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LOCAL POINTWISE A POSTERIORI GRADIENT ERROR BOUNDS FOR THE STOKES EQUATIONS

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ABSTRACT. We consider the standard Taylor-Hood finite element method for the stationary Stokes system on polyhedral domains. We prove *local* a posteriori error estimates for the maximum error in the gradient of the velocity field. Because the gradient of the velocity field blows up near re-entrant corners and edges, such local error control is necessary when pointwise control of the gradient error is desirable. Computational examples confirm the utility of our estimates in adaptive codes.

1. Introduction

We consider finite element methods for the stationary Stokes equations

$$
\begin{aligned}
-\Delta \mathbf{u} + \nabla p &= \mathbf{f}, && \text{in } \Omega,\\
\nabla \cdot \mathbf{u} &= g, && \text{in } \Omega,\\
u &= 0, && \text{on } \partial\Omega.
\end{aligned}
\tag{1.1}
$$

Here we assume that $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, is a polygonal ($n = 2$) or polyhedral ($n = 3$) domain. We note that our main results also include the case of polyhedral crack domains. We also require that $\int_\Omega g \,\mathrm{d}x = 0$ in order to ensure existence and $\int_\Omega p \,\mathrm{d}x = 0$ in order to ensure uniqueness of solutions. With

$$V = (H_0^1(\Omega))^n, \quad X = L_2(\Omega),$$

we introduce the bilinear form

$$\mathcal{L}((\mathbf{u}, p), (\mathbf{v}, \lambda)) = a(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) - b(\mathbf{u}, \lambda), \tag{1.2}$$

where

$$a(\mathbf{u}, \mathbf{v}) = \int_\Omega \sum_{i,j=1}^n \frac{\partial \mathbf{u}_i}{\partial x_j} \frac{\partial \mathbf{v}_i}{\partial x_j} \,\mathrm{d}x, \quad b(\mathbf{v}, p) = -\int_\Omega (\nabla \cdot \mathbf{v}) p \,\mathrm{d}x. \tag{1.3}$$

Writing also

$$(\mathbf{f}, \mathbf{v}) = \int_\Omega \sum_{i,j=1}^n \mathbf{f}_i \mathbf{v}_i \,\mathrm{d}x, \quad (g, \lambda) = \int_\Omega g\lambda \,\mathrm{d}x,$$

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we obtain the weak formulation of (1.1): find $(\mathbf{u}, p) \in V \times X$ such that

$$\mathcal{L}((\mathbf{u}, p), (\mathbf{v}, \lambda)) = (\mathbf{f}, \mathbf{v}) + (g, \lambda) \quad \forall (\mathbf{v}, \lambda) \in V \times X.$$

Let $\{\mathcal{T}_h\}$ be a regular family of triangulations of Ω, generated for example by an adaptive bisection algorithm. We assume that $V_h \subset V$ and $X_h \subset X$ are standard Taylor-Hood finite element spaces corresponding to a mesh $\mathcal{T}_h$; properties are given below. The finite element method for (1.1) is: find $(\mathbf{u}_h, p_h) \in V_h \times X_h$ such that

$$\mathcal{L}((\mathbf{u}_h, p_h), (\mathbf{v}_h, \lambda_h)) = (\mathbf{f}, \mathbf{v}_h) + (g, \lambda_h) \quad \forall (\mathbf{v}_h, \lambda_h) \in V_h \times X_h. \tag{1.4}$$

We enforce $\int_\Omega p_h \, \mathrm{d}x = 0$ in order to ensure uniqueness.

Our goal in this paper is to prove *local* a posteriori error estimates for $\nabla(\mathbf{u} - \mathbf{u}_h)$ in the maximum norm. More precisely, let $D \subset \Omega$ be a given target subdomain. We seek a posteriori control of

$$\|\nabla(\mathbf{u} - \mathbf{u}_h)\|_{L_\infty(D)} = \sup_{x \in D} \max_{i,j} |D_i(\mathbf{u}_j - \mathbf{u}_{h,j})(x)|.$$

In order to motivate our results, we briefly describe the PhD thesis [Sve06] by E. D. Svensson. It considers the problem of computationally characterizing mixing in incompressible flows. Let $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, be an open set containing a fluid. The motion $\mathbf{v} : [0, \infty) \times \Omega \to \Omega$ of the fluid is described by the system

$$\frac{\partial \mathbf{v}(t, x)}{\partial t} = \mathbf{u}(\mathbf{v}(t, x)), \ t > 0; \quad \mathbf{v}(0, x) = x, \tag{1.5}$$

of ordinary differential equations. Here $x \in \Omega$ is the starting point of the particle path and $\mathbf{u}$ is the velocity field obtained by solving (1.1). In order to solve (1.5) computationally, Svensson discretized both (1.5) and (1.1) by the finite element method. In order to control the error in approximating (1.5), it is necessary to control $\|\mathbf{u} - \mathbf{u}_h\|_{L_\infty(\Omega)}$. Such bounds can be found for arbitrary polyhedral domains in $\mathbb{R}^2$ or $\mathbb{R}^3$ in [SL06].

Svensson also developed a shadowing error estimate for (1.5), that is, an estimate for the distance between the computed path and a true path not necessarily having the same starting point. This estimate involves a linearization of (1.5), which in turn requires pointwise error bounds for $\nabla(\mathbf{u} - \mathbf{u}_h)$. Global pointwise gradient bounds for the case when Ω is convex can be found in [SL06]. However, (1.5) is often naturally formulated in nonconvex domains, and $\nabla \mathbf{u}$ is generally not bounded near reentrant corners and edges of $\partial\Omega$. For this application, it is sufficient to provide a posteriori control of $\|\nabla(\mathbf{u} - \mathbf{u}_h)\|_{L_\infty(D)}$ on regions $D \subset \Omega$ not abutting nonconvex parts of $\partial\Omega$. In this work we employ techniques developed in [Dem07] in the context of Poisson's problem in order to provide local pointwise gradient error control for the Stokes system. We note that while [Dem07] provides a roadmap for the current work, our proofs here also involve significant technical challenges not present in scalar elliptic problems.

More precisely, let $D \subset \Omega$ with D lying a distance $d > 0$ from any reentrant vertex (when $n = 2, 3$) or edge (when $n = 3$) of $\partial\Omega$. Let $D_d = \{x \in \Omega : \mathrm{dist}(D, x) < d\}$. We define the W^1_∞-type residual error indicator

$$\begin{aligned} \eta_{1,\infty}(T) = h_T \|\mathbf{f} + \Delta \mathbf{u}_h - \nabla p_h\|_{L_\infty(T)} \\ + \|[\![\nabla \mathbf{u}_h]\!]\|_{L_\infty(\partial T)} + \|g - \nabla \cdot \mathbf{u}_h\|_{L_\infty(T)}, \quad T \in \mathcal{T}_h. \end{aligned}$$

Here $[\![\nabla \mathbf{u}_h]\!]$ is the jump in the (componentwise) normal derivative of $\mathbf{u}_h$ across the element boundary ∂T and $h_T = \mathrm{diam}(T)$. Let also $\underline{h} = \min_{T \in \mathcal{T}_h} h_T$. Finally, let

$\mathcal{T}_{D_d} = \{T \in \mathcal{T}_h : T \cap D_d \neq \emptyset\}$. We use the standard norms $\|\cdot\|_{L_p(D)}$, $\|\cdot\|_{W_p^m(D)}$ and seminorms $|\cdot|_{W_p^m(D)}$, $|\cdot|_{C^{1,\beta}(\overline{D})}$. Our main result is then the following.

Theorem 1. *Let $\rho \leq c_0 \min\{d, \underline{h}\}$ for a sufficiently small constant c_0, and assume that $\mathbf{u} \in C^{1,\beta}(\overline{D_\rho})$ for some $\beta > 0$. Under the above assumptions, we then have*

$$\|\nabla(\mathbf{u}-\mathbf{u}_h)\|_{L_\infty(D)} \leq C \ln \frac{d}{\rho} \max_{T \in \mathcal{T}_{D_d}} \frac{h_T}{h_T + \mathrm{dist}(T,D)} \eta_{1,\infty}(T) + C\rho^\beta |\mathbf{u}|_{C^{1,\beta}(\overline{D_\rho})} + C\frac{1}{d}\|\mathbf{u}-\mathbf{u}_h\|_{L_\infty(\Omega)}.$$

The above estimate consists of a *local residual term*, a *regularization penalty*, and a *global pollution term*. The *local residual term* $\max_{T\in\mathcal{T}_{D_d}} \frac{h_T}{h_T+\mathrm{dist}(T,D)}\eta_{1,\infty}(T)$ measures local error contributions from a neighborhood of the target region D. Since $\frac{h_T}{h_T+\mathrm{dist}(T,D)} = 1$ when $T \cap D \neq \emptyset$, error contributions from D are measured by the W^1_∞-type residual indicator $\eta_{1,\infty}$. Error contributions from elements intersecting D_d but not touching D are measured by the term $\frac{h_T}{h_T+\mathrm{dist}(T,D)}\eta_{1,\infty}(T)$. The strength of this contribution decays smoothly from $\eta_{1,\infty}(T)$ to $\frac{h_T}{d}\eta_{1,\infty}(T)$ as $h_T + \mathrm{dist}(T,D)$ increases from h_T to $O(d)$. The extra factor h_T present in the latter error indicator significantly de-emphasizes these contributions as the mesh is refined, and in particular effectively measures the error contribution from these regions in L_∞ instead of W^1_∞. The *regularization penalty* $\rho^\beta|\mathbf{u}|_{C^{1,\beta}(\overline{D_\rho})}$ is due to technicalities associated with bounding maximum norms. Note that $\mathbf{u} \in C^{1,\beta}(\overline{D_\rho})$ for some $\beta > 0$ in the situation comtemplated here (cf. Section 5 of [MR06]), and that we generally select $\rho = \underline{h}^\gamma$ with γ sufficiently large. Thus, though a priori in nature and therefore undesirable, this regularization penalty is generally of higher order and asymptotically negligible. It may also be removed if a certain nondegeneracy condition holds; we discuss this in more detail below.

The *pollution term* $C\frac{1}{d}\|\mathbf{u}-\mathbf{u}_h\|_{L_\infty(\Omega)}$ measures the influence of global solution properties on the local solution quality. We next state a corollary in which the pollution error is bounded a posteriori by means of Lemma 3 (stated below).

