{h,2}\rangle - \langle \nabla\theta_u, \nabla\varphi_{h,2}\rangle = -\langle \rho_v, \varphi_{h,2}\rangle + \langle \nabla\rho_u, \nabla\varphi_{h,2}\rangle && \forall \varphi_{h,2} \in S_h.
\end{aligned}$$

In the second equation we set $\varphi_{h,2} = A_h\varphi_{h,1}$ to get

$$\langle \nabla\theta_v, \nabla\varphi_{h,1}\rangle = \langle A_h^2\theta_u, \varphi_{h,1}\rangle - \langle A_hP_h\rho_v, \varphi_{h,1}\rangle + \langle A_h^2R_h\rho_u, \varphi_{h,1}\rangle.$$

Inserting this into the first equation gives

$$\langle \dot\theta_u, \varphi_{h,1}\rangle + \langle A_h^2\theta_u, \varphi_{h,1}\rangle = \langle A_hP_h\rho_v, \varphi_{h,1}\rangle - \langle A_h^2R_h\rho_u, \varphi_{h,1}\rangle,$$

so the strong form is

$$\dot\theta_u + A_h^2\theta_u = A_hP_h\rho_v - A_h^2R_h\rho_u, \quad t > 0; \quad \theta_u(0) = 0,$$

with the mild solution

$$\theta_u(t) = \int_0^t \mathrm{e}^{-(t-s)A_h^2} A_hP_h\rho_v(s)\,\mathrm{d}s - \int_0^t \mathrm{e}^{-(t-s)A_h^2} A_h^2R_h\rho_u(s)\,\mathrm{d}s.$$

Taking norms here gives

$$\begin{aligned}
\|\theta_u(t)\| \le{}& \Big\|\int_0^t \mathrm{e}^{-(t-s)A_h^2} A_hP_h\rho_v(s)\,\mathrm{d}s\Big\|\\
&+ \Big\|\int_0^t \mathrm{e}^{-(t-s)A_h^2} A_h^2R_h\rho_u(s)\,\mathrm{d}s\Big\| = I + II.
\end{aligned} \tag{5.9}$$

For I we define

$$w_h(t) = \int_0^t \mathrm{e}^{-(t-s)A_h^2} P_h\rho_v(s)\,\mathrm{d}s,$$

which satisfies the equation

$$\dot w_h + A_h^2 w_h = P_h\rho_v, \quad t > 0; \quad w_h(0) = 0.$$

Multiply by $\dot w_h$ to get

$$\|\dot w_h\|^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|A_hw_h\|^2 = \langle P_h\rho_v, \dot w_h\rangle \le \|\rho_v\|\|\dot w_h\| \le \frac{1}{2}\|\rho_v\|^2 + \frac{1}{2}\|\dot w_h\|^2.$$

So we get

$$\|\dot w_h\|^2 + \frac{\mathrm{d}}{\mathrm{d}t}\|A_hw_h\|^2 \le \|\rho_v\|^2.$$

Integrate and ignore $\int_0^t \|\dot{w}_h(s)\|^2\,\mathrm{d}s$ to get

$$\Big\|A_h\int_0^t \mathrm{e}^{-(t-s)A_h^2}P_h\rho_v(s)\,\mathrm{d}s\Big\| = \|A_h w_h(t)\| \le \Big(\int_0^t \|\rho_v(s)\|^2\,\mathrm{d}s\Big)^{\frac12},$$

where, from (2.7),

$$\|\rho_v\| = \|(R_h - I)v\| \le Ch^2|v|_2.$$

So we get

$$\Big\|A_h\int_0^t \mathrm{e}^{-(t-s)A_h^2}P_h\rho_v(s)\,\mathrm{d}s\Big\| \le Ch^2\Big(\int_0^t |v(s)|_2^2\,\mathrm{d}s\Big)^{\frac12}. \tag{5.10}$$

