Adaptive Finite Element Method for a Coefficient Inverse Problem for the Maxwell’s System

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Abstract. We consider a coefficient inverse problems for the Maxwell’s system in 3-D. The coefficient of interest is the dielectric permittivity function. Only backscattering single measurement data are used. The problem is formulated as an optimization problem. The key idea is to use the adaptive finite element method for the solution. Both analytical and numerical results are presented. Similar ideas for inverse problems for the complete time dependent Maxwell’s system were not considered in the past.

1. Introduction

In this work we consider an adaptive hybrid finite element/difference method for an electromagnetic coefficient inverse problem (CIP) in the form of a parameter identification problem. Our goal is reconstruct dielectric permittivity \( \epsilon \) of the media under condition that magnetic permeability \( \mu = 1 \). We consider the case of a single measurement and use the backscattering data only to reconstruct this coefficient \( \epsilon \). Potential applications of our algorithm are in airport security, imaging of land mines, imaging of defects in non-destructive testing, etc.. This is because the dielectric constants of explosives are much higher than ones of regular materials, see tables in http://www.clippercontrols.com/info/dielectric_constants.html.

To solve our inverse problem numerically, we seek to minimize the Tikhonov functional:

\[
F(E, \epsilon) = \frac{1}{2} \| E - \tilde{E} \|^2 + \frac{1}{2} \gamma \| \epsilon - \epsilon_0 \|^2.
\]

Here \( E \) is the vector of the electric field satisfying Maxwell’s equations and \( \tilde{E} \) is observed data at a finite set of observation points at the backscattering side of the boundary, \( \epsilon_0 \) is the initial guess for \( \epsilon \), \( \gamma \) is regularization parameter (Tikhonov regularization), and \( \| \cdot \| \) is the discrete \( L_2 \) norm. The data \( \tilde{E} \) in our computations are generated in experiments, where short electromagnetic impulses are emitted on the part of the boundary of the surrounding media. The goal is to recover the unknown spatially distributed function \( \epsilon \) from the recorded boundary data \( \tilde{E} \).

The minimization problem is reformulated as the problem of finding a stationary point of a Lagrangian involving a forward equation (the state equation), a backward equation (the adjoint equation) and an equation expressing that the gradient with respect to the coefficient \( \epsilon \) vanishes. To approximately obtain the value of \( \epsilon \), we arrange an iterative process via solving in each step the forward and backward equations and updating the coefficient \( \epsilon \). In our numerical example the regularization parameter \( \gamma \) [13, 32, 33] is chosen experimentally on the basis of the best performance. An analytical study of the question of the choice of the

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regularization parameter is outside of the scope of this publication. We refer to [17] for a
detailed analysis of this interesting topic for the adaptivity technique.

The aim of this work is to derive a posteriori error estimate for our CIP and present
a numerical example of an accurate reconstruction using adaptive error control. Following
Johnson et al. [5, 4, 16, 14, 22], and related works, we shall derive a posteriori error estimate
for the Lagrangian involving the residuals of the state equation, adjoint state equation and
the gradient with respect to $\epsilon$. In this work we use the called all-at-once approach to find
Frechét derivative for the Tikhonov functional. Rigorous derivation of the Frechét derivatives
for state and adjoint problems as well as of the Frechét derivative of the Tikhonov functional
with respect to the coefficient can be performed similarly with [7, 8] and will be done in a
forthcoming publication.

Given a finite element mesh, a posteriori error analysis shows subdomains where the biggest
error of the computed solution is. Thus, one needs to refine mesh in those subdomains. It is
important that a posteriori error analysis does not need a priori knowledge of the solution.
Instead it uses only an upper bound of the solution. In the case of classic forward problems,
upper bounds are obtained from a priori estimates of solutions [1]. In the case of CIPs,
upper bounds are assumed to be known in advance, which goes along well with the Tikhonov
concept for ill-posed problems [13, 33].

A posteriori error analysis addresses the main question of the adaptivity: Where to refine
the mesh? In the case of classic forward problems this analysis provides upper estimates
for differences between computed and exact solutions locally, in subdomains of the original
domain, see, e.g. [1, 14, 15, 16, 31]. In the case of a forward problem, the main factor
enabling to conduct a posteriori error analysis is the well-posedness of this problem. However,
every CIP is non-linear and ill-posed. Because of that, an estimate of the difference between
computed and exact coefficients is replaced by a posteriori estimate of the accuracy of either
the Lagrangian [6, 3, 17] or of the Tikhonov functional [7]. Nevertheless, it was shown in
the recent publications [4, 8] that an estimate of the accuracy of the reconstruction of the
unknown coefficient is possible in CIPs (in particular, see subsection 2.3 and Theorems 7.3
and 7.4 of [8]).

An outline of the work is following: in Section 2.1 we recall Maxwell’s equations and in
Section 2.2 we present the constrained formulation of Maxwell’s equations. In Section 3
we formulate our CIP and in Section 4 we introduce the finite element discretization. In
Section 5 we present a fully discrete version used in the computations. Next, in Section 6
we establish a posteriori error estimate and formulate the adaptive algorithm. Finally, in
Section 7 we present computational results demonstrating the effectiveness of the adaptive
finite element/difference method on an inverse scattering problem in three dimensions.

2. Statements of forward and inverse problems

2.1. Maxwell’s equations. The electromagnetic equations in an inhomogeneous isotropic
case in the bounded domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ with boundary $\partial \Omega$ are described by the first
order system of partial differential equations
\[
\begin{align*}
\frac{\partial D}{\partial t} - \nabla \times H &= -J, \quad \text{in } \Omega \times (0,T), \\
\frac{\partial B}{\partial t} + \nabla \times E &= 0, \quad \text{in } \Omega \times (0,T),
\end{align*}
\]

(2.1) \quad D = \varepsilon E, \\
B = \mu H, \\
E(x,0) = E_0(x), \\
H(x,0) = H_0(x),

where \(E(x,t), H(x,t), D(x,t), B(x,t)\) are the electric and magnetic fields and the electric and magnetic inductions, respectively, while \(\varepsilon(x) > 0\) and \(\mu(x) > 0\) are the dielectric permittivity and magnetic permeability that depend on \(x \in \Omega\), \(t\) is the time variable, \(T\) is some final time, and \(J(x,t) \in \mathbb{R}^d\) is a (given) current density.

The electric and magnetic inductions satisfy the relations

(2.2) \quad \nabla \cdot D = \rho, \quad \nabla \cdot B = 0 \quad \text{in } \Omega \times (0,T),

where \(\rho(x,t)\) is a given charge density.

Eliminating \(B\) and \(D\) from (2.1), we obtain two independent second order systems of partial differential equations

(2.3) \quad \varepsilon \frac{\partial^2 E}{\partial t^2} + \nabla \times (\mu^{-1} \nabla \times E) = -j, \\
(2.4) \quad \mu \frac{\partial^2 H}{\partial t^2} + \nabla \times (\varepsilon^{-1} \nabla \times H) = \nabla \times (\varepsilon^{-1} J),

where \(j = \frac{\partial J}{\partial t}\). System (2.3)-(2.4) should be completed with appropriate initial and boundary conditions.