Corollary 2. *In addition to the above assumptions, let Ω have a Lipschitz boundary. Then*

$$\|\nabla(\mathbf{u}-\mathbf{u}_h)\|_{L_\infty(D)} \leq C \ln \frac{d}{\rho} \max_{T \in \mathcal{T}_{D_d}} \frac{h_T}{h_T + \mathrm{dist}(T,D)} \eta_{1,\infty}(T) + C\rho^\beta |\mathbf{u}|_{C^{1,\beta}(\overline{D_\rho})} + C\Big(\ln\frac{1}{\underline{h}}\Big)^{\alpha_n}\frac{1}{d}\max_{T\in\mathcal{T}_h} h_T\eta_{1,\infty}(T). \tag{1.6}$$

Here $\alpha_2 = 2$ and $\alpha_3 = 4/3$.

Because we expect convergence to be faster in L_∞ than in W^1_∞, the pollution term is of higher order when viewed from the perspective of a priori convergence rates. Adaptive algorithms based on (1.6) can correspondingly be expected to generate coarser meshes in $\Omega \setminus D_d$ than in D_d. Note also that the restriction in Corollary 2, that $\partial\Omega$ is Lipschitz, excludes the case of crack domains. As we remark below, this restriction is likely unnecessary.

We finally give a brief discussion of related literature. As mentioned above, [SL06] contains global a posteriori error estimates in the maximum norm for the Stokes equation. Local a posteriori estimates for maximum gradient errors for

Poisson's problem are proved in [Dem07]; related global maximum gradient error estimators are developed in [Dem06]. Local a priori error estimates for the Stokes equation are developed in [HXZL08] and used to justify a local parallel finite element algorithm. Finally, global W^1_∞ a priori error estimates for the Stokes system on convex polygonal and polyhedral domains can be found in the recent papers [GNS04], [GNS05], and [GL10]; we also refer to these works for a more comprehensive overview of previous literature on maximum norm a priori analysis for the Stokes problem.

An outline of the paper is as follows. In Section 2 we give several preliminary definitions. In Section 3 we use a duality argument in order to represent the local pointwise gradient error, and in Section 4 we complete the proof of Theorem 1 by proving a number of regularity estimates. Section 5 contains discussion of refinements and extensions; in particular, we give a condition under which the regularization penalty in (1.6) can be removed and discuss the possibility of proving local pointwise estimates for the pressure error $p - p_h$. Finally, in Section 6 we present a computational example.

2. Preliminaries

2.1. **Finite element spaces and interpolants.** We employ the standard Taylor-Hood finite element spaces. Let V_h be the continuous piecewise polynomials of degree k and let X_h be the continuous piecewise polynomials of degree $k-1$, $k \geq 2$. With this definition, existence of solutions to (1.4) is well known to hold, as well as uniqueness so long as $\int_\Omega p_h \, \mathrm{d}x = 0$ is enforced.

We also assume the existence of interpolation operators $I_h : V \to V_h$, $J_h : X \to X_h$ such that, for $1 \leq p \leq \infty$,

$$\|I_h v - v\|_{W^j_p(T)} \leq C h_T^{m-j} |v|_{W^m_p(P_T)}, \quad j = 0, 1, \ m = 1, 2, \tag{2.1}$$

$$\|J_h p - p\|_{L_p(T)} \leq C h_T^m |p|_{W^m_p(P_T)}, \quad m = 0, 1. \tag{2.2}$$

These are standard properties of interpolants of Clément or Scott-Zhang type. Also, P_T is the patch of elements touching T, with a corresponding hierarchy of neighbor patches of a simplex $T \in \mathcal{T}_h$ defined by:

$$\begin{aligned} P_T &= \cup\{S \in \mathcal{T}_h : \bar{S} \cap \bar{T} \neq \emptyset\}, \\ P_T' &= \cup\{S \in \mathcal{T}_h : \bar{S} \cap \bar{P}_T \neq \emptyset\}, \\ P_T'' &= \cup\{S \in \mathcal{T}_h : \bar{S} \cap \bar{P}_T' \neq \emptyset\}, \quad \text{etc.} \end{aligned}$$

We finally note that I_h and J_h may be defined so that, if $\mathbf{v} \in (H^1_0(D_d))^n$ and $q \in L_2(\Omega)$ is supported in D_d, then $I_h \mathbf{v}$ and $J_h q$ have support in $\mathcal{T}_{D_d}$.

2.2. **Reference domains.** Our proofs involve carrying out duality arguments over subdomains B of Ω. To describe these we recall that the target domain D is a fixed distance away from reentrant corners or edges but may touch the remaining parts of $\partial\Omega$. We fix a point $x_0 \in D$ where the maximum gradient error over D is attained. When $\mathrm{dist}(x_0, \partial\Omega) \geq d$ we can choose our subdomain B to be a square or cube with diameter d centered at x_0. When x_0 is close to $\partial\Omega$ we must carefully control the size and shape of these subdomains in order to ensure that regularity constants appearing in our estimates are uniformly bounded. We thus define reference domains to which we may map portions of Ω lying near $\partial\Omega$. A similar approach was used in [Dem07], and we refer to §2.2 of that work for more detail. In particular, it is shown there that there exists a set $\{\tilde{B}_1, ..., \tilde{B}_M\}$ of reference domains of unit diameter, each of which is a convex polyhedron, with the following properties.

There exists a constant $d_0 \leq 1$ depending on Ω such that whenever $d \leq d_0$, the following hold: Assume that $x_0 \in \Omega$ with $\text{dist}(x_0, e) \geq d$ for all reentrant corners or edges e of $\partial\Omega$. Then there exist constants $c_1 > 0$ and $0 < c_2 \leq 1$, independent of x_0, and a subdomain B of Ω such that $x_0 \in B$, $\text{dist}(x_0, \partial B \setminus \partial\Omega) \geq c_1 d$, and such that for some $1 \leq i \leq M$ there is an affine bijection $A_i : \tilde{B}_i \to B$, where A_i consists only of translation and scaling by cd for some $c_2 \leq c \leq 1$.

We will also use a cut-off function ω. With x_0 and c_1 as above, let $\omega \in C_0^\infty(B_{c_1 d}(x_0))$ satisfy

$$\omega \equiv 1 \text{ on } B_{c_1 d/2}(x_0). \tag{2.3}$$

Here $B_r(x)$ denotes the open ball with radius r and center x. If $B_{c_1 d}(x_0) \cap \Omega$ is not connected, then we assume that $\omega \equiv 0$ on any component of $B_{c_1 d}(x_0) \cap \Omega$ not containing x_0. Note that $\omega = 0$ on $\partial B \setminus \partial\Omega$ and that ω may be defined so that

$$\|\omega\|_{W_\infty^j(B)} \leq C_j d^{-j}, \quad j = 0, 1, 2, \ldots \tag{2.4}$$

2.3. **A posteriori estimates in L_∞.** We quote a result from [SL06].

Lemma 3. *Assume that Ω is a polyhedral domain in $\mathbb{R}^n$, $n = 2, 3$. Then*

$$\|\mathbf{u} - \mathbf{u}_h\|_{L_\infty(\Omega)} \leq C\Big(\ln \frac{1}{\underline{h}}\Big)^{\alpha_n} \max_{T \in \mathcal{T}_h} h_T \eta_{1,\infty}(T).$$

Here $\alpha_2 = 2$ and $\alpha_3 = 4/3$.

Remark 4. The arguments of [SL06] are valid for polyhedral domains with Lipschitz boundary, which excludes the case of crack domains. However, similar estimates for Poisson's problem can also be obtained for polyhedral domains with cracks (cf. [DG10]), and we expect the same to be true for the Stokes system. Our numerical examples are carried out on a two-dimensional domain with a crack.

3. Error representation

In this section we represent $\|\nabla(\mathbf{u} - \mathbf{u}_h)\|_{L_\infty(D)}$ by employing a regularized Green's function.

3.1. **Pointwise gradient error.** Let

$$\mathbf{e_u} = \mathbf{u}_h - \mathbf{u}, \quad e_p = p_h - p.$$

We begin by selecting $x_0 \in D$ and i, j so that

$$\|\nabla \mathbf{e_u}\|_{L_\infty(D)} = \sup_{x \in D} \max_{k,l} |D_k \mathbf{e_u}_l(x)| = |D_i \mathbf{e_u}_j(x_0)| = |D_i \mathbf{e_u}(x_0) \cdot \mathbf{k}_j|.$$

Here $\mathbf{k}_j$ denotes the j-th basis vector. We shall express this by means of an approximate "delta function": $D_i \mathbf{e_u}(x_0) \cdot \mathbf{k}_j \approx (D_i \mathbf{e_u} \cdot \mathbf{k}_j, \delta) = (D_i \mathbf{e_u}, \delta \mathbf{k}_j)$.

In order to do so, we select a simplex $T_{x_0} \in \mathcal{T}_h$ such that

$$x_0 \in \overline{T_{x_0}}.$$

Furthermore, we define a regularized "delta function" δ corresponding to the point x_0 (cf. [SW95]). Following §2.3 of [Dem07], we let ρ be as in Theorem 1 and fix a shape-regular simplex T_0 of diameter ρ such that $x_0 \in T_0 \subset T_{x_0}$. More precisely, T_0 should satisfy the same regularity assumption as the simplices in the mesh family

$\{\mathcal{T}_h\}$. Then $\delta \in C_0^\infty(T_0)$ may be defined so that for any polynomial P of degree at most $k-1$ (where k is the polynomial degree of V_h),

$$P(x_0) = \int_{T_0} \delta P \, \mathrm{d}x,$$

$$\|\delta\|_{W_p^j(T_0)} \le C\rho^{-j-n(1-\frac{1}{p})}, \quad 1 \le p \le \infty, \; j = 0, 1. \tag{3.1}$$

Following precisely the arguments in Proposition 2.3 of [Dem07], we obtain the following.

Lemma 5. *Assume that* $\mathbf{u} \in C^{1,\beta}(\overline{T_0})$ *for some* $\beta \in (0,1]$ *and* $\mathbf{v}_h \in V_h$. *Then*

$$|D_i(\mathbf{u}_j - \mathbf{v}_{hj})(x_0)| \le |(D_i(\mathbf{u}_j - \mathbf{v}_{hj}), \delta)| + C\rho^\beta |\mathbf{u}|_{C^{1,\beta}(\overline{T_0})}. \tag{3.2}$$

Employing (3.2), we obtain the following error representation.

