



UNIVERSITY OF GOTHENBURG

PREPRINT 2010:31

A Posteriori Error Estimates for Continuous/Discontinuous Galerkin Approximations of the Kirchhoff– Love Buckling Problem

PETER HANSBO MATS G. LARSON

Department of Mathematical Sciences Division of Mathematics CHALMERS UNIVERSITY OF TECHNOLOGY UNIVERSITY OF GOTHENBURG Gothenburg Sweden 2010

Preprint 2010:31

A Posteriori Error Estimates for Continuous/ Discontinuous Galerkin Approximations of the Kirchhoff–Love Buckling Problem

Peter Hansbo, Mats G. Larson

Department of Mathematical Sciences Division of Mathematics Chalmers University of Technology and University of Gothenburg SE-412 96 Gothenburg, Sweden Gothenburg, May 2010

Preprint 2010:31 ISSN 1652-9715

Matematiska vetenskaper Göteborg 2010

A Posteriori Error Estimates for Continuous/Discontinuous Galerkin Approximations of the Kirchhoff–Love Buckling Problem

Peter Hansbo^a Mats G. Larson^b

^aDepartment of Mathematical Sciences, Chalmers University of Technology and University of Gothenburg, SE-41296 Göteborg, Sweden

^bDepartment of Mathematics and Mathematical Statistics, Umeå University, SE-901 87 Umeå, Sweden

Abstract

Second order buckling theory involves a one-way coupled coupled problem where the stress tensor from a plane stress problem appears in an eigenvalue problem for the fourth order Kirchhoff plate. In this paper we present an a posteriori error estimate for the critical buckling load and mode corresponding to the smallest eigenvalue and associated eigenvector. A particular feature of the analysis is that we take the effect of approximate computation of the stress tensor and also provide an error indicator for the plane stress problem. The Kirchhoff plate is discretized using a continuous/discontinuous finite element method based on standard continuous piecewise polynomial finite element spaces which can also be used to solve the plane stress problem.

 $Key\ words:$ discontinuous Galerkin, adaptivity, a posteriori error estimate, Kirchhoff plate, buckling

1 Introduction

Buckling of thin plates can be modeled by an eigenvalue problem involving the stress tensor of the plane stress problem corresponding to a given load situation tangential to the plate. The smallest eigenvalues corresponds to the critical parameter multiplying the given plane stress load that results in buckling.

Thin plates are modeled by fourth order differential equations according to the Kirchhoff–Love theory and require special attention when discretized using the finite element method. In this paper we use the continuous/discontinuous Galerkin (c/dG) method proposed by Engel et al. [1] which is based on standard continuous piecewise polynomial spaces of order greater or equal to two inserted into a discontinuous Galerkin formulation, see Hansbo and Larson [3], of the fourth order plate equation. We refer also to Wells and Dung [8] for a method closely related to the one presented here.

The c/dG formulation has the advantage that it uses standard finite element spaces, is easy to implement, and extends naturally to higher order polynomials. Another important advantage in this particular problem is that we may solve the plane stress problem using the same finite element spaces. Note that this would not be the case if we, for instance, used nonconforming Morley elements for the plate problem since this element can not be used for the plane stress problem.

In this paper we derive a posteriori error estimates for the critical buckling load and mode corresponding to the first eigenpair. The error estimates are derived using duality techniques and are based on Larson [6] where a posteriori error estimates for the Laplacian was presented. A particular feature of the estimates presented herein is that we also take the effect of discretization of the plane stress problem into account. The error analysis of the buckling problem results in a specific goal functional which should be controlled in the plane stress solver. Here we follow the general approach to error estimation for one-way coupled problems developed by Larson and Bengzon [7], and adapted to second order plate theory in [2]. In this context we also mention the work [5] by Heuveline and Rannacher where a posteriori error estimates for a nonsymmetric eigenvalue problem related to the linearized stability of the Navier-Stokes equations is presented. These estimates also involves the effect of the accuracy in the computed flow field on the eigenvalue problem and are thus related to our approach.

This paper is organized as follows: in Section 2 we present the Kirchhoff– Love buckling problems and the continuous/discontinuous Galerkin method, in Section 3 we derive the a posteriori error estimates, in Section 4 we present some numerical results, and in Section 5 we present some conclusions.

2 The Buckling Problem and Finite Element Method

2.1 The Kirchhoff-Love Buckling Eigenvalue Problem

Let a plate occupy a domain Ω in \mathbf{R}^2 with boundary $\partial \Omega = \partial \Omega_C \cup \partial \Omega_S$ partitioned into a closed subset $\partial \Omega_C$ and an open subset $\partial \Omega_S$ where clamped and

simply supported boundary conditions are respectively applied. The Kirchhoff-Love buckling problem then takes the form: find the plate displacements $u_{\rm P}$ (orthogonal to the plate) such that

div div
$$\boldsymbol{\sigma}_{\mathrm{P}}(\nabla u_{\mathrm{P}}) - \operatorname{div} t(\boldsymbol{\sigma}_{\mathrm{M}} \nabla u_{\mathrm{P}}) = f_{\mathrm{P}}$$
 in Ω (1)

$$u_{\rm P} = 0 \quad \text{on } \partial\Omega \tag{2}$$

$$\boldsymbol{n} \cdot \nabla u_{\mathrm{P}} = 0 \quad \text{on } \partial \Omega_C$$
 (3)

$$\boldsymbol{n} \cdot \boldsymbol{\sigma}_{\mathrm{P}}(\nabla u_{\mathrm{P}})\boldsymbol{n} = 0 \quad \text{on } \partial\Omega_{S} \tag{4}$$

where t denotes the thickness of the plate, $f_{\rm P}$ is a given load, and the membrane stress tensor

$$\boldsymbol{\sigma}_{\mathrm{M}} = \frac{Y}{1+\nu} \boldsymbol{\varepsilon}(\boldsymbol{u}_{\mathrm{M}}) + \frac{Y\nu}{1-\nu^{2}} \mathrm{tr} \; \boldsymbol{\varepsilon}(\boldsymbol{u}_{\mathrm{M}}) \boldsymbol{I}$$
(5)

is determined by the following plane stress problem: find the membrane displacements $u_{\rm M}$ (tangential to the plate) such that

$$-\operatorname{div} \boldsymbol{\sigma}_{\mathrm{M}}(\boldsymbol{u}_{\mathrm{M}}) = \boldsymbol{f}_{\mathrm{M}} \qquad \text{in } \boldsymbol{\Omega}$$
(6)

