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BACKWARD-EULER AND MIXED DISCONTINUOUS GALERKIN METHODS FOR THE VLASOV-POISSON SYSTEM PART I: CONVERGENCE ANALYSIS

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Abstract. We construct a numerical method for the two-dimensional Vlasov-Poisson system based on backward-Euler approximation in the time combined with, a mixed finite element method for discretization of the Poisson equation in spatial domain, and a discontinuous Galerkin (DG) finite element approximation in the phase-space variables for the Vlasov equation. We prove stability estimates and derive optimal convergence rates depending upon the compatibility of the finite element meshes, used for the discretizations of the spatial variable in Poisson (mixed) and Vlasov (DG) equations, respectively. The error estimates for the Poisson equation are based on using Brezzi-Douglas-Marini elements in L_2 and $H^{-s},\ s>0$, norms.

1. Introduction

In this paper we study the approximate solution for the deterministic two-dimensional Vlasov-Poisson (VP) system described below: Given the initial distribution of particles density $f_0(x,v)$, $(x,v)\in\Omega_x\times\mathbb{R}^2\subset\mathbb{R}^2\times\mathbb{R}^2$, find the evolution of a plasma formed by charged particles (ions and electrons), at time t, in a bounded open set $\Omega_x\subset\mathbb{R}^2$ with a phase space density $f(x,v,t)$ satisfying

$$\begin{cases} \partial_t f + v\cdot\nabla_x f - \nabla_x\varphi\cdot\nabla_v f = 0, & \text{in } \Omega\times[0,T],\\ f(x,v,0)=f_0(x,v), & \text{in } \Omega=\Omega_x\times\mathbb{R}^2,\\ -\Delta_x\varphi = \displaystyle\int_{\mathbb{R}^2} f(x,v,t)\,dv, & \text{in } \Omega_x\times[0,T],\\ \varphi(x,t)=0, & \text{on } \partial\Omega_x\times[0,T], \end{cases} \tag{1.1}$$

where $\cdot$ denotes the scalar product. To construct numerical methods we shall restrict the velocity variable v to a bounded domain $\Omega_v\subset\mathbb{R}^2$ and provide the equation with a Dirichlet type, inflow boundary condition, in the velocity variable. We also split the equation system to separate Poisson and Vlasov equations coupled with the potential φ. Thus we reformulate the problem (1.1) as follows: given the initial data $f_0(x,v)$, $(x,v)\in\Omega_x\times\Omega_v\subset\mathbb{R}^2\times\mathbb{R}^2$, find the density function $f(x,v,t)$ of the initial-boundary value problem for the Vlasov equation

$$\begin{cases} \partial_t f + v\cdot\nabla_x f - \nabla_x\varphi\cdot\nabla_v f = 0, & \text{in } \ \Omega\times[0,T],\\ f(x,v,0)=f_0(x,v), & \text{in } \ \Omega=\Omega_x\times\Omega_v,\\ f(x,v,t)=0, & \text{on } \ \Gamma_v^-\times\Omega_v\times[0,T], \end{cases} \tag{1.2}$$

where the potential φ satisfies the following Dirichlet problem for Poisson equation

$$\begin{cases} -\Delta_x\varphi = \displaystyle\int_{\Omega_v} f(x,v,t)\,dv, & \text{in } \ \Omega_x\times[0,T],\\ \varphi(x,t)=0, & \text{on } \ \partial\Omega_x\times[0,T]. \end{cases} \tag{1.3}$$

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and $\Gamma_v^- := \{x \in \partial\Omega_x : \mathbf{n}(x) \cdot v < 0\}$, is the inflow boundary of Ω_x, with respect to v and $\mathbf{n}(x)$ is the outward unit normal to $\partial\Omega_x$ at $x \in \partial\Omega_x$. We solve the problem (1.3), replacing f by a given suitable function g. Then inserting the solution, say φ_g, in (1.2) we obtain a new Vlasov equation as (1.2) with φ replaced by φ_g. Assume that we can solve this new Vlasov equation. Then in this way we link its solution f_g to the given function g via, $f_g = \Lambda[g]$. Now a solution f for the original Vlasov equation is a fixed point of the operator Λ: $f = \Lambda[f]$, provided that Λ fulfills the conditions of a Schauder fixed point operator, see [21] for the details. For a discrete version, in a finite dimensional space, the argument relies on the Brouwer fixed point theorem, as in [2] and the reference therein.

Positivity, existence, uniqueness and regularity properties for the continuous problem (1.1) in the full space $\mathbb{R}^{2d}$, $d = 2, 3$, are inherited from those derived for a bounded positive initial data $f_0 \in L_\infty(\mathbb{R}^{2d}) \geq 0$, with the bounded second phase-space moment: $\int_{\mathbb{R}^{2d}}(1 + |x|^2 + |v|^2) f_0 \, dxdv < \infty$, see [6]. Further analytic approaches are given, e.g. by Horst in [15].

We consider the two-dimensional case and study convergence of a numerical scheme consisting of

(i) A mixed Brezzi-Douglas-Marini (BDM) finite element method for the spatial discretization for the Poisson equation (1.3).

(ii) A discontinuous Galerkin (DG) method for space-velocity, variables for the Vlasov equation (1.2).

(iii) A backward-Euler (BE) discretization in time for the Vlasov equation (1.2).

Problem (ii) being hyperbolic would require a finer mesh than the more regular elliptic problem (i). We shall correlate these meshes at the final combined step. However, the numerical approaches for the problems (i) and (ii) are chosen independently, therefore they are presented with different (distinguishable) meshes $\hbar$ and h, respectively.