2.2. Constrained formulation of Maxwell’s equations. To discretize Maxwell’s equations are available different formulation. Examples are the edge elements of Nédélec [27], the node-based first-order formulation of Lee and Madsen [24], the node-based curl-curl formulation with divergence condition of Paulsen and Lynch [29], the node-based interior-penalty discontinuous Galerkin FEM [18]. Edge elements are probably the most satisfactory from a theoretical point of view [25]; in particular, they correctly represent singular behavior at reentrant corners. However, they are less attractive for time dependent computations, because the solution of a linear system is required at every time iteration. Indeed, in the case of triangular or tetrahedral edge elements, the entries of the diagonal matrix resulting from mass-lumping are not necessarily strictly positive [12]; therefore, explicit time stepping cannot be used in general. In contrast, nodal elements naturally lead to a fully explicit scheme when mass-lumping is applied [12, 23].

In this work we consider Maxwell’s equations in convex geometry without reentrant corners and with smooth coefficient \(\varepsilon\) where value of \(\varepsilon\) does not vary much. Since we consider applications of our method in airport security and imaging of land mines such assumptions are natural. Thus, we are able use the node-based curl-curl formulation with divergence condition of Paulsen and Lynch [29]. Direct application of standard piecewise continuous \([H^1(\Omega)]^3\)-conforming FE for the numerical solution of Maxwell’s equations can result in spurious solutions.
Following [29] we supplement divergence equations for electric and magnetic fields to enforce the divergence condition and reformulate Maxwell equations as a constrained system:

\[
\frac{\epsilon}{c^2} \frac{\partial^2 E}{\partial t^2} + \nabla \times (\nabla \times E) - s \nabla (\nabla \cdot E) = -j, \tag{2.5}
\]

and

\[
\frac{\partial^2 H}{\partial t^2} + \nabla \times (\epsilon^{-1} \nabla \times H) - s \nabla (\epsilon^{-1} \nabla \cdot H) = \nabla \times (\epsilon^{-1} J), \tag{2.6}
\]

respectively, where \( s > 0 \) denotes the penalty factor. Here and below we assume that electric permeability \( \mu = 1 \).

For simplicity, we consider the system (2.5) – (2.6) with homogeneous initial conditions

\[
\frac{\partial E}{\partial t} (x, 0) = E(x, 0) = 0, \quad \text{in } \Omega, \tag{2.7}
\]

\[
\frac{\partial H}{\partial t} (x, 0) = H(x, 0) = 0, \quad \text{in } \Omega, \tag{2.8}
\]

and perfectly conducting boundary conditions

\[
E \times n = 0, \quad \text{on } \partial \Omega \times (0, T), \tag{2.9}
\]

\[
H \cdot n = 0, \quad \text{on } \partial \Omega \times (0, T), \tag{2.10}
\]

where \( n \) is the outward normal vector on \( \partial \Omega \). The choice of the parameter \( s \) depends on how much emphasis one places on the gauge condition; the optimal choice is \( s = 1 \) [21, 29].

2.3. Statements of forward and inverse problems. In this work as the forward problem we consider Maxwell equation for electric field with homogeneous initial conditions and perfectly conducting boundary conditions

\[
\frac{\epsilon}{c^2} \frac{\partial^2 E}{\partial t^2} + \nabla \times (\nabla \times E) - s \nabla (\nabla \cdot E) = -j, \quad x \in \Omega, \quad 0 < t < T, \tag{2.11}
\]

\[
\nabla \cdot (\epsilon E) = 0, \quad x \in \Omega, \quad 0 < t < T, \quad \nabla \cdot n = 0, \quad \text{on } \partial \Omega \times (0, T).
\]

The inverse problem for (2.6), (2.8), (2.10) can be formulated similarly and is not considered in this work. Let \( \Omega \subset \mathbb{R}^3 \) be a convex bounded domain with the boundary \( \partial \Omega \subset C^3 \). We assume that the coefficient \( \epsilon (x) \) of equation (2.11) is such that

\[
\epsilon (x) \in [1, d], \quad d = \text{const.} > 1, \quad \epsilon (x) = 1 \quad \text{for } x \in \mathbb{R}^3 \setminus \Omega, \tag{2.12}
\]

\[
\epsilon (x) \in C^2 (\mathbb{R}^3). \tag{2.13}
\]

We consider the following

**Inverse Problem.** Suppose that the coefficient \( \epsilon (x) \) satisfies (2.12) and (2.13), where the number \( d > 1 \) is given. Assume that the function \( \epsilon (x) \) is unknown in the domain \( \Omega \). Determine the function \( \epsilon (x) \) for \( x \in \Omega \), assuming that the following function \( \tilde{E} (x, t) \) is known

\[
E (x, t) = \tilde{E} (x, t), \quad \forall (x, t) \in \partial \Omega \times (0, \infty). \tag{2.14}
\]

A priori knowledge of upper and lower bounds of the coefficient \( \epsilon (x) \) corresponds well with the Tikhonov concept about the availability of a priori information for an ill-posed problem
In applications the assumption \( \epsilon(x) = 1 \) for \( x \in \mathbb{R}^3 \setminus \Omega \) means that the target coefficient \( \epsilon(x) \) has a known constant value outside of the medium of interest \( \Omega \). The function \( \tilde{E}(x,t) \) models time dependent measurements of the electric wave field at the boundary of the domain of interest. In practice measurements are performed at a number of detectors. In this case the function \( \tilde{E}(x,t) \) can be obtained via one of standard interpolation procedures, a discussion of which is outside of the scope of this publication.

3. Tikhonov functional and optimality conditions

We reformulate our inverse problem as an optimization problem, where one seek the permittivity \( \epsilon(x) \), which result in a solution of equations (2.11) with best fit to time domain observations \( \tilde{E} \), measured at a finite number of observation points. Denote \( Q_T = \Omega \times (0,T) \), \( S_T = \partial \Omega \times (0,T) \). Our goal is minimize Tikhonov functional

\[
F(E, \epsilon) = \frac{1}{2} \int_{S_T} (E|_{S_T} - \tilde{E})^2 z_\delta(t) dx dt + \frac{1}{2} \gamma \int_{\Omega} (\epsilon - \epsilon_0)^2 dx,
\]

where \( \tilde{E} \) is the observed electric field, \( E \) satisfies the equations (2.11) and thus depends on \( \epsilon \), and \( \gamma \) is regularization parameter. Here \( z_\delta(t) \) is a cut-off function, which is introduced to ensure that compatibility conditions at \( S_T \cap \{ t = T \} \) are satisfied, and \( \delta > 0 \) is a small number. So, we choose such a function \( z_\delta \) that

\[
z_\delta \in C^\infty [0,T], \quad z_\delta(t) = \begin{cases} 1 & \text{for } t \in [0,T - \delta], \\ 0 & \text{for } t \in (T - \frac{\delta}{2}, T], \\ 0 < z_\delta < 1 & \text{for } t \in (T - \delta, T - \frac{\delta}{2}). \end{cases}
\]

To solve this minimization problem we introduce the Lagrangian

\[
L(u) = F(E, \epsilon) - \int_{\Omega_T} \epsilon \frac{\partial \lambda}{\partial t} \frac{\partial E}{\partial t} dx dt + \int_{\Omega_T} (\nabla \times E)(\nabla \times \lambda) dx dt
\]

\[
+ \int_{\Omega_T} \nabla \cdot (\epsilon E) \lambda dx dt + s \int_{\Omega_T} (\nabla \cdot E)(\nabla \cdot \lambda) dx dt + \int_{\Omega_T} j \lambda dx dt,
\]

where \( u = (E, \lambda, \epsilon) \), and search for a stationary point with respect to \( u \) satisfying \( \forall \bar{u} = (\tilde{E}, \tilde{\lambda}, \tilde{\epsilon}) \)

\[
L'(u; \bar{u}) = 0,
\]

where \( L'(u; \cdot) \) is the Jacobian of \( L \) at \( u \).