Lemma 6. *Under the assumptions of Theorem 1 we have*

$$\|\nabla \mathbf{e_u}\|_{L_\infty(D)} \le |(D_i \mathbf{e_u}, \delta \mathbf{k}_j)| + C\rho^\beta |\mathbf{u}|_{C^{1,\beta}(\overline{D_\rho})}. \tag{3.3}$$

3.2. **Localization.** As above, we let d be the distance from the target subdomain D to the nearest reentrant edge or vertex in $\partial\Omega$. Let $x_0 \in D$ be a point where the maximum gradient error is attained and choose the subdomain B and the cut-off function ω as in Section 2.2. As in Theorem 1 we assume that $\rho \le c_0 d$ for c_0 sufficiently small. Hence, we may achieve $\omega \equiv 1$ on $\operatorname{supp}(\delta)$ and $\operatorname{supp}(\delta) \subset T_0 \subset B$, so that the main term in (3.3) becomes

$$(D_i \mathbf{e_u}, \delta \mathbf{k}_j) = -(\mathbf{e_u}, D_i \delta \mathbf{k}_j) = -(\omega \mathbf{e_u}, D_i \delta \mathbf{k}_j). \tag{3.4}$$

We introduce a localized adjoint problem: find $(\mathbf{v}, q) \in V_B \times X_B$ such that

$$\mathcal{L}_B((\mathbf{w}, \lambda), (\mathbf{v}, q)) = (\mathbf{w}, D_i \delta \mathbf{k}_j) \quad \forall (\mathbf{w}, \lambda) \in V_B \times X_B, \tag{3.5}$$

where the form $\mathcal{L}_B$ is defined as in (1.2)–(1.3) but with integrals extending only over B and $V_B \times X_B = (H_0^1(B))^n \times L_2(B)$. The strong form is:

$$\begin{aligned} -\Delta \mathbf{v} - \nabla q &= D_i \delta \mathbf{k}_j, && \text{in } B, \\ \nabla \cdot \mathbf{v} &= 0, && \text{in } B, \\ \mathbf{v} &= 0, && \text{on } \partial B. \end{aligned}$$

We extend $\mathbf{v}$ by zero outside of B. Then $\mathbf{v} \in (W_\infty^1(\Omega))^n$ but has no higher global regularity. Note also that $\omega \mathbf{e_u} \in V_B$ because $\omega = 0$ on $\partial B \setminus \partial\Omega$ and $\mathbf{e_u} = 0$ on $\partial\Omega$.

3.3. **Duality argument.** We choose $(\mathbf{w}, \lambda) = (\omega \mathbf{e_u}, e_p) \in V_B \times X_B$ in (3.5), use (3.4), and recall that $\mathbf{v} = 0$ outside B. Thus,

$$\begin{aligned} (\mathbf{e_u}, D_i \delta \mathbf{k}_j) &= (\omega \mathbf{e_u}, D_i \delta \mathbf{k}_j) = \mathcal{L}_B((\omega \mathbf{e_u}, e_p), (\mathbf{v}, q)) \\ &= a(\omega \mathbf{e_u}, \mathbf{v}) + b(\mathbf{v}, e_p) - b(\omega \mathbf{e_u}, q) \\ &= a(\mathbf{e_u}, \mathbf{v}) - a((1-\omega)\mathbf{e_u}, \mathbf{v}) + b(\mathbf{v}, e_p) + (\nabla\omega \cdot \mathbf{e_u}, q) - b(\mathbf{e_u}, \omega q) \\ &= \mathcal{L}((\mathbf{e_u}, e_p), (\mathbf{v}, \omega q)) - a((1-\omega)\mathbf{e_u}, \mathbf{v}) + (\nabla\omega \cdot \mathbf{e_u}, q), \end{aligned} \tag{3.6}$$

since

$$\begin{aligned} -b(\omega \mathbf{e_u}, q) &= (\nabla \cdot (\omega \mathbf{e_u}), q) = (\nabla\omega \cdot \mathbf{e_u}), q) + (\omega \nabla \cdot \mathbf{e_u}, q) \\ &= (\nabla\omega \cdot \mathbf{e_u}, q) - b(\mathbf{e_u}, \omega q). \end{aligned}$$

Writing $a(\mathbf{u},\mathbf{v}) = \int_\Omega \mathbf{u}_{i,j}\mathbf{v}_{i,j}\,\mathrm{d}x$, setting $B_{d/2} = B_{c_1 d/2}(x_0) \cap B$, and recalling that $\omega = 0$ on $\partial B \setminus \partial\Omega$, $\mathbf{e_u} = 0$ on $\partial\Omega$, and (2.3), we have

$$\begin{aligned}
-a((1-\omega)\mathbf{e_u},\mathbf{v}) &= -\int_B ((1-\omega)\mathbf{e_u}_i)_{,j}\mathbf{v}_{i,j}\,\mathrm{d}x \\
&= -\int_{\partial B}(1-\omega)n_j\mathbf{e_u}_i\mathbf{v}_{i,j}\,\mathrm{d}x + \int_B (1-\omega)\mathbf{e_u}_i\mathbf{v}_{i,jj}\,\mathrm{d}x \\
&= -\int_{\partial B\setminus\partial\Omega} n_j\mathbf{e_u}_i\mathbf{v}_{i,j}\,\mathrm{d}x + \int_{B\setminus B_{d/2}}(1-\omega)\mathbf{e_u}_i\mathbf{v}_{i,jj}\,\mathrm{d}x.
\end{aligned}$$

Employing (3.6) and (2.4), and recalling that $\nabla\omega = 0$ on $B_{d/2}$, we conclude that

$$\begin{aligned}
|(\mathbf{e_u}, D_i\delta\mathbf{k}_j)| \le |\mathcal{L}((\mathbf{e_u},e_p),(\mathbf{v},\omega q))| + \|\mathbf{e_u}\|_{L_\infty(B)}\big(\|\Delta\mathbf{v}\|_{L_1(B\setminus B_{d/2})} \\
+ \|\partial\mathbf{v}/\partial n\|_{L_1(\partial B\setminus\partial\Omega)} + Cd^{-1}\|q\|_{L_1(B\setminus B_{d/2})}\big).
\end{aligned} \tag{3.7}$$

We now consider the term $|\mathcal{L}((\mathbf{e_u},e_p),(v,\omega q))|$. Galerkin orthogonality implies

$$\mathcal{L}((\mathbf{e_u},e_p),(\mathbf{v},\omega q)) = \mathcal{L}((\mathbf{e_u},e_p),(\mathbf{v}-I_h\mathbf{v},\omega q - J_h(\omega q)). \tag{3.8}$$

Recalling that $\mathrm{supp}(\mathbf{v}-I_h\mathbf{v}) \subset \mathcal{T}_{D_d}$ and employing standard techniques for proving residual error estimates (cf. [Dem07] for similar computations), we next compute

$$\begin{aligned}
\mathcal{L}((\mathbf{e_u},e_p),(\mathbf{v}-I_h\mathbf{v},\omega q - J_h(\omega q)) &\le C\sum_{T\in\mathcal{T}_{D_d}}\eta_{1,\infty}(T) \\
&\times\big(h_T^{-1}\|\mathbf{v}-I_h\mathbf{v}\|_{L_1(T)} + \|\nabla(\mathbf{v}-I_h\mathbf{v})\|_{L_1(T)} + \|\omega q - J_h(\omega q)\|_{L_1(T)}\big) \\
&\le C\max_{T\in\mathcal{T}_{D_d}}\frac{h_T}{h_T+\mathrm{dist}(T,D)}\eta_{1,\infty}(T)\sum_{T\in\mathcal{T}_{D_d}}\frac{h_T+\mathrm{dist}(T,T_{x_0})}{h_T} \\
&\times\big(h_T^{-1}\|\mathbf{v}-I_h\mathbf{v}\|_{L_1(T)} + \|\nabla(\mathbf{v}-I_h\mathbf{v})\|_{L_1(T)} + \|\omega q - J_h(\omega q)\|_{L_1(T)}\big).
\end{aligned} \tag{3.9}$$

Here we used that $h_T + \mathrm{dist}(T,T_{x_0}) \ge C(h_T+\mathrm{dist}(T,D))$, so that $\frac{h_T}{h_T+\mathrm{dist}(T,T_{x_0})} \le C\frac{h_T}{h_T+\mathrm{dist}(T,D)}$.

Recall that we assume $\rho \le c_0\underline{h}$. We choose c_0 so small that $B_{2\rho}(x_0) \subset P'_{T_{x_0}}$ and then apply the interpolation estimates (2.1) and (2.2) to the terms in (3.9) as follows. Recall that $x_0 \in T_{x_0}$. For $T \in P''_{T_{x_0}}$ and for T with $T\cap(\partial B\setminus\partial\Omega) \ne \emptyset$ we apply (2.1) with $m=1$, while for all other $T\in\mathcal{T}_{D_d}$ we apply (2.1) with $m=2$. Similarly, for $T\in P''_{T_{x_0}}$ we apply (2.2) with $m=0$, while for all other $T\in\mathcal{T}_{D_d}$ we apply (2.2) with $m=1$. We also note that $h_T+\mathrm{dist}(T,T_{x_0}) \simeq h_T$ for $T\in P''_{T_{x_0}}$ and that $h_T+\mathrm{dist}(T,T_{x_0}) \simeq \mathrm{dist}(x,x_0)$ for $x\in T\in\mathcal{T}_{D_d}\setminus P'_{T_{x_0}}$.

