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# A POSTERIORI ERROR ANALYSIS FOR THE CAHN-HILLIARD EQUATION

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ABSTRACT. The Cahn-Hilliard equation is discretized by a Galerkin finite element method based on continuous piecewise linear functions in space and discontinuous piecewise constant functions in time. A posteriori error estimates are proved by using the methodology of dual weighted residuals.

## 1. INTRODUCTION

We consider the Cahn-Hilliard equation

$$(1.1) \quad \begin{aligned} u_t - \Delta w &= 0 && \text{in } \Omega \times [0, T], \\ w + \epsilon \Delta u - f(u) &= 0 && \text{in } \Omega \times [0, T], \\ \frac{\partial u}{\partial \nu} = 0, \frac{\partial w}{\partial \nu} &= 0 && \text{on } \partial\Omega \times [0, T], \\ u(\cdot, 0) &= g_0 && \text{in } \Omega, \end{aligned}$$

where  $\Omega$  is a polygonal domain in  $\mathbf{R}^d$ ,  $d = 1, 2, 3$ ,  $u = u(x, t)$ ,  $w = w(x, t)$ ,  $\Delta = \sum_{i=1}^d \frac{\partial^2}{\partial x_i^2}$ ,  $u_t = \frac{\partial u}{\partial t}$ ,  $\nu$  is the exterior unit normal to  $\partial\Omega$ , and  $\epsilon > 0$  is a small parameter. The Cahn-Hilliard equation is a model for phase separation and spinodal decomposition [3]. The nonlinearity  $f$  is the derivative of a double-well potential. A typical example is  $f(u) = u^3 - u$ .

We discretize (1.1) by a Galerkin finite element method, which is based on continuous piecewise linear functions with respect to  $x$  and discontinuous piecewise constant functions with respect to  $t$ . This numerical method is the same as the implicit Euler time stepping combined with spatial discretization by a standard finite element method.

We perform an a posteriori error analysis within the framework of dual weighted residuals [2]. If  $J(u)$  is a given goal functional, this results in an

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error estimate essentially of the form

$$|J(u) - J(U)| \leq \sum_{n=1}^N \sum_{K \in \mathbf{T}_n} \left\{ \rho_{u,K} \omega_{u,K} + \rho_{w,K} \omega_{w,K} \right\} + \mathcal{R},$$

where  $U$  denotes the numerical solution and  $\mathbf{T}_n$  is the spatial mesh at time level  $t_n$ . The terms  $\rho_{u,K}, \rho_{w,K}$  are local residuals from the first and second equations in (1.1), respectively. The weights  $\omega_{u,K}, \omega_{w,K}$  are derived from the solution of the linearized adjoint problem. The remainder  $\mathcal{R}$  is quadratic in the error.

There is an extensive literature on numerical methods for the Cahn-Hilliard equation; see, for example, [5] and [4] for a priori error estimates. Adaptive methods based on a posteriori estimates are presented in [1] and [6]. However, these estimates are restricted to spatial discretization. We are not aware of any completely discrete a posteriori error analysis.

## 2. PRELIMINARIES

Here we present the methodology of dual weighted residuals [2] in an abstract form.

Let  $A(\cdot; \cdot)$  be a semilinear form; that is, it is nonlinear in the first and linear in the second variable, and  $J(\cdot)$  be an output functional, not necessarily linear, defined on some function space  $V$ . Consider the variational problem: Find  $u \in V$  such that

$$(2.1) \quad A(u; \psi) = 0 \quad \forall \psi \in V,$$

and the corresponding finite element problem: Find  $u_h \in V_h \subset V$  such that

$$(2.2) \quad A(u_h; \psi_h) = 0 \quad \forall \psi_h \in V_h.$$

We suppose that the derivatives of  $A$  and  $J$  with respect to the first variable  $u$  up to order three exist and are denoted by

$$A'(u; \varphi), A''(u; \psi, \varphi), A'''(u; \xi, \psi, \varphi),$$

and

$$J'(u; \varphi), J''(u; \psi, \varphi), J'''(u; \xi, \psi, \varphi),$$

respectively, for increments  $\varphi, \psi, \xi \in V$ . Here we use the convention that the forms are linear in the variables after the semicolon.

We want to estimate  $J(u) - J(u_h)$ . Introduce the dual variable  $z \in V$  and define the Lagrange functional

$$\mathcal{L}(u; z) := J(u) - A(u; z)$$

and seek the stationary points  $(u, z) \in V \times V$  of  $\mathcal{L}(\cdot; \cdot)$ ; that is,

$$(2.3) \quad \mathcal{L}'(u; z, \varphi, \psi) = J'(u; \varphi) - A'(u; z, \varphi) - A(u; \psi) = 0 \quad \forall \varphi, \psi \in V.$$

By choosing  $\varphi = 0$ , we retrieve (2.1). By taking  $\psi = 0$ , we identify the linearized adjoint equation to find  $z \in V$  such that

$$(2.4) \quad J'(u; \varphi) - A'(u; z, \varphi) = 0 \quad \forall \varphi \in V.$$

The corresponding finite element problem is: Find  $(u_h, z_h) \in V_h \times V_h$  such that

$$(2.5) \quad \begin{aligned} \mathcal{L}'(u_h; z_h, \varphi_h, \psi_h) &= J'(u_h; \varphi_h) - A'(u_h; z_h, \varphi_h) - A(u_h; \psi_h) \\ &= 0 \quad \forall \varphi_h, \psi_h \in V_h. \end{aligned}$$

By choosing  $\varphi_h = 0$ , we retrieve (2.2). By taking  $\psi_h = 0$ , we identify the linearized adjoint equation to find  $z_h \in V_h$  such that

$$(2.6) \quad J'(u_h; \varphi_h) - A'(u_h; z_h, \varphi_h) = 0 \quad \forall \varphi_h \in V_h.$$

We quote three propositions from [2, Ch. 6].

**Proposition 2.1.** *Let  $L(\cdot)$  be a three times differentiable functional defined on a vector space  $X$ , which has a stationary point  $x \in X$ , that is,*

$$L'(x; y) = 0 \quad \forall y \in X.$$

*Suppose that on a finite dimensional subspace  $X_h \subset X$  the corresponding Galerkin approximation,*

$$L'(x_h; y_h) = 0 \quad \forall y_h \in X_h,$$

*has a solution,  $x_h \in X_h$ . Then there holds the error representation*

$$L(x) - L(x_h) = \frac{1}{2}L'(x_h; x - y_h) + \mathcal{R} \quad \forall y_h \in X_h,$$

*with a remainder term  $\mathcal{R}$ , which is cubic in the error  $e := x - x_h$ ,*

$$\mathcal{R} := \frac{1}{2} \int_0^1 L'''(x_h + se; e, e, e) s(s-1) ds.$$

Since

$$\mathcal{L}(u; z) - \mathcal{L}(u_h; z_h) = J(u) - J(u_h),$$

at stationary points  $(u, z), (u_h, z_h)$ , Proposition 2.1 yields the following result for the Galerkin approximation (2.2) of the variational equation (2.1).

**Proposition 2.2.** *For any solutions  $u$  and  $u_h$  of equations (2.1) and (2.2) we have the error representation*

$$J(u) - J(u_h) = \frac{1}{2}\rho(u_h; z - \varphi_h) + \frac{1}{2}\rho^*(u_h; z_h, u - \psi_h) + \mathcal{R}^{(3)} \quad \forall \varphi_h, \psi_h \in V_h,$$

*where  $z$  and  $z_h$  are solutions of the adjoint problems (2.4) and (2.6) and*

$$\rho(u_h; \cdot) = -A(u_h; \cdot),$$

$$\rho^*(u_h; z_h, \cdot) = J'(u_h; \cdot) - A'(u_h; z_h, \cdot),$$

*and, with  $e_u = u - u_h$ ,  $e_z = z - z_h$ , the remainder is*

$$\begin{aligned} \mathcal{R}^{(3)} &= \frac{1}{2} \int_0^1 \left( J'''(u_h + se_u; e_u, e_u, e_u) - A'''(u_h + se_u; z_h + se_z, e_u, e_u, e_u) \right. \\ &\quad \left. - 3A''(u_h + se_u; e_u, e_u, e_z) \right) s(s-1) ds. \end{aligned}$$

The forms  $\rho(\cdot; \cdot), \rho^*(\cdot; \cdot, \cdot)$  are the residuals of (2.1) and (2.4), respectively. The remainder  $\mathcal{R}^{(3)}$  is cubic in the error. The following proposition shows that the residuals are equal up to a quadratic remainder.