$$\boldsymbol{u}_{\mathrm{M}} = 0 \qquad \text{on } \partial \Omega_D \qquad (7)$$

$$\boldsymbol{n} \cdot \boldsymbol{\sigma}_{\mathrm{M}}(\boldsymbol{u}_{\mathrm{M}}) = \boldsymbol{g}_{\mathrm{M}} \qquad \text{on } \partial \Omega_{N}$$

$$\tag{8}$$

where $\boldsymbol{f}_{\mathrm{M}}$ and $\boldsymbol{g}_{\mathrm{M}}$ are given loads and $\partial \Omega = \partial \Omega_D \cup \partial \Omega_N$ is a partition of the boundary into a closed set $\partial \Omega_D$ and an open set $\partial \Omega_N$ where Dirichlet and Neumann boundary conditions, respectively, are applied. Further

$$\boldsymbol{\sigma}_{\mathrm{P}}(\boldsymbol{\varepsilon}) = \frac{Yt^3}{12(1-\nu^2)} \left((1-\nu)\boldsymbol{\varepsilon} + \nu \mathrm{tr}(\boldsymbol{\varepsilon}) \, \boldsymbol{I} \right) \tag{9}$$

is the plate stress tensor, $\boldsymbol{\varepsilon}(\boldsymbol{v}) = (\nabla \boldsymbol{v} + (\nabla \boldsymbol{v})^T)/2$ is the strain tensor, Y is the Young's modulus, and ν is the Poisson ratio.

Scaling the membrane loads by a parameter $\lambda_{\rm P}$, i.e., replacing the loads by $\lambda_{\rm P} \boldsymbol{f}_{\rm M}$ and $\lambda_{\rm P} \boldsymbol{g}_{\rm M}$ we note that by linearity $\boldsymbol{\sigma}_{\rm M}$ is replaced by $\lambda_{\rm P} \boldsymbol{\sigma}_{\rm M}$. The critical buckling loads are then determined by the eigenvalue problem: find $u_{\rm P}$ and $\lambda_{\rm P}$ such that

div div
$$\boldsymbol{\sigma}_{\mathrm{P}}(\nabla u_{\mathrm{P}}) - \operatorname{div} t(\lambda_{\mathrm{P}}\boldsymbol{\sigma}_{\mathrm{M}}\nabla u_{\mathrm{P}}) = 0$$
 in Ω (10)

$$u_{\rm P} = 0 \quad \text{on } \partial \Omega \tag{11}$$

$$\boldsymbol{n} \cdot \nabla u_{\mathrm{P}} = 0 \quad \text{on } \partial \Omega \tag{12}$$

The corresponding variational formulation reads: find the plate displacement $u_{\rm P} \in \mathcal{V}_{\rm P} = \{v \in H^2(\Omega) \mid v = 0 \text{ on } \partial\Omega \text{ and } \boldsymbol{n} \cdot \nabla v = 0 \text{ on } \partial\Omega_C\}$ and eigenvalue $\lambda_{\rm P} \in \mathbf{R}$ such that

$$a_{\rm P}(\nabla u_{\rm P}, \nabla v) + \lambda_{\rm P} t(\boldsymbol{\sigma}_{\rm M} \nabla u_{\rm P}, \nabla v) = 0 \quad \forall v \in \mathcal{V}_{\rm P}$$
(13)

and $\boldsymbol{\sigma}_{\mathrm{M}} \in [L^2(\Omega)]^{2 \times 2}$ is defined by (5), with $\boldsymbol{u}_{\mathrm{M}} \in \mathcal{V}_{\mathrm{M}} = [H_0^1(\Omega)]^2$ the solution of

$$a_{\mathrm{M}}(\boldsymbol{u}_{\mathrm{M}}, \boldsymbol{v}) = l_{\mathrm{M}}(\boldsymbol{v}), \quad \forall \boldsymbol{v} \in \mathcal{V}_{\mathrm{M}}$$
(14)

Here the forms $a_{\rm P}(\cdot, \cdot)$, $a_{\rm M}(\cdot, \cdot)$, and $l_{\rm M}(\cdot, \cdot)$ are defined by

$$a_{\rm P}(\boldsymbol{\theta}, \boldsymbol{\vartheta}) = (\boldsymbol{\sigma}_{\rm P}(\boldsymbol{\theta}), \boldsymbol{\varepsilon}(\boldsymbol{\vartheta})) \tag{15}$$

$$a_{\rm M}(\boldsymbol{\theta},\boldsymbol{\vartheta}) = (\boldsymbol{\sigma}_{\rm M}(\boldsymbol{\theta}),\boldsymbol{\varepsilon}(\boldsymbol{\vartheta})) \tag{16}$$

$$l_{\rm M}(\boldsymbol{v}) = (\boldsymbol{f}_{\rm M}, \boldsymbol{v}) + (\boldsymbol{g}_{\rm M}, \boldsymbol{v})_{\partial \Omega_N}$$
(17)

where $(\cdot, \cdot)_{\omega}$ is the $L^2(\omega)$ inner product and, for brevity, we write $(\cdot, \cdot)_{\Omega} = (\cdot, \cdot)$.

2.2 The Mesh and Finite Element Spaces

We consider a subdivision $\mathcal{T} = \{T\}$ of Ω into a geometrically conforming finite element mesh that respects the two partitions of the boundary, $\partial \Omega =$ $\partial \Omega_C \cup \partial \Omega_S$ and $\partial \Omega = \partial \Omega_D \cup \partial \Omega_N$. We assume that the elements are shape regular, i.e., the quotient of the diameter of the smallest circumscribed sphere and the largest inscribed sphere is uniformly bounded. We denote by h_T the diameter of element T and by $h = \max_{T \in \mathcal{T}} h_T$ the global mesh size parameter.

Let the space of continuous piecewise polynomials of order k on be defined by

$$\mathcal{CP}_k = \{ v \in C^0(\Omega) : v|_T \in \mathcal{P}_k(T) \; \forall T \in \mathcal{T} \}$$
(18)

where $\mathcal{P}_k(T)$ is the space of polynomials of order k defined on T.

We introduce the Scott-Zhang type interpolation operators $\pi_{\rm P} : \mathcal{V}_{\rm P} \to \mathcal{CP}_k \cap \mathcal{V}_{\rm P}$ and recall the following elementwise interpolation error estimate

$$|u - \pi_{\mathcal{P}} u|_{T,m} \le C h_T^{s-m} |u|_{\mathcal{N}(T),s} \tag{19}$$

where $0 \leq m \leq s \leq k+1$ and $\mathcal{N}(T)$ is the union of all elements which are neighbors to element T. The corresponding interpolation operator for the membrane problem is denoted by π_{M} .