More precisely, let

$$\begin{aligned}
I(T) = \frac{h_T+\mathrm{dist}(T,T_{x_0})}{h_T} \\
\times\big(h_T^{-1}\|\mathbf{v}-I_h\mathbf{v}\|_{L_1(T)} + \|\nabla(\mathbf{v}-I_h\mathbf{v})\|_{L_1(T)} + \|\omega q - J_h(\omega q)\|_{L_1(T)}\big).
\end{aligned}$$

When bounding the sum $\sum_{T\in\mathcal{T}_{D_d}} I(T)$ in (3.9), we have for the terms with $T\in P''_{T_{x_0}}$

$$\begin{aligned}
\sum_{T\in P''_{T_{x_0}}} I(T) &\le C\sum_{T\in P''_{T_{x_0}}}\big(\|\nabla\mathbf{v}\|_{L_1(P_T)} + \|\omega q\|_{L_1(P_T)}\big) \\
&\le C\big(\|\nabla\mathbf{v}\|_{L_1(P'''_{T_{x_0}})} + \|q\|_{L_1(P'''_{T_{x_0}})}\big).
\end{aligned}$$

For the terms with $T \notin P''_{T_{x_0}}$, we use $B_{2\rho}(x_0) \subset P'_{T_{x_0}}$, (2.4), and that $\mathbf{v}$, ωq vanish outside B to obtain

$$\begin{aligned}\sum_{\substack{T\cap(\partial B\setminus\partial\Omega)=\emptyset\\ T\notin P''_{T_{x_0}}}} I(T) &\le C \sum_{T\notin P''_{T_{x_0}}} \big(h_T + \mathrm{dist}(T, T_{x_0})\big)\big(\|D^2\mathbf{v}\|_{L_1(P_T)} + \|\nabla(\omega q)\|_{L_1(P_T)}\big)\\ &\le C \int_{\mathcal{T}_{D_d}\setminus P'_{T_{x_0}}} |x - x_0|\big(|\nabla\omega||q| + |\omega||\nabla q| + |D^2\mathbf{v}|\big)\,\mathrm{d}x,\\ &\le C\int_B |q|\,\mathrm{d}x + C\int_{B\setminus B_{2\rho}(x_0)} |x-x_0|\big(|\nabla q| + |D^2\mathbf{v}|\big)\,\mathrm{d}x.\end{aligned}$$

For the terms with $T \cap (\partial B \setminus \partial\Omega) \neq \emptyset$, we have instead

$$\begin{aligned}\sum_{\substack{T\cap(\partial B\setminus\partial\Omega)\neq\emptyset\\ T\notin P''_{T_{x_0}}}} I(T) &\le C \sum_{\substack{T\cap(\partial B\setminus\partial\Omega)\neq\emptyset\\ T\notin P''_{T_{x_0}}}} \big(h_T + \mathrm{dist}(T, T_{x_0})\big) h_T^{-1}\|\nabla\mathbf{v}\|_{L_1(P_T)}\\ &\le C \sum_{\substack{T\cap(\partial B\setminus\partial\Omega)\neq\emptyset\\ T\notin P''_{T_{x_0}}}} h_T^{-1}\int_{P_T} |x-x_0||\nabla\mathbf{v}|\,\mathrm{d}x.\end{aligned}$$

Thus,

$$\begin{aligned}&\sum_{T\in\mathcal{T}_{D_d}} I(T)\\ &\quad\le C\Big(\|\nabla\mathbf{v}\|_{L_1(B)} + \|q\|_{L_1(B)} + \int_{B\setminus B_{2\rho}(x_0)} |x-x_0|\big(|\nabla q| + |D^2\mathbf{v}|\big)\,\mathrm{d}x\\ &\qquad + \sum_{\substack{T\cap(\partial B\setminus\partial\Omega)\neq\emptyset\\ T\notin P''_{T_{x_0}}}} h_T^{-1}\int_{P_T} |x-x_0||\nabla\mathbf{v}|\,\mathrm{d}x\Big).\end{aligned} \tag{3.10}$$

Collecting the previous results, we obtain the following error representation.

Lemma 7. *Under the assumptions of Theorem 1 we have*

$$\begin{aligned}\|\nabla(\mathbf{u}-\mathbf{u}_h)\|_{L_\infty(D)} &\le C\max_{T\in\mathcal{T}_{D_d}} \frac{h_T}{h_T + \mathrm{dist}(T, D)}\eta_{1,\infty}(T)\\ &\quad\times\Big(\|\mathbf{v}\|_{W_1^1(B)} + \|q\|_{L_1(B)} + \int_{B\setminus B_{2\rho(x_0)}} |x-x_0|\big(|\nabla q| + |D^2\mathbf{v}|\big)\,\mathrm{d}x\\ &\qquad + \sum_{\substack{T\cap(\partial B\setminus\partial\Omega)\neq\emptyset\\ T\notin P'_{T_{x_0}}}} h_T^{-1}\int_T |x-x_0||\nabla\mathbf{v}|\,\mathrm{d}x\Big)\\ &\quad + \|\mathbf{u}-\mathbf{u}_h\|_{L_\infty(\Omega)}\Big(\|\Delta\mathbf{v}\|_{L_1(B\setminus B_{d/2})} + \|\partial\mathbf{v}/\partial n\|_{L_1(\partial B\setminus\partial\Omega)}\\ &\qquad + Cd^{-1}\|q\|_{L_1(B\setminus B_{d/2})}\Big) + C\rho^\beta\|\mathbf{u}\|_{C^{1,\beta}(\overline{D_\rho})}.\end{aligned} \tag{3.11}$$

Proof. We collect (3.10) into (3.9) and subsequently into (3.8) and (3.7). Combining the result with (3.3) yields (3.11). ∎

3.4. **Regularity estimates.** The proofs of Theorem 1 and Corollary 2 will be complete after the following regularity estimates are proved. The proof of these estimates is carried out in Section 4. We remind the reader that $B_{d/2} = B_{c_1 d/2}(x_0) \cap B$; see (2.3).

Lemma 8. *Let* $(\mathbf{v}, q) \in V_B \times X_B$ *be the solution of* (3.5). *Then*

$$\|\mathbf{v}\|_{W_1^1(B)} + \|q\|_{L_1(B)} \leq C \ln \frac{d}{\rho}, \tag{3.12}$$

$$\|\Delta \mathbf{v}\|_{L_1(B \setminus B_{d/2})} \leq C d^{-1}, \tag{3.13}$$

$$\|\partial \mathbf{v}/\partial n\|_{L_1(\partial B \setminus \partial \Omega)} \leq C d^{-1}, \tag{3.14}$$

$$\|q\|_{L_1(B \setminus B_{d/2})} \leq C, \tag{3.15}$$

$$\int_{B \setminus B_{2\rho}(x_0)} |x - x_0| \big(|\nabla q(x)| + |D^2 \mathbf{v}(x)| \big) \, \mathrm{d}x \leq C \ln \frac{d}{\rho}, \tag{3.16}$$

$$\sum_{\substack{T \cap (\partial B \setminus \partial \Omega) \neq \emptyset \\ T \notin P''_{T_{x_0}}}} h_T^{-1} \int_{P_T} |x - x_0| |\nabla \mathbf{v}(x)| \, \mathrm{d}x \leq C. \tag{3.17}$$

4. Regularity estimates

In this section we prove Lemma 8. First we collect some basic regularity results and properties of the Green's function for Stokes' problem from the literature and then use them to analyze regularized Green's functions.

4.1. **Regularity and Green's matrix estimates for the Stokes system.** We begin by stating a standard regularity result. Let $\tilde{B} = \tilde{B}_i$ be one of the reference domains $\tilde{B}_i$ defined in Subsection 2.2. Assume that $(\tilde{\mathbf{v}}, \tilde{q}) \in V_{\tilde{B}} \times X_{\tilde{B}}$ solves the adjoint problem

$$\mathcal{L}_{\tilde{B}}((\mathbf{w}, \lambda), (\tilde{\mathbf{v}}, \tilde{q})) = (\mathbf{w}, \tilde{\mathbf{f}})_{\tilde{B}} + (\lambda, \tilde{g})_{\tilde{B}}, \quad \forall (\mathbf{w}, \lambda) \in V_{\tilde{B}} \times X_{\tilde{B}}, \tag{4.1}$$

where $\tilde{\mathbf{f}} \in (H^{-1}(\tilde{B}))^n$ and $\tilde{g} \in X_{\tilde{B}}/\mathbb{R}$. Then

$$\|\tilde{\mathbf{v}}\|_{H_0^1(\tilde{B})} + \|\tilde{q}\|_{L_2(\tilde{B})} \leq C(\|\tilde{\mathbf{f}}\|_{H^{-1}(\tilde{B})} + \|\tilde{g}\|_{L_2(\tilde{B})}). \tag{4.2}$$

Since $\tilde{B}$ is convex, we also have (cf. [MR07])

$$\|\tilde{\mathbf{v}}\|_{H^2(\tilde{B})} + \|\tilde{q}\|_{H^1(\tilde{B})} \leq C(\|\tilde{\mathbf{f}}\|_{L_2(\tilde{B})} + \|\tilde{g}\|_{H^1(\tilde{B})}). \tag{4.3}$$

Next we state results for the Green's matrix for the adjoint problem (4.1).

Lemma 9. *Assume that* $\tilde{B} \subset \mathbb{R}^n$, $n = 2, 3$, *is a convex polyhedral domain and let* $(\tilde{\mathbf{v}}, \tilde{q})$ *be the solution of* (4.1). *There exists* $\{G_{lj}(\tilde{x}, \tilde{\xi})\}_{1 \leq l,j \leq n+1}$, $(\tilde{x}, \tilde{\xi}) \in \tilde{B} \times \tilde{B}$, *such that for* $\tilde{x} \in \tilde{B}$ *and* $1 \leq l \leq n$,

$$\begin{aligned} \tilde{\mathbf{v}}_l(\tilde{x}) &= \int_{\tilde{B}} \sum_{j=1}^n G_{lj}(\tilde{x}, \tilde{\xi}) \tilde{\mathbf{f}}_j(\tilde{\xi}) \, \mathrm{d}\tilde{\xi} + \int_{\tilde{B}} G_{l,n+1}(\tilde{x}, \tilde{\xi}) \tilde{g}(\tilde{\xi}) \, \mathrm{d}\tilde{\xi}, \\ \tilde{q}(\tilde{x}) &= \int_{\tilde{B}} \sum_{j=1}^n G_{n+1,j}(\tilde{x}, \tilde{\xi}) \tilde{\mathbf{f}}_j(\tilde{\xi}) \, \mathrm{d}\tilde{\xi} + \int_{\tilde{B}} G_{n+1,n+1}(\tilde{x}, \tilde{\xi}) \tilde{g}(\tilde{\xi}) \, \mathrm{d}\tilde{\xi}. \end{aligned} \tag{4.4}$$

Moreover, there is a constant C such that, for $\delta_{l,n+1}+|\alpha| \le 1$ and $\delta_{n+1,j}+|\beta|\le 1$,

$$|D^\alpha_{\tilde x} D^\beta_{\tilde\xi} G_{lj}(\tilde x,\tilde\xi)| \le \begin{cases} C|\tilde x-\tilde\xi|^{-\kappa}, & \text{if } \kappa>0,\\ C\ln|\tilde x-\tilde\xi|, & \text{if } \kappa=0,\end{cases} \tag{4.5}$$

where $\delta_{l,j}$ is Kronecker's delta and

$$\kappa = n+\delta_{l,n+1}+|\alpha|+\delta_{n+1,j}+|\beta|-2.$$

In case of three space dimension, these estimates can be found in [Ros10b], [Ros10b], and [MR10]; cf. [MR05]. In the case $n=2$, Lemma 9 does not appear directly in the literature to our knowledge. In the case of Poisson's problem, a similar estimate for $n=2$ is found in [Fro93] for generic convex domains. [NP94] contains Green's function estimates for elliptic scalar equations of order $2m$ on (polygonal) cones in two space dimensions. The correct asymptotics for convex polygonal domains may be derived from these estimates (cf. [Ros10a]), and the authors also state that "The passage to the case of elliptic systems entails only some notational complications". We shall thus also assume the results of Lemma 9 in the two dimensional case.