**Proposition 2.3.** *With the notation from above, we have*

$$\rho^*(u_h; z_h, u - \psi_h) = \rho(u_h; z - \varphi_h) + \delta\rho \quad \forall \varphi_h, \psi_h \in V_h,$$

with

$$\delta\rho = \int_0^1 \left( A''(u_h + se_u; z_h + se_z, e_u, e_u) - J''(u_h + se_u; e_u, e_u) \right) ds.$$

Moreover, we have the simplified error representation

$$J(u) - J(u_h) = \rho(u_h; z - \varphi_h) + \mathcal{R}^{(2)} \quad \forall \varphi_h \in V_h,$$

with quadratic remainder

$$\mathcal{R}^{(2)} = \int_0^1 \left( A''(u_h + se_u; z, e_u, e_u) - J''(u_h + se_u; e_u, e_u) \right) ds.$$

### 3. GALERKIN DISCRETIZATION AND DUAL PROBLEM

In this section, we apply the dual weighted residuals methodology to the Cahn-Hilliard equation (1.1). We denote  $I = [0, T]$ ,  $Q = \Omega \times I$ , and

$$\langle v, w \rangle_{\mathcal{D}} = \int_{\mathcal{D}} vw \, dz, \quad \|v\|_{\mathcal{D}}^2 = \int_{\mathcal{D}} v^2 \, dz$$

for subsets  $\mathcal{D}$  of  $Q$  or  $\Omega$  with the relevant Lebesgue measure  $dz$ . Let  $V = H^1(\Omega)$  and  $\mathcal{W} = C^1([0, T], V)$ . By multiplying the first equation by  $\psi_u \in V$  and the second equation by  $\psi_w \in V$ , integrating over  $\Omega$  and using Green's formula, we obtain the weak formulation: Find  $u, w \in \mathcal{W}$  such that  $u(0) = g_0$  and

$$(3.1) \quad \begin{aligned} \langle u_t, \psi_u \rangle_{\Omega} + \langle \nabla w, \nabla \psi_u \rangle_{\Omega} &= 0 \quad \forall \psi_u \in V, t \in [0, T], \\ \langle w, \psi_w \rangle_{\Omega} - \epsilon \langle \nabla u, \nabla \psi_w \rangle_{\Omega} - \langle f(u), \psi_w \rangle_{\Omega} &= 0 \quad \forall \psi_w \in V, t \in [0, T]. \end{aligned}$$

Split the interval  $I = [0, T]$  into subintervals  $I_n = [t_{n-1}, t_n)$  of lengths  $k_n = t_n - t_{n-1}$ ,

$$0 = t_0 < t_1 < \dots < t_n < \dots < t_N = T.$$

For each time level  $t_n, n \geq 1$ , let  $\mathcal{V}_n$  be the space of continuous piecewise linear functions with respect to regular spatial meshes  $\mathbf{T}_n = \{K\}$ , which may vary from time level to time level. By extending the spatial meshes  $\mathbf{T}_n$  as constant in time to the time slab  $\Omega \times I_n$ , we obtain meshes  $\mathcal{T}_k$  of the space-time domain  $Q = \Omega \times I$ , which consist of  $(d+1)$ -dimensional prisms  $Q_K^n := K \times \bar{I}_n$ . Define the finite element space

$$\mathcal{V} := \left\{ \varphi: \bar{Q} \rightarrow \mathbf{R} : \varphi(\cdot, t)|_{\bar{\Omega}} \in \mathcal{V}_n, t \in I_n, \varphi(x, \cdot)|_{I_n} \in \Pi_0, x \in \bar{\Omega} \right\}.$$

Here  $\Pi_0$  denotes the polynomials of degree 0. For functions from this space and their continuous analogues, we define

$$v_n^+ = \lim_{t \downarrow t_n} v(t), \quad v_n^- = \lim_{t \uparrow t_n} v(t), \quad [v]_n = v_n^+ - v_n^-.$$



For all  $u, w, \psi_u, \psi_w \in \mathcal{V}$  or  $\mathcal{W}$ , consider the semilinear form

$$\begin{aligned} A(u, w; \psi_u, \psi_w) &= \sum_{n=1}^N \int_{I_n} \left\{ \langle u_t, \psi_u \rangle_{\Omega} + \langle \nabla w, \nabla \psi_u \rangle_{\Omega} + \langle w, \psi_w \rangle_{\Omega} \right. \\ &\quad \left. - \epsilon \langle \nabla u, \nabla \psi_w \rangle_{\Omega} - \langle f(u), \psi_w \rangle_{\Omega} \right\} dt \\ &\quad + \sum_{n=2}^N \langle [u]_{n-1}, \psi_{u,n-1}^+ \rangle_{\Omega} + \langle u_0^+ - g_0, \psi_{u,0}^+ \rangle_{\Omega}. \end{aligned}$$

Solutions  $u, w \in \mathcal{W}$  of (1.1) satisfy the variational problem

$$(3.2) \quad A(u, w; \psi_u, \psi_w) = 0 \quad \forall \psi_u, \psi_w \in \mathcal{W}$$

and the finite element problem can be formulated: Find  $U, W \in \mathcal{V}$  such that

$$(3.3) \quad A(U, W; \psi_u, \psi_w) = 0 \quad \forall \psi_u, \psi_w \in \mathcal{V}.$$

We now show that this is a standard time-stepping method. Since  $U(t) = U_n = U_n^- = U_{n-1}^+$ ,  $W(t) = W_n$  for  $t \in I_n$ , we have

$$\begin{aligned} A(U, W; \psi_u, \psi_w) &= \sum_{n=1}^N \int_{I_n} \left\{ \langle \nabla W_n, \nabla \psi_u \rangle_{\Omega} + \langle W_n, \psi_w \rangle_{\Omega} \right. \\ (3.4) \quad &\quad \left. - \epsilon \langle \nabla U_n, \nabla \psi_w \rangle_{\Omega} - \langle f(U_n), \psi_w \rangle_{\Omega} \right\} dt \\ &\quad + \sum_{n=2}^N \langle U_n - U_{n-1}, \psi_{u,n-1}^+ \rangle_{\Omega} + \langle U_1 - g_0, \psi_{u,0}^+ \rangle_{\Omega}. \end{aligned}$$

By taking

$$\psi_u(t) = \begin{cases} \chi_u \in \mathcal{V}_n, & t \in I_n, \\ 0, & \text{otherwise,} \end{cases} \quad \psi_w(t) = \begin{cases} \chi_w \in \mathcal{V}_n, & t \in I_n, \\ 0, & \text{otherwise,} \end{cases}$$

we see that (3.3) amounts to the implicit Euler time-stepping,

$$\begin{aligned} \langle U_0 - g_0, \chi_u \rangle_{\Omega} &= 0 \quad \forall \chi_u \in \mathcal{V}_1, \\ k_n \langle \nabla W_n, \nabla \chi_u \rangle_{\Omega} + \langle U_n - U_{n-1}, \chi_u \rangle_{\Omega} &= 0 \quad \forall \chi_u \in \mathcal{V}_n, n \geq 1, \\ \langle W_n, \chi_w \rangle_{\Omega} - \epsilon \langle \nabla U_n, \nabla \chi_w \rangle_{\Omega} - \langle f(U_n), \chi_w \rangle_{\Omega} &= 0 \quad \forall \chi_w \in \mathcal{V}_n, n \geq 1. \end{aligned}$$