We assume that \( \lambda(x,T) = \partial_t \lambda(x,T) = 0 \) and seek to impose such conditions on the function \( \lambda \) that in (3.2) \( L(E, \lambda, \epsilon) := L(u) = F(E, \epsilon) \). In other words, the sum of integral terms in (3.2) should be equal to zero. Then we will come up with the formulation of the so-called adjoint problem for the function \( \lambda \).

To proceed further we use the fact that \( \lambda(x,T) = \partial_t \lambda(x,T) = 0 \) and \( E(x,0) = \partial_{tt} E(x,0) = 0 \), together with perfectly conducting boundary conditions \( n \times E = n \times \lambda = 0 \) and \( n \cdot (\nabla \cdot E) = n \cdot E = n \cdot (\epsilon E) = 0 \) and \( n \cdot (\nabla \cdot \lambda) = n \cdot \lambda = 0 \) on \( \partial \Omega \). The equation (3.3) expresses that for all \( \bar{u} \),

\[
0 = \frac{\partial L}{\partial \lambda}(u)(\bar{\lambda}) = - \int_{\Omega_T} \epsilon \frac{\partial \lambda}{\partial t} \frac{\partial E}{\partial t} dx dt + \int_{\Omega_T} (\nabla \times E)(\nabla \times \lambda) dx dt
\]

\[
+ s \int_{\Omega_T} (\nabla \cdot E)(\nabla \cdot \lambda) dx dt + \int_{\Omega_T} \nabla \cdot (\epsilon E) \lambda dx dt + \int_{\Omega_T} j \lambda dx dt,
\]
0 = \frac{\partial L}{\partial E}(u)(\bar{E}) = \int_{\Omega_T} (E - \bar{E}) \bar{E} z_{\delta} \, dxdt \\
(3.5) \quad - \int_{\Omega_T} \epsilon \frac{\partial \lambda}{\partial t} \frac{\partial \bar{E}}{\partial t} \, dxdt + \int_{\Omega_T} (\nabla \times \lambda)(\nabla \times \bar{E}) \, dxdt \\
+ s \int_{\Omega_T} (\nabla \cdot \lambda)(\nabla \cdot \bar{E}) \, dxdt - \int_{\Omega_T} \epsilon \nabla \lambda \bar{E} \, dxdt,
(3.6) \quad 0 = \frac{\partial L}{\partial \epsilon}(u)(\bar{\epsilon}) = - \int_{\Omega_T} \frac{\partial \lambda}{\partial t} \frac{\partial \bar{E}}{\partial t} \, dxdt - \int_{\Omega_T} E \nabla \lambda \bar{E} \, dxdt + \gamma \int_{\Omega} (\epsilon - \epsilon_0) \bar{\epsilon} \, dx, \, x \in \Omega,

The equation (3.4) is the weak formulation of the state equation (2.5) and the equation (3.5) is the weak formulation of the following adjoint problem

\epsilon \frac{\partial^2 \lambda}{\partial t^2} + \nabla \times (\nabla \times \lambda) - s \nabla (\nabla \cdot \lambda) = -(E - \bar{E}) z_{\delta}, \, x \in \Omega, \, 0 < t < T,
\n\n\n(3.7) \quad \nabla \cdot (\epsilon \lambda) = 0, \, x \in \Omega, \, 0 < t < T,
\n\lambda(\cdot, T) = \frac{\partial \lambda}{\partial t}(\cdot, T) = 0,
\n\lambda \times n = 0 \text{ on } S_T.

Further, (3.6) expresses stationarity with respect to \( \epsilon \).

4. Finite element discretization

We discretize \( \Omega \times (0, T) \) denoting by \( K_h = \{ K \} \) a partition of the domain \( \Omega \) into tetrahedra \( K \) \( (h = h(x)) \) being a mesh function defined as \( h|_K = h_K \) representing the local diameter of the elements, and we let \( J_k \) be a partition of the time interval \( (0, T) \) into time intervals \( J = (t_{k-1}, t_k) \) of uniform length \( \tau = t_k - t_{k-1} \). We assume also a minimal angle condition on the \( K_h \) \([9]\).

To formulate the finite element method for (3.3) we introduce the finite element spaces \( V_h \), \( W_h^E \) and \( W_h^\lambda \) defined by :

\( V_h := \{ v \in L_2(\Omega) : v \in P_0(K), \forall K \in K_h \}, \)
\( W_h^E := \{ w \in [H^1(\Omega \times J)]^3 : w(\cdot, 0) = 0, w \times n|_{\partial \Omega} = 0 \}, \)
\( W_h^\lambda := \{ w \in [H^1(\Omega \times J)]^3 : w(\cdot, T) = 0, w \times n|_{\partial \Omega} = 0 \}, \)
\( W_h^\lambda := \{ w \in W_h^\lambda : w|_{K \times J} \in [P_1(K) \times P_1(J)]^3, \forall K \in K_h, \forall J \in J \}, \)
\( W_h^\lambda := \{ w \in W_h^\lambda : w|_{K \times J} \in [P_1(K) \times P_1(J)]^3, \forall K \in K_h, \forall J \in J \}, \)

where \( P_1(K) \) and \( P_1(J) \) are the set of continuous piecewise linear functions on \( K \) and \( J \), respectively.

We define \( U_h = W_h^E \times W_h^\lambda \times V_h \). The finite element method now reads: Find \( u_h \in U_h \), such that

\( 4.1 \quad L'(u_h)(\bar{u}) = 0 \, \forall \bar{u} \in U_h. \)
5. Fully discrete scheme

We expand $E, \lambda$ in terms of the standard continuous piecewise linear functions $\varphi_i(x)$ in space and $\psi_i(t)$ in time and substitute this into (2.11) and (3.7) to obtain the following system of linear equations:

\begin{align}
M(\mathbf{E}^{k+1} - 2\mathbf{E}^k + \mathbf{E}^{k-1}) &= -\tau^2 F^k - \tau^2 K\mathbf{E}^k - s\tau^2 C\mathbf{E}^k - \tau^2 B\mathbf{E}^k, \\
M(\lambda^{k+1} - 2\lambda^k + \lambda^{k-1}) &= -\tau^2 S^k - \tau^2 K\lambda^k - s\tau^2 C\lambda^k - \tau^2 B\lambda^k,
\end{align}

with initial conditions:

\begin{align}
E(\cdot, 0) &= \frac{\partial E}{\partial t}(\cdot, 0) = 0, \\
\lambda(\cdot, T) &= \frac{\partial \lambda}{\partial t}(\cdot, T) = 0.
\end{align}

Here, $M$ is the block mass matrix in space, $K$ is the block stiffness matrix corresponding to the rotation term, $C$ and $B$ are the stiffness matrices corresponding to the divergence terms, $F^k$ and $S^k$ are the load vectors at time level $t_k$, $\mathbf{E}^k$ and $\lambda^k$ denote the nodal values of $E(\cdot, t_k)$ and $\lambda(\cdot, t_k)$, respectively, $\tau$ is the time step.