To define our method we introduce the set of edges in the mesh, $\mathcal{E} = \{E\}$, and we split \mathcal{E} into three disjoint subsets

$$\mathcal{E} = \mathcal{E}_I \cup \mathcal{E}_C \cup \mathcal{E}_D \tag{20}$$

where \mathcal{E}_I is the set of edges in the interior of Ω , \mathcal{E}_D is the set of edges on $\partial \Omega_D$, and \mathcal{E}_S is the set of edges on $\partial \Omega_S$. Further, with each edge we associate a fixed unit normal \boldsymbol{n} such that for edges on the boundary \boldsymbol{n} is the exterior

unit normal. We denote the jump of a function $\boldsymbol{v} \in \boldsymbol{\Gamma}_h$ at an edge E by $[\boldsymbol{v}] = \boldsymbol{v}^+ - \boldsymbol{v}^-$ for $E \in \mathcal{E}_I$ and $[\boldsymbol{v}] = \boldsymbol{v}^+$ for $E \in \mathcal{E}_B$, and the average $\langle \boldsymbol{v} \rangle = (\boldsymbol{v}^+ + \boldsymbol{v}^-)/2$ for $E \in \mathcal{E}_I$ and $\langle \boldsymbol{v} \rangle = \boldsymbol{v}^+$ for $E \in \mathcal{E}_B$, where $\boldsymbol{v}^{\pm} = \lim_{\epsilon \downarrow 0} \boldsymbol{v}(\boldsymbol{x} \mp \epsilon \boldsymbol{n})$ with $\boldsymbol{x} \in E$.

2.3 The Continuous/Discontinuous Galerkin Method

We shall solve the membrane equation using standard continuous Galerkin and the plate problem with the continuous/discontinuous Galerkin method. The method takes the form: find $U_{\rm P} \in C\mathcal{P}_{k_{\rm P}} \cap \mathcal{V}_{\rm P}$ and $\Lambda_{\rm P} \in \mathbf{R}$ such that

$$A_{\rm P}(\nabla U_{\rm P}, \nabla v) + \Lambda_{\rm P} t(\boldsymbol{\Sigma}_{\rm M} \nabla U_{\rm P}, \nabla v) = 0 \quad \forall v \in \mathcal{CP}_{k_{\rm P}} \cap \mathcal{V}_{\rm P}$$
(21)

where $\boldsymbol{\Sigma}_{\mathrm{M}} = Y/(1+\nu)\boldsymbol{\varepsilon}(\boldsymbol{U}_{\mathrm{M}}) + Y\nu/(1-\nu^2) \mathrm{tr} \ \boldsymbol{\varepsilon}(\boldsymbol{U}_{\mathrm{M}})\boldsymbol{I}$ with $\boldsymbol{U}_{\mathrm{M}} \in [\mathcal{CP}_{k_{\mathrm{M}}}]^2 \cap \mathcal{V}_{\mathrm{M}}$ determined by

$$a_{\mathrm{M}}(\boldsymbol{U}_{\mathrm{M}},\boldsymbol{v}) = (\boldsymbol{f}_{\mathrm{M}},\boldsymbol{v}) \quad \forall \boldsymbol{v} \in [\mathcal{CP}_{k_{\mathrm{M}}}]^2 \cap \mathcal{V}_{\mathrm{M}}$$
 (22)

The bilinear form $A_{\rm P}(\cdot, \cdot)$ is defined by

$$A_{\mathrm{P}}(\boldsymbol{\theta}, \boldsymbol{\vartheta}) = \sum_{T \in \mathcal{T}} (\boldsymbol{\sigma}_{\mathrm{P}}(\boldsymbol{\theta}), \boldsymbol{\varepsilon}(\boldsymbol{\vartheta}))_{T} - \sum_{E \in \mathcal{E}_{I} \cup \mathcal{E}_{C}} (\langle \boldsymbol{n} \cdot \boldsymbol{\sigma}_{\mathrm{P}}(\boldsymbol{\theta}) \rangle, [\boldsymbol{\vartheta}])_{E} - \sum_{E \in \mathcal{E}_{I} \cup \mathcal{E}_{C}} ([\boldsymbol{\theta}], \langle \boldsymbol{n} \cdot \boldsymbol{\sigma}_{\mathrm{P}}(\boldsymbol{\vartheta}) \rangle)_{E} + \frac{Yt^{3}}{12(1-\nu)} \gamma \sum_{E \in \mathcal{E}_{I} \cup \mathcal{E}_{C}} h_{E}^{-1}([\boldsymbol{\theta}], [\boldsymbol{\vartheta}])_{E}$$
(23)

for all $\boldsymbol{\theta}, \boldsymbol{\vartheta} \in \bigoplus_{T \in \mathcal{T}} [H^1(T)]^2$. Here γ is a positive parameter and h_E is defined by

$$h_E = \left(|T^+| + |T^-| \right) / (2|E|) \quad \text{for } E = \partial T^+ \cap \partial T^-$$
(24)

with |T| the area of T, on each edge E. See [4] for details on the value of γ .

3 A Posteriori Error Estimates

3.1 Preliminaries

We first define a projector onto the eigenspace $\operatorname{Eig}(\lambda_{\rm P})$ (corresponding to the eigenvalue $\lambda_{\rm P}$) which is associated with the natural scalar products involved

in the variational statement. We define $\mathcal{P}_{\lambda_{\mathrm{P}}} : \mathcal{V}_{\mathrm{P}} \to \mathrm{Eig}(\lambda_{\mathrm{P}})$ as follows

$$A_{\rm P}(\nabla \mathcal{P}_{\lambda_{\rm P}} v, \nabla w) = A_{\rm P}(\nabla v, \nabla w) \quad \forall w \in \operatorname{Eig}(\lambda_{\rm P})$$
(25)

Note that since w are eigenfunctions associated with $\lambda_{\rm P}$ the projection also satisfies the following equation

$$(\sigma_{\mathrm{M}} \nabla \mathcal{P}_{\lambda_{\mathrm{P}}} v, \nabla w) = (\sigma_{\mathrm{M}} \nabla v, \nabla w) \quad \forall w \in \mathrm{Eig}(\lambda_{\mathrm{P}})$$
(26)