Now take a goal functional  $J(u)$ , which depends only on  $u$ , and set

$$\mathcal{L}(v; z) = J(u) - A(v; z),$$

where  $v = (u, w)$ ,  $z = (z_u, z_w)$ . With  $\varphi = (\varphi_u, \varphi_w)$ ,  $\psi = (\psi_u, \psi_w)$ , stationary points are given by

$$\mathcal{L}'(v; z, \varphi, \psi) = J'(u; \varphi_u) - A'(v; z, \varphi) - A(v; \psi) = 0 \quad \forall \varphi, \psi \in \mathcal{W} \times \mathcal{W}.$$

With  $\psi = 0$  we obtain  $A'(v; z, \varphi) = J'(u; \varphi_u)$ , the adjoint problem. So we should compute  $A'(u, w; z_u, z_w, \varphi_u, \varphi_w)$  and  $J'(u; \varphi_u)$ . To this end we write

$$\begin{aligned} A(u, w; \psi_u, \psi_w) &= \langle u_t, \psi_u \rangle_Q + \langle \nabla w, \nabla \psi_u \rangle_Q + \langle w, \psi_w \rangle_Q - \epsilon \langle \nabla u, \nabla \psi_w \rangle_Q \\ &\quad - \langle f(u), \psi_w \rangle_Q + \langle u(0) - g_0, \psi_u(0) \rangle_{\Omega}. \end{aligned}$$

Hence,

$$\begin{aligned} A'(u, w; z_u, z_w, \varphi_u, \varphi_w) &= \langle \varphi_{u,t}, z_u \rangle_Q + \langle \nabla \varphi_w, \nabla z_u \rangle_Q + \langle \varphi_w, z_w \rangle_Q \\ &\quad - \epsilon \langle \nabla \varphi_u, \nabla z_w \rangle_Q - \langle \varphi_u, z_w \rangle_Q + \langle \varphi_u(0), z_u(0) \rangle_\Omega. \end{aligned}$$

By integration by parts in  $t$ ,

$$\langle \varphi_{u,t}, z_u \rangle_Q = -\langle \varphi_u, z_{u,t} \rangle_Q + \langle \varphi_u(T), z_u(T) \rangle_\Omega - \langle \varphi_u(0), z_u(0) \rangle_\Omega,$$

we obtain

$$\begin{aligned} A'(u, w; z_u, z_w, \varphi_u, \varphi_w) &= -\langle \varphi_u, z_{u,t} \rangle_Q + \langle \nabla \varphi_w, \nabla z_u \rangle_Q \\ &\quad + \langle \varphi_w, z_w \rangle_Q + \epsilon \langle \nabla \varphi_u, \nabla z_w \rangle_Q \\ &\quad - \langle \varphi_u, f'(u)z_w \rangle_Q + \langle \varphi_u(T), z_u(T) \rangle_\Omega. \end{aligned}$$

The adjoint problem is thus to find  $z_u, z_w \in \mathcal{W}$  such that

$$\begin{aligned} (3.5) \quad &\langle \varphi_u, -z_{u,t} \rangle_Q - \epsilon \langle \nabla \varphi_u, \nabla z_w \rangle_Q \\ &- \langle \varphi_u, f'(u)z_w \rangle_Q + \langle \varphi_u(T), z_u(T) \rangle_\Omega \\ &+ \langle \nabla \varphi_w, \nabla z_w \rangle_Q + \langle \varphi_w, z_w \rangle_Q = J'(u; \varphi_u) \quad \forall \varphi_u, \varphi_w \in \mathcal{W}. \end{aligned}$$

We now specialize to the case of a linear goal functional of the form

$$J(u) = \langle u, g \rangle_Q + \langle u(T), g_T \rangle_\Omega,$$

for some  $g \in L_2(Q)$ ,  $g_T \in L_2(\Omega)$ . Then

$$(3.6) \quad J'(u; \varphi_u) = \langle \varphi_u, g \rangle_Q + \langle \varphi_u(T), g_T \rangle_\Omega.$$

The adjoint problem then becomes: Find  $z_u, z_w \in \mathcal{W}$  such that

$$\begin{aligned} (3.7) \quad &\langle \varphi_u, -z_{u,t} - f'(u)z_w - g \rangle_Q - \epsilon \langle \nabla \varphi_u, \nabla z_w \rangle_Q \\ &+ \langle \varphi_u(T), z_u(T) - g_T \rangle_\Omega = 0 \quad \forall \varphi_u \in \mathcal{W}, \\ &\langle \varphi_w, z_w \rangle_Q + \langle \nabla \varphi_w, \nabla z_w \rangle_Q = 0 \quad \forall \varphi_w \in \mathcal{W}. \end{aligned}$$

The strong form of this is

$$\begin{aligned} (3.8) \quad &-\partial_t z_u + \epsilon \Delta z_w - f'(u)z_w = g \quad \text{in } Q, \\ &z_w - \Delta z_u = 0 \quad \text{in } Q, \\ &\frac{\partial z_u}{\partial \nu} = 0, \quad \frac{\partial z_w}{\partial \nu} = 0 \quad \text{on } \partial\Omega \times I, \\ &z_u(T) = g_T \quad \text{in } \Omega. \end{aligned}$$

#### 4. A POSTERIORI ERROR ESTIMATES

From Proposition 2.3 we have the error representation

$$(4.1) \quad J(u) - J(U) = -A(U, W; z_u - \pi z_u, z_w - \pi z_w) + \mathcal{R}^{(2)},$$

where  $z = (z_u, z_w)$  is the solution of the adjoint problem (3.5) and  $\pi z_u, \pi z_w \in \mathcal{V}$  are appropriate approximations to be defined below. The remainder is quadratic in the error.

In order to write this as a sum of local contributions we must rewrite  $A(U, W; \psi_u, \psi_w)$  in (3.4). First we compute  $\int_{I_n} \langle \nabla W, \nabla \psi_u \rangle_\Omega dt$ . By using Green's formula elementwise, we have

$$\begin{aligned} \int_{I_n} \langle \nabla W, \nabla \psi_u \rangle_\Omega dt &= \int_{I_n} \sum_{K \in \mathbf{T}_n} \langle \nabla W, \nabla \psi_u \rangle_K dt \\ &= \int_{I_n} \sum_{K \in \mathbf{T}_n} -\langle \Delta W, \psi_u \rangle_K dt + \int_{I_n} \sum_{K \in \mathbf{T}_n} \langle \partial_\nu W, \psi_u \rangle_{\partial K} dt, \end{aligned}$$

where  $\partial_\nu W = \nu \cdot \nabla W$ . We divide the boundary  $\partial K \in \mathbf{T}_n$  into two parts: internal edges, denoted by  $\mathcal{E}_I^n$ , and edges on the boundary  $\partial\Omega$ , denoted by  $\mathcal{E}_{\partial\Omega}^n$ . So we get, with  $[\ ]$  denoting the jump across the edge,

$$\begin{aligned} &\int_{I_n} \sum_{K \in \mathbf{T}_n} \langle \partial_\nu W, \psi_u \rangle_{\partial K} dt \\ &= \int_{I_n} \sum_{E \in \mathcal{E}_I^n} \langle \partial_\nu W, \psi_u \rangle_E dt + \int_{I_n} \sum_{E \in \mathcal{E}_{\partial\Omega}^n} \langle \partial_\nu W, \psi_u \rangle_E dt \\ &= \int_{I_n} \sum_{K \in \mathbf{T}_n} -\frac{1}{2} \langle [\partial_\nu W], \psi_u \rangle_{\partial K \setminus \partial\Omega} dt + \int_{I_n} \sum_{K \in \mathbf{T}_n} \langle \partial_\nu W, \psi_u \rangle_{\partial K \cap \partial\Omega} dt. \end{aligned}$$