For II we use

$$R_h\rho_u = R_h(P_hu - u) = P_hu - R_hu = P_h(u - R_hu).$$

Then

$$\begin{aligned}\Big\|\int_0^t A_h^2\mathrm{e}^{-(t-s)A_h^2}R_h\rho_u(s)\,\mathrm{d}s\Big\| &\le \int_0^t \|A_h^2\mathrm{e}^{-(t-s)A_h^2}P_h(u(s) - R_hu(s))\|\,\mathrm{d}s\\ &\le \int_0^t \|A_h^2\mathrm{e}^{-(t-s)A_h^2}P_h\|\,\mathrm{d}s \max_{0\le s\le t}\|u(s) - R_hu(s)\|\,\mathrm{d}s.\end{aligned}$$

Here we use $\|A_h\| \le Ch^{-2}$ from (2.8) and (2.5) to get

$$\begin{aligned}\int_0^t \|A_h^2\mathrm{e}^{-(t-s)A_h^2}P_h\|\,\mathrm{d}s &= \int_0^{h^4} \|A_h\|^2\|\mathrm{e}^{-sA_h^2}\|\,\mathrm{d}s + \int_{h^4}^t \|A_h^2\mathrm{e}^{-sA_h^2}\|\,\mathrm{d}s\\ &\le Ch^{-4}h^4 + C\int_{h^4}^t s^{-1}\mathrm{e}^{-cs}\,\mathrm{d}s \le C(1+\log(1/h)) \le C|\log(h)|.\end{aligned}$$

Hence, by (2.7), we have

$$\Big\|\int_0^t A_h^2\mathrm{e}^{-(t-s)A_h^2}R_h\rho_u(s)\,\mathrm{d}s\Big\| \le Ch^2|\log(h)|\max_{0\le s\le t}|u(s)|_2. \tag{5.11}$$

Putting (5.10) and (5.11) in (5.9) gives

$$\|\theta_u(t)\| \le Ch^2\Big\{\Big(\int_0^t |v(s)|_2^2\,\mathrm{d}s\Big)^{\frac12} + |\log(h)|\max_{0\le s\le t}|u(s)|_2\Big\}. \tag{5.12}$$

Finally, by the best approximation property of P_h,

$$\|\rho_u(t)\| = \|P_hu - u\| \le \|R_hu - u\| \le Ch^2|u(t)|_2. \tag{5.13}$$

Putting (5.12) and (5.13) in (5.8) gives the desired result (5.3). □

In the next lemma we prove a stabilty estimate for the deterministic Cahn-Hilliard equation (5.1).

Lemma 5.2. *Assume that u, v are the solutions of* (5.1). *Then*

$$|u(t)|_2^2 + \int_0^t |v(s)|_2^2\,\mathrm{d}s \le |u_0|_2^2 + \int_0^t |f(s)|_2^2\,\mathrm{d}s.$$

Proof. Multiply the first equation in (5.1) by A^2u to get

$$\frac{1}{2}|u|_2^2 + \langle A^2 v, Au\rangle = 0.$$

The second equation of (5.1) gives $Au = v - f$, so we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|u|_2^2 + \langle A^2 v, v\rangle = \langle A^2 v, f\rangle \le |v|_2|f|_2 \le \frac{1}{2}|v|_2^2 + \frac{1}{2}|f|_2^2,$$

so that

$$\frac{\mathrm{d}}{\mathrm{d}t}|u|_2^2 + |v|_2^2 \le |f|_2^2.$$

The proof is finished by integration. □

5.2. **Error estimate for the stochastic Cahn-Hilliard equation.** In the next theorem we prove an error estimate for the nonlinear Cahn-Hilliard-Cook equation.

Theorem 5.3. *Assume that* $\|A^{\frac{1}{2}}Q^{\frac{1}{2}}\|_{\mathrm{HS}} < \infty$ *and* X, X_h *are the solutions of* (3.3) *and* (3.5) *with* X_0 $\mathcal{F}_0$*-measurable with values in* H^3 *and* $\|X_0\|^2_{L_2(\Omega,H^1)} + \|X_0\|^4_{L_4(\Omega,L_4)} < \infty$*. Let* $T>0$*,* $\epsilon \in (0,1)$*, and let* $\Omega_\epsilon \subset \Omega$ *and* K_T *be as in Corollary 3.2. Then we have*

$$\|X_h(t) - X(t)\| \le C(\|X_0\|_3, \epsilon^{-1}K_T, T)h^2|\log(h)|, \quad \text{on } \Omega_\epsilon,\ t\in[0,T].$$

The constant $C(\|X_0\|_3, \epsilon^{-1}K_T, T)$ grows rapidly with $\epsilon^{-1}K_T$ and T due to the use of Gronwall's lemma in the proof.