We introduce the norm

$$|||v|||^{2} = A_{\mathrm{P}}(\nabla v, \nabla v), \quad \forall v \in H_{0}^{2}(\Omega) \cup \mathcal{CP}_{k_{\mathrm{P}},0}$$

$$(27)$$

and normalize computed eigenvectors $U_{\rm P}$ as follows

$$\||\nabla U_{\rm P}\||^2 = A_{\rm P}(\nabla U_{\rm P}, \nabla U_{\rm P}) = |\Lambda_{\rm P}(\Sigma_{\rm M} \nabla U_{\rm P}, \nabla U_{\rm P})| = 1$$
(28)

3.2 Error Representation Formulas

The Dual Problem. 3.2.1

To derive error representation formulas we introduce the following dual problem: find $\phi_{\rm P}$ such that

div div
$$\boldsymbol{\sigma}_{\mathrm{P}}(\nabla\phi_{\mathrm{P}}) - \operatorname{div} t\lambda_{\mathrm{P}}(\boldsymbol{\sigma}_{\mathrm{M}}\nabla\phi_{\mathrm{P}}) = \psi_{\mathrm{P}} \operatorname{in} \Omega$$
 (29)

 $p(\boldsymbol{\sigma}_{\mathrm{M}}, \boldsymbol{\psi}_{\mathrm{Yr}}) = 0 \text{ on } \partial\Omega$ $\boldsymbol{\nabla}_{\mathrm{V}} = 0 \text{ on } \partial\Omega$ (30)

$$\boldsymbol{n} \cdot \nabla \phi_{\mathrm{P}} = 0 \text{ on } \partial \Omega_C \tag{31}$$

$$\boldsymbol{n} \cdot \boldsymbol{\sigma}_{\mathrm{P}}(\nabla \phi_{\mathrm{P}}) \boldsymbol{n} = 0 \text{ on } \partial \Omega_{S}$$
(32)

Different choices of $\psi_{\rm P}$ will lead to estimates for the errors in eigenvalues and eigenvectors; they will be chosen in such a way that the solution to the dual problem is well defined. We return to these issues below.

Multiplying with the error $e_{\rm P} = u_{\rm P} - U_{\rm P}$ and integrating by parts we obtain

$$(e_{\rm P}, \psi_{\rm P}) = (e_{\rm P}, \operatorname{div} \operatorname{div} \boldsymbol{\sigma}_{\rm P}(\nabla \phi_{\rm P})) - (e_{\rm P}, \operatorname{div} t\lambda_{\rm P}(\boldsymbol{\sigma}_{\rm M} \nabla \phi_{\rm P}))$$

$$= \sum_{T \in \mathcal{T}} (\boldsymbol{\sigma}_{\rm P}(\nabla e_{\rm P}), \boldsymbol{\varepsilon}(\nabla \phi_{\rm P}))_{T} - \sum_{E \in \mathcal{E}_{I} \cup \mathcal{E}_{C}} ([\nabla e_{\rm P}], \boldsymbol{n} \cdot \boldsymbol{\sigma}_{\rm P}(\nabla \phi_{\rm P}))_{E} + \lambda_{\rm P} t(\boldsymbol{\sigma}_{\rm M} \nabla e_{\rm P}, \nabla \phi_{\rm P})$$

$$= A_{\rm P}(\nabla e_{\rm P}, \nabla \phi_{\rm P}) + t\lambda_{\rm P}(\boldsymbol{\sigma}_{\rm M} \nabla e_{\rm P}, \nabla \phi_{\rm P})$$

$$(33)$$

$$= -A_{\rm P}(\nabla U_{\rm P}, \nabla \phi_{\rm P}) - t\lambda_{\rm P}(\boldsymbol{\sigma}_{\rm M} \nabla U_{\rm P}, \nabla \phi_{\rm P})$$

$$= -A_{\rm P}(\nabla U_{\rm P}, \nabla (\phi_{\rm P} - \pi_{\rm P} \phi_{\rm P}))$$

$$(36)$$

$$= -A_{\rm P}(\nabla U_{\rm P}, \nabla(\phi_{\rm P} - \pi_{\rm P}\phi_{\rm P})) -\Lambda_{\rm P}t(\Sigma_{\rm M}\nabla U_{\rm P}, \nabla(\phi_{\rm P} - \pi_{\rm P}\phi_{\rm P})) + (\Lambda_{\rm P} - \lambda_{\rm P})t(\boldsymbol{\sigma}_{\rm M}\nabla U_{\rm P}, \nabla\phi_{\rm P}) -\Lambda_{\rm P}t((\boldsymbol{\sigma}_{\rm M} - \Sigma_{\rm M})\nabla U_{\rm P}, \nabla\phi_{\rm P})$$
(37)

where in (34) we used the fact that $[\nabla e_{\rm P}]$ is parallel to the normal at each edge and in particular for $E \in \mathcal{E}_S$ we thus have $([\nabla e_{\rm P}], \boldsymbol{n} \cdot \boldsymbol{\sigma}_{\rm P}(\nabla \phi_{\rm P}))_E =$ $([\nabla e_{\rm P}], \boldsymbol{n} \cdot \boldsymbol{\sigma}_{\rm P}(\nabla \phi_{\rm P})\boldsymbol{n})_E = 0$; in (35) we used the fact that $[\nabla \phi_{\rm P}] = 0$; in (36) we eliminated the exact solution; and in (37) we rearranged the terms using the identity $\lambda_{\rm P} \boldsymbol{\sigma}_{\rm M} = \Lambda_{\rm P} \boldsymbol{\Sigma}_{\rm M} - (\Lambda_{\rm P} - \lambda_{\rm P}) \boldsymbol{\sigma}_{\rm M} + \Lambda_{\rm P}(\boldsymbol{\sigma}_{\rm M} - \boldsymbol{\Sigma}_{\rm M})$ and finally used Galerkin orthogonality (21) to subtract $\pi_{\rm P} \phi_{\rm P}$.