Let  $\partial_x$  denote the spatial boundary and define  $\partial_x Q = \partial\Omega \times I$  and  $\partial_x Q_K^n = \partial K \times I_n$ . Hence,

$$\begin{aligned} \int_{I_n} \langle \nabla W, \nabla \psi_u \rangle_\Omega dt &= \sum_{K \in \mathbf{T}_n} \left\{ -\langle \Delta W, \psi_u \rangle_{Q_K^n} - \frac{1}{2} \langle [\partial_\nu W], \psi_u \rangle_{\partial_x Q_K^n \setminus \partial_x Q} \right. \\ &\quad \left. + \langle \partial_\nu W, \psi_u \rangle_{\partial_x Q_K^n \cap \partial_x Q} \right\}, \end{aligned}$$

and in the same way

$$\begin{aligned} \epsilon \int_{I_n} \langle \nabla U, \nabla \psi_w \rangle_\Omega dt &= \sum_{K \in \mathbf{T}_n} \left\{ -\epsilon \langle \Delta U, \psi_w \rangle_{Q_K^n} - \frac{1}{2} \epsilon \langle [\partial_\nu U], \psi_w \rangle_{\partial_x Q_K^n \setminus \partial_x Q} \right. \\ &\quad \left. + \epsilon \langle \partial_\nu U, \psi_w \rangle_{\partial_x Q_K^n \cap \partial_x Q} \right\}. \end{aligned}$$

Note that  $\Delta W = \Delta U = 0$  on  $Q_K^n$  for piecewise linear functions, but we find it instructive to keep these terms. Inserting this into (3.4) and noting that

$$\int_{I_n} \langle W, \psi_w \rangle_\Omega dt = \sum_{K \in \mathbf{T}_n} \langle W, \psi_w \rangle_{Q_K^n},$$

and

$$\int_{I_n} \langle f(U), \psi_w \rangle_\Omega dt = \sum_{K \in \mathbf{T}_n} \langle f(U), \psi_w \rangle_{Q_K^n},$$

gives

$$\begin{aligned}
A(U, W; \psi_u, \psi_w) &= \sum_{n=1}^N \sum_{K \in \mathbf{T}_n} \left\{ - \langle \Delta W, \psi_u \rangle_{Q_K^n} \right. \\
&\quad + \langle \epsilon \Delta U + W - f(U), \psi_w \rangle_{Q_K^n} - \frac{1}{2} \langle [\partial_\nu W], \psi_u \rangle_{\partial_x Q_K^n \setminus \partial_x Q} \\
&\quad + \frac{1}{2} \epsilon \langle [\partial_\nu U], \psi_w \rangle_{\partial_x Q_K^n \setminus \partial_x Q} + \langle \partial_\nu W, \psi_u \rangle_{\partial_x Q_K^n \cap \partial_x Q} \\
&\quad \left. - \epsilon \langle \partial_\nu U, \psi_w \rangle_{\partial_x Q_K^n \cap \partial_x Q} + \langle [U]_{n-1}, \psi_{u, n-1}^+ \rangle_K \right\},
\end{aligned}$$

where we have set  $U_0^- = g_0$  for simplicity. Hence (4.1) becomes

$$\begin{aligned}
(4.2) \quad J(u) - J(U) &= \sum_{n=1}^N \sum_{K \in \mathbf{T}_n} \left\{ \langle R_u, z_u - \pi z_u \rangle_{Q_K^n} + \langle R_w, z_w - \pi z_w \rangle_{Q_K^n} \right. \\
&\quad + \langle r_u, z_u - \pi z_u \rangle_{\partial_x Q_K^n} + \langle r_w, z_w - \pi z_w \rangle_{\partial_x Q_K^n} \\
&\quad \left. - \langle [U]_{n-1}, (z_u - \pi z_u)_{n-1}^+ \rangle_K \right\} + \mathcal{R}^{(2)},
\end{aligned}$$

with the interior residuals

$$R_u = \Delta W, \quad R_w = -\epsilon \Delta U - W + f(U),$$

the edge residuals

$$\begin{aligned}
r_w|_\Gamma &= \begin{cases} -\frac{1}{2} \epsilon [\partial_\nu U], & \Gamma \subset \partial_x Q_K^n \setminus \partial_x Q, \\ 0, & \text{otherwise,} \end{cases} \\
r_u|_\Gamma &= \begin{cases} \frac{1}{2} [\partial_\nu W], & \Gamma \subset \partial_x Q_K^n \setminus \partial_x Q, \\ 0, & \text{otherwise,} \end{cases}
\end{aligned}$$

and the boundary residuals

$$\begin{aligned}
r_w|_\Gamma &= \begin{cases} \epsilon \partial_\nu U, & \Gamma \subset \partial_x Q_K^n \cap \partial_x Q, \\ 0, & \text{otherwise,} \end{cases} \\
r_u|_\Gamma &= \begin{cases} -\partial_\nu W, & \Gamma \subset \partial_x Q_K^n \cap \partial_x Q, \\ 0, & \text{otherwise.} \end{cases}
\end{aligned}$$

Here the subscript  $u$  refers to residuals from the first equation in (3.1) and the subscript  $w$  to residuals from the second equation.

We now define  $\pi z_u, \pi z_w \in \mathcal{V}$ . Let

$$(P_n v)(t) = \frac{1}{k_n} \int_{I_n} v(s) \, ds$$

be the orthogonal projector onto constants. Let  $\pi_n: C(\bar{\Omega}) \rightarrow \mathcal{V}_n$  be the nodal interpolator; that is, it is defined by

$$(\pi_n v)(a) = v(a),$$

for all nodal points  $a$  in  $\mathbf{T}_n$ . Then we define  $\pi: C(\bar{Q}) \rightarrow \mathcal{V}$  by  $\pi v|_{I_n} = P_n \pi_n v$ . Since  $R_u, R_w, r_u$ , and  $r_w$  are piecewise constant in  $t$ , we have

$$\begin{aligned}
 & J(u) - J(U) \\
 (4.3) \quad &= \sum_{n=1}^N \sum_{K \in \mathbf{T}_n} \left\{ \langle R_u, P_n(z_u - \pi_n z_u) \rangle_{Q_K^n} + \langle R_w, P_n(z_w - \pi_n z_w) \rangle_{Q_K^n} \right. \\
 & \quad + \langle r_u, P_n(z_u - \pi_n z_u) \rangle_{\partial_x Q_K^n} + \langle r_w, P_n(z_w - \pi_n z_w) \rangle_{\partial_x Q_K^n} \\
 & \quad \left. - \langle [U]_{n-1}, (z_u - \pi z_u)_{n-1}^+ \rangle_K \right\} + \mathcal{R}^{(2)}.
 \end{aligned}$$

Applying the Cauchy-Schwartz inequality to each term gives

$$\begin{aligned}
 |J(u) - J(U)| &\leq \sum_{n=1}^N \sum_{K \in \mathbf{T}_n} \left\{ \|R_u\|_{Q_K^n} \|P_n(z_u - \pi_n z_u)\|_{Q_K^n} \right. \\
 & \quad + h_K^{-\frac{1}{2}} \|r_u\|_{\partial_x Q_K^n} h_K^{\frac{1}{2}} \|P_n(z_u - \pi_n z_u)\|_{\partial_x Q_K^n} \\
 & \quad + \|R_w\|_{Q_K^n} \|P_n(z_w - \pi_n z_w)\|_{Q_K^n} \\
 & \quad + h_K^{-\frac{1}{2}} \|r_w\|_{\partial_x Q_K^n} h_K^{\frac{1}{2}} \|P_n(z_w - \pi_n z_w)\|_{\partial_x Q_K^n} \\
 & \quad \left. + k_n^{-\frac{1}{2}} \|[U]_{n-1}\|_K k_n^{\frac{1}{2}} \|(z_u - \pi z_u)_{n-1}^+\|_K \right\} + |\mathcal{R}^{(2)}|.
 \end{aligned}$$