The explicit formulas for the entries in system (5.1) at each element $e$ can be given as:

\begin{align}
M_{i,j}^e &= (\epsilon \varphi_i, \varphi_j)_e, \\
K_{i,j}^e &= \left( \frac{1}{\mu} \nabla \times \varphi_i, \nabla \times \varphi_j \right)_e, \\
C_{i,j}^e &= \left( \frac{1}{\mu} \nabla \cdot \varphi_i, \nabla \cdot \varphi_j \right)_e, \\
B_{i,j}^e &= \left( \nabla \cdot (\epsilon \varphi_i), \varphi_j \right)_e, \\
F_{j,m}^e &= \left( (j, \varphi_j \psi_m) \right)_{e \times J}, \\
S_{j,m}^e &= \left( (E - \bar{E}, \varphi_j \psi_m) \right)_{e \times J},
\end{align}

where $(\cdot, \cdot)_e$ denotes the $L_2(e)$ scalar product.

To obtain an explicit scheme we approximate $M$ with the lumped mass matrix $M^L$ – see [10, 20, 23]. Next, we multiply (5.1) with $(M^L)^{-1}$ and get the following explicit method:

\begin{align}
\mathbf{E}^{k+1} &= -\tau^2(M^L)^{-1} F^k + 2\mathbf{E}^k - \tau^2(M^L)^{-1} K\mathbf{E}^k \\
&\quad - s\tau^2(M^L)^{-1} C\mathbf{E}^k - \tau^2(M^L)^{-1} B\mathbf{E}^k - \mathbf{E}^{k-1}, \\
\lambda^{k+1} &= -\tau^2(M^L)^{-1} S^k + 2\lambda^k - \tau^2(M^L)^{-1} K\lambda^k \\
&\quad - s\tau^2(M^L)^{-1} C\lambda^k - \tau^2(M^L)^{-1} B\lambda^k - \lambda^{k-1}.
\end{align}

Finally, to approximate coefficient $\epsilon$ can be used one of the gradient-like methods with an appropriate initial guess value $\epsilon_0$. The discrete version of gradient with respect to the coefficient (3.6) takes the form:

\begin{align}
gh = -\int_0^T \frac{\partial \lambda_h^k}{\partial t} \frac{\partial E_k^t}{\partial t} \, dx dt - \int_0^T E_{h}^k \nabla \lambda_h^k dt + \gamma(\epsilon_h^k - \epsilon_0).
\end{align}

Here, $\lambda_h^k$ and $E_{h}^k$ are computed values of the adjoint and forward problems at time moment $k$ using explicit scheme (5.5), and $\epsilon_h^k$ is approximated value of the coefficient.
6. An a posteriori error estimate for the Lagrangian and an adaptive algorithm

6.1. A posteriori error estimate. Following [3] we now present the main framework in the proof of an a posteriori error estimate for the Lagrangian. Let $C$ denote various constants of moderate size. We write an equation for the error $e$ in the Lagrangian as

$$
e = L(u) - L(u_h)$$

$$= \int_0^1 \frac{d}{d\epsilon} L(u\epsilon + (1 - \epsilon)u_h) d\epsilon$$

$$= \int_0^1 L'(u\epsilon + (1 - \epsilon)u_h)(u - u_h) d\epsilon$$

$$= L'(u_h)(u - u_h) + R,$$

where $R$ denotes (a small) second order term. For full details of the arguments we refer to [2] and [14].

Next, we use the splitting $u - u_h = (u - u_h^I) + (u_h^I - u_h)$ where $u_h^I$ denotes an interpolant of $u$, the Galerkin orthogonality (4.1) and neglect the term $R$ to get the following error representation:

$$e \approx L'(u_h)(u - u_h^I) = (I_1 + I_2 + I_3),$$

where

$$I_1 = - \int_{\Omega_T} \left( \epsilon_h \frac{\partial}{\partial t} \frac{\partial}{\partial \epsilon} E_h (\lambda - \lambda_h^I) \right) dxdt + \int_{\Omega_T} (\nabla \times (\lambda - \lambda_h^I))(\nabla \times E_h) dxdt$$

$$+ s \int_{\Omega_T} (\nabla \cdot E_h)(\nabla \cdot (\lambda - \lambda_h^I)) dxdt$$

$$+ \int_{\Omega_T} \nabla \cdot (\epsilon_h E_h)(\lambda - \lambda_h^I) dxdt + \int_{\Omega_T} j(\lambda - \lambda_h^I) dxdt,$$

$$I_2 = \int_{\partial \Omega_T} (E_h - \tilde{E})(E - E_h^I) \frac{\partial}{\partial n} dxdt - \int_{\Omega_T} \epsilon_h \frac{\partial}{\partial t} \frac{\partial}{\partial \epsilon} E_h^I dxdt$$

$$+ \int_{\Omega_T} (\nabla \times \lambda_h)(\nabla \times (E - E_h^I)) dxdt$$

$$- \int_{\Omega_T} \epsilon_h \nabla \lambda_h(E - E_h^I) dxdt + s \int_{\Omega_T} (\nabla \cdot \lambda_h)(\nabla \cdot (E - E_h^I)) dxdt,$$

$$I_3 = - \int_{\Omega_T} \frac{\partial}{\partial t} \frac{\partial}{\partial \epsilon} E_h (\epsilon - \epsilon_h^I) dxdt - \int_{\Omega_T} E_h \nabla \lambda_h(\epsilon - \epsilon_h^I) dxdt + \gamma \int_{\Omega} (\epsilon_h - \epsilon_0)(\epsilon - \epsilon_h^I) dx.$$
To estimate (6.3) we integrate by parts in the first, second and third terms to get:

\[(6.6)\]

\[
I_1 = \int_{\Omega_T} \left( \epsilon_h \frac{\partial^2 E_h}{\partial t^2} + \nabla \times (\nabla \times E_h) - s \nabla (\nabla \cdot E_h) + \nabla \cdot (\epsilon_h E_h) + j \right) (\lambda - \lambda_h^I) \, dx \, dt
\]

\[
+ \sum_k \int_{\Omega} \epsilon_h \left[ \frac{\partial E_h(t_k)}{\partial t} \right] (\lambda - \lambda_h^I(t_k)) \, dx - \int_0^T \int_{\partial K} \left( n_K \times (\nabla \times E_h) \right) (\lambda - \lambda_h^I) \, ds \, dt
\]

\[
+ s \sum_k \int_0^T \int_{\partial K} (\nabla \cdot E_h) \left( n_K \cdot (\lambda - \lambda_h^I) \right) \, dS \, dt = J_1 + J_2 + J_3 + J_4,
\]

where \( J_i, i = 1, \ldots, 4 \) denote integrals that appear on the right of (6.6). In particular, \( J_2, J_3 \) result from integration by parts in space, whereas \( \frac{\partial E_h}{\partial t} \) appears during the integration by parts in time and denotes the jump of the derivative of \( E_h \) in time. Here \( n_K \) denotes the exterior unit normal to element \( K \).