Note that the constants C in (4.2), (4.3), and (4.5) do not depend on the choice of the reference domain $\tilde B=\tilde B_i$, since $\tilde B_i$ lies in the finite set $\{\tilde B_1,\dots,\tilde B_M\}$.

We finally remark that we only employ (4.5) with $\kappa=n$ and $\kappa=n-1$. The logarithmic estimate occurring when $\kappa=0$ is thus not used here; we only include it for the sake of completeness.

Remark 10. An alternative to using Green's function estimates is to employ W^1_q-type regularity estimates as $q\downarrow 1$. This was the approach taken in [SL06] for proving gradient estimates for the Stokes system in the global maximum norm on convex polyhedral domains. The disadvantage of this approach is that it requires an unnatural restriction on the maximum interior dihedral angle when $n=3$. The a priori estimates of [GNS04], [GNS05] suffer from the same restriction, which was subsequently overcome in [GL10] by the use of sharp Green's function estimates. We similarly avoid this restriction by employing sharp Green's function estimates. Note also that the techniques we employ here could be easily used to extend the global W^1_∞ a posteriori estimates of [SL06] to include convex polyhedral domains with no restriction on the maximum interior dihedral angle. In the context of Poisson's problem, we refer to [Dem06] for a posteriori estimates and to [GLRS09] for a priori estimates which similarly use sharp Green's function estimates to obtain pointwise gradient bounds on any convex polyhedral domain.

In two space dimensions, the use of L_q-type regularity estimates leads to optimal results with respect to domain geometry and thus provides a reasonable alternative to using Green's functions. The techniques of [SL06] could be extended to prove local error estimates for $n=2$ as well.

4.2. **Transformation to a reference domain.** We now map B to a reference domain $\tilde B$ by translating and scaling $\tilde B$ by a factor cd for some $c_2\le c\le 1$ as in Subsection 2.2. Let $(\mathbf{v},q)$ be the solution of (3.5). With slight abuse of notation, we assume $c=1$ and define $\tilde{\mathbf{v}}(\tilde x)=d^{-1}\mathbf{v}(d\tilde x)$, $\tilde q(\tilde x)=q(d\tilde x)$, and $\tilde\delta(\tilde x)=\delta(d\tilde x)$. Note that $\tilde\delta$ has radius of support ρ/d, since δ has radius of support ρ. We also have

$$\mathcal{L}_{\tilde B}((\mathbf{w},\lambda),(\tilde{\mathbf{v}},\tilde q)) = (\mathbf{w}, D_{\tilde x_i}\tilde\delta\mathbf{k}_j), \quad \forall(\mathbf{w},\lambda)\in V_{\tilde B}\times X_{\tilde B}. \tag{4.6}$$

4.3. **Proof of Lemma 8.** We can now prove regularity estimates for the localized adjoint problem (3.5).

Proof of (3.12). We begin by bounding $\|\mathbf{v}\|_{W_1^1(B)} + \|q\|_{L_1(B)}$. Letting $\tilde{x}_0 \in \tilde{B}$ be the image of $x_0 \in B$, we first compute

$$\begin{aligned}\|\mathbf{v}\|_{W_1^1(B)} + \|q\|_{L_1(B)} &\le Cd^n\big(\|\tilde{\mathbf{v}}\|_{W_1^1(\tilde{B})} + \|\tilde{q}\|_{L_1(\tilde{B})}\big)\\ &\le Cd^n\Big(\|\tilde{\mathbf{v}}\|_{W_1^1(B_{2\rho/d}(\tilde{x}_0))} + \|\tilde{q}\|_{L_1(B_{2\rho/d}(\tilde{x}_0))}\\ &\quad + \|\tilde{\mathbf{v}}\|_{W_1^1(\tilde{B}\setminus B_{2\rho/d}(\tilde{x}_0))} + \|\tilde{q}\|_{L_1(\tilde{B}\setminus B_{2\rho/d}(\tilde{x}_0))}\Big).\end{aligned} \tag{4.7}$$

Using (4.2) with $\tilde{\mathbf{f}} = D_{\tilde{x}_i}\tilde{\delta}\mathbf{k}_j$, $\tilde{g} = 0$, and (3.1), we next find that

$$\begin{aligned}&d^n\big(\|\tilde{\mathbf{v}}\|_{W_1^1(B_{2\rho/d}(\tilde{x}_0))} + \|\tilde{q}\|_{L_1(B_{2\rho/d}(\tilde{x}_0))}\big)\\ &\quad\le Cd^n\Big(\frac{\rho}{d}\Big)^{n/2}\big(\|\tilde{\mathbf{v}}\|_{H^1(B_{2\rho/d}(\tilde{x}_0))} + \|\tilde{q}\|_{L_2(B_{2\rho/d}(\tilde{x}_0))}\big)\\ &\quad\le C(\rho d)^{n/2}\big(\|\tilde{\mathbf{v}}\|_{H_0^1(\tilde{B})} + \|\tilde{q}\|_{L_2(\tilde{B})}\big)\\ &\quad\le C(\rho d)^{n/2}\|\tilde{\delta}\|_{L_2(\tilde{B})} \le C\rho^{n/2}\|\delta\|_{L_2(B)} \le C.\end{aligned} \tag{4.8}$$

Employing (4.4) and (4.5), we have for $\tilde{x} \in \tilde{B} \setminus B_{2\rho/d}(\tilde{x}_0)$, that

$$\begin{aligned}D_{\tilde{x}_k}\tilde{\mathbf{v}}_l(\tilde{x}) &= D_{\tilde{x}_k}\int_{\tilde{B}} G_{lj}(\tilde{x},\tilde{\xi})D_{\tilde{\xi}_i}\tilde{\delta}(\tilde{\xi})\,\mathrm{d}\tilde{\xi} = -\int_{B_{\rho/d}(\tilde{x}_0)}\tilde{\delta}(\tilde{\xi})D^2_{\tilde{x}_k\tilde{\xi}_i}G_{lj}(\tilde{x},\tilde{\xi})\,\mathrm{d}\tilde{\xi}\\ &\le \|\tilde{\delta}\|_{L_1(\tilde{B})}\|D^2_{\tilde{x}_k\tilde{\xi}_i}G_{lj}(\tilde{x},\cdot)\|_{L_\infty(B_{\rho/d}(\tilde{x}_0))}\\ &\le d^{-n}\|\delta\|_{L_1(B)}C\sup_{\tilde{\xi}\in B_{\rho/d}(\tilde{x}_0)}|\tilde{x}-\tilde{\xi}|^{-n} \le Cd^{-n}|\tilde{x}-\tilde{x}_0|^{-n}.\end{aligned} \tag{4.9}$$

Here we used that $1 \le l, j \le n$, $|\alpha| = |\beta| = 1$, so that $\kappa = n$, and that $|\tilde{x}-\tilde{x}_0| \le |\tilde{x}-\tilde{\xi}| + |\tilde{\xi}-\tilde{x}_0| \le |\tilde{x}-\tilde{\xi}| + \rho/d \le 2|\tilde{x}-\tilde{\xi}|$. Similarly,

$$\tilde{q}(\tilde{x}) \le Cd^{-n}|\tilde{x}-\tilde{x}_0|^{-n} \tag{4.10}$$

and

$$\tilde{\mathbf{v}}(\tilde{x}) \le Cd^{-n}|\tilde{x}-\tilde{x}_0|^{1-n}. \tag{4.11}$$

Thus, setting $r = |\tilde{x}-\tilde{x}_0|$ and transforming to polar coordinates leads to

$$\begin{aligned}&d^n\big(\|\tilde{\mathbf{v}}\|_{W_1^1(\tilde{B}\setminus B_{2\rho/d}(\tilde{x}_0))} + \|\tilde{q}\|_{L_1(\tilde{B}\setminus B_{2\rho/d}(\tilde{x}_0))}\big)\\ &\quad\le Cd^nd^{-n}\int_{2\rho/d}^{\mathrm{diam}(\tilde{B})} r^{n-1}r^{-n}\,\mathrm{d}r \le C\ln\frac{d}{\rho}.\end{aligned} \tag{4.12}$$

Collecting (4.12) and (4.8) into (4.7) completes the proof of (3.12).