We start with a continuous time variable and a coarse spatial mesh of size $\hbar$ and solve (i) to obtain $\varphi_\hbar$. Then, in (ii), we replace φ by $\varphi_\hbar$; refine the mesh iteratively (viz, $h = 2^{-j}\hbar$, $j = 0, \dots, M$) and obtain the discrete solution f_h. At each iteration step this procedure yields a, continuous in time, linearized Vlasov equation for f. At the final step we choose the correlated mesh sizes $h \sim \hbar$. The approximation (f_h, φ_h) in (ii) may, roughly, be viewed as a two step procedure viz $(f_h, \varphi_h) \approx (f, \varphi_\hbar) \approx (f, \varphi)$. To perform (ii) we may formulate a linearized Vlasov equation:

$$\partial_t f + v \cdot \nabla_x f - \nabla_x \varphi_\hbar \cdot \nabla_v f = 0, \quad \text{with} \qquad -\Delta_{\hbar,x}\varphi_\hbar = \int_{\Omega_v} f_\hbar(x, v, t)\, dv, \quad (1.4)$$

where $\Delta_{\hbar,x}$ is the discrete Laplacian operator defined by $(-\Delta_{\hbar,x}\varphi_\hbar, u) = (\nabla\varphi_\hbar, \nabla u)$.

Backward Euler approximation in (iii) yields yet an another iteration procedure. It starts from the initial data $f_0(x, t)$ and provides phase-space solutions at each time level t_n, $n = 1, \dots, N$. In performing the time iterations φ_h^{n-1} (depending on f_h^{n-1}) is used to compute f_h^n on the next time level, which yields a fully linearized, discrete in time, Vlasov equation. Observe that the mixed finite elements in BDM-spaces in (i), as described, are for spatial approximation and do not involve time discretizations.

We derive sharp error bounds for (i) and (ii). The convergence rates for the discontinuous Galerkin (ii) and backward Euler (iii) methods are equivalent. Then, combining (i)-(iii) an optimal fully discrete method is constructed by assuming compatibility conditions on the mesh parameters. For sharp approximations, regularity requirements of type, e.g. $\varphi \in W^{r,\infty}(\Omega_x), r \geq 1$, and also a mesh compatibility relation like $h \sim \hbar \sim \Delta t$, will be necessary.

Numerical studies for the Vlasov-Poisson and related equations have been dominated by the particle methods studied, e.g. by Cottet and Raviart in [11]; Ganguly, Lee and Victory in [14]; and Wollman, Ozizmir and Narasimhan in [23].

On the other hand, Raviart Thomas and BDM approaches are extensively used in the finite element approximation of the elliptic, parabolic, and parabolic integro-differential equations with memory. Some related studies in this part are, e.g. the optimal L_∞ study of finite element methods for irregular meshes by Scott in [20]; the two families of mixed finite element methods for the second order elliptic problems by Brezzi, Douglas and Marini in [8], where the BDM spaces are introduced; maximum norm estimates for the finite element approximation of the Stokes problem in 2D by Duran, Nochetto and Wang in [12]; the asymptotic expansions and L_∞ estimates for mixed finite element methods for the second order elliptic problems by Wang in [22], the maximum norm estimates for Ritz-Volterra projection by Lin in [17]; the global superconvergence analysis in $W^{1,\infty}$-norm for Galerkin FEMs of integro-differential equations by Liu, Liu, Rao and Zhang in [19]; the L_∞-error estimates and superconvergence in maximum norm of mixed FEMs for nonfickian flows in porous media by Ewing, Lin, Wang and Zhang in [13].

In our step (i) in the present approach we use the results in [20], [8], and [22]. As for the discontinuous Galerkin approximation relevant in the Vlasov-Poisson estimates we refer to the articles by Brezzi, Manzini, Marini and Russo for elliptic problem in [9], Johnson and Saranen for the Euler and Navier-Stokes equations in [16]; Asadzadeh for the Vlasov-Poisson equations in [2] and Asadzadeh and Kowalczyk for the Vlasov-Fokker-Planck system in [3].

An outline of this paper is as follows: in Section 2 we state notations and preliminaries and derive L_2-norm error estimates for the Poisson equation (1.3) in mixed BDM spaces. In section 3 we derive L_2-stability for the DG method for the time discretized system at each time level. Section 4 is devoted to error estimates for the time iteration in the Vlasov equation and the DG method for the Vlasov-Poisson system. Here we also discuss the compatibility between the mixed BDM approach for the Poisson equation and the DG method for the Vlasov-Poisson system.

In a forthcoming paper (part II) we shall construct two versions of a numerical algorithm for this problem. The first one is a simplified scheme combining a finite difference discretization in velocity with discontinuous Galerkin method in space and backward Euler in time variables. The second algorithm is devoted to tackle the scheme in the present paper.

2. Mixed method for the Poisson equation

We shall discretize the Poisson equation (1.3) using BDM spaces. To this end, we use the notation (vector functions will be denoted by bold face)

$$\rho(x,t) := -\Delta_x \varphi(x,t) = \int_{\Omega_v} f(x,v,t)\,dv, \qquad \Psi(x,t) := -\nabla_x \varphi(x,t),$$

and define a mixed form for (Ψ,φ), (in an abstract form $\mathcal{L}\varphi := \rho$) as

$$\begin{cases} \Psi + \nabla_x \varphi = 0, & \text{in} \quad \Omega_x, \\ \operatorname{div} \Psi = \rho, & \text{in} \quad \Omega_x, \\ \varphi = \tilde{g}, & \text{on} \quad \partial\Omega_x, \end{cases} \tag{2.1}$$

where, to begin, we ignore the time dependence in φ and Ψ. We shall use the following Hilbert space

$$S := H(\operatorname{div}, \Omega_x) = \{\mathbf{u} \in [L_2(\Omega_x)]^2 : \operatorname{div}\,\mathbf{u} \in L_2(\Omega_x)\},$$

associated with the norm

$$||\mathbf{u}||_S^2 = \|\mathbf{u}\|_2^2 + \|\operatorname{div}\mathbf{u}\|_2^2.$$