To estimate \( J_3 \) we sum over the element boundaries where each internal side \( S \in S_h \) occurs twice. Let \( E_s \) denote the function \( E_h \) in one of the normal directions of each side \( S \) and \( n_s \) is outward normal vector on \( S \). Then we can write

\[(6.7)\]

\[
\sum_k \int_{\partial K} \left( n_K \times (\nabla \times E_h) \right) (\lambda - \lambda_h^I) \, dS = \sum_s \int_S \left[ n_s \times (\nabla \times E_s) \right] (\lambda - \lambda_h^I) \, dS,
\]

where \( \left[ n_s \times (\nabla \times E_s) \right] \) is the tangential jump of \( \nabla \times E_h \) computed from the two elements sharing \( S \). We distribute each jump equally to the two sharing triangles and return to a sum over all element edges \( \partial K \) as:

\[(6.8)\]

\[
\sum_s \int_S \left[ n_s \times (\nabla \times E_s) \right] (\lambda - \lambda_h^I) \, dS = \sum_k \frac{1}{2} h_K^{-1} \int_{\partial K} \left[ n_s \times (\nabla \times E_s) \right] (\lambda - \lambda_h^I) h_K \, dS.
\]

We formally set \( dx = h_K dS \) and replace the integrals over the element boundaries \( \partial K \) by integrals over the elements, to get:

\[(6.9)\]

\[
\left| \sum_k \frac{1}{2} h_K^{-1} \int_{\partial K} \left[ n_s \times (\nabla \times E_s) \right] (\lambda - \lambda_h^I) h_K \, dS \right| \leq C \int_{\Omega} \max_{S \in \partial K} \left[ n_s \times (\nabla \times E_s) \right] \left[ n_K \times (\nabla \times E_h) \right] \, \lambda - \lambda_h^I \, dx,
\]

with \( \left[ n_K \times (\nabla \times E_h) \right]_K = \max_{S \in \partial K} \left[ n_s \times (\nabla \times E_s) \right]_S \). Here and later we denote by \( C \) different constants of moderate size.

In a similar way we can estimate \( J_4 \) in (6.6):

\[
J_4 = s \sum_k \int_{\partial K} (\nabla \cdot E_h) \left( n_K \cdot (\lambda - \lambda_h^I) \right) \, dS
\]

\[
= s \sum_s \int_S \left[ \nabla \cdot E_s \right] \left[ n_s \cdot (\lambda - \lambda_h^I) \right] \, dS
\]

\[
= s \sum_k \frac{1}{2} h_K^{-1} \int_{\partial K} \left[ \nabla \cdot E_s \right] \left[ n_s \cdot (\lambda - \lambda_h^I) \right] h_K \, dS.
\]
Again, replacing the integrals over the boundaries by integrals over the elements we get the following estimate for $J_4$:

\begin{equation}
|J_4| \leq s C \int_{\Omega} \max_{S \subset \partial K} h_K^{-1} ||\nabla \cdot E_h|| \cdot \left| \left[ n_K \cdot (\lambda - \lambda_h^f) \right] \right| \, dx,
\end{equation}

with $||\nabla \cdot E_h||_K = \max_{S \subset \partial K} [\nabla \cdot E_h] \cdot J_2$ is estimated similarly with $J_3, J_4$.

We substitute expressions for $J_2, J_3$ and $J_4$ in (6.6) to get:

\begin{equation}
|I_1| \leq \int_{\Omega_T} \left( \epsilon_h \frac{\partial^2 E_h}{\partial t^2} + \nabla \times (\nabla \times E_h) - s \nabla (\nabla \cdot E_h) + \nabla \cdot (\epsilon_h E_h) + j \right) \cdot \left| \lambda - \lambda_h^f \right| \, dx \, dt
\end{equation}

\begin{align*}
+ C \int_{\Omega_T} \epsilon_h t^{-1} \cdot |[\partial E_{ht}]| \cdot |\lambda - \lambda_h^f| \, dx \, dt \\
+ C \int_{\Omega_T} \max_{S \subset \partial K} h_K^{-1} \cdot |\left[ n_K \times (\nabla \times E_h) \right]| \cdot |\lambda - \lambda_h^f| \, dx \, dt \\
+ s C \int_{\Omega_T} \max_{S \subset \partial K} h_K^{-1} \cdot |[\nabla \cdot E_h]| \cdot \left| \left[ n_K \cdot (\lambda - \lambda_h^f) \right] \right| \, dx \, dt.
\end{align*}

where

\[ [\partial E_{ht}] = [\partial E_{ht_k}] \text{ on } J_k \]

and $[\partial E_{ht_k}]$ is defined as the maximum of the two jumps in time on each time interval $J_k$:

\begin{equation}
[\partial E_{ht_k}] = \max_{J_k} \left( \left[ \frac{\partial E_h}{\partial t}(t_k) \right], \left[ \frac{\partial E_h}{\partial t}(t_{k+1}) \right] \right).
\end{equation}

Next, we use a standard interpolation estimate [3] for $\lambda - \lambda_h^f$ to get

\begin{equation}
|I_1| \leq \int_{\Omega_T} \left( \epsilon_h \frac{\partial^2 E_h}{\partial t^2} + \nabla \times (\nabla \times E_h) - s \nabla (\nabla \cdot E_h) + \nabla \cdot (\epsilon_h E_h) + j \right) \cdot \left( \tau^2 \left| \frac{\partial^2 \lambda}{\partial t^2} \right| + h^2 |D^2 \lambda| \right) \, dx \, dt
\end{equation}

\begin{align*}
+ C \int_{\Omega_T} \epsilon_h t^{-1} \cdot |[\partial E_{ht}]| \cdot \left( \tau^2 \left| \frac{\partial^2 \lambda}{\partial t^2} \right| + h^2 |D^2 \lambda| \right) \, dx \, dt \\
+ C \int_{\Omega_T} \max_{S \subset \partial K} h_K^{-1} \cdot |\left[ n_K \times (\nabla \times E_h) \right]| \cdot \left( \tau^2 \left| \frac{\partial^2 \lambda}{\partial t^2} \right| + h^2 |D^2 \lambda| \right) \, dx \, dt \\
+ s C \int_{\Omega_T} \max_{S \subset \partial K} h_K^{-1} \cdot |[\nabla \cdot E_h]| \cdot \left| \left[ n_K \cdot (\tau^2 \left| \frac{\partial^2 \lambda}{\partial t^2} \right| + h^2 |D^2 \lambda|) \right] \right| \, dx \, dt.
\end{align*}
Next, in (6.14) the terms \(\frac{\partial^2 E_h}{\partial t^2}, \nabla \times (\nabla \times E_h), \nabla (\nabla \cdot E_h)\) vanish, since \(E_h\) is continuous piecewise linear function. We then estimate \(\frac{\partial^2 \lambda}{\partial t^2} \approx \frac{[\partial \lambda_h]}{\tau} \) and \(D_x^2 \lambda \approx \frac{[\partial \lambda_h]}{h}\) to get:

\[
|I_1| \leq C \int_{\Omega_T} |j + \nabla \cdot (\epsilon_h E_h)\cdot (\tau^2 \frac{|\partial \lambda_h|}{\tau} + h^2 \frac{|\partial \lambda_h|}{h}) | \ dx \ dt \\
+ C \int_{\Omega_T} \epsilon_h \tau^{-1} |[\partial E_h]| \cdot (\tau^2 \frac{|\partial \lambda_h|}{\tau} + h^2 \frac{|\partial \lambda_h|}{h}) | \ dx \ dt \\
+ C \int_{\Omega_T} \max_{S \subset \partial K} h^{-1}_K \cdot \left| n_k \times (\nabla \times E_h) \right| \cdot \left( \tau^2 \frac{|\partial \lambda_h|}{\tau} + h^2 \frac{|\partial \lambda_h|}{h} \right) \ dx \ dt \\
+ s \ C \int_{\Omega_T} \max_{S \subset \partial K} h^{-1}_K \cdot \left| [\nabla \cdot E_h] \cdot \left[ n_k \cdot \left( \tau^2 \frac{|\partial \lambda_h|}{\tau} + h^2 \frac{|\partial \lambda_h|}{h} \right) \right] \right) \ dx \ dt.
\]