Here  $h_K = \text{diam}(K)$ . For  $a, b, c, d \geq 0$  we have

$$(ab + cd) \leq (a^2 + c^2)^{\frac{1}{2}} (b^2 + d^2)^{\frac{1}{2}}.$$

We use this inequality for each term in the previous inequality and set

$$\begin{aligned}
 \rho_{u,K} &= \left( \|R_u\|_{Q_K^n}^2 + h_K^{-1} \|r_u\|_{\partial_x Q_K^n}^2 \right)^{\frac{1}{2}}, \\
 \omega_{u,K} &= \left( \|P_n(z_u - \pi_n z_u)\|_{Q_K^n}^2 + h_K \|P_n(z_u - \pi_n z_u)\|_{\partial_x Q_K^n}^2 \right)^{\frac{1}{2}}, \\
 \rho_{w,K} &= \left( \|R_w\|_{Q_K^n}^2 + h_K^{-1} \|r_w\|_{\partial_x Q_K^n}^2 \right)^{\frac{1}{2}}, \\
 \omega_{w,K} &= \left( \|P_n(z_w - \pi_n z_w)\|_{Q_K^n}^2 + h_K \|P_n(z_w - \pi_n z_w)\|_{\partial_x Q_K^n}^2 \right)^{\frac{1}{2}}, \\
 \rho_K &= \left( k_n^{-1} \|[U]_{n-1}\|_K^2 \right)^{\frac{1}{2}}, \\
 \omega_K &= \left( k_n \|(z_u - \pi z_u)_{n-1}^+\|_K^2 \right)^{\frac{1}{2}}.
 \end{aligned}$$

Note that, since  $R_u = \Delta W = 0$  for piecewise linear functions, the first term in  $\rho_{u,K}$  and  $\omega_{u,K}$  can actually be removed. So we have

$$|J(u) - J(U)| \leq \sum_{n=1}^N \sum_{K \in \mathbf{T}_n} \left\{ \rho_{u,K} \omega_{u,K} + \rho_{w,K} \omega_{w,K} + \rho_K \omega_K \right\} + |\mathcal{R}^{(2)}|.$$

We have proved the following theorem:

**Theorem 4.1.** *We have the a posteriori error estimate*

$$(4.4) \quad |J(u) - J(U)| \leq \sum_{n=1}^N \sum_{K \in \mathbf{T}_n} \left\{ \rho_{u,K} \omega_{u,K} + \rho_{w,K} \omega_{w,K} + \rho_K \omega_K \right\} + |\mathcal{R}^{(2)}|.$$

Note that on each space-time cell  $Q_K^n$ , the terms  $\rho_{u,K} \omega_{u,K}$  and  $\rho_{w,K} \omega_{w,K}$  can be used to control the spatial mesh and the term  $\rho_K \omega_K$  to control the time step  $k_n$  in an adaptive algorithm; see [2]. We do not pursue this here.

In the following we want to obtain a weight free a posteriori error estimate where the weights in (4.4) are replaced by a global stability constant. We need the following interpolation error estimate, see [2, Lemma 9.4].

**Lemma 4.2.** *With  $\pi$  and  $\pi_n$  as defined as before, there holds*

$$(4.5) \quad \|P_n(z - \pi_n z)\|_{Q_K^n} + h_K^{\frac{1}{2}} \|P_n(z - \pi_n z)\|_{\partial_x Q_K^n} \leq Ch_K^2 \|D^2 z\|_{Q_K^n},$$

$$(4.6) \quad \|z(t_{n-1}) - P_n z\|_K \leq Ch_n^{\frac{1}{2}} \|\partial_t z\|_{Q_K^n}.$$

Here  $\|D^2 z\|_{Q_K^n}$  denotes the seminorm  $\left( \sum_{|\alpha|=2} \|D^\alpha z\|_{Q_K^n}^2 \right)^{\frac{1}{2}}$ .

In the following we assume that  $J(\cdot)$  is a linear functional given by (3.6) and  $\Omega$  is such that we have the elliptic regularity estimate

$$(4.7) \quad \|D^2 v\|_\Omega \leq C \|\Delta v\|_\Omega \quad \forall v \in H^2(\Omega) \text{ with } \frac{\partial v}{\partial \nu} \Big|_\Gamma = 0.$$

We also assume a global bound for  $f'(u)$ , which is reasonable since it is known that  $\|u\|_{L_\infty(Q)} \leq C$  (c.f. [5]).

In particular, with

$$g = (u - U)/\|u - U\|_Q \text{ and } g_T = (u_N - U_N)/\|u_N - U_N\|_\Omega$$

the following theorem provides bounds for the norms of the error,  $\|u - U\|_Q$  and  $\|u_N - U_N\|_\Omega$ .

**Theorem 4.3.** *Assume that  $\|f'(u)\|_{L_\infty} \leq \beta$  and that (4.7) holds. Let  $z_u, z_w$  be the solutions of (3.8). Then there is  $C = C(\beta)$  such that the following a posteriori error estimates hold.*

(i) *Let  $g \in L_2(Q)$  with  $\|g\|_Q = 1$  and  $g_T = 0$ . Then*

$$(4.8) \quad \begin{aligned} & |\langle u - U, g \rangle_Q| \\ & \leq CC_S \sum_{n=1}^N \sum_{K \in \mathbf{T}_n} \left\{ h_K^4 (\rho_{u,K}^2 + \rho_{w,K}^2) + (h_K^4 + k_n^2) \rho_K^2 \right\}^{\frac{1}{2}} + |\mathcal{R}^{(2)}|, \end{aligned}$$

where

$$C_S = \sup_{g \in L_2(Q)} \frac{\left( \|D^2 z_u\|_Q^2 + \|\partial_t z_u\|_Q^2 + \|D^2 z_w\|_Q^2 \right)^{\frac{1}{2}}}{\|g\|_Q}.$$

(ii) Let  $g_T \in L_2(\Omega)$  with  $\|g_T\|_\Omega = 1$  and  $g = 0$ . Then

$$(4.9) \quad \begin{aligned} & |\langle u - U, g_T \rangle_\Omega| \\ & \leq CC_S \sum_{n=1}^N \sum_{K \in \mathbf{T}_n} \left\{ h_K^4 (\rho_{u,K}^2 + \sigma_n^{-1} \rho_{w,K}^2 + \sigma_n^{-1} \rho_K^2) + k_n^2 \sigma^{-1} \rho_K^2 \right\}^{\frac{1}{2}} \\ & \quad + |\mathcal{R}^{(2)}|, \end{aligned}$$

where  $\sigma(t) = T - t$ ,

$$\sigma_n = \begin{cases} \sigma(t_n) = T - t_n, & n = 1, \dots, N-1, \\ k_N, & n = N, \end{cases}$$

and

$$\begin{aligned} C_S = \sup_{g_T \in L_2(\Omega)} & \left( \epsilon^{-1} \max_I \|z_u\|_\Omega^2 + \epsilon^{-1} \|z_w\|_Q^2 \right. \\ & \left. + \|D^2 z_u\|_Q^2 + \|\sigma^{\frac{1}{2}} \partial_t z_u\|_Q^2 + \epsilon^2 \|\sigma^{\frac{1}{2}} D^2 z_w\|_Q^2 \right)^{\frac{1}{2}} / \|g_T\|_\Omega. \end{aligned}$$

*Proof.* Part (i). From Theorem 4.2 we have

$$\begin{aligned} \omega_{u,K} &= \left( \|P_n(z_u - \pi_n z_u)\|_{Q_K^n}^2 + h_K \|P_n(z_u - \pi_n z_u)\|_{\partial_x Q_K^n}^2 \right)^{\frac{1}{2}} \\ &\leq Ch_K^2 \|D^2 z_u\|_{Q_K^n}, \\ \omega_{w,K} &= \left( \|P_n(z_w - \pi_n z_w)\|_{Q_K^n}^2 + h_K \|P_n(z_w - \pi_n z_w)\|_{\partial_x Q_K^n}^2 \right)^{\frac{1}{2}} \\ &\leq Ch_K^2 \|D^2 z_w\|_{Q_K^n}, \end{aligned}$$

and

$$\begin{aligned} \omega_K &= k_n^{\frac{1}{2}} \|(z_u - \pi_n z_u)_{n-1}^+\|_K \\ &\leq k_n^{\frac{1}{2}} \|P_n(z_u - \pi_n z_u)\|_K + k_n^{\frac{1}{2}} \|z_u(t_{n-1}) - P_n z_u\|_K \\ &\leq Ch_K^2 \|D^2 z_u\|_{Q_K^n} + Ck_n \|\partial_t z_u\|_{Q_K^n} + |\mathcal{R}^{(2)}|. \end{aligned}$$