The weak form for (2.1) reads as follows: find $(\Psi,\varphi)\in S\times L_2(\Omega_x)$ such that

$$\begin{cases} (\Psi,\mathbf{u})-(\text{div }\mathbf{u},\varphi)=-\langle\tilde{g},\mathbf{u}\cdot\mathbf{n}\rangle, & \forall\mathbf{u}\in S,\\ (\text{div }\Psi,w)=(\rho,w), & \forall w\in L_2(\Omega_x),\end{cases} \tag{2.2}$$

where $(\cdot,\cdot)$ is the usual inner product in either $S=\Big[L_2(\Omega_x)\Big]^2$ or $L_2(\Omega_x)$ and $\langle\cdot,\cdot\rangle$ is the inner product in $L_2(\partial\Omega_x)$ and $\mathbf{n}$ is the outward unit normal to $\partial\Omega_x$. For $\tilde{g}\equiv 0$, the problems (1.3) and (2.2) are equivalent and the solvability of (2.2) is based on the inf-sup condition

$$\inf_{\Psi\in S}\sup_{w\in L_2}\frac{(\text{div }\Psi,w)}{\|\Psi\|_S\|w\|_2}\ge\lambda, \tag{2.3}$$

due to Babuška [4] and Brezzi [7] (known as Babuška-Brezzi condition) where λ is a positive constant.

We consider a quasi-uniform triangulation of Ω_x as $\Omega_x^h:\mathcal{T}_\hbar^x=\{\tau\}$. For a positive integer k, we let $\mathbf{P}_k(\tau)$ denote the restriction of the set of all vector polynomials of total degree not greater than k to τ, and define

$$\begin{aligned} S_\hbar=S_\hbar^k&:=\{\mathbf{u}\in S(\Omega_x):\mathbf{u}|_\tau\in\mathbf{P}_k(\tau),\ \ \tau\in\mathcal{T}_\hbar^x\},\\ W_\hbar=W_\hbar^{k-1}&:=\{w\in L_2(\Omega_x):w|_\tau\in P_{k-1}(\tau),\ \tau\in\mathcal{T}_\hbar^x\}.\end{aligned} \tag{2.4}$$

Then, $S_\hbar\times W_\hbar\subset S(\Omega_x)\times L_2(\Omega_x)$ is a mixed finite element space on the triangulation $\mathcal{T}_\hbar^x$ of Ω_x, for which the discrete version of the Babuška-Brezzi condition holds true:

$$\inf_{\Psi_\hbar\in S_\hbar}\sup_{w_\hbar\in W_\hbar}\frac{(\text{div }\Psi_\hbar,w_\hbar)}{\|\Psi_\hbar\|_S\|w_\hbar\|_2}\ge\tilde{\lambda}, \tag{2.5}$$

where $\tilde{\lambda}$ is independent of $\hbar$. Note that $W_\hbar$ is the space of piecewise polynomials of degree not greater than $k-1$. The mixed finite element method for (2.2) is now formulated as follows, see [8] and [22]: find $(\Psi_\hbar,\varphi_\hbar)\in S_\hbar\times W_\hbar$ such that

$$\begin{cases} (\Psi_\hbar,\mathbf{u})-(\text{div }\mathbf{u},\varphi_\hbar)=-\langle\tilde{g},\mathbf{u}\cdot\mathbf{n}\rangle, & \forall\mathbf{u}\in S_\hbar^k,\\ (\text{div }\Psi_\hbar,w)=(\rho,w), & \forall w\in W_\hbar^{k-1}.\end{cases} \tag{2.6}$$

To simplify (2.6) it is customary, see [1], to employ a Lagrange multiplier to enforce continuity of normal components of $\Psi_\hbar$ across interelement boundaries. To this end, let $\{e\}$ denote the collection of edges of elements in $\mathcal{T}_\hbar^x$ and set

$$\begin{aligned} &\mathcal{E}_\hbar^k:=\{\chi:\chi|_e\in\mathbf{P}_k(e)\text{ if }e\subset\Omega_x,\ \text{ while }\chi|_e=0\text{ for }e\subset\partial\Omega_x\},\\ &\mathcal{S}_\hbar^k:=\{\mathbf{u}:\mathbf{u}|_\tau\in S_\hbar,\ \ \tau\in\mathcal{T}_\hbar^x\},\qquad\text{i.e.,}\ \sum_\tau\langle\mathbf{u}\cdot\mathbf{n}_\tau,\chi\rangle_{\partial\tau}=0,\quad\chi\in\mathcal{E}_\hbar^k,\end{aligned} \tag{2.7}$$

and reformulate (2.6) as: to find $\{\Psi_\hbar,\varphi_\hbar,\chi_\hbar\}\in\mathcal{S}_\hbar^k\times W_\hbar^{k-1}\times\mathcal{E}_\hbar^k$ such that

$$\begin{cases} (\Psi_\hbar,\mathbf{u})-\sum_\tau(\text{div }\mathbf{u},\varphi_\hbar)_\tau+\sum_\tau\langle\mathbf{u}\cdot\mathbf{n}_\tau,\chi_\hbar\rangle_{\partial\tau}=-\langle\tilde{g},\mathbf{u}\cdot\mathbf{n}\rangle, & \mathbf{u}\in\mathcal{S}_\hbar^k,\\ \sum_\tau(\text{div }\Psi_\hbar,w)=(\rho,w), & w\in W_\hbar^{k-1},\\ \sum_\tau\langle\mathbf{u}\cdot\mathbf{n}_\tau,\zeta\rangle_{\partial\tau}=0, & \zeta\in\mathcal{E}_\hbar^k.\end{cases}$$

Note that, a formal subtraction of the equations (2.6) and (2.2), in the subspaces $S_\hbar\times W_\hbar^{k-1}\subset S\times L_2$, yields a Galerkin orthogonality for the mixed method as:

$$\begin{cases} (\Psi-\Psi_\hbar,\mathbf{u})-(\text{div }\mathbf{u},\varphi-\varphi_\hbar)=0, & \forall\mathbf{u}\in S_\hbar,\\ (\text{div }(\Psi-\Psi_\hbar),w)=0, & \forall w\in W_\hbar^{k-1}(\Omega_x).\end{cases} \tag{2.8}$$