We estimate \(I_2\) similarly:

\[
|I_2| \leq \int_{\Omega_T} \left| \left( \epsilon_h \frac{\partial^2 \lambda_h}{\partial t^2} + \nabla \times (\nabla \times \lambda_h) - \epsilon_h \lambda_h - s \nabla (\nabla \cdot \lambda_h) \right) \right| \cdot |E - E_h^I| \ dx \ dt \\
+ \int_{S_T} |E_h - \tilde{E}| \cdot |E - E_h^I| \ z_\delta \ dx \ dt + \left| \sum_k \int_{\Omega_T} \epsilon_h \frac{\partial \lambda_h}{\partial t} (t_k) \ (E - E_h^I)(t_k) \ dx \right| \\
+ \left| \sum_k \int_0^T \int_{\partial K} \left( n_k \times (\nabla \times \lambda_h) \right) \ (E - E_h^I) \ dS \ dt \right| \\
+ s \ \sum_k \int_0^T \int_{\partial K} (\nabla \cdot \lambda_h) \ (n_k \cdot (E - E_h^I)) \ dS \ dt.
\]

Next, we can estimate (6.16) similarly with (6.15) as

\[
|I_2| \leq \int_{\Omega_T} \left| \left( \epsilon_h \frac{\partial^2 \lambda_h}{\partial t^2} + \nabla \times (\nabla \times \lambda_h) - \epsilon_h \nabla \lambda_h - s \nabla (\nabla \cdot \lambda_h) \right) \right| \cdot |E - E_h^I| \ dx \ dt \\
+ \int_{S_T} |E_h - \tilde{E}| \cdot |E - E_h^I| \ z_\delta \ dx \ dt \\
+ C \int_{\Omega_T} \epsilon_h \tau^{-1} \cdot \left| [\partial \lambda_h] \right| \cdot |E - E_h^I| \ dx \ dt \\
+ C \int_{\Omega_T} \max_{S \subset \partial K} h^{-1}_K \cdot \left| n_k \times (\nabla \times \lambda_h) \right| \cdot |E - E_h^I| \ dx \ dt \\
+ s \ C \int_{\Omega_T} \max_{S \subset \partial K} h^{-1}_K \cdot \left| [\nabla \cdot \lambda_h] \cdot \left[ n_k \cdot (E - E_h^I) \right] \right) \ dx \ dt.
\]
Again, the terms $\frac{\partial^2 \lambda_h}{\partial t^2}, \nabla \times (\nabla \times \lambda_h), \nabla (\nabla \cdot \lambda_h)$ vanish, since $\lambda_h$ is also continuous piecewise linear function. Finally we get

$$|I_2| \leq \int_{\Omega_T} |\epsilon_h \nabla \lambda_h| \cdot \left( \tau^2 \left| \frac{\partial E_h}{\partial t} \right| + h^2 \left| \frac{\partial E_h}{\partial n} \right| \right) \, dx \, dt$$

$$+ \int_{S_T} |E_h - \bar{E}| \cdot \left( \tau^2 \left| \frac{\partial E_h}{\partial t} \right| + h^2 \left| \frac{\partial E_h}{\partial n} \right| \right) z_\delta \, dx \, dt$$

$$+ C \int_{\Omega_T} \epsilon_h \tau^{-1} \cdot \left[ [\partial \lambda_h t] \cdot \left( \tau^2 \left| \frac{\partial E_h}{\partial t} \right| + h^2 \left| \frac{\partial E_h}{\partial n} \right| \right) \right] \, dx \, dt$$

$$+ C \int_{\Omega_T} \max_{S \subset \partial K} h^{-1}_K \cdot \left[ n_K \times (\nabla \times \lambda_h) \right] \cdot \left( \tau^2 \left| \frac{\partial E_h}{\partial t} \right| + h^2 \left| \frac{\partial E_h}{\partial n} \right| \right) \, dx \, dt$$

$$+ s \, C \int_{\Omega_T} \max_{S \subset \partial K} h^{-1}_K \cdot |\nabla \cdot \lambda_h| \cdot |n_K \cdot \left( \tau^2 \left| \frac{\partial E_h}{\partial t} \right| + h^2 \left| \frac{\partial E_h}{\partial n} \right| \right) \, dx \, dt.$$

To estimate $I_3$ we use a standard approximation estimate in the form $\epsilon - \epsilon_h \approx h D_x \epsilon$ to get:

$$|I_3| \leq C \int_{\Omega_T} \left| \frac{\partial \lambda_h}{\partial t} \right| \cdot \left| \frac{\partial E_h}{\partial t} \right| \cdot h \, |D_x \epsilon| \, dx \, dt + C \int_{\Omega_T} |E_h| \cdot |\nabla \lambda_h| \cdot h \, |D_x \epsilon| \, dx \, dt$$

$$+ \gamma \, C \int_{\Omega_T} |\epsilon_h - \epsilon_0| \cdot h \, |D_x \epsilon| \, dx$$

$$\leq C \int_{0}^{T} \int_{\Omega} \left| \frac{\partial \lambda_h}{\partial t} \right| \cdot \left| \frac{\partial E_h}{\partial t} \right| \cdot h \left| \frac{\epsilon_h}{h} \right| \, dx \, dt + C \int_{\Omega_T} |E_h| \cdot |\nabla \lambda_h| \cdot h \left| \frac{\epsilon_h}{h} \right| \, dx \, dt$$

$$+ \gamma \, C \int_{\Omega_T} |\epsilon_h - \epsilon_0| \cdot \left| \frac{\epsilon_h}{h} \right| \, dx$$

$$\leq C \int_{0}^{T} \int_{\Omega} \left| \frac{\partial \lambda_h}{\partial t} \right| \cdot \left| \frac{\partial E_h}{\partial t} \right| \cdot \left| \frac{\epsilon_h}{h} \right| \, dx \, dt + C \int_{\Omega_T} |E_h| \cdot |\nabla \lambda_h| \cdot \left| \frac{\epsilon_h}{h} \right| \, dx \, dt$$

$$+ \gamma \, C \int_{\Omega_T} |\epsilon_h - \epsilon_0| \cdot \left| \frac{\epsilon_h}{h} \right| \, dx.$$
Theorem 5.1 Let $L(u) = L(E, \lambda, \epsilon)$ be the Lagrangian defined in (3.2), and let $L(u_h) = L(E_h, \lambda_h, \epsilon_h)$ be the approximation of $L(u)$. Then the following error representation formula for the error $e = L(u) - L(u_h)$ in the Lagrangian holds:

$$
|e| \leq \sum_{i=1}^{3} \int_{\Omega_T} R_{E_i} \sigma_{\lambda_i} \, dx dt + \int_{\Omega_T} R_{E_4} \sigma_{\lambda_2} \, dx dt + \int_{\Omega_T} R_{\lambda_4} \sigma_{E_1} \, dx dt \\
+ \sum_{i=2}^{4} \int_{\Omega_T} R_{\lambda_i} \sigma_{E_1} \, dx dt + \int_{\Omega_T} R_{\lambda_5} \sigma_{E_2} \, dx dt + \sum_{i=1}^{3} \int_{\Omega_T} R_{\epsilon_i} \sigma_{\epsilon} \, dx dt,
$$