Hence,

$$\begin{aligned} |\langle u - U, g \rangle_Q| &\leq \sum_{n=1}^N \sum_{K \in \mathbf{T}_n} \left\{ \rho_{u,K} \omega_{u,K} + \rho_{w,K} \omega_{w,K} + \rho_K \omega_K \right\} \\ &\leq \sum_{n=1}^N \sum_{K \in \mathbf{T}_n} \left\{ Ch_K^2 \rho_{u,K} \|D^2 z_u\|_{Q_K^n} + Ch_K^2 \rho_{w,K} \|D^2 z_w\|_{Q_K^n} \right. \\ &\quad \left. + \rho_K (Ch_K^2 \|D^2 z_u\|_{Q_K^n} + Ck_n \|\partial_t z_u\|_{Q_K^n}) \right\} \end{aligned}$$

and the desired estimate (4.8) follows by the Cauchy-Schwartz inequality

$$\begin{aligned}
& \sum_{n=1}^N \sum_{K \in \mathbf{T}_n} h_K^2 \rho_{u,K} \|D^2 z_u\|_{Q_K^n} \\
& \leq \left( \sum_{n=1}^N \sum_{K \in \mathbf{T}_n} h_K^4 \rho_{u,K}^2 \right)^{\frac{1}{2}} \left( \sum_{n=1}^N \sum_{K \in \mathbf{T}_n} h_K^2 \rho_{u,K} \|D^2 z_u\|_{Q_K^n}^2 \right)^{\frac{1}{2}} \\
& = \left( \sum_{n=1}^N \sum_{K \in \mathbf{T}_n} h_K^4 \rho_{u,K}^2 \right)^{\frac{1}{2}} \|D^2 z_u\|_Q \leq C_S \left( \sum_{n=1}^N \sum_{K \in \mathbf{T}_n} h_K^4 \rho_{u,K}^2 \right)^{\frac{1}{2}} \|g\|_Q,
\end{aligned}$$

and similarly for the other terms.

Part (ii). The previous bound for  $\sum_{n=1}^N \sum_{K \in \mathbf{T}_n} \rho_{u,K} \omega_{u,K}$  applies here also. Consider then

$$\sum_{n=1}^N \sum_{K \in \mathbf{T}_n} \rho_{w,K} \omega_{w,K} \leq \sum_{n=1}^{N-1} \sum_{K \in \mathbf{T}_n} \rho_{w,K} C h_K^2 \|D^2 z_w\|_{Q_K^n} + \sum_{K \in \mathbf{T}_N} \rho_{w,K} \omega_{w,K}.$$

Here,

$$\begin{aligned}
& \sum_{n=1}^{N-1} \sum_{K \in \mathbf{T}_n} \rho_{w,K} C h_K^2 \|D^2 z_w\|_{Q_K^n} \\
& = \sum_{n=1}^{N-1} \sum_{K \in \mathbf{T}_n} \rho_{w,K} C h_K^2 \|\sigma^{-\frac{1}{2}} \sigma^{\frac{1}{2}} D^2 z_w\|_{Q_K^n} \\
& \leq C \sum_{n=1}^{N-1} \sum_{K \in \mathbf{T}_n} \rho_{w,K} \sigma_n^{-\frac{1}{2}} h_K^2 \|\sigma^{\frac{1}{2}} D^2 z_w\|_{Q_K^n} \\
& \leq C \left( \sum_{n=1}^{N-1} \sum_{K \in \mathbf{T}_n} \sigma_n^{-1} h_K^4 \rho_{w,K}^2 \right)^{\frac{1}{2}} \left( \sum_{n=1}^{N-1} \sum_{K \in \mathbf{T}_n} \|\sigma^{\frac{1}{2}} D^2 z_w\|_{Q_K^n}^2 \right)^{\frac{1}{2}} \\
& \leq C \left( \sum_{n=1}^{N-1} \sum_{K \in \mathbf{T}_n} \sigma_n^{-1} h_K^4 \rho_{w,K}^2 \right)^{\frac{1}{2}} \|\sigma_n^{\frac{1}{2}} D^2 z_w\|_Q \\
& \leq C_S C \left( \sum_{n=1}^{N-1} \sum_{K \in \mathbf{T}_n} \sigma_n^{-1} h_K^4 \rho_{w,K}^2 \right)^{\frac{1}{2}} \|g_T\|_\Omega.
\end{aligned}$$

The term with  $n = N$  is special. We go back to (4.3) and replace it by

$$\begin{aligned}
\sum_{K \in \mathbf{T}_N} \langle R_w, z_w - \pi_N z_w \rangle_{Q_K^N} & = \sum_{K \in \mathbf{T}_N} \left\langle R_w, (I - \pi_N) \int_{I_N} z_w \, dt \right\rangle_K \\
& \leq \sum_{K \in \mathbf{T}_N} \|R_w\|_K C h_K^2 \left\| D^2 \int_{I_N} z_w \, dt \right\|_K.
\end{aligned}$$



Here, by the regularity estimate (4.7),  $\epsilon \Delta z_w = \partial_t z_u + f'(u)z_w$  from the first equation in (3.8), and  $\|f'(u)\|_{L^\infty} \leq \beta$ , we have

$$\begin{aligned} \left\| \mathbf{D}^2 \int_{I_N} z_w \, dt \right\|_K &\leq C \left\| \int_{I_N} \Delta z_w \, dt \right\|_K \\ &= C \epsilon^{-1} \left\| \int_{I_N} (\partial_t z_u + f'(u)z_w) \, dt \right\|_K \\ &\leq C \epsilon^{-1} \left( \|z_u(t_N)\|_K + \|z_u(t_{N-1})\|_K + \beta k_N^{\frac{1}{2}} \|z_w\|_{Q_K^N} \right). \end{aligned}$$

Hence, since  $\rho_{w,K} = \|R_w\|_{Q_K^N} = k_N^{\frac{1}{2}} \|R_w\|_K$ , we have

$$\begin{aligned} &\sum_{K \in \mathbf{T}_N} \langle R_w, z_w - \pi_N z_w \rangle_{Q_K^N} \\ &\leq \sum_{K \in \mathbf{T}_N} \|R_w\|_K C h_K^2 \epsilon^{-1} \left( \|z_u(t_N)\|_K + \|z_u(t_{N-1})\|_K + k_N^{\frac{1}{2}} \|z_w\|_{Q_K^N} \right) \\ &= C \epsilon^{-1} \sum_{K \in \mathbf{T}_N} k_N^{-\frac{1}{2}} h_K^2 \rho_{w,K} \left( \|z_u(t_N)\|_K + \|z_u(t_{N-1})\|_K + k_N^{\frac{1}{2}} \|z_w\|_{Q_K^N} \right) \\ &\leq C \epsilon^{-1} \left( \sum_{K \in \mathbf{T}_N} k_N^{-1} h_K^4 \rho_{w,K}^2 \right)^{\frac{1}{2}} \left( \|z_u(t_N)\|_\Omega + \|z_u(t_{N-1})\|_\Omega + k_N^{\frac{1}{2}} \|z_w\|_Q \right) \\ &\leq C \epsilon^{-1} C_S \|g_T\|_\Omega \left( \sum_{K \in \mathbf{T}_N} \sigma_N^{-1} h_K^4 \rho_{w,K}^2 \right)^{\frac{1}{2}}, \end{aligned}$$

where we have used  $\sigma_N = k_N$ . So we have

$$(4.10) \quad \sum_{n=1}^N \sum_{K \in \mathbf{T}_n} \rho_{w,K} \omega_{w,K} \leq C C_S \|g_T\|_\Omega \left( \sum_{n=1}^N \sum_{K \in \mathbf{T}_n} \sigma_n^{-1} h_K^4 \rho_{w,K}^2 \right)^{\frac{1}{2}}.$$