Proof. Let $\omega \in \Omega_\epsilon$ be fixed. Set

$$X(t) = Y(t) + W_A(t), \tag{5.14}$$

where $W_A(t)$ is the stochastic convolution

$$W_A(t) = \int_0^t \mathrm{e}^{-(t-s)A^2}\,\mathrm{d}W(s), \tag{5.15}$$

and $Y(t)$ is the mild solution (4.3) of (1.3). Also set

$$X_h(t) = Z_h(t) + W_{A_h}(t), \tag{5.16}$$

where $W_{A_h}(t)$ is the stochastic convolution

$$W_{A_h}(t) = \int_0^t \mathrm{e}^{-(t-s)A_h^2}P_h\,\mathrm{d}W(s), \tag{5.17}$$

and

$$Z_h(t) = \mathrm{e}^{-tA_h^2}P_hX_0 - \int_0^t \mathrm{e}^{-(t-s)A_h^2}A_hP_hf(X_h(s))\,\mathrm{d}s, \tag{5.18}$$

is the mild solution of

$$\dot{Z}_h + A_h^2 Z_h = -A_hP_hf(X_h), \quad t>0; \quad Z_h(0) = P_hX_0. \tag{5.19}$$

Finally, let

$$Y_h(t) = \mathrm{e}^{-tA_h^2} P_h X_0 - \int_0^t \mathrm{e}^{-(t-s)A_h^2} A_h P_h f(X(s))\,\mathrm{d}s, \tag{5.20}$$

be the mild solution of

$$\dot{Y}_h + A_h^2 Y_h = -A_h P_h f(X), \quad t > 0; \quad Y_h(0) = P_h X_0. \tag{5.21}$$

We subtract (5.14) from (5.16),

$$\begin{aligned} X_h - X &= (Z_h + W_{A_h}) - (Y + W_A) \\ &= (W_{A_h} - W_A) + (Y_h - Y) + (Z_h - Y_h), \end{aligned}$$

and take norms,

$$\|X_h - X\| \le \|W_{A_h} - W_A\| + \|Y_h - Y\| + \|Z_h - Y_h\|. \tag{5.22}$$

We compute the three norms on the right hand side.

First we compute $\|W_{A_h}(t) - W_A(t)\|$. Since $\|A^{\frac12} Q^{\frac12}\|_{\mathrm{HS}} < \infty$, we have that $\|Q^{\frac12}\|_{\mathrm{HS}} < \infty$ and hence, by Theorem 2.2 and Chebyshev's inequality, we get

$$\begin{aligned} \|W_{A_h}(t) - W_A(t)\| &\le \epsilon^{-\frac12} \big(\mathbf{E}[\|W_{A_h}(t) - W_A(t)\|^2]\big)^{\frac12} \\ &\le \epsilon^{-\frac12} C h^2 |\log(h)| \|Q^{\frac12}\|_{\mathrm{HS}} \le C(\epsilon^{-1} K_Q)^{\frac12} h^2 |\log(h)|, \end{aligned}$$

see (3.13). Since $K_Q \le K_T$, we conclude

$$\|W_{A_h}(t) - W_A(t)\| \le C(\epsilon^{-1} K_T)^{\frac12} h^2 |\log(h)|. \tag{5.23}$$

Now we consider $\|Y_h(t) - Y(t)\|$ and use Theorem (5.1) to get

$$\|Y_h(t) - Y(t)\| \le C h^2 \Big\{ |\log(h)| \max_{0\le s\le t} |Y(s)|_2 + \Big(\int_0^t |V(s)|_2^2 \,\mathrm{d}s \Big)^{\frac12} \Big\}, \tag{5.24}$$

where $Y(t)$ and $V(t)$ are the solutions of

$$\begin{aligned} &\dot{Y} + AV = 0, && t > 0, \\ &V = AY + f(X), && t > 0, \\ &Y(0) = X_0. \end{aligned} \tag{5.25}$$