3.2.2 Representation of the Error in the Eigenvalue

Setting $\psi_{\rm P} = 0$ and denoting the solution to the dual problem by $\phi_{{\rm P},\lambda_{\rm P}}$ we get

$$(\Lambda_{\rm P} - \lambda_{\rm P})t(\boldsymbol{\sigma}_{\rm M}\nabla U_{\rm P}, \nabla\phi_{\rm P,\lambda_{\rm P}}) = A_{\rm P}(\nabla U_{\rm P}, \nabla(\phi_{\rm P,\lambda_{\rm P}} - \pi_{\rm P}\phi_{\rm P,\lambda_{\rm P}})) + \Lambda_{\rm P}t(\boldsymbol{\Sigma}_{\rm M}\nabla U_{\rm P}, \nabla(\phi_{\rm P,\lambda_{\rm P}} - \pi_{\rm P}\phi_{\rm P,\lambda_{\rm P}})) + \Lambda_{\rm P}t((\boldsymbol{\Sigma}_{\rm M} - \boldsymbol{\sigma}_{\rm M})\nabla U_{\rm P}, \nabla\phi_{\rm P,\lambda_{\rm P}})$$
(38)

In this case the solution to the dual problem is an arbitrary eigenfunction associated with $\lambda_{\rm P}$, i.e., $\phi_{\rm P,\lambda_{\rm P}} \in {\rm Eig}(\lambda_{\rm P})$. Choosing $\phi_{\rm P,\lambda_{\rm P}} = \mathcal{P}_{\lambda_{\rm P}} U_{\rm P} / ||| \mathcal{P}_{\lambda_{\rm P}} U_{\rm P} |||$ we obtain the following estimate

$$\begin{aligned} |(\Lambda_{\rm P} - \lambda_{\rm P})t(\boldsymbol{\sigma}_{\rm M}\nabla U_{\rm P}, \nabla\phi_{\rm P})| \\ &= |(\Lambda_{\rm P} - \lambda_{\rm P})\lambda_{\rm P}^{-1}A_{\rm P}(\nabla U_{\rm P}, \nabla\phi_{{\rm P},\lambda_{\rm P}})| \\ &= |(\Lambda_{\rm P} - \lambda_{\rm P})\lambda_{\rm P}^{-1}| |A_{\rm P}(\nabla \mathcal{P}_{\lambda_{\rm P}}U_{\rm P}, \nabla\phi_{{\rm P},\lambda_{\rm P}})| \\ &= |(\Lambda_{\rm P} - \lambda_{\rm P})\lambda_{\rm P}^{-1}| |||\nabla \mathcal{P}_{\lambda_{\rm P}}U_{\rm P}||| \end{aligned}$$
(39)

We now assume that the computed eigenvalue $\Lambda_{\rm P}$ approximates the exact eigenvalue $\lambda_{\rm P}$ and that there are constants $0 \leq \delta < 1$ and h_0 such that

$$\||\nabla (I - \mathcal{P}_{\lambda_{\mathrm{P}}})U_{\mathrm{P}}\|| \le \delta \tag{40}$$

for all meshes with $\max_{T \in \mathcal{T}} h_K \leq h_0$. We remark that the validity of this assumption follows from standard a priori convergence theory. Using (39), (40), and the scaling (28) together with Pythagoras identity we obtain

$$|(\Lambda_{\rm P} - \lambda_{\rm P})t(\boldsymbol{\sigma}_{\rm M}\nabla U_{\rm P}, \nabla\phi_{\rm P})| \ge |(\Lambda_{\rm P} - \lambda_{\rm P})\lambda_{\rm P}^{-1}|(1 - \delta^2)^{1/2}$$
(41)

Finally, combining (38), (41), and using the triangle inequality we arrive at

$$(1 - \delta^{2})^{1/2} |(\Lambda_{\rm P} - \lambda_{\rm P})\lambda_{\rm P}^{-1}| \leq |A_{\rm P}(\nabla U_{\rm P}, \nabla(\phi_{{\rm P},\lambda_{\rm P}} - \pi_{\rm P}\phi_{{\rm P},\lambda_{\rm P}})) + \Lambda_{\rm P}t(\boldsymbol{\Sigma}_{\rm M}\nabla U_{\rm P}, \nabla(\phi_{{\rm P},\lambda_{\rm P}} - \pi_{\rm P}\phi_{{\rm P},\lambda_{\rm P}}))| + |\Lambda_{\rm P}t((\boldsymbol{\sigma}_{\rm M} - \boldsymbol{\Sigma}_{\rm M})\nabla U_{\rm P}, \nabla\phi_{{\rm P},\lambda_{\rm P}})|$$
(42)

3.2.3 Representation of the Error in the Eigenvector.

Following Larson [6] we define the error in an eigenvector to be the component orthogonal to the exact eigenspace which it approximates an element in. Note that this definition has the advantage that it covers also multiple eigenvectors. More precisely we will estimate the error in the $H^m(\Omega)$ seminorm for m = 0, 1. We then define the error e_M as

$$e_{\rm M} = (I - P_{\rm M})U_{\rm P} \tag{43}$$

where $P_{\rm M}$ is the orthogonal projection $H^m(\Omega) \to \operatorname{Eig}(\lambda_{\rm P})$ defined by $(v - P_0 v, w) = 0$ and $(\nabla (v - P_1 v, \nabla w)$ for all $w \in \operatorname{Eig}(\lambda_{\rm P})$ and m = 0, 1, respectively. To represent the semi norm $|e_{\rm P}|_{\rm M}$ we let $\psi_{\rm P} = \psi_{{\rm P},m} = (-\Delta)^m e_{{\rm P},m}/|e_{{\rm P},m}|_{\rm M}$ with m = 0, 1 and we denote the corresponding solution to the dual problem by $\phi_{{\rm P},m}, m = 0, 1$. We then get

$$|e_{\mathrm{P},m}|_{\mathrm{M}} = -A_{\mathrm{P}}(\nabla U_{\mathrm{P}}, \nabla(\phi_{\mathrm{P},m} - \pi_{\mathrm{P}}\phi_{\mathrm{P},m})) -\Lambda_{\mathrm{P}}t(\boldsymbol{\Sigma}_{\mathrm{M}}\nabla U_{\mathrm{P}}, \nabla(\phi_{\mathrm{P},u} - \pi_{\mathrm{P}}\phi_{\mathrm{P},m})) + (\Lambda_{\mathrm{P}} - \lambda_{\mathrm{P}})t(\boldsymbol{\sigma}_{\mathrm{M}}\nabla U_{\mathrm{P}}, \nabla\phi_{\mathrm{P},m}) + \lambda_{\mathrm{P}}t((\boldsymbol{\Sigma}_{\mathrm{M}} - \boldsymbol{\sigma}_{\mathrm{M}})\nabla U_{\mathrm{P}}, \nabla\phi_{\mathrm{P},m})$$
(44)

In this case we require the solution $\phi_{\mathbf{P},m}$ to be orthogonal to $\operatorname{Eig}(\lambda_{\mathbf{P}})$ to achieve uniqueness.