Now we compute  $\sum_{n=1}^N \sum_{K \in \mathbf{T}_n} \rho_K \omega_K$ . For  $K \in \mathbf{T}_N$  we use

$$\begin{aligned} \omega_K &= k_N^{\frac{1}{2}} \|(z_u - \pi_N z_u)_{N-1}^+\|_K \\ &\leq k_N^{\frac{1}{2}} \|P_N(z_u - \pi_N z_u)\|_K + k_N^{\frac{1}{2}} \|z_u(t_{N-1}) - P_N z_u\|_K \\ &= \|P_N(z_u - \pi_N z_u)\|_{Q_K^N} + k_N^{\frac{1}{2}} \|z_u(t_{N-1}) - P_N z_u\|_K \\ &\leq C h_K^2 \|\mathbf{D}^2 z_u\|_{Q_K^N} + k_N^{\frac{1}{2}} \|z_u(t_{N-1}) - P_N z_u\|_K. \end{aligned}$$

Then we have

$$\begin{aligned}
& \sum_{n=1}^N \sum_{K \in \mathbf{T}_n} \rho_K \omega_K \\
&= C \sum_{n=1}^N \sum_{K \in \mathbf{T}_n} \rho_K h_K^2 \|D^2 z_u\|_{Q_K^n} + C \sum_{n=1}^{N-1} \sum_{K \in \mathbf{T}_n} \rho_K k_n \sigma_n^{-\frac{1}{2}} \|\sigma^{\frac{1}{2}} \partial_t z_u\|_{Q_K^n} \\
&\quad + \sum_{K \in \mathbf{T}_N} \rho_K k_N^{\frac{1}{2}} \|z_u(t_{N-1}) - P_N z_u\|_K \\
&\leq C \left( \sum_{n=1}^N \sum_{K \in \mathbf{T}_n} h_K^4 \rho_K^2 \right)^{\frac{1}{2}} \|D^2 z_u\|_Q + C \left( \sum_{n=1}^{N-1} \sum_{K \in \mathbf{T}_n} \rho_K^2 k_n^2 \sigma_n^{-1} \right)^{\frac{1}{2}} \|\sigma^{\frac{1}{2}} \partial_t z_u\|_Q \\
&\quad + C \left( \sum_{K \in \mathbf{T}_N} k_N \rho_K^2 \right)^{\frac{1}{2}} \|z_u(t_{N-1}) - P_N z_u\|_\Omega.
\end{aligned}$$

Using  $\sigma_N = k_N$  and

$$\|z_u(t_{N-1}) - P_N z_u\|_\Omega \leq 2 \max_I \|z_u\|_\Omega \leq 2C_S \|g_T\|_\Omega,$$

gives

$$\begin{aligned}
& \sum_{n=1}^N \sum_{K \in \mathbf{T}_n} \rho_K \omega_K \leq C \left( \sum_{n=1}^N \sum_{K \in \mathbf{T}_n} h_K^4 \rho_K^2 \right)^{\frac{1}{2}} C_S \|g_T\|_\Omega \\
&\quad + C \left( \sum_{n=1}^{N-1} \sum_{K \in \mathbf{T}_n} \rho_K^2 k_n^2 \sigma_n^{-1} \right)^{\frac{1}{2}} C_S \|g_T\|_\Omega + C \left( \sum_{K \in \mathbf{T}_N} k_N \rho_K^2 \right)^{\frac{1}{2}} C_S \|g_T\|_\Omega \\
&= CC_S \left( \sum_{n=1}^N \sum_{K \in \mathbf{T}_n} h_K^4 \rho_K^2 \right)^{\frac{1}{2}} \|g_T\|_\Omega + CC_S \left( \sum_{n=1}^N \sum_{K \in \mathbf{T}_n} \rho_K^2 k_n^2 \sigma_n^{-1} \right)^{\frac{1}{2}} \|g_T\|_\Omega.
\end{aligned}$$

This completes the proof.  $\square$

Finally, we prove a priori bounds for the stability constants  $C_S$ .

**Theorem 4.4.** *Assume that  $\|f'(u)\|_{L_\infty(Q)} \leq \beta$  and  $\epsilon \in (0, 1]$  and that (4.7) holds. Then the solution of (3.8) admits the following a priori bounds, where  $C = C(\beta)$ . If  $g_T = 0$ , then*

$$(4.11) \quad \|D^2 z_u\|_Q^2 + \|\partial_t z_u\|_Q^2 + \epsilon^2 \|D^2 z_w\|_Q^2 \leq C \|g\|_Q^2 e^{C\epsilon^{-1}T}.$$

If  $g = 0$ , then, with  $\sigma(t) = T - t$ ,

$$\begin{aligned}
(4.12) \quad & \epsilon^{-1} \max_I \|z_u\|_\Omega^2 + \|z_w\|_Q^2 + \|D^2 z_u\|_Q^2 + \|\sigma^{\frac{1}{2}} \partial_t z_u\|_Q^2 + \epsilon^2 \|\sigma^{\frac{1}{2}} D^2 z_w\|_Q^2 \\
& \leq C \epsilon^{-1} \|g_T\|_\Omega^2 e^{C\epsilon^{-1}T}.
\end{aligned}$$

*Proof.* We first estimate  $\|z_w\|_Q^2$ . To this end we use  $\Delta z_u = z_w$  from the second equation of (3.8) to get

$$\langle \Delta z_w, z_u \rangle_\Omega = \langle z_w, \Delta z_u \rangle_\Omega = \|z_w\|_\Omega^2.$$

Then we multiply the first equation of (3.8) by  $z_u$ , and integrate over  $[t, T]$ ,  $\int_t^T \langle -\partial_t z_u, z_u \rangle_\Omega ds + \epsilon \int_t^T \|z_w\|_\Omega^2 ds - \int_t^T \langle f'(u)z_w, z_u \rangle_\Omega ds = \int_t^T \langle g, z_u \rangle_\Omega ds$ .

By assumption we know that  $\|f'(u)\|_{L^\infty(Q)} \leq \beta$ , so we have

$$\begin{aligned} & \frac{1}{2} \|z_u(t)\|_\Omega^2 - \frac{1}{2} \|z_u(T)\|_\Omega^2 + \epsilon \int_t^T \|z_w\|_\Omega^2 ds \\ & \leq \int_t^T \|f'(u)\|_{L^\infty(Q)} \|z_w\|_\Omega \|z_u\|_\Omega ds + \int_t^T \|g\|_\Omega \|z_u\|_\Omega ds \\ & \leq \int_t^T \left( \frac{\beta^2}{2\epsilon} \|z_u\|_\Omega^2 + \frac{\epsilon}{2} \|z_w\|_\Omega^2 \right) ds + \int_t^T \left( \frac{\epsilon}{2} \|g\|_\Omega^2 + \frac{1}{2c} \|z_u\|_\Omega^2 \right) ds \\ & \leq \frac{\beta^2}{\epsilon} \int_t^T \|z_u\|_\Omega^2 ds + \frac{\epsilon}{2} \int_t^T \|z_w\|_\Omega^2 ds + \int_t^T \left( \frac{\epsilon}{2} \|g\|_\Omega^2 + \frac{1}{2c} \|z_u\|_\Omega^2 \right) ds. \end{aligned}$$

Hence, with  $z_u(T) = g_T$  and  $c = \frac{\epsilon}{\beta^2}$ ,

$$\begin{aligned} & \|z_u(t)\|_\Omega^2 + \epsilon \int_t^T \|z_w\|_\Omega^2 ds \\ & \leq \frac{\epsilon}{\beta^2} \|g\|_Q^2 + \|g_T\|_\Omega^2 + 2\beta^2\epsilon^{-1} \int_t^T \|z_u\|_\Omega^2 ds \\ & \leq \frac{C}{\epsilon} \|g\|_Q^2 + \|g_T\|_\Omega^2 + C\epsilon^{-1} \int_t^T \|z_u\|_\Omega^2 ds. \end{aligned}$$

Define

$$\Phi(t) = \|z_u(t)\|_\Omega^2 + \epsilon \int_t^T \|z_w(s)\|_\Omega^2 ds.$$

Obviously we have  $\|z_u(s)\|_\Omega^2 \leq \Phi(s)$ , so that

$$\Phi(t) \leq C\epsilon \|g\|_Q^2 + \|g_T\|_\Omega^2 + C\epsilon^{-1} \int_t^T \Phi(s) ds.$$