By using Lemma 5.2, (2.10), and (3.19), we get

$$\begin{aligned} \int_0^t |V(s)|_2^2 \,\mathrm{d}s &\le |X_0|_2^2 + \int_0^t |f(X(s))|_2^2 \,\mathrm{d}s \\ &\le \|X_0\|_2^2 + C \int_0^t (1 + \|X(s)\|_1^2) \|X(s)\|_3 \,\mathrm{d}s \\ &\le \|X_0\|_3^2 + C \int_0^t \big(1 + \|X(s)\|_3^3\big) \,\mathrm{d}s \\ &\le \|X_0\|_3^2 + CT \Big(1 + (\epsilon^{-1} K_T)^{\frac32})\Big). \end{aligned}$$

So

$$\int_0^t |V(s)|_2^2 \, \mathrm{d}s \leq C(\|X_0\|_3, \epsilon^{-1}K_T, T). \tag{5.26}$$

Now we bound $|Y(t)|_2$. By Theorem 4.2 we have

$$|Y(t)|_2 \leq \|Y(t)\|_3 \leq C(\|X_0\|_3, \epsilon^{-1}K_T, T). \tag{5.27}$$

Using (5.26) and (5.27) in (5.24) gives

$$\|Y_h(t) - Y(t)\| \leq C(\|X_0\|_3, \epsilon^{-1}K_T, T)h^2|\log(h)|. \tag{5.28}$$

Finally we compute $\|e_h(t)\| = \|Z_h(t) - Y_h(t)\|$. By subtraction of (5.18) and (5.20), we obtain

$$\begin{aligned}
\|e_h(t)\| &\leq \int_0^t \|\mathrm{e}^{-(t-s)A_h^2} A_h P_h P(f(X_h(s)) - f(X(s))\| \, \mathrm{d}s \\
&= \int_0^t \|A_h^{\frac{3}{2}} \mathrm{e}^{-(t-s)A_h^2} A_h^{-\frac{1}{2}} P_h P(f(X_h(s)) - f(X(s))\| \, \mathrm{d}s \\
&\leq \int_0^t \|A_h^{\frac{3}{2}} \mathrm{e}^{-(t-s)A_h^2} P_h\| \|A_h^{-\frac{1}{2}} P(f(X_h(s)) - f(X(s))\| \, \mathrm{d}s,
\end{aligned}$$

since the constant eigenmodes cancel (cf. (2.9)). Using (2.11) and (2.5) gives

$$\|e_h(t)\| \leq C \int_0^t (t-s)^{-\frac{3}{4}} (1 + \|X_h(s)\|_1^2 + \|X(s)\|_1^2) \|X_h(s) - X(s)\| \, \mathrm{d}s.$$

By Corollary (3.2) we have

$$\begin{aligned}
\|e_h(t)\| &\leq C \int_0^t (t-s)^{-\frac{3}{4}} \Big(1 + \epsilon^{-1}K_T\Big) \Big(\|W_{A_h}(s) - W_A(s)\| \\
&\qquad + \|Y_h(s) - Y(s)\| + \|e_h(s)\|\Big) \, \mathrm{d}s \\
&\leq C\Big(1 + \epsilon^{-1}K_T\Big) T^{\frac{1}{4}} \max_{0 \leq s \leq T} \Big(\|W_{A_h}(s) - W_A(s)\| + \|Y_h(s) - Y(s)\|\Big) \\
&\qquad + C\Big(1 + \epsilon^{-1}K_T\Big) \int_0^t (t-s)^{-\frac{3}{4}} \|e_h(s)\| \, \mathrm{d}s.
\end{aligned}$$