(6.20)

where residuals are defined by

$$
R_{E_1} = |j + \nabla \cdot (\epsilon_h E_h)|, \quad R_{E_2} = \epsilon_h^{-1} \left| \left[ \frac{\partial E_h t}{\partial t} \right] \right|, \quad R_{E_3} = \max_{S \subset \partial K} h_K^{-1} \left| n_K \times (\nabla \times E_h) \right|,
$$

$$
R_{E_4} = s \max_{S \subset \partial K} h_K^{-1} \left| [\nabla \cdot E_h] \right|,
$$

$$
R_{\lambda_1} = |E_h - \bar{E}|, \quad R_{\lambda_2} = |\epsilon_h \nabla \lambda_h|, \quad R_{\lambda_3} = \epsilon_h^{-1} \left| \left[ \frac{\partial \lambda_h t}{\partial t} \right] \right|,
$$

$$
R_{\lambda_4} = \max_{S \subset \partial K} h_K^{-1} \left| n_K \times (\nabla \times \lambda_h) \right|, \quad R_{\lambda_5} = s \max_{S \subset \partial K} h_K^{-1} \left| [\nabla \cdot \lambda_h] \right|,
$$

$$
R_{\epsilon_1} = \left| \frac{\partial \lambda_h}{\partial t} \right|, \quad R_{\epsilon_2} = |E_h| \cdot |\nabla \lambda_h|, \quad R_{\epsilon_3} = |\epsilon_h - \epsilon_0|,
$$

and interpolation errors are

$$
\sigma_{\lambda_1} = C \left( \left| \frac{\partial \lambda_h}{\partial t} \right| + h \left| \frac{\partial \lambda_h}{\partial n} \right| \right), \quad \sigma_{\lambda_2} = C \left[ n_K \cdot \left( \frac{\partial \lambda_h}{\partial t} \right) + h \left| \frac{\partial \lambda_h}{\partial n} \right| \right],
$$

$$
\sigma_{E_1} = C \left( \left| \frac{\partial E_h}{\partial t} \right| + h \left| \frac{\partial E_h}{\partial n} \right| \right), \quad \sigma_{E_2} = C \left[ n_K \cdot \left( \frac{\partial E_h}{\partial t} \right) + h \left| \frac{\partial E_h}{\partial n} \right| \right],
$$

$$
\sigma_{\epsilon} = C |\epsilon_h|.
$$

Remark 5.1

If solutions $\lambda_h$ and $E_h$ to the adjoint and state equations are computed with good accuracy, then we can neglect terms $\sum_{i=1}^{4} \int_{\Omega_T} R_{E_i} \sigma_{\lambda_i} \, dx dt + \sum_{i=1}^{5} \int_{\Omega_T} R_{\lambda_i} \sigma_{E_1} \, dx dt + \int_{\Omega_T} R_{\epsilon_i} \sigma_{\epsilon} \, dx dt$ in a posteriori error estimation (6.20). Thus the term

$$
N(\epsilon_h) = \int_{0}^{T} \left| \frac{\partial \lambda_h}{\partial t} \right| \frac{\partial E_h}{\partial t} \, dt + \gamma (\epsilon_h - \epsilon_0)
$$

(6.21)

dominates. This fact is also observed numerically (see next section) and will be explained analytically in forthcoming publication.

Mesh refinement recommendation

From the Theorem 5.1 and Remark 5.1 follows that the mesh should be refined in such subdomain of the domain $\Omega$ where values of the function $N(\epsilon_h)$ are close to the number

$$
\max_{\Omega} N(\epsilon_h) = \max_{\Omega} \left| \int_{0}^{T} \left[ \frac{\partial \lambda_h}{\partial t} \right] \frac{\partial E_h}{\partial t} \, dt + \gamma (\epsilon_h - \epsilon_0) \right|.
$$

(6.22)

6.2. The Adaptive algorithm. In this section we outline our adaptive algorithm using the mesh refinement recommendation of section 5. So, on each mesh we should find an
approximate solution of the equation \( N(\epsilon_h) = 0 \). In other words, we should approximately solve the following equation with respect to the function \( \epsilon_h(x) \),

\[
\int_0^T \frac{\partial \lambda_h}{\partial t} \frac{\partial E_h}{\partial t} dt + \gamma (\epsilon_h - \epsilon_0) = 0.
\]

For each new mesh we first linearly interpolate the function \( \epsilon_0(x) \) on it. On every mesh we iteratively update approximations \( \epsilon^m_h \) of the function \( \epsilon_h \), where \( m \) is the number of iteration in optimization procedure. To do so, we use the quasi-Newton method with the classic BFGS update formula with the limited storage [28]. Denote

\[
g^m(x) = \gamma(\epsilon^m_h - \epsilon_0)(x) + \int_0^T (E_{ht}\lambda_{ht})(x, t, \epsilon^m_h) dt,
\]

where functions \( E_h(x, t, \epsilon^m_h), \lambda_h(x, t, \epsilon^m_h) \) are computed via solving state and adjoint problems with \( \epsilon := \epsilon^m_h \).

Based on the mesh refinement recommendation of section 5, we use the following adaptivity algorithm in our computations:

**Adaptive algorithm**

Step 0. Choose an initial mesh \( K_h \) in \( \Omega \) and an initial time partition \( J_0 \) of the time interval \( (0, T) \). Start with the initial approximation \( \epsilon^0_h = \epsilon_0 \) and compute the sequence of \( \epsilon^m_h \) via the following steps:

Step 1. Compute solutions \( E_h(x, t, \epsilon^m_h) \) and \( \lambda_h(x, t, \epsilon^m_h) \) of state and adjoint problems of (2.11) and (3.7) on \( K_h \) and \( J_k \).

Step 2. Update the coefficient \( \epsilon_h := \epsilon^{m+1}_h \) on \( K_h \) and \( J_k \) using the quasi-Newton method, see details in [3, 28]

\[
\epsilon^{m+1}_h = \epsilon^m_h + \alpha g^m(x),
\]

where \( \alpha \) is step-size in gradient update [30].

Step 3. Stop computing \( \epsilon^m_h \) and obtain the function \( \epsilon_h \) if either \( ||g^m||_{L_2(\Omega)} \leq \theta \) or norms \( ||g^m||_{L_2(\Omega)} \) are stabilized. Otherwise set \( m := m + 1 \) and go to step 1. Here \( \theta \) is the tolerance in quasi-Newton updates.

Step 4. Compute the function \( B_h(x) \),

\[
B_h(x) = \left| \int_0^T \frac{\partial \lambda_h}{\partial t} \frac{\partial E_h}{\partial t} dt + \gamma (\epsilon_h - \epsilon_0) \right|.
\]

Next, refine the mesh at all points where

\[
(6.24) \quad B_h(x) \geq \beta_1 \max_{\Omega} B_h(x).
\]

Here the tolerance number \( \beta_1 \in (0, 1) \) is chosen by the user.

Step 5. Construct a new mesh \( K_h \) in \( \Omega \) and a new time partition \( J_k \) of the time interval \( (0, T) \). On \( J_k \) the new time step \( \tau \) should be chosen in such a way that the CFL condition is satisfied. Interpolate the initial approximation \( \epsilon_0 \) from the previous mesh to the new mesh. Next, return to step 1 and perform all above steps on the new mesh.