We apply Gronwall's lemma to get

$$\Phi(t) \leq C(\epsilon \|g\|_Q^2 + \|g_T\|_\Omega^2) e^{C\epsilon^{-1}(T-t)}.$$

This means

$$\|z_u(t)\|_\Omega^2 + \epsilon \int_t^T \|z_w\|_\Omega^2 ds \leq C(\epsilon \|g\|_Q^2 + \|g_T\|_\Omega^2) e^{C\epsilon^{-1}(T-t)}.$$

We conclude

$$\max_I \|z_u\|_\Omega^2 \leq C(\epsilon \|g\|_Q^2 + \|g_T\|_\Omega^2) e^{C\epsilon^{-1}T}.$$

$$(4.13) \quad \|z_w\|_Q^2 \leq C(\|g\|_Q^2 + \epsilon^{-1} \|g_T\|_\Omega^2) e^{C\epsilon^{-1}T}.$$

From the second equation we know  $z_w = \Delta z_u$ . So, by (4.7) and (4.13),

$$(4.14) \quad \|D^2 z_u\|_Q^2 \leq C \|\Delta z_u\|_Q^2 = C \|z_w\|_Q^2 \leq C(\|g\|_Q^2 + \epsilon^{-1} \|g_T\|_\Omega^2) e^{C\epsilon^{-1}T}.$$

This takes care of the first terms in (4.11) and (4.12).

Now assume that  $g_T = 0$ . Consider the dual problem (3.8) and multiply the first equation by  $-\partial_t z_u$  and integrate over  $Q$  to get

$$(4.15) \quad \langle \partial_t z_u, \partial_t z_u \rangle_Q - \epsilon \langle \Delta z_w, \partial_t z_u \rangle_Q - \langle f'(u) z_w, \partial_t z_u \rangle_Q = -\langle g, \partial_t z_u \rangle_Q.$$

So, by using  $z_w = \Delta z_u$  from the second equation, we get

$$\langle \Delta z_w, \partial_t z_u \rangle_Q = \langle z_w, \partial_t \Delta z_u \rangle_Q = \langle \Delta z_u, \partial_t \Delta z_u \rangle_Q = \frac{1}{2} \int_0^T \frac{d}{dt} \|\Delta z_u\|_\Omega^2 dt.$$

By putting this in (4.15) and using that  $\|f'(u)\|_{L^\infty(Q)} \leq \beta$ , we have

$$\begin{aligned} & \|\partial_t z_u\|_Q^2 - \frac{\epsilon}{2} \|\Delta z_u(T)\|_\Omega^2 + \frac{\epsilon}{2} \|\Delta z_u(0)\|_\Omega^2 \\ & \leq \|f'(u)\|_{L^\infty(Q)} \|z_w\|_Q \|\partial_t z_u\|_Q + \|g\|_Q \|\partial_t z_u\|_Q \\ & \leq \frac{c\beta^2}{2} \|z_w\|_Q^2 + \frac{1}{2c} \|\partial_t z_u\|_Q^2 + \frac{\epsilon}{2} \|g\|_Q^2 + \frac{1}{2c} \|\partial_t z_u\|_Q^2. \end{aligned}$$

Put  $c = 2$  and kick back  $\|\partial_t z_u\|_Q^2$  to get, with  $z_u(T) = g_T = 0$ ,

$$\frac{1}{2} \|\partial_t z_u\|_Q^2 + \frac{\epsilon}{2} \|\Delta z_u(0)\|_\Omega^2 \leq \beta^2 \|z_w\|_Q^2 + \|g\|_Q^2.$$

Hence, by (4.13) with  $C = C(\beta)$ ,

$$(4.16) \quad \|\partial_t z_u\|_Q^2 \leq C \|z_w\|_Q^2 + C \|g\|_Q^2 \leq C \|g\|_Q^2 e^{-C\epsilon^{-1}T}.$$

It remains to bound  $\|D^2 z_w\|_Q^2$ . From the first equation of (3.8) we get  $\epsilon \Delta z_w = g + \partial_t z_u + f'(u) z_w$ . Taking norms and using (4.7), (4.13), and (4.16) gives

$$\begin{aligned} \epsilon^2 \|D^2 z_w\|_Q^2 & \leq \epsilon^2 C \|\Delta z_w\|_Q^2 = C \|g + \partial_t z_u + f'(u) z_w\|_Q^2 \\ & \leq C \left( \|g\|_Q^2 + \|\partial_t z_u\|_Q^2 + \|f'(u)\|_{L^\infty(Q)}^2 \|z_w\|_Q^2 \right) \\ & \leq C \|g\|_Q^2 e^{C\epsilon^{-1}T}. \end{aligned}$$

This completes the proof of (4.11)

Now let  $g = 0$  and set  $\sigma(t) = T - t$ . Multiply the first equation of (3.8) by  $-\sigma \partial_t z_u$  to get

$$\langle \partial_t z_u, \sigma \partial_t z_u \rangle_Q - \epsilon \langle \Delta z_w, \sigma \partial_t z_u \rangle_Q - \langle f'(u) z_w, \sigma \partial_t z_u \rangle_Q = 0.$$

Here, since  $z_w = \Delta z_u$  and  $\sigma'(t) = -1$ ,

$$\begin{aligned}
 \langle \Delta z_w, \sigma \partial_t z_u \rangle_Q &= \langle z_w, \sigma \Delta \partial_t z_u \rangle_Q \\
 &= \langle \Delta z_u, \sigma \Delta \partial_t z_u \rangle_Q \\
 &= \frac{1}{2} \int_0^T \frac{d}{dt} (\sigma \|\Delta z_u\|_\Omega^2) dt - \frac{1}{2} \int_0^T \sigma' \|\Delta z_u\|_\Omega^2 dt \\
 &= \frac{1}{2} \sigma(T) \|\Delta z_u(T)\|_\Omega^2 - \frac{1}{2} \sigma(0) \|\Delta z_u(0)\|_\Omega^2 + \frac{1}{2} \int_0^T \|z_w\|_\Omega^2 dt \\
 &= -\frac{1}{2} T \|\Delta z_u(0)\|_\Omega^2 + \frac{1}{2} \|z_w\|_Q^2.
 \end{aligned}$$

Hence,

$$\begin{aligned}
 \|\sigma^{\frac{1}{2}} \partial_t z_w\|_Q^2 + \|\Delta z_u(0)\|_\Omega^2 &\leq \frac{\epsilon}{2} \|z_w\|_Q^2 + \|f'(u)\|_{L^\infty} \|\sigma^{\frac{1}{2}} z_w\|_Q \|\sigma^{\frac{1}{2}} \partial_t z_u\|_Q \\
 &\leq \frac{1}{2} (\epsilon + \beta^2 T) \|z_w\|_Q^2 + \frac{1}{2} \|\sigma^{\frac{1}{2}} \partial_t z_u\|_Q^2.
 \end{aligned}$$

So by (4.13) we have

$$\|\sigma^{\frac{1}{2}} \partial_t z_u\|_Q \leq (\epsilon + \beta^2 T) \|z_w\|_Q^2 C \epsilon^{-1} \|g_T\|_\Omega^2 e^{C\epsilon^{-1}T}.$$

Finally, from (4.7) and  $\epsilon \Delta z_w = \partial_t z_u + f'(u) z_w$  we get

$$\begin{aligned}
 \epsilon^2 \|\sigma^{\frac{1}{2}} D^2 z_w\|_Q^2 &\leq \epsilon^2 C \|\sigma^{\frac{1}{2}} \Delta z_w\|_Q^2 = C \|\sigma^{\frac{1}{2}} (\partial_t z_u + f'(u) z_w)\|_Q^2 \\
 &\leq C \left( \|\sigma^{\frac{1}{2}} \partial_t z_u\|_Q^2 + T \|z_w\|_Q^2 \right) \\
 &\leq C \epsilon^{-1} \|g_T\|_\Omega^2 e^{C\epsilon^{-1}T}.
 \end{aligned}$$

This completes the proof of (4.12).  $\square$

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