Proof of (3.13). Next we bound $\|\Delta\mathbf{v}\|_{L_1(B\setminus B_{d/2})} = \|\Delta\mathbf{v}\|_{L_1(B\setminus B_{c_1d/2}(x_0))}$, recalling that $B_{d/2} = B_{c_1d/2}(x_0)\cap B$, see (2.3). We set $c_1 = 1$ for simplicity. Let $\tilde{\omega}$ be a cut-off function with $\tilde{\omega} = 1$ on $\tilde{B}\setminus B_{1/2}(\tilde{x}_0)$, $\tilde{\omega} = 0$ on $B_{1/4}(\tilde{x}_0)$, and with $|\tilde{\omega}|_{W^j_\infty(\tilde{B})} \le C$, $j = 0, 1, 2$. Noting that $-\Delta\tilde{\mathbf{v}} + \nabla\tilde{q} = 0$ in $\tilde{B}\setminus B_{1/4}(\tilde{x}_0)$, we compute that

$$\begin{aligned}-\Delta(\tilde{\omega}\tilde{\mathbf{v}}) + \nabla(\tilde{\omega}\tilde{q}) &= \tilde{\omega}(-\Delta\tilde{\mathbf{v}} + \nabla\tilde{q}) + (-\mathbf{v}\Delta\tilde{\omega} - 2\nabla\tilde{\omega}\cdot\nabla\tilde{\mathbf{v}} + \tilde{q}\nabla\tilde{\omega})\\ &= -\mathbf{v}\Delta\tilde{\omega} - 2\nabla\tilde{\omega}\cdot\nabla\tilde{\mathbf{v}} + \tilde{q}\nabla\tilde{\omega},\\ \nabla\cdot(\tilde{\omega}\tilde{\mathbf{v}}) &= \tilde{\omega}\nabla\cdot\tilde{\mathbf{v}} + \tilde{\mathbf{v}}\cdot\nabla\tilde{\omega} = \tilde{\mathbf{v}}\cdot\nabla\tilde{\omega}.\end{aligned}$$

Thus, using (4.3) along with the Poincaré inequality $\|\tilde{\mathbf{v}}\|_{L_2(\tilde{B})} \le \|\nabla\tilde{\mathbf{v}}\|_{L_2(\tilde{B})}$, we have

$$\begin{aligned}\|\Delta\mathbf{v}\|_{L_1(B\setminus B_{d/2}(x_0))} &\le Cd^{n-1}\|\Delta\tilde{\mathbf{v}}\|_{L_2(\tilde{B}\setminus B_{1/2}(\tilde{x}_0))} \le Cd^{n-1}\|\Delta(\tilde{\omega}\tilde{\mathbf{v}})\|_{L_2(\tilde{B})}\\ &\le Cd^{n-1}\big(\|\mathbf{v}\Delta\tilde{\omega} + 2\nabla\tilde{\omega}\cdot\nabla\tilde{\mathbf{v}} - \tilde{q}\nabla\tilde{\omega}\|_{L_2(\tilde{B})} + \|\tilde{\mathbf{v}}\cdot\nabla\tilde{\omega}\|_{H^1(\tilde{B})}\big)\\ &\le Cd^{n-1}\big(\|\nabla\tilde{\mathbf{v}}\|_{L_\infty(\tilde{B}\setminus B_{1/4}(\tilde{x}_0))} + \|\tilde{q}\|_{L_\infty(\tilde{B}\setminus B_{1/4}(\tilde{x}_0))}\big).\end{aligned}$$

Employing (4.9) and (4.10) with $|\tilde{x}-\tilde{x}_0| \ge \frac{1}{4}$ then finally yields

$$\|\Delta\mathbf{v}\|_{L_1(B\setminus B_{d/2}(x_0))} \le Cd^{-1}.$$

Proof of (3.14). Since B is one of the unit reference domains $\tilde{B}_i$ scaled by a multiple of d, we have

$$\|\partial\mathbf{v}/\partial n\|_{L_1(\partial B\setminus\partial\Omega)} \le Cd^{n-1}\|\partial\mathbf{v}/\partial n\|_{L_\infty(\partial B\setminus\partial\Omega)}. \tag{4.13}$$

Let $x \in \partial B\setminus\partial\Omega$ and let $\tilde{x}, \tilde{x}_0 \in \tilde{B}$ be the images of x and x_0. Then $|\tilde{x}-\tilde{x}_0| \ge C$ for some constant C. Also $\nabla\tilde{\mathbf{v}}(\tilde{x}) = \nabla\mathbf{v}(x)$, so that (4.9) immediately yields

$$\Big|\frac{\partial\mathbf{v}}{\partial n}(x)\Big| \le Cd^{-n}. \tag{4.14}$$

Combining (4.13) and (4.14) completes the proof of (3.14).
Proof of (3.15). We first compute, again with $c_1 = 1$ for simplicity,

$$\|q\|_{L_1(B\setminus B_{d/2})} \le Cd^n\|\tilde{q}\|_{L_\infty(\tilde{B}\setminus\tilde{B}_{1/2}(\tilde{x}_0))}. \tag{4.15}$$

Using (4.4) and (4.5) with $\delta_{l,n+1}+|\alpha| = \delta_{j,n+1}+|\beta| = 1$, $\kappa = n$, we have, for $\tilde{x}\in\tilde{B}\setminus\tilde{B}_{1/2}(\tilde{x}_0)$ and with j as in (4.6), that

$$\begin{aligned}\tilde{q}(\tilde{x}) &= \int_{\mathrm{supp}(\tilde{\delta})} G_{n+1,j}(\tilde{x},\tilde{\xi})D_{\tilde{\xi}_i}\tilde{\delta}(\tilde{\xi})\,\mathrm{d}\tilde{\xi} = -\int_{\mathrm{supp}(\tilde{\delta})}\tilde{\delta}(\tilde{\xi})D_{\tilde{\xi}_i}G_{n+1,j}(\tilde{x},\tilde{\xi})\,\mathrm{d}\tilde{\xi}\\ &\le \|\tilde{\delta}\|_{L_1(\tilde{B})}\|D_{\tilde{\xi}_i}G_{n+1,j}(\tilde{x},\cdot)\|_{L_\infty(\mathrm{supp}(\tilde{\delta}))} \le Cd^{-n}.\end{aligned} \tag{4.16}$$

Here we have used the fact that $\mathrm{dist}(\mathrm{supp}(\tilde{\delta}), \tilde{B}\setminus B_{1/2}(\tilde{x}_0)) \ge 1/4$. Combining (4.15) and (4.16) completes the proof of (3.15).
Proof of (3.16). Let $d_0 = \frac{3}{2}\frac{\rho}{d}$ and $d_j = 2^j\frac{\rho}{d}$, $j \ge 1$. Let also $\Omega_j = \{\tilde{x}\in\tilde{B} : d_j < |\tilde{x}-\tilde{x}_0| \le d_{j+1}\}$, $j\ge 0$ and $\Omega_j' = \Omega_{j-1}\cup\Omega_j\cup\Omega_{j+1}$; note that $\tilde{B}\setminus B_{2\rho/d}(\tilde{x}_0) = \cup_{j=1}^J\Omega_j$ with $J \approx \ln\frac{d}{\rho}$. Finally, we let ω_j be a smooth cutoff function which is 1 on Ω_j, 0 outside of $\Omega_{j-1}\cup\Omega_j\cup\Omega_{j+1}$, and which satisfies $\|D^\ell\omega_j\|_{L_\infty(\tilde{B})} \le Cd_j^{-\ell}$, $\ell = 0,1,2$. Transforming to $\tilde{B}$, we then have

$$\begin{aligned}&\int_{B\setminus B_{2\rho}(x_0)} |x-x_0|\big(|\nabla q(x)| + |D^2\mathbf{v}(x)|\big)\,\mathrm{d}x\\ &\quad\le Cd^n\int_{\tilde{B}\setminus B_{2\rho/d}(\tilde{x}_0)} |\tilde{x}-\tilde{x}_0|\big(|\nabla\tilde{q}(\tilde{x})| + |D^2\tilde{\mathbf{v}}(\tilde{x})|\big)\,\mathrm{d}\tilde{x}\\ &\quad\le Cd^n\sum_{j=1}^J d_j^{n/2+1}\left(\|\nabla\tilde{q}\|_{L_2(\Omega_j)} + \|D^2\tilde{\mathbf{v}}\|_{L_2(\Omega_j)}\right)\\ &\quad\le Cd^n\sum_{j=1}^J d_j^{n/2+1}\left(\|\nabla(\omega_j\tilde{q})\|_{L_2(\tilde{B})} + \|D^2(\omega_j\tilde{\mathbf{v}})\|_{L_2(\tilde{B})}\right).\end{aligned}$$

We apply (4.3) with $\tilde{\mathbf{v}}$ replaced by $\omega_j \tilde{\mathbf{v}}$ and $\tilde{q}$ by $\omega_j \tilde{q}$. Then

$$\tilde{\mathbf{f}} = -\Delta(\omega_j \tilde{\mathbf{v}}) - \nabla(\omega_j \tilde{q}) = -\tilde{\mathbf{v}}\Delta\omega_j - 2\nabla\tilde{\mathbf{v}}\nabla\omega_j - \tilde{q}\nabla\omega_j,$$

since $\omega_j(-\Delta\tilde{\mathbf{v}} - \nabla\tilde{q}) = 0$. Also,

$$\tilde{g} = -\nabla\cdot(\omega_j\tilde{\mathbf{v}}) = -\tilde{\mathbf{v}}\cdot\nabla\omega_j,$$

where integration by parts yields $\int_{\tilde{B}} \tilde{g}\,\mathrm{d}\tilde{x} = \int_{\tilde{B}} \omega_j \nabla\cdot\tilde{\mathbf{v}}\,\mathrm{d}\tilde{x} = 0$, since $\tilde{\mathbf{v}}$ is divergence-free. Employing (4.3), bounds for the derivatives of ω_j, and Hölder's inequality then yields

$$\begin{aligned} Cd^n &\sum_{j=1}^{J} d_j^{n/2+1}\left(\|\nabla(\omega_j\tilde{q})\|_{L_2(\tilde{B})} + \|D^2(\omega_j\tilde{\mathbf{v}})\|_{L_2(\tilde{B})}\right) \\ &\le Cd^n \sum_{j=1}^{J} d_j^{n/2+1}\Big(\|\tilde{\mathbf{v}}\Delta\omega_j\|_{L_2(\tilde{B})} + 2\|\nabla\tilde{\mathbf{v}}\nabla\omega_j\|_{L_2(\tilde{B})} \\ &\quad + \|\tilde{q}\nabla\omega_j\|_{L_2(\tilde{B})} + \|\tilde{\mathbf{v}}\cdot\nabla\omega_j\|_{H^1(\tilde{B})}\Big) \\ &\le Cd^n \sum_{j=1}^{J} d_j^n \left(d_j^{-1}\|\tilde{\mathbf{v}}\|_{L_\infty(\Omega_j')} + \|\nabla\tilde{\mathbf{v}}\|_{L_\infty(\Omega_j')} + \|\tilde{q}\|_{L_\infty(\Omega_j')}\right). \end{aligned}$$

Inequalities (4.9), (4.10), and (4.11) then yield

$$\begin{aligned} Cd^n &\sum_{j=1}^{J} d_j^n \left(d_j^{-1}\|\tilde{\mathbf{v}}\|_{L_\infty(\Omega_j')} + \|\nabla\tilde{\mathbf{v}}\|_{L_\infty(\Omega_j')} + \|\tilde{q}\|_{L_\infty(\Omega_j')}\right) \\ &\le Cd^n \sum_{j=1}^{J} d^{-n}(d_j^{n-1}d_j^{1-n} + d_j^n d_j^{-n}) \le CJ \le C\ln\frac{d}{\rho}. \end{aligned}$$

Collecting the previous statements completes the proof of (3.16).