We apply Gronwall's Lemma 2.3 with $\alpha = 1$, $\beta = \frac{1}{4}$ and

$$\begin{aligned}
A &= C(1 + \epsilon^{-1}K_T) T^{\frac{1}{4}} \max_{0 \leq s \leq T} \Big(\|W_{A_h}(s) - W_A(s)\| + \|Y_h(s) - Y(s)\|\Big), \\
B &= C(1 + \epsilon^{-1}K_T),
\end{aligned}$$

to get

$$\|Z_h(t) - Y_h(t)\| = \|e_h(t)\| \leq AC(B, T), \quad t \in [0, T]. \tag{5.29}$$

But we bounded $\|W_{A_h}(t) - W_A(t)\|$ and $\|Y_h(t) - Y(t)\|$ in (5.23) (5.28). By putting these values and (5.29) in (5.22) we get the desired result. □

We finally show that X_h converges strongly to X. More precisely, we show that $X_h(t) \to X(t)$ in $L_2(\Omega, H)$ uniformly on $[0, T]$ as $h \to 0$.

Theorem 5.4. *Assume that* $\|A^{\frac{1}{2}}Q^{\frac{1}{2}}\|_{\mathrm{HS}} < \infty$ *and* X, X_h *are the solutions of* (3.3) *and* (3.5) *with* X_0 $\mathcal{F}_0$*-measurable with values in* H^3 *and* $\|X_0\|^2_{L_2(\Omega,H^1)} + \|X_0\|^4_{L_4(\Omega,L_4)} < \infty$*. Then*

$$\max_{t\in[0,T]} \big(\mathbf{E}[\|X_h(t) - X(t)\|^2]\big)^{\frac{1}{2}} \to 0 \quad \text{as } h \to 0.$$

Proof. From Theorem 3.1 it follows that

$$\mathbf{E}\left[\|X(t)\|^4_{L_4}\right] \le K_T, \quad \mathbf{E}\left[\|X_h(t)\|^4_{L_4}\right] \le K_T, \quad t \in [0,T],$$

with K_T as in Corollary 3.2. Let $\epsilon \in (0,1)$ and let Ω_ϵ be as in Corollary 3.2. Then

$$\begin{aligned}\mathbf{E}\left[\|X_h(t) - X(t)\|^2\right] \le &\int_{\Omega_\epsilon} \|X_h(t) - X(t)\|^2 \,\mathrm{d}\mathbf{P} \\ &+ 2\int_{\Omega_\epsilon^c} \left(\|X_h(t)\|^2 + \|X(t)\|^2\right) \mathrm{d}\mathbf{P}.\end{aligned}$$

Here, by Hölder's inequality, we have

$$\begin{aligned}\int_{\Omega_\epsilon^c} \|X(t)\|^2 \,\mathrm{d}\mathbf{P} &\le \Big(\int_{\Omega_\epsilon^c} 1^2 \,\mathrm{d}\mathbf{P}\Big)^{\frac{1}{2}} \Big(\int_{\Omega_\epsilon^c} \|X(t)\|^4_{L_4} \,\mathrm{d}\mathbf{P}\Big)^{\frac{1}{2}} \\ &\le \epsilon^{\frac{1}{2}} \left(\mathbf{E}\left[\|X(t)\|^4_{L_4}\right]\right)^{\frac{1}{2}} \le \epsilon^{\frac{1}{2}} K_T^{\frac{1}{2}}.\end{aligned}$$

Therefore, by Theorem 5.3,

$$\max_{t\in[0,T]} \left(\mathbf{E}\left[\|X_h(t) - X(t)\|^2\right]\right)^{\frac{1}{2}} \le C(\epsilon^{-1}K_T, T)h^2|\log(h)| + CK_T^{\frac{1}{4}}\epsilon^{\frac{1}{4}}.$$

Since $\frac{\epsilon^{\frac{1}{4}}}{C(\epsilon^{-1}K_T,T)} \to 0$ monotonically as $\epsilon \to 0$, we may choose ϵ, depending on h, such that the two terms are equal. □

Since $C(\epsilon^{-1}K_T, T)$ grows rapidly with ϵ^{-1}, it is not possible to obtain a rate of convergence from this proof.

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