Step 6. Stop mesh refinements if norms defined in step 3 either increase or stabilize, compared with the previous mesh.
7. Numerical example

We test the performance of the adaptive algorithm formulated above on the solution of an inverse electromagnetic scattering problem in three dimensions. In our computational example we consider the domain $\Omega = [-9.0, 9.0] \times [-10.0, -12.0] \times [-9.0, 9.0]$ with an unstructured mesh consisting of tetrahedra. The domain $\Omega$ is split into inner domain $\Omega_1$ which contains scatterer, and surrounding outer domain $\Omega_2$ such that $\Omega = \Omega_1 \cup \Omega_2$. The spherical part of the boundary of the domain $\Omega_1$ we denote as $\partial \Omega_1$ and the boundary of the domain $\Omega$ we denote as $\partial \Omega$. The domain $\Omega_1$ is a cylinder covered by spherical surface from top, see Figure 1-a). We set $\epsilon(x) = 10$ inside of the inclusion depicted on Figure 1-b) and $\epsilon(x) = 1$ outside of it. Hence, the inclusion/background contrast in the dielectric permittivity coefficient is 10 : 1.

In our computational test we chose a time step $\tau$ according to the Courant-Friedrichs-Levy (CFL) stability condition

$$\tau \leq \frac{\sqrt{\epsilon_{\text{max}}}/h}{\sqrt{3}},$$  

where $h$ is the minimal local mesh size, $\epsilon_{\text{max}}$ is an upper bound for the coefficient $\epsilon$.

The forward problem in our test is

$$\frac{\partial^2 E}{\partial t^2} + \nabla \times (\nabla \times E) - s \nabla (\nabla \cdot E) = 0, \quad x \in \Omega, \quad 0 < t < T,$$

$$\nabla \cdot (\epsilon E) = 0, \quad x \in \Omega, \quad 0 < t < T,$$

$$\frac{\partial E}{\partial t}(x, 0) = E(x, 0) = 0, \quad \text{in} \ \Omega,$$

$$E \times n = f(t), \quad \text{on} \ \partial \Omega_1 \times [0, t_1],$$

$$E \times n = 0, \quad \text{on} \ \partial \Omega_1 \times (t_1, T],$$

$$E \times n = 0, \quad \text{on} \ \partial \Omega \times [0, T].$$  

Let $\Omega_1 \subset \mathbb{R}^3$ be a convex bounded domain which is split into upper $\Omega_{\text{up}}$ and lower $\Omega_{\text{down}}$ domains such that $\Omega_1 = \Omega_{\text{up}} \cup \Omega_{\text{down}}$. We assume that we need to reconstruct coefficient $\epsilon(x)$ only in $\Omega_{\text{down}}$, from back reflected data at $\partial \Omega_1$. In other words we assume that the coefficient $\epsilon(x)$ of equation (7.2) is such that

$$\epsilon(x) \in [1, d], \quad d = \text{const.} > 1, \quad \epsilon(x) = 1 \quad \text{for} \ x \in \mathbb{R}^3 \setminus \Omega_{\text{down}},$$

$$\epsilon(x) \in C^2(\mathbb{R}^3).$$  

In the following example we consider the electrical field which is given as

$$f(t) = -((\sin(100t - \pi/2) + 1)/10) \times n, \quad 0 \leq t \leq \frac{2\pi}{100}.$$  

We initialize (7.5) at the spherical boundary $\partial \Omega_1$ and propagate it into $\Omega$. The observation points are placed on $\partial \Omega_1$. We note, that in actual computations applying adaptive algorithm the number of the observations points on $\partial \Omega_1$ increases from coarse to finer mesh.

As follows from Theorem 5.1, to estimate the error in the Lagrangian we need to compute approximated values of $(E_h, \lambda_h, \epsilon_h)$ together with residuals and interpolation errors. Since the residuals $R_{\epsilon_1}, R_{\epsilon_3}$ dominate we neglect computations of others terms in a posteriori error estimate appearing in (6.20), see also Remark 5.1. We seek the solution of the optimization problem in an iterative process, where we start with a coarse mesh shown in Fig. 1, refine
this mesh as in step 6 of Algorithm in section 6, and construct a new mesh and a new time
partition.

To generate the data at the observation points, we solve the forward problem (7.2), with
function \( f(t) \) given by (7.5) in the time interval \( t = [0,36.0] \) with the exact value of the
parameters \( \epsilon = 10.0, \mu = 1 \) inside scatterer, and \( \epsilon = \mu = 1.0 \) everywhere else in \( \Omega \). We start
the optimization algorithm with guess values of the parameter \( \epsilon = 1.0 \) at all points in \( \Omega \).
The solution of the inverse problem needs to be regularized since different coefficients can
 correspond to similar wave reflection data on \( \partial \Omega_1 \). We regularize the solution of the inverse
problem by introducing an regularization parameter \( \gamma \) (small).

The computations were performed on four adaptively refined meshes. In Fig. 2-b) we show
a comparison of \( R_{\epsilon_1} \) over the time interval \([25,36]\) on different adaptively refined meshes.
Here, the smallest values of the residual \( R_{\epsilon_1} \) are shown on the corresponding meshes.

The \( L_2 \)-norms in space of the adjoint solution \( \lambda_h \) on different optimization iterations on
adaptively refined meshes are shown in Fig. 2-a). Here, we solved the adjoint problem back-
ward in time from \( t = 36.0 \) down to \( t = 0.0 \). The \( L_2 \)-norms are presented on the time interval
\([25,36]\) since the solution does not vary much on the time interval \([0,25]\). We observe, that
the norm of the adjoint solution decreases faster on finer meshes.

The reconstructed parameter \( \epsilon \) on different adaptively refined meshes at the final optimiza-
tion iteration is presented in Fig. 3. We show isosurfaces of the parameter field \( \epsilon(x) \) with a
given parameter value. We observe that the qualitative value of the reconstructed parameter
\( \epsilon \) is acceptable only using adaptive error control on finer meshes although the shape of the
inclusion is reconstructed sufficiently good on the coarse mesh.

However, since the quasi-Newton method is only locally convergent, the values of the iden-
tified parameters are very sensitive to the guess values of the parameters in the optimization
algorithm and also to the values of the regularization parameter \( \gamma \). We use cut-off constrain
on the computed parameter \( \epsilon \), as also a smoothness indicator to update new values of the
parameter \( \epsilon \) by local averaging over the neighbouring elements. Namely, minimal and max-
imal values of the coefficient \( \epsilon \) in box constraints belongs to the following set of admissible
parameters \( \epsilon \in P = \{ \epsilon \in C(\Omega) | 1 \leq \epsilon(x) \leq 10\} \).

8. Conclusions

We present an adaptive finite element method for an inverse electromagnetic scattering
problem. The adaptivity is based on a posteriori error estimate for the associated Lagrangian
in the form of space-time integrals of residuals multiplied by weights. We illustrate usefulness of a posteriori error indicator on an inverse electromagnetic scattering problem in three
dimensions.

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Figure 1: Computational domain $\Omega$

Figure 2: Comparison of a) $||\lambda_h||$ and b) $R_{e1}$ on different adaptively refined meshes. Here the horizontal x-axis denotes time steps.
Figure 3: Isosurfaces of the parameter $\epsilon$ on different adaptively refined meshes.
References