Proof of (3.17). Let $\mathcal{T}_{\partial B} = \{T \in \mathcal{T}_h : T \cap (\partial B \setminus \partial\Omega) \neq \emptyset, T \notin P'_{T_{x_0}}\}$. We assert that there exists $c > 0$ such that for any $T \in \mathcal{T}_{\partial B}$ and for any $x \in P_T$, $\mathrm{dist}(x, \mathrm{supp}(\delta)) \ge cd$. Assume first that $h_T \le \tilde{c}d$ for a sufficiently small constant $\tilde{c}$. Assume also that $c_0 \le \frac{1}{8}$ in the hypotheses of Theorem 1, so that $\rho \le \frac{d}{8}$ and $\mathrm{dist}(x, \mathrm{supp}(\delta)) \ge |x - x_0| - \frac{d}{4}$. Since $P_T \cap \partial B \neq \emptyset$, $|x - x_0| \ge \mathrm{dist}(P_T, x_0) \ge \mathrm{dist}(\partial B, x_0) - Ch_T \ge d - C\tilde{c}d \ge Kd$, where C is chosen so that $\mathrm{diam}(P_T) \le Ch_T$. This completes the proof in the case that h_T is sufficiently small relative to d. In the case $h_T \sim d$, note that there must in any case be at least one ring of elements $P'_T \setminus P_T$ between P_T and T_{x_0}, since $T \notin P'_{T_{x_0}}$. By shape regularity, elements in this ring must have diameter proportional to h_T, and more generally this ring must have thickness γh_T for some γ depending on the shape regularity of the mesh. Thus, here also $\mathrm{dist}(P_T, \mathrm{supp}(\delta)) \ge cd$ for any $T \in \mathcal{T}_{\partial B}$.

Computing as in (4.8)–(4.9), we then have for $x \in P_T$, $T \in \mathcal{T}_{\partial B}$ that

$$|\nabla\mathbf{v}(x)| = |\nabla\tilde{\mathbf{v}}(\tilde{x})| \le C|x - x_0|^{-n} \le Cd^{-n}.$$

Applying Hölder's inequality, we thus have for $T \in \mathcal{T}_{\partial B}$,

$$h_T^{-1}\|(\cdot - x_0)\nabla\mathbf{v}\|_{L_1(P_T)} \le Ch_T^{n-1}\||\cdot - x_0|\nabla\mathbf{v}\|_{L_\infty(P_T)} \le Ch_T^{n-1}d^{1-n}.$$

Note also that if $T \cap \partial B \neq \emptyset$, then $\mathrm{vol}_{n-1}(P_T \cap \partial B) \sim h_T^{n-1}$. This is because for any $x \in T$, $B_{ch_T}(x) \cap \Omega \subset P_T$ for some fixed $c > 0$ depending only on the shape regularity of $\mathcal{T}_h$. For any $x \in T \cap \partial B$, we thus have $\text{vol}_{n-1}(B_{ch_T} \cap \partial B) \geq Ch_T^{n-1}$. Thus,

$$\sum_{T \in \mathcal{T}_{\partial B}} h_T^{-1} \|(\cdot - x_0)\nabla \mathbf{v}\|_{L_1(P_T)} \leq C \sum_{T \in \mathcal{T}_{\partial B}} h_T^{n-1} d^{1-n}$$
$$\leq C \sum_{T \in \mathcal{T}_{\partial B}} \text{vol}(P_T \cap \partial B) d^{1-n}.$$

Finally, we note that there is finite overlap of the sets $P_T \cap \partial B$, so that

$$C \sum_{T \in \mathcal{T}_{\partial B}} \text{vol}(P_T \cap \partial B) d^{1-n} \leq C d^{1-n} \,\text{vol}_{n-1} \partial B \leq C.$$

This completes the proof of (3.17).

The proof of Lemma 8, and hence also the proof of Theorem 1, is complete.

5. Extensions

5.1. **Removing the regularization penalty.** In [Dem07] a condition is given under which the regularization penalty $C\rho^\beta |\mathbf{u}|_{C^{1,\beta}(\overline{D_\rho})}$ in (1.6) can be removed. Precisely as in Corollary 1.2 of that work, we can prove the following.

Corollary 11. *Let Ω, D, d, and D_d be as in Theorem 1. Assume also that there exist a point $x_1 \in D$ and a radius $\xi > 0$ such that $|D^\gamma \mathbf{u}_i(x_1)| \geq C^* > 0$ for some $1 \leq i \leq n$ and multi-index γ with $|\gamma| = k+1$, and such that $\|\mathbf{u}_i\|_{W_\infty^{k+2}(B_\xi(x_1))} \leq C^{**}$. Finally, assume that $\mathbf{u} \in C^{1,\beta}(\overline{D_{\tilde{\xi}}})$ for some $0 < \beta < 1$ and $\tilde{\xi} > 0$. Then*

$$\|\nabla(\mathbf{u} - \mathbf{u}_h)\|_{L_\infty(D)} \leq C\ell_{h,d} \max_{T \in \mathcal{T}_{D_d}} \frac{h_T}{h_T + dist(T, D)} \eta_{1,\infty}(T)$$
$$+ C\Big(\ln \frac{1}{\underline{h}}\Big)^{\alpha_n} \frac{1}{d} \max_{T \in \mathcal{T}_h} h_T \eta_{1,\infty}(T).$$

Here

$$\ell_{h,d} = \Big| \ln \min \Big\{ \frac{1}{d} \Big(\frac{C^*}{C^{**} + C(d)\|\mathbf{u}\|_{C^{1,\beta}(\overline{D_{\tilde{\xi}}})}} \Big)^{\frac{k+1}{\beta}}, \frac{\underline{h}^{\frac{k+1}{\beta}}}{d}, \frac{\xi^{\frac{k+1}{\beta}}}{d}, \frac{\tilde{\xi}^{\frac{k+1}{\beta}}}{d} \Big\} \Big|. \tag{5.1}$$

The conditions of Corollary 11 guarantee a lower bound for $\nabla(\mathbf{u} - \mathbf{u}_h)$ near the point x_1. By choosing ρ properly in Theorem 1, we are then able to bound the regularization penalty by an appropriate factor of $\|\nabla(\mathbf{u} - \mathbf{u}_h)\|_{L_\infty(D)}$, which in turn is multiplied by a small constant and thus can be reabsorbed. The arguments are entirely based on approximation theory and thus apply here precisely as in Section 3.2 of [Dem07], which we refer to for further details.

We also note that the logarithmic factor $\ell_{h,d}$ defined in (5.1) is dependent upon a priori quantities in the pre-asymptotic range, but it becomes a standard logarithmic factor of the form $\ln(d/\underline{h})$ as $\underline{h} \to 0$. Note also that when bounding maximum errors in function values instead of in gradients, it is possible to remove the regularization penalty without introducing any a priori information into the upper bound; cf. [NSSV06], [SL06], [DG10]. There is thus a substantial technical difference between L_∞ and W_∞^1 in the context of a posteriori error estimation.

5.2. **Estimates for the pressure.** Because $\|e_p\|_{L_p(\Omega)}$ essentially scales like the quantity $\|\mathbf{e_u}\|_{W_p^1(\Omega)}$, it is reasonable to expect that $\|p-p_h\|_{L_\infty(D)}$ will be bounded by the right hand side of (1.6) with appropriate adjustments to the regularization penalty. However, the fact that $\int_D e_p \,\mathrm{d}x \neq 0$ provides additional technical challenges. We provide a brief sketch of a proof of such an estimate.

We let $x_0 \in D$ be such that $\|e_p\|_{L_\infty(D)} = |e_p(x_0)|$ and otherwise retain the definitions of B, D, d, ω, ρ, and δ. Let also $\overline{\delta}_B = \frac{1}{|B|}\int_B \delta \,\mathrm{d}x$. Lemma 5 and elementary manipulations yield

$$|e_p(x_0)| \leq |(\omega e_p, \delta - \overline{\delta}_B) + (e_p, \omega\overline{\delta}_B)| + C\rho^\beta |p|_{C^{0,\beta}(\overline{D_\rho})}. \tag{5.2}$$

Next let $(\mathbf{v}, q) \in V_B \times X_B$ solve

$$\mathcal{L}_B((\mathbf{w},\lambda),(\mathbf{v},q)) = (\lambda, \delta - \overline{\delta}_B) \quad \forall (\mathbf{w},\lambda) \in V_B \times X_B.$$

Elementary calculations as in Section 3.3 yield

$$\begin{aligned}(\omega e_p, \delta - \overline{\delta}_B) &= \mathcal{L}_B((\omega \mathbf{e_u}, \omega e_p),(\mathbf{v},q)) \\ &\leq \mathcal{L}((\mathbf{e_u}, e_p),(\omega\mathbf{v},\omega q)) + |(\mathbf{v}\cdot\nabla\omega, e_p)| \\ &\quad + \frac{1}{d}\|\mathbf{e_u}\|_{L_\infty(\Omega)}\big(\|\nabla\mathbf{v}\|_{L_1(B\setminus B_{d/2})} + \|q\|_{L_1(B\setminus B_{d/2})} + d^{-1}\|v\|_{L_1(B\setminus B_{d/2})}\big).\end{aligned} \tag{5.3}$$

The term $\mathcal{L}((\mathbf{e_u}, e_p),(\omega\mathbf{v},\omega q))$ in (5.3) may be handled much as in Section 3.3, with some technical differences arising because $\mathbf{v}$ is multiplied by the cut-off function ω in the present case, but not in Section 3.3. Similarly, the term in (5.3) involving $\frac{1}{d}\|\mathbf{e_u}\|_{L_\infty(\Omega)}$ can be bounded in much the same way as before.