2.1. **Error estimates for the mixed method.** Our mail tools are existence of local projections $\Pi_\hbar = \Pi_\hbar^k : H(\text{div}, \Omega_x) \to S_\hbar$, and $\pi_\hbar = \pi_\hbar^{k-1} : L_2(\Omega_x) \to W_\hbar^{k-1}$: such that

$$\text{div} \circ \Pi_\hbar^k = \pi_\hbar^{k-1} \circ \text{div}, \tag{2.9}$$

and we have, the local, orthogonality

$$(w - \pi_\hbar^{k-1} w, \phi)_\tau = 0, \qquad \phi \in W_\hbar^{k-1}(\tau), \quad \tau \in \mathcal{T}_\hbar^x, \tag{2.10}$$

and, under certain conditions, the global orthogonality relations

$$(\text{div } (\mathbf{u} - \Pi_\hbar^k \mathbf{u}), w) = 0, \qquad w \in W_\hbar^{k-1}. \tag{2.11}$$

Further, since div $S_\hbar^k = W_\hbar^{k-1}$,

$$(\text{div } \mathbf{u}, w - \pi_\hbar^{k-1} w) = 0, \qquad \mathbf{u} \in S_\hbar^k. \tag{2.12}$$

Then, it is well known that, for $0 \le s \le k$ and $0 \le j \le k$,

$$\|w - \pi_\hbar^{k-1} w\|_{H^{-s}(\Omega_x)} \le C\Big(\sum_\tau \hbar_\tau^{2(s+j)} \|w\|_{j,\tau}^2\Big)^{1/2}. \tag{2.13}$$

Moreover, for $1 \le r \le k+1$,

$$\|\mathbf{u} - \Pi_\hbar^k \mathbf{u}\|_{L_2(\Omega_x)} \le C\Big(\sum_\tau \hbar_\tau^{2r} \|\mathbf{u}\|_r^2\Big)^{1/2}. \tag{2.14}$$

We shall use the following global form of the estimates (2.13) (with $s = 0$) and (2.13) in $L_2(\Omega_x)$-norm, justified by the construction of π_h^{k-1} and Π_h^k,

$$\|w - \pi_h^{k-1} w\|_{L_2(\Omega_x)} \le Ch^k \|D^k w\|_{L_2(\Omega_x)}, \qquad \forall w \in H^k(\Omega_x), \tag{2.15}$$

$$\|\mathbf{u} - \Pi_\hbar^k \mathbf{u}\|_{L_2(\Omega_x)} \le Ch^{k+1} \|D^{k+1} \mathbf{u}\|_{L_2(\Omega_x)}, \qquad \forall \mathbf{u} \in \Big[H^{k+1}(\Omega_x)\Big]^d. \tag{2.16}$$

Below we gather the main error estimates of this approximation in $L_2(\Omega_x)$-norm. We shall prove some of this estimates (for more detailed estimates see [8] and [22]).

Theorem 2.1. *Let* $\{\Psi_\hbar, \varphi_\hbar\} \in S_\hbar^k \times W_\hbar^{k-1}$ *be the solution of the mixed finite element scheme* (2.6)*. Then, we have the following* $L_2(\Omega_x)$ *error estimates:*

$$\|\Psi - \Psi_\hbar\|_2 \le C\|\Psi - \Pi_\hbar^k \Psi\|_2 \le C\hbar^r \|\Psi\|_r, \quad 1 \le r \le k+1. \tag{2.17}$$

$$\|\rho - \rho_\hbar\|_2 = \|\rho - \pi_\hbar^{k-1} \rho\|_2 \le C\hbar^r \|\rho\|_r, \quad 0 \le r \le k. \tag{2.18}$$

$$\|\varphi_\hbar - \pi_\hbar^{k-1} \varphi_\hbar\|_2 \le C\hbar \|\Psi - \Pi_\hbar^k \Psi\|_2 + C\hbar^{\min(2,k)} \|\rho - \pi_\hbar^{k-1} \rho\|_2. \tag{2.19}$$

$$\|\varphi - \varphi_\hbar\|_2 \le C\hbar^r \Big(\|\rho\|_{r-2} + |g|_{r-1/2}\Big), \qquad 2 \le r \le k+2. \tag{2.20}$$

Proof. Using (2.12) we may rewrite the first equation in (2.8) as

$$(\Psi - \Psi_\hbar, \mathbf{u}) - (\text{div}\, \mathbf{u}, \pi_\hbar^{k-1} \varphi - \varphi_\hbar) = 0, \qquad \mathbf{u} \in S_\hbar^k. \tag{2.21}$$

Let now $\tilde{\mathbf{e}}_\hbar := \Pi_\hbar^k \Psi - \Psi_\hbar$ and in (2.21) take $\mathbf{u} = \tilde{\mathbf{e}}_\hbar$. Then,

$$\begin{aligned} \|\tilde{\mathbf{e}}_\hbar\|_{L_2(\Omega_x)}^2 &= (\Pi_\hbar \Psi - \Psi_\hbar, \tilde{\mathbf{e}}_\hbar) = (\Psi - \Psi_\hbar, \tilde{\mathbf{e}}_\hbar) - (\Psi - \Pi_\hbar \Psi, \tilde{\mathbf{e}}_\hbar) \\ &= (\text{div}\, \tilde{\mathbf{e}}_\hbar, \pi_\hbar^{k-1} \varphi - \varphi_\hbar) - (\Psi - \Pi_\hbar \Psi, \tilde{\mathbf{e}}_\hbar) = -(\Psi - \Pi_\hbar \Psi, \tilde{\mathbf{e}}_\hbar), \end{aligned} \tag{2.22}$$

where, we used (2.21) and (2.12). Thus, using Cauchy-Schwarz inequality

$$\|\tilde{\mathbf{e}}_\hbar\|_{L_2(\Omega_x)} \le \|\Psi - \Pi_\hbar \Psi\|_{L_2(\Omega_x)}, \tag{2.23}$$

and hence, using the well-known estimates for projection error, we get

$$\begin{aligned} \|\Psi - \Psi_\hbar\|_{L_2(\Omega_x)} &\le \|\tilde{\mathbf{e}}_\hbar\|_{L_2(\Omega_x)} + \|\Psi - \Pi_\hbar \Psi\|_{L_2(\Omega_x)} \\ &\le 2\|\Psi - \Pi_\hbar \Psi\|_{L_2(\Omega_x)} \le C\hbar^r \|\Psi\|_r, \qquad 1 \le r \le k+1. \end{aligned} \tag{2.24}$$

This proofs the first estimate (2.17) of the theorem.

Next, note that by successive use of (2.11) and the second relation in (2.8),

$$(\operatorname{div}\tilde{\mathbf{e}}_\hbar, w) = (\operatorname{div}(\Psi - \Psi_\hbar), w) = 0, \qquad \forall\, w \in W_\hbar. \tag{2.25}$$

Taking $w = \operatorname{div}\tilde{\mathbf{e}}_\hbar$ we get $\operatorname{div}\tilde{\mathbf{e}}_\hbar = 0$. Thus, by the same calculations as in (2.24), and using projection error

$$\begin{aligned}\|\rho - \rho_\hbar\|_{L_2(\Omega_x)} &= \|\operatorname{div}(\Psi - \Psi_\hbar)\|_{L_2(\Omega_x)} = \|\operatorname{div}(\Psi - \Pi_\hbar\Psi)\|_{L_2(\Omega_x)} \\ &\le C\hbar^r\|\operatorname{div}\Psi\|_r = C\hbar^r\|\rho\|_r, \qquad 0 \le r \le k,\end{aligned} \tag{2.26}$$

which yields the second assertion (2.18) of the theorem.

Further, let $\mathcal{L}^\star\phi = \varrho$, where $\mathcal{L}^\star$ the adjoint operator for $\mathcal{L}$, $\varrho \in L_2(\Omega_x)$ and $\phi \in H^2(\Omega_x) \cap H_0^1(\Omega_x)$. Then, we may write, see [8],

$$(\pi_\hbar^{k-1}\varphi - \varphi_\hbar\,,\, \varrho) = (\Psi - \Psi_\hbar, \nabla\phi - \Pi_\hbar(\nabla_x\phi)) + (\operatorname{div}(\Psi - \Psi_\hbar), \phi - \pi_\hbar\phi). \tag{2.27}$$

Then, by (2.15)-(2.16), together with elliptic regularity of $\mathcal{L}^\star$, (2.27) yields (2.19).

Finally, using (2.17)-(2.19), and the projection error estimates (2.15) and (2.16),

$$\begin{aligned}\|\varphi - \varphi_\hbar\|_{L_2(\Omega_x)} &\le \|\pi_\hbar^{k-1}\varphi - \varphi_\hbar\|_{L_2(\Omega_x)} + \|\varphi - \pi_\hbar^{k-1}\varphi\|_{L_2(\Omega_x)} \\ &\le C\Big(h^{r+2}\|\Psi\|_{r+1} + h^{\min(r+2,k)}\|\rho\|_r + h^{\min(r,k)}\|\varphi\|_r\Big),\end{aligned} \tag{2.28}$$

which, using elliptic regularity of $\mathcal{L}^\star$ is simplified to (2.20), (we omit the details), and the proof is complete. □

Below, we gather some of the L_∞ results, due to Wang [22], for the error $\Psi - \Psi_\hbar$, based on the regularized Greens function approach. These are, intermediate steps in the L_∞ studies, that are relevant in our L_2-error estimates.

Proposition 2.1. *Let* (Ψ, φ) *and* $(\Psi_\hbar, \varphi_\hbar)$ *be the exact solution for* (2.2) *and the mixed finite element approximations in BDM space, respectively, and assume that* $\varphi \in W^{1,\infty}(\Omega_x)$. *Then*

$$\|\Psi - \Pi_\hbar\Psi\|_\infty \le C|\log\hbar|^{1/2}\Big(\|\Psi - \Pi_\hbar^k\Psi\|_\infty + \hbar\,|\log_\hbar|^{\delta_{1k}/2}\|\rho - \pi_\hbar^{k-1}\rho\|_\infty\Big), \qquad k \ge 1$$

where δ_{1k} *is the Kronecker function. An improved version of the above estimate for sufficiently smooth* $\partial\Omega$ *and* $k > 1$ *is given by*

$$\|\Psi - \Pi_\hbar\Psi\|_\infty \le C\Big(|\log\hbar|^{1/2}\|\Psi - \Pi_\hbar^k\Psi\|_\infty + \hbar\|\rho - \pi_\hbar^{k-1}\rho\|_\infty\Big). \tag{2.29}$$

If in addition, $\varphi \in W^{k+2,\infty}(\Omega_x)$, *then*

$$\|\Psi - \Psi_\hbar\|_\infty \le C\hbar^{k+1}|\log\hbar|^{1/2}\left(\|\varphi\|_{k+2,\infty} + |\log\hbar|^{\delta_{k1}/2}\|\rho\|_{k,\infty}\right). \tag{2.30}$$

The estimates in Proposition 2.1 are used to derive projection and finite element error estimates for $\|\varphi_\hbar - \pi_\hbar^{k-1}\varphi\|_\infty$ and $\|\varphi - \varphi_\hbar\|_\infty$. We use (2.30) in our estimates.

3. The discontinuous Galerkin method for Vlasov equation

In this section we consider the Vlasov equation (1.2), and insert the computed value $\varphi_\hbar$, from the previous section, for the potential function φ. Thus, we study the following linearized version of the Vlasov equation: (1.4),

$$\begin{cases} \partial_t f + v\cdot\nabla_x f - \nabla_x\varphi_\hbar\cdot\nabla_v f = 0, & \text{in } \Omega\times[0,T], \\ f(x,v,0) = f_0(x,v), & \text{in } \Omega = \Omega_x\times\Omega_v, \\ f(x,v,t) = 0, & \text{on } \Gamma_v^-\times\Omega_v\times[0,T], \end{cases} \tag{3.1}$$

where, we discretize (3.1) by the discontinuous Galerkin (DG) finite element method in (x, v), combined with the backward Euler (BE) method in t. Since the DG, in time, is equivalent to the BE method, therefore, compared to DG method in phase-space-time, this split simplifies the analysis in the DG approach without affecting the final estimates.

Let now, $\mathcal{C}_h := \{K\} = \{\tau_x \times \tau_v\}$ be a family of quasi-uniform triangulation of the phase space domain $\Omega = \Omega_x \times \Omega_v$, with the mesh parameter $h(\sim h_x \sim h_v)$, refined by a nested iteration strategy: $(i) \to (ii) \to (i)$, where in (I) we use the refinement $h_x = 2^{-j}\hbar$ for some $j = 0, 1, 2, \dots, M$, and in (ii) we let $\hbar = h_x$.

For the remaining part of the paper we let k be a positive integer and introduce triangular finite element spaces of test and trial functions as

$$\begin{aligned}
&V_h = V_h^k := \{w \in L_2(\Omega) : w|_K \in P_k(K),\ \forall K \in \mathcal{C}_h\}\\
&V_h^{0,v} = V_h^{0,v,k} := \{w \in L_2(\Omega) : w|_K \in P_k(K),\ w|_{\partial K \cap \Gamma_v^-} = 0,\ \forall K \in \mathcal{C}_h\}\\
&\tilde{V}_h = \tilde{V}_h^{\,k} := \{w \in C([0,T], L_2(\Omega)) : w(t)|_K \in P_k(K),\ \forall K \in \mathcal{C}_h\}\\
&\tilde{W}_h = \tilde{W}_h^{k-1} := \{w \in C([0,T], L_2(\Omega_x)) : w(\cdot,t)|_{\tau_x} \in P_{k-1}(\tau_x),\ \forall \tau_x \in \mathcal{T}_{h_x}^x\}\\
&\tilde{S}_h = \tilde{S}_h^k := \{\mathbf{u} \in C([0,T], [L_2(\Omega_x)]^2) : \mathbf{u}(\cdot,t)|_{\tau_x} \in \mathbf{P}_k(\tau_x),\ \ \forall \tau_x \in \mathcal{T}_{h_x}^x\},
\end{aligned}$$

where in $\tilde{S}_h$; $\mathbf{u} \cdot \mathbf{n}_e$ are continuous across all interior edges e for $\tau_x \in \mathcal{T}_{h_x}^x$.

Next, we formulate the discontinuous Galerkin approximation of the Vlasov equation (3.1) in x, v-variables as: given the initial data f_0, and an approximate potential $\varphi_\hbar \in \tilde{W}_\hbar$ (computed in Section 2), find $f_h \in \tilde{V}_h$ such that, for all $g \in V_h^{0,v}$,

$$(\partial_t f_h + G(\varphi_\hbar)\nabla f_h, g + hG(\varphi_\hbar)\nabla g)_\Omega + \sum_{K \in \mathcal{C}_h} \int_{\partial K_G^-} [f_h] g_+ |G(\varphi_\hbar)\ \cdot \mathbf{n}|\, d\nu = 0, \quad (3.2)$$

where for $(x, v) \in \partial K$, we use the jump notation $[w] = w^+ - w^-$ with

$$w_\pm(x, v) = \lim_{|s| \to 0} w((x, v) \pm s \cdot G(\varphi_\hbar)), \qquad s = (s_x, s_v), \quad s_x > 0, \quad s_v > 0,$$

and we suppress the inner product sign "$\cdot$", e.g. $G\nabla := G \cdot \nabla$, and $(\cdot,\cdot)_\mathcal{D}$ denotes the scalar product over the domain $\mathcal{D}$. Further we use the notation $G(\varphi_\hbar) := (v, -\nabla_x \varphi_\hbar) = (v, \Psi_\hbar)$, $\nabla f = (\nabla_x f, \nabla_v f)$ and $\partial K_G^- = \{(x, v) \in \partial K : G \cdot \mathbf{n}(x, v) < 0\}$. Note that, in practice, the steps (i) and (ii), are performed as follows: starting with $\varphi_\hbar$ (computed in Section 2), first we compute f_h using (3.2) with $h_x = 2^{-j}\hbar$ and $h \sim h_x \sim h_v$, then we let $\hbar = h_x$ and compute a new $\varphi_\hbar$ by BDM mixed method of Section 2, and insert the result in (3.2) and compute a new f_h. We may iterate this procedure until a certain stopping criterion. Note that, at the final step we choose $h = \hbar$ (i.e., $j = 0$).

The boundary term in (3.2) is the sum of jump terms over interelement boundaries in (x, v)-variables. In case of no confusion we use ∂K_- and ∂K_+ for ∂K_G^- and ∂K_G^+, respectively. Finally, (3.2) is valid continuous in t.

Combining (3.2) and (2.6) we get the mixed discontinuous Galerkin method for the system (1.1) in x, v-variables: find $(f_h, \Psi_\hbar, \varphi_\hbar) \in \tilde{V}_h \times \tilde{S}_\hbar \times \tilde{W}_\hbar$ such that

$$\begin{cases}
(\partial_t f_h + G(\varphi_\hbar)\nabla f_h, g + hG(\varphi_\hbar)\nabla g)_\Omega + \displaystyle\sum_{K \in \mathcal{C}_h} \int_{\partial K_G^-} [f_h] g_+ |G(\varphi_\hbar) \cdot \mathbf{n}|\, d\nu = 0,\\
(\Psi_\hbar, \mathbf{u}) - (\text{div } \mathbf{u}, \varphi_\hbar) = 0,\\
\qquad\qquad (\text{div } \Psi_h, w) = (\rho_\hbar, w), \qquad \text{for all } \ g \in V_h^{0,v},\ \ \mathbf{u} \in S_\hbar \quad \text{and} \quad w \in W_\hbar.
\end{cases}$$

Finally, we apply the backward Euler scheme in time which gives a discrete in time formulation, i.e. for each $n = 1, 2, \dots, N$, we have a variational formulation for a modified stationary Vlasov-Poisson system in (x, v)-domain, where the data for the Poisson equation as well as the source term (initial data) of the Vlasov equation, both are equal to the solution of the Vlasov equation at the previous time level $n-1$.

Then, the discrete system at the time step n reads: given $f_h^{n-1} \in V_h$, find, first $(\Psi_h^{n-1}, \varphi_h^{n-1}) \in S_h \times W_h$, and then $f_h^n \in V_h$ such that

$$\begin{cases} \Big((f_h^n - f_h^{n-1})/\Delta t + G(\varphi_h^{n-1})\nabla f_h^n\,,\, g + hG(\varphi_h^{n-1})\nabla g\Big)_\Omega \\ \qquad\qquad + \displaystyle\sum_{K\in\mathcal{C}_h}\int_{\partial K_G^-} [f_h^n] g_+ |G(\varphi_h^{n-1})\,\cdot \mathbf{n}\,|\, d\nu = 0, \\ (\Psi_h^{n-1}, \mathbf{u}) - (\text{div}\,\mathbf{u}, \varphi_h^{n-1}) = 0, \\ \quad (\text{div}\,\Psi_h^{n-1}, w) = (\rho_h^{n-1}, w), \qquad \forall (g, \mathbf{u}, w) \in V_h \times S_h \times W_h. \end{cases} \tag{3.3}$$

The scheme (3.3) operates as follows: given $f_h^{n-1} \in V_h$; ρ_h^{n-1}, Ψ_h^{n-1} and φ_h^{n-1} are computed from the last two equations. Then f_h^n is computes from the first equation of (3.3).

The first equation in problem (3.3) can be formulated in a concise form as

$$b(G(\varphi_h^{n-1}); f_h^n, g) = L(g), \qquad \forall g \in V_h^{0,v}, \tag{3.4}$$

where b and L are, respectively, bilinear and linear forms defined by:

$$L(g) := \Big(f_h^{n-1}, g + hG(\varphi_h^{n-1})\cdot\nabla g\Big)_\Omega, \qquad \text{and} \tag{3.5}$$

$$\begin{aligned} b(G(\varphi_h^{n-1}); f, g) := &\Big(f + \Delta t(G(\varphi_h^{n-1})\cdot\nabla f), g + hG(\varphi_h^{n-1})\cdot\nabla g)\Big) \\ &+ \Delta t \sum_{K\in\mathcal{C}_h}\int_{\partial K_G^-} [f] g_+ |G(\varphi_h^{n-1})\,\cdot \mathbf{n}|\, d\nu. \end{aligned} \tag{3.6}$$

In contrary to $b(G(\varphi); f, g)$, which is nonlinear (φ depends on f), $b(G(\varphi_h^{n-1}); f, g)$, with φ_h^{n-1} depending on f_h^{n-1}, is now linear. Recall that, for the composite phase-space schemes (3.3) the final meshes are chosen as $h = h_x$ and $h_v \sim h_x$. Therefore, in the sequel, we shall only use h as our phase-space parameter. Finally, we introduce a triple norm, viz

$$\begin{aligned} |||g|||_\omega^2 := &\|g\|_\Omega^2 + h\Delta t\|G(\omega)\cdot\nabla g\|_\Omega^2 + \frac{h+\Delta t}{2}\times \\ &\times\Big(\sum_{K\in\mathcal{C}_h}\int_{\partial K_G^-} [g]^2 |G(\omega)\,\cdot \mathbf{n}|\, d\nu + \int_{\partial\Omega_+} g^2 |G(\omega)\cdot\mathbf{n}|\, d\nu\Big). \end{aligned} \tag{3.7}$$

Below, we prove, L_2-based, stability estimates for (3.4), at an arbitrary time step n, in $|||g|||_\varphi$-norm. In section 4, we shall derive error estimates in $|||\cdot|||_{\varphi_h^{n-1}}$-norm.

3.1. L_2-stability estimates.

Lemma 3.1. *Assume that the function g satisfies the homogeneous inflow boundary condition: $g|_{\Gamma_-} = 0$ and that $h \approx \Delta t$. Then, the bilinear form $b(\cdot;\cdot,\cdot)$ is coercive (elliptic) with respect to $|||\cdot|||_\varphi$-norm, i.e.,*

$$b(G(\varphi); g, g) \geq (1 - h/2)|||g|||_\varphi^2, \qquad \forall g \in V_h^0,$$

where, for simplicity, we restrict the domain of g to

$V_h^0 := \{g \in L_2(\Omega) : g|_K \in H^1(K),\ g|_{\Gamma^-} = 0,\ g$ *is piecewise discontinuous on* $\mathcal{C}_h\}$.

Proof. Assume that φ^{n-1} is known from the previous steps (suppress all n), then

$$\begin{aligned} b(G(\varphi); f, g) = &(f, g)_\Omega + (f, hG(\varphi)\cdot\nabla g)_\Omega + \Delta t(g, G(\varphi)\cdot\nabla f)_\Omega \\ &+ \Delta t(G(\varphi)\cdot\nabla f, hG(\varphi)\cdot\nabla g)_\Omega + \Delta t\sum_{K\in\mathcal{C}_h}\int_{\partial K_-(G)} [f] g_+ |G(\varphi)\cdot\mathbf{n}|\, d\nu, \end{aligned}$$

which, with $f=g$, yields

$$\begin{aligned} b(G(\varphi);g,g) = \sum_{K\in\mathcal{C}_h} \Big[\|g\|_K + (h+\Delta t)(g, G(\varphi)\cdot\nabla g)_K + \\ + \Delta t h \|G(\varphi)\cdot\nabla g\|_K^2 + \Delta t \int_{\partial K_G^-} [g] g_+ |G(\varphi)\cdot\mathbf{n}|\, d\nu \Big] := \sum_{i=1}^4 T_i. \end{aligned} \tag{3.8}$$

Hence we only need to estimate the terms T_2 and T_4. Now, using Green's formula

$$\begin{aligned} (g, G(\varphi)\cdot\nabla g)_K =& \frac{1}{2}\int_{\partial K} (G(\varphi)\cdot\mathbf{n}) g^2\, d\nu \\ =& \frac{1}{2}\int_{\partial K_+} g_-^2 |G(\varphi)\cdot\mathbf{n}|\, d\nu - \frac{1}{2}\int_{\partial K_-} g_+^2 |G(\varphi)\cdot\mathbf{n}|\, d\nu. \end{aligned} \tag{3.9}$$

Next, we write $[g]g_+ = g_+^2 - g_-g_+$ to get

$$\int_{\partial K_G^-} [g] g_+ |G(\varphi)\cdot\mathbf{n}|\, d\nu = \int_{\partial K_G^-} g_+^2 |G(\varphi)\cdot\mathbf{n}|\, d\nu - \int_{\partial K_G^-} g_- g_+ |G(\varphi)\cdot\mathbf{n}|\, d\nu. \tag{3.10}$$

Combining (3.9), (3.10) and the identity

$$\sum_{K\in\mathcal{C}_h} \int_{\partial K_+} g_-^2 \bullet = \sum_{K\in\mathcal{C}_h} \int_{\partial K_-} g_-^2 \bullet - \int_{\Gamma_-} g_-^2 \bullet + \int_{\Gamma_+} g_-^2 \bullet, \tag{3.11}$$

we can write (note the added $(h+\Delta t)$-term is zero)

$$\begin{aligned} T_2 + T_4 = \sum_{K\in\mathcal{C}_h} \Big[& \frac{h+\Delta t}{2} \Big(\int_{\partial K_-} [g_-^2 |G(\varphi)\cdot\mathbf{n}| - g_+^2 |G(\varphi)\cdot\mathbf{n}|]\, d\nu \Big) \\ & + \Delta t \int_{\partial K_-} g_+^2 |G(\varphi)\cdot\mathbf{n}|\, d\nu - \Delta t \int_{\partial K_-} g_+ g_- |G(\varphi)\cdot\mathbf{n}|\, d\nu \\ & + (h+\Delta t) \Big(\int_{\partial K_-} g_+ g_- |G(\varphi)\cdot\mathbf{n}|\, d\nu - \int_{\partial K_-} g_+ g_- |G(\varphi)\cdot\mathbf{n}|\, d\nu \Big) \Big] \\ & + \frac{h+\Delta t}{2} \Big(\int_{\Gamma_+} g_-^2 |G(\varphi)\cdot\mathbf{n}|\, d\nu - \int_{\Gamma_-} g_-^2 |G(\varphi)\cdot\mathbf{n}|\, d\nu \Big). \end{aligned}$$

By the assumption $g|_{\Gamma_-} = 0$, the identity above can be written as

$$\begin{aligned} T_2 + T_4 = \sum_{K\in\mathcal{C}_h} \Big[& \frac{h+\Delta t}{2} \int_{\partial K_-} [g]^2 |G(\varphi)\cdot\mathbf{n}| - (h+\Delta t) \int_{\partial K_-} g_+^2 |G(\varphi)\cdot\mathbf{n}|\, d\nu \\ & + \Delta t \int_{\partial K_-} g_+^2 |G(\varphi)\cdot\mathbf{n}|\, d\nu - \Delta t \int_{\partial K_-} g_+ g_- |G(\varphi)\cdot\mathbf{n}|\, d\nu \\ & + (h+\Delta t) \int_{\partial K_-} g_+ g_- |G(\varphi)\cdot\mathbf{n}|\, d\nu \Big] + \frac{h+\Delta t}{2} \int_{\Gamma_+} g_-^2 |G(\varphi)\cdot\mathbf{n}|\, d\nu \\ = & \sum_{K\in\mathcal{C}_h} \Big[\frac{h+\Delta t}{2} \int_{\partial K_-} [g]^2 |G(\varphi)\cdot\mathbf{n}| - h \int_{\partial K_-} [g] g_+ |G(\varphi)\cdot\mathbf{n}|\, d\nu \Big] \\ & + \frac{h+\Delta t}{2} \int_{\Gamma_+} g_-^2 |G(\varphi)\cdot\mathbf{n}|\, d\nu. \end{aligned}$$

Using $-[g]g_+ \geq -[g]^2/2 - g_+^2/2$, the negative term above is bounded below, viz

$$-h \int_{\partial K_-} [g] g_+ |G(\varphi)\cdot\mathbf{n}| \geq -\frac{h}{2} \int_{\partial K_-} [g]^2 |G(\varphi)\cdot\mathbf{n}| - \frac{h}{2} \int_{\partial K_-} g_+^2 |G(\varphi)\cdot\mathbf{n}|.$$

Now we use the trace estimate, see, e.g.[18], $\int_{\partial K_-} g_+^2|G(\varphi)\cdot \mathbf{n}| \le C_K\|g\|_K^2$, (where for a convex domain K; $C_K<1$), to obtain the bound

$$-h\int_{\partial K_-}[g]g_+|G(\varphi)\cdot\mathbf{n}| \ge -\frac{h}{2}\int_{\partial K_-}[g]^2|G(\varphi)\cdot\mathbf{n}| - C_K\frac{h}{2}\|g\|_K^2. \tag{3.12}$$

Inserting (3.12) in the last equality for T_2+T_4 we get the estimate

$$T_2+T_4 \ge \sum_{K\in\mathcal{C