Next we estimate the second term on the right hand side as follows

$$\lambda_{\mathrm{P}}t(\boldsymbol{\sigma}_{\mathrm{M}}\nabla U_{\mathrm{P}}, \nabla\phi_{\mathrm{P},m}) = A_{\mathrm{P}}(\nabla U_{\mathrm{P}}, \nabla\phi_{\mathrm{P},m})$$

$$= A_{\mathrm{P}}(\nabla e_{\mathrm{P},m}, \nabla\phi_{\mathrm{P},m})$$

$$\leq |e_{\mathrm{P},m}|_{\mathrm{M}}|\phi_{\mathrm{P},m}|_{4-m}$$

$$\leq C_{\mathrm{M}}|e_{\mathrm{P},m}|_{\mathrm{M}}|\psi_{\mathrm{P},m}|_{-m}$$

$$\leq C_{\mathrm{M}}|e_{\mathrm{P},m}|_{\mathrm{M}} \qquad (45)$$

where we used the stability estimate $|\phi_{P,m}|_{4-m} \leq C_M |\psi_{P,m}|_{-m}$ and at last the identity $|\psi_{P,m}|_{-m} = 1$ which follows from the definition of $\psi_{P,m}$. Thus we have

$$|(\Lambda_{\rm P} - \lambda_{\rm P})t(\boldsymbol{\sigma}_{\rm M}\nabla U_{\rm P}, \nabla\phi_{{\rm P},m})| \le |(\Lambda_{\rm P} - \lambda_{\rm P})\lambda_{\rm P}^{-1}|C_{\rm M}|e_{{\rm P},m}|_{\rm M}$$
(46)

Now again assuming that the computed eigenvalue $\Lambda_{\rm P}$ approximates the exact eigenvalue $\lambda_{\rm P}$ and that there are constants $0 \leq \delta < 1$ and h_0 such that

$$|(\Lambda_{\rm P} - \lambda_{\rm P})\lambda_{\rm P}^{-1}|C_{\rm M} \le \delta \tag{47}$$

for all meshes with $\max_{T \in \mathcal{T}} h_K \leq h_0$. We note again that the validity of this assumption follows from standard a priori convergence theory. Combining (44), (46), and (47) and using the triangle inequality we obtain the estimate

$$(1 - \delta)|e_{\mathrm{P}}|_{\mathrm{M}} \leq |A_{\mathrm{P}}(\nabla U_{\mathrm{P}}, \nabla(\phi_{\mathrm{P},m} - \pi_{\mathrm{P}}\phi_{\mathrm{P},m})) + \Lambda_{\mathrm{P}}t(\boldsymbol{\Sigma}_{\mathrm{M}}\nabla U_{\mathrm{P}}, \nabla(\phi_{\mathrm{P},m} - \pi_{\mathrm{P}}\phi_{\mathrm{P},m}))| + |\lambda_{\mathrm{P}}t((\boldsymbol{\Sigma}_{\mathrm{M}} - \boldsymbol{\sigma}_{\mathrm{M}})\nabla U_{\mathrm{P}}, \nabla\phi_{\mathrm{P},m})|$$

$$(48)$$

Remark. The constant $C_{\rm M}$ is of the form

$$C_{\rm M} = \frac{c_{\rm M}}{\text{gap}(\lambda_{\rm P})} \tag{49}$$

where $gap(\lambda_P)$ is the distance between λ_P and the closest eigenvalue. Thus assumption (47) guarantees satisfactory resolution of the spectrum in the vicinity of λ_P .

3.2.4 Representation of the Modeling Error.

Introducing the dual problem: find $\phi_{M,X} \in \mathcal{V}_M$ such that

$$a_{\rm M}(\boldsymbol{v}, \boldsymbol{\phi}_{{\rm M}, X}) = (\boldsymbol{\sigma}_{\rm M}(\boldsymbol{v}) \nabla U_{\rm P}, \nabla \phi_{{\rm P}, X}), \tag{50}$$

for all $\boldsymbol{v} \in \mathcal{V}_{\mathrm{M}}, X \in \{0, 1, \lambda_{\mathrm{P}}\}$, we get, by setting $\boldsymbol{v} = \boldsymbol{e}_{\mathrm{M}}$ and using Galerkin orthogonality (22) for the membrane equation, the following error representation formula

$$((\boldsymbol{\Sigma}_{\mathrm{M}} - \boldsymbol{\sigma}_{\mathrm{M}}) \nabla U_{\mathrm{P}}, \nabla \phi_{\mathrm{P},X}) = (\boldsymbol{\sigma}_{\mathrm{M}}(\boldsymbol{e}_{\mathrm{M}}) \nabla U_{\mathrm{P}}, \nabla \phi_{\mathrm{P},X})$$
$$= a_{\mathrm{M}}(\boldsymbol{e}_{\mathrm{M}}, \boldsymbol{\phi}_{\mathrm{M},X})$$
$$= a_{\mathrm{M}}(\boldsymbol{e}_{\mathrm{M}}, \boldsymbol{\phi}_{\mathrm{M},X} - \boldsymbol{\pi}_{\mathrm{M}}\boldsymbol{\phi}_{\mathrm{M},X})$$
$$= l_{\mathrm{M}}(\boldsymbol{\phi}_{\mathrm{M},X} - \boldsymbol{\pi}_{\mathrm{M}}\boldsymbol{\phi}_{\mathrm{M},X})$$
$$- a_{\mathrm{M}}(\boldsymbol{U}_{\mathrm{M}}, \boldsymbol{\phi}_{\mathrm{M},X} - \boldsymbol{\pi}_{\mathrm{M}}\boldsymbol{\phi}_{\mathrm{M},X})$$
(51)

where at the final step we eliminated the exact solution using (6)

Combining the estimates above we obtain the following abstract error estimates. For the error in the eigenvalue

$$(1 - \delta^{2})^{1/2} |(\Lambda_{\rm P} - \lambda_{\rm P})\lambda_{\rm P}^{-1}| \leq |A_{\rm P}(\nabla U_{\rm P}, \nabla(\phi_{{\rm P},\lambda_{\rm P}} - \pi_{\rm P}\phi_{{\rm P},\lambda_{\rm P}})) + \Lambda_{\rm P}t(\boldsymbol{\sigma}_{\rm M}\nabla U_{\rm P}, \nabla(\phi_{{\rm P},\lambda_{\rm P}} - \pi_{\rm P}\phi_{{\rm P},\lambda_{\rm P}}))| + |l_{\rm M}(\phi_{{\rm M},\lambda_{\rm P}} - \boldsymbol{\pi}_{\rm M}\phi_{{\rm M},\lambda_{\rm P}}) - a_{\rm M}(\boldsymbol{U}_{\rm M}, \boldsymbol{\phi}_{{\rm M},\lambda_{\rm P}} - \boldsymbol{\pi}_{\rm M}\phi_{{\rm M},\lambda_{\rm P}})|$$
(52)

and for the error in the eigenvector

$$(1-\delta)|e_{\mathrm{P}}|_{\mathrm{M}} \leq |A_{\mathrm{P}}(\nabla U_{\mathrm{P}}, \nabla(\phi_{\mathrm{P},m} - \pi_{\mathrm{P}}\phi_{\mathrm{P},m})) + \Lambda_{\mathrm{P}}t(\boldsymbol{\Sigma}_{\mathrm{M}}\nabla U_{\mathrm{P}}, \nabla(\phi_{\mathrm{P},m} - \pi_{\mathrm{P}}\phi_{\mathrm{P},m}))| + |l_{(\boldsymbol{\phi}_{\mathrm{M},m}} - \boldsymbol{\pi}_{\mathrm{M}}\boldsymbol{\phi}_{\mathrm{M},m}) - a_{\mathrm{M}}(\boldsymbol{U}_{\mathrm{M}}, \boldsymbol{\phi}_{\mathrm{M},m} - \boldsymbol{\pi}_{\mathrm{M}}\boldsymbol{\phi}_{\mathrm{M},m})|$$

$$(53)$$

for m = 0, 1.

3.4 Error Estimates Using the Dual Weighted Residual Approach

Using standard procedures, involving integration by parts, the Cauchy-Schwartz inequality, a trace inequality, and the interpolation error estimate (19), we obtain the following estimate

$$|A_{\mathrm{P}}(\nabla U_{\mathrm{P}}, \nabla(\phi_{\mathrm{P},X} - \pi_{\mathrm{P}}\phi_{\mathrm{P},X})) + \Lambda_{\mathrm{P}}t(\boldsymbol{\Sigma}_{\mathrm{M}}\nabla U_{P}, \nabla(\phi_{\mathrm{P},X} - \pi_{\mathrm{P}}\phi_{\mathrm{P},X}))| \leq \sum_{T \in \mathcal{T}_{h}} R_{\mathrm{P},T}W_{\mathrm{P},X,T}$$
(54)

where the plate element residual $R_{P,T}$ and weight $W_{P,X,T}$ are defined by

$$R_{\mathrm{P},T}^{2} = \|f_{\mathrm{P}} - \operatorname{div} \operatorname{div} \boldsymbol{\sigma}_{\mathrm{P}}(\nabla U_{\mathrm{P}})\|_{T}^{2} + h_{T}^{-1} \|[\boldsymbol{n} \cdot \operatorname{div} \boldsymbol{\sigma}_{\mathrm{P}}(\nabla U_{\mathrm{P}})]\|_{\partial T}^{2} + h_{T}^{-3} \|[\boldsymbol{n} \cdot \boldsymbol{\sigma}_{\mathrm{P}}(\nabla U_{\mathrm{P}})]\|_{\partial T}^{2} + \gamma^{2} h_{E}^{-5} \|[\nabla U_{\mathrm{P}}]\|_{\partial T}^{2}$$
(55)

$$W_{\mathbf{P},X,T} = h_T^{\alpha_{\mathbf{P},X}} |\phi_{\mathbf{P},X}|_{\mathcal{N}(K),\alpha_{\mathbf{P}}}, \quad 0 \le \alpha_{\mathbf{P},X} \le k_{\mathbf{P}} + 1$$
(56)

Here the regularity parameter $\alpha_{P,X}$ reflects the regularity properties of the solutions to the dual problems. For the membrane problem we have the corresponding estimate

$$|a_{\mathrm{M}}(\boldsymbol{e}_{\mathrm{M}}, \boldsymbol{\phi}_{\mathrm{M},X} - \boldsymbol{\pi}_{\mathrm{M}}\boldsymbol{\phi}_{\mathrm{M},X})| \leq \sum_{T \in \mathcal{T}_{h}} R_{\mathrm{M},T} W_{\mathrm{M},X,T}$$
(57)

where the residual and weight are defined by

$$R_{\mathrm{M},T}^{2} = \|\boldsymbol{f}_{\mathrm{M}} + \operatorname{div} \boldsymbol{\sigma}_{\mathrm{M}}(\boldsymbol{U}_{\mathrm{M}})\|_{T}^{2} + h_{T}^{-1} \|[\boldsymbol{n} \cdot \operatorname{div} \boldsymbol{\sigma}_{\mathrm{M}}(\boldsymbol{U}_{\mathrm{M}})]\|_{\partial T \setminus \partial \Omega}^{2} + h_{T}^{-1} \|\boldsymbol{g}_{\mathrm{M}} - \boldsymbol{n} \cdot \operatorname{div} \boldsymbol{\sigma}_{\mathrm{M}}(\boldsymbol{U}_{\mathrm{M}})\|_{\partial T \cap \partial \Omega_{N}}^{2}$$
(58)

and

$$W_{\mathrm{M},X,T} = h_T^{\alpha_{\mathrm{M}}} |\phi_{\mathrm{M},X}|_{\mathcal{N}(K),\alpha_{\mathrm{M},X}}$$
(59)

for $0 \leq \alpha_{M,X} \leq k_M + 1, X \in \{\lambda_P, 0, 1\}$. Collecting these estimates and the abstract a posteriori error estimates we finally arrive at the following dual weighted residual a posteriori error estimates

$$(1 - \delta^2)^{1/2} |(\Lambda_{\rm P} - \lambda_{\rm P})\lambda_{\rm P}^{-1}| \le \sum_{T \in \mathcal{T}_h} R_{{\rm P},T} W_{{\rm P},\lambda,T} + \sum_{T \in \mathcal{T}_h} R_{{\rm M},T} W_{{\rm M},\lambda,T}$$
(60)

and for the error in the eigenvector

$$(1-\delta)|e_{\mathbf{P}}|_{m} \leq \sum_{T \in \mathcal{T}_{h}} R_{\mathbf{P},T} W_{\mathbf{P},m,T} + \sum_{T \in \mathcal{T}_{h}} R_{\mathbf{M},T} W_{\mathbf{M},m,T}$$
(61)

m = 0, 1. Considering the expected optimal regularity of the dual problems we get

$$\alpha_{\mathrm{P},\lambda} = k_{\mathrm{P}} + 1, \quad \alpha_{\mathrm{P},m} = 4 - m, \quad \alpha_{\mathrm{M},m} = \alpha_{\mathrm{M},\lambda} = 2 \tag{62}$$

3.5 Residual Based Estimates

Using stability estimates for the solutions to the dual problems we obtain the residual based estimates

$$(1-\delta^2)|(\Lambda_{\rm P}-\lambda_{\rm P})\lambda_{\rm P}^{-1}|^2 \le C\left(\sum_{T\in\mathcal{T}_h} h_T^{2\alpha_{\rm P,\lambda_{\rm P}}} R_{\rm P,T}^2 + \sum_{T\in\mathcal{T}_h} h_T^{2\alpha_{\rm M,\lambda}} R_{{\rm M},T}^2\right)$$
(63)

and for the error in the eigenvector

$$(1-\delta)^{2}|e_{\mathrm{P}}|_{m}^{2} \leq C\left(\sum_{T\in\mathcal{T}_{h}}h_{T}^{2\alpha_{\mathrm{P},m}}R_{\mathrm{P},T}^{2} + \sum_{T\in\mathcal{T}_{h}}h_{T}^{2\alpha_{\mathrm{M},m}}R_{\mathrm{M},T}^{2}\right)$$
(64)

m = 0, 1.

4 Numerical Examples

4.1 Known Stress Tensor

We consider the L-shaped domain $(0,1) \times (0,1) \setminus (1/2,1) \times (0,1/2)$. The plate is simply supported on all boundaries (u = 0), and the in-plane stress tensor is chosen as the unit tensor. Thus, we have no error contribution from the membrane problem. We set Y = 1, $\nu = 1/4$, and t = 1. We use the adaptive algorithm for the computation of the lowest three eigenvalues. The singularity in the inward-pointing corner is excited for the first two but not for the third, which is also clearly visible in the adaptation of the meshes shown in Figures 1–3. In Figure 4, we give the corresponding eigensolution, and in Figure 5 we give the corresponding effectivity indices (approximate error in eigenvalue divided by exact error). The third eigenvalue can be computed analytically, the first two have been estimated by an approximate solution on a dense mesh. The effectivity indices have been computed on a sequence of meshes obtained using a fixed ratio refinement technique where the elements with the highest 25% element error indicators have been refined in each step. The unknown constant in the error representation formula has been set so that the effectivity index is of medium size; the same constant has been used for all three eigenvalues.

4.2 Computed Stress Tensor

For our second example, we use the same domain, material data, and boundary conditions for the plate. For the elasticity computations, we use a body force f = (r, -9r/10), where r denotes the distance from the inward pointing corner. The boundary conditions were: clamped conditions at x = 1/2, $y \le 1/2$, at y = 0, at x = 1, and at y = 1/2, $x \ge 0$. The remaining boundaries were traction free.

In Figure 5 we give the adapted mesh using the full estimate, and, for comparison, we also give, in Figures 6–7, the corresponding meshes when only partial estimates, plate residual and stress residual, respectively, are used. In Figure 9 we show the lowest buckling mode for which the estimate is aiming. Finally, we show, in Figure 10, how the different residuals behave asymptotically as estimates of the eigenvalue error. Clearly, in order to obtain an effectivity index that does not increase or decrease, we need the full residual, though we concede that the balance between these two residuals may be difficult to ascertain. Here we have willfully chosen the balance in order to obtain a reasonably constant effectivity index for the full residual. However, if we instead use the dual weighted residual method and solve the dual problems numerically we can compute the weights numerically and obtain the proper weighting between the plate and membrane residuals.

5 Conclusions

We have formulated a continuous/discontinuous Galerkin method for the buckling problem with second order effects. The method has the advantage that we can solve both the membrane and plate problem with the same standard finite element spaces of continuous piecewise polynomials defined on triangles or bricks. Furthermore, we proved a posteriori error estimates for both the error in the eigenvalue (critical buckling load) and the eigenvectors (buckling modes) with the special feature that also the effect of approximate solution of the membrane problem is taken into account. Based on the estimates we constructed an adaptive algorithm for adaptive mesh refinement.

References

- Engel, G., Garikipati, K., Hughes, T. J. R., Larson, M. G., Mazzei, L., and Taylor, R. L. (2002) Continuous/discontinuous approximations of fourth-order elliptic problems in structural and continuum mechanics with applications to thin bending elements and strain gradient elasticity, Comput. Methods Appl. Mech. Engrg. 191, 3669–3750.
- [2] Hansbo, P., Heintz, D., and Larson, M. G. (2010) An adaptive finite element method for second-order plate theory, Int. J. Numer. Meth. Engrg. 81, 584–603.
- [3] Hansbo, P. and Larson, M. G. (2002) A discontinuous Galerkin method for the plate problem, Calcolo 39, 41–59.
- [4] Hansbo P. and Larson, M. G. (2008) A posteriori error estimates for continuous/discontinuous Galerkin approximations of the Kirchhoff–Love plate. Preprint 2008:10, Department of Mathematical Sciences, Chalmers University of Technology and University of Gothenburg,
- [5] Heuveline, V. and Rannacher, R. (2006) Adaptive FEM for eigenvalue problems with application in hydrodynamic stability analysis, In: 'Advances in Numerical Mathematics', Proc. Int. Conf., Sept. 16-17, 2005, Moscow, Moscow: Institute of Numerical Mathematics RAS.
- [6] Larson, M. G. (2000) A posteriori and a priori error analysis for finite element approximations of self-adjoint elliptic eigenvalue problems, SIAM J. Numer. Anal. 38, 608–625.

- [7] Larson, M. G. and Bengzon, F. (2008) Adaptive finite element approximation of multiphysics problems. Comm. Numer. Methods Engrg. 24, 505–521.
- [8] Wells, G. N. and Dung, N. T. (2007) A C⁰ discontinuous Galerkin formulation for Kirchhoff plates. Comput. Methods Appl. Mech. Engrg. 196, 3370–3380.



Fig. 1. Adapted mesh for the first eigenvalue



Fig. 2. Adapted mesh for the second eigenvalue



Fig. 3. Adapted mesh for the third eigenvalue



Fig. 4. The first three eigensolutions



Fig. 5. Computed effectivity indices for the first three eigenvalue computations.



Fig. 6. Adapted mesh using the full estimate.



Fig. 7. Adapted mesh for a partial estimate (only the plate residual).



Fig. 8. Adapted mesh for a partial estimate (only the stress residual).



Fig. 9. Buckling mode.



Fig. 10. Computed effectivity indices for the full estimate and the partial estimates (plate residual and stress residual).