The major difference arises in the terms $(\mathbf{v}\cdot\nabla\omega, e_p)$ in (5.3) and $(e_p, \omega\overline{\delta}_B)$ in (5.2). Note first that these terms can both be bounded by $\frac{1}{d}\|e_p\|_{W_\infty^{-1}(\Omega)}$, where $\|z\|_{W_\infty^{-1}(\Omega)} = \sup_{\|\psi\|_{W_1^1(\Omega)}=1}(z,\psi)$. Thus, we may bound the pollution error here by $\frac{1}{d}(\|\mathbf{e_u}\|_{L_\infty(\Omega)} + \|e_p\|_{W_\infty^{-1}(\Omega)})$. This bound for the pollution error is consistent with previously published local a priori energy error bounds, which contain a term of the form $\|e_p\|_{H^{-1}(\Omega)}$ (cf. [HXZL08]). Our technical development above completely avoids global negative norm terms involving e_p, which are relatively difficult to bound. A similar observation was also recently made in [GL10], where local a priori energy estimates for Stokes are proved with no factors of e_p appearing in the upper bound.

Employing negative norms is not necessarily the best way to bound the e_p pollution terms $(\mathbf{v}\cdot\nabla\omega, e_p)$ and $(e_p, \omega\overline{\delta}_B)$, however. One may instead attack these terms directly via a duality argument. For example, letting $(\mathbf{z},\psi) \in V \times X$ be the solution of $\mathcal{L}((\mathbf{w},\lambda),(\mathbf{z},\psi)) = (\lambda, \omega\overline{\delta}_B - \frac{1}{|\Omega|}\int_\Omega \omega\overline{\delta}_B \,\mathrm{d}x)$ $\forall$ $(\mathbf{w},\lambda) \in V \times X$, we have

$$\begin{aligned}(e_p, \omega\overline{\delta}_B) &= \mathcal{L}((\mathbf{e_u}, e_p),(\mathbf{z},\psi)) \\ &\leq C \max_{T\in\mathcal{T}_h}\big(h_T\eta_{1,\infty}(T)\big)\big(\|\mathbf{z}\|_{W_1^2(\Omega)} + \|\psi\|_{W_1^1(\Omega)}\big).\end{aligned} \tag{5.4}$$

Bounds for $\|\mathbf{z}\|_{W_1^2(\Omega)} + \|\psi\|_{W_1^1(\Omega)}$ may be obtained as in Section 3.2 of [SL06], that is, by proving bounds for $\|\mathbf{z}\|_{W_q^2(\Omega)} + \|\psi\|_{W_q^1(\Omega)}$ as $q \downarrow 1$. Here the right-hand-side data $\omega\overline{\delta}_B$ of the adjoint problem functions essentially as a scaled unit mass with radius of support d instead of ρ, so the logarithmic factors arising in (5.4) will depend on d instead of on ρ. We do not pursue the details or give a precise statement of results.

6. Computational example

6.1. **Algorithm.** In our tests we employ the standard adaptive finite element algorithm given by

$$\mathsf{solve} \to \mathsf{estimate} \to \mathsf{mark} \to \mathsf{refine}.$$

Given D, d, and D_d as above, we let

$$\eta(T) = \begin{cases} \dfrac{h_T}{h_T + \mathrm{dist}(T, D)} \eta_{1,\infty}(T), & T \cap D_d \neq \emptyset, \\ \dfrac{h_T}{d} \eta_{1,\infty}(T), & T \cap D_d = \emptyset. \end{cases} \tag{6.1}$$

We employ a maximum strategy in the "mark" step of the algorithm. More precisely, we mark an element $T \in \mathcal{T}_h$ for refinement if

$$\eta(T) \geq 0.5 \max_{T' \in \mathcal{T}_h} \eta(T').$$

Note that greater efficiency can at times be obtained by calibrating constants more carefully in (6.1), i.e., by weighting residual contributions $\frac{h_T}{d}\eta_{1,\infty}(T)$ from elements $T \in \mathcal{T}_h \setminus \mathcal{T}_{D_d}$ by a different constant than residual contributions from elements in $\mathcal{T}_{D_d}$. Fine-tuning the algorithm in this fashion does not affect rates of convergence and is explored more thoroughly in [Dem07], so we do not consider it further here.

We use the polynomial degree $k = 2$ in our tests. The computations were carried out using the finite element toolbox ALBERTA (cf. [SS05]).

6.2. **Test function and subdomain.** In our tests we let $\Omega = (-1, 1) \times (-1, 1) \setminus [0, 1) \times \{0\}$, that is, the unit square with a "crack" consisting of the right half of the x-axis removed. We let $D = B_{1/4}(-1, -1)$, $D_d = B_{\sqrt{2}}(-1, -1)$, and $d = \sqrt{2} - \frac{1}{4}$.

Let

$$w(x) = \begin{cases} 1 + 384(x - 1)(x - 0.5)^5 - 64(x - 0.5)^6, & x \geq 0.5, \\ 1, & 0 \leq x < 0.5, \\ w(-x), & x < 0. \end{cases}$$

Note that $w \in W^3_\infty(\mathbb{R})$. Letting (r, ϕ) be polar coordinates, we also define

$$\gamma(r, \theta) = r^{1.5}\big(3\sin(\theta/2) - \sin(3\theta/2)\big).$$

Finally, we let

$$\begin{aligned} u_1(x, y) &= \frac{\partial}{\partial y}\Big(w(x)w(y)\gamma(r(x, y), \theta(x, y))\Big), \\ u_2(x, y) &= -\frac{\partial}{\partial x}\Big(w(x)w(y)\gamma(r(x, y), \theta(x, y))\Big), \end{aligned}$$

and

$$\mathbf{u}(x, y) = (u_1(x, y), u_2(x, y)).$$

Then $\mathbf{u}$ satisfies homogenous Dirichlet boundary conditions on $\partial\Omega$ and also satisfies $\nabla \cdot \mathbf{u} = 0$ in Ω. Finally, we let

$$p(r, \theta) = -6r^{-0.5}\cos(\theta/2).$$

Then $-\Delta\mathbf{u} + \nabla p = 0$ for $r < 0.5$. For $r > 0.5$, we set $\mathbf{f} = -\Delta\mathbf{u} + \nabla p$.

6.3. **Results.** In Figure 1 we display a logarithmic error plot showing optimal-order decrease of the target error quantity $\|\nabla(\mathbf{u}-\mathbf{u}_h)\|_{L_\infty(D)}$. Note that the pollution error $\|\mathbf{u}-\mathbf{u}_h\|_{L_\infty(\Omega)}$ only decreases at the same rate as $\|\nabla(\mathbf{u}-\mathbf{u}_h)\|_{L_\infty(D)}$, which is suboptimal for the L_∞ norm but sufficient to maintain optimality for the target error quantity. In Figure 2 we display a computational mesh having 28974 degrees of freedom. Note that the heaviest refinement occurs in the lower left hand corner in the target subdomain D and also near the singularity at the origin (crack tip).

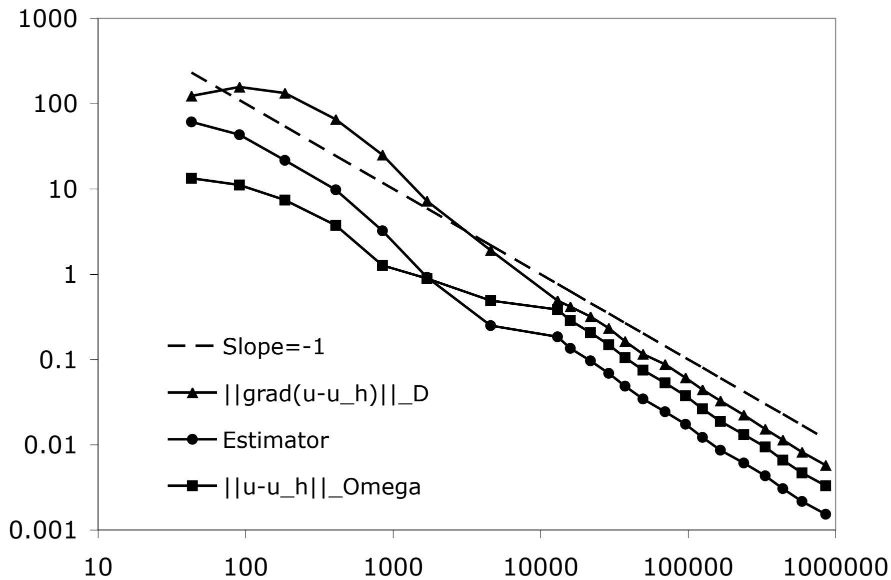

FIGURE 1. Target error $\|\nabla(\mathbf{u}-\mathbf{u}_h)\|_{L_\infty(D)}$, error estimator $\max_{T\in\mathcal{T}_h}\eta(T)$, and pollution error $\|\mathbf{u}-\mathbf{u}_h\|_{L_\infty(\Omega)}$.

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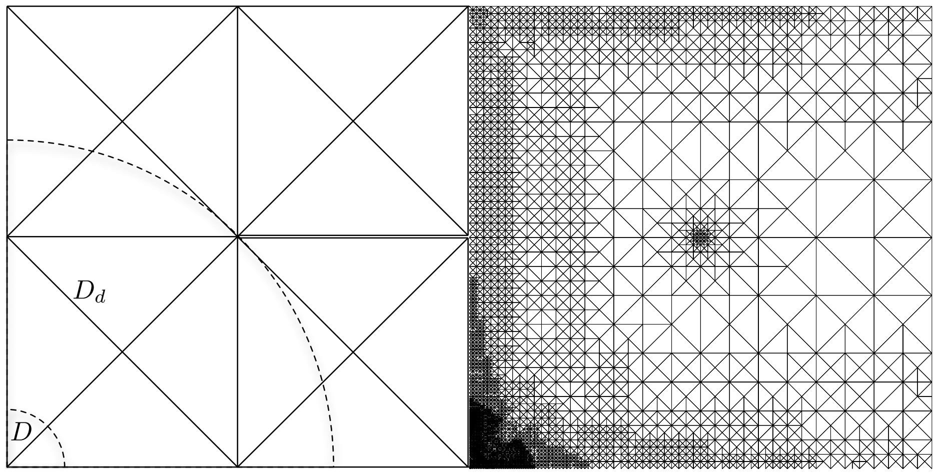

FIGURE 2. Initial mesh with $D = B_{1/4}(-1,1)$ and $D_d = B_{\sqrt{2}}(-1,1)$ outlined (left). Computational mesh with 28974 degrees of freedom (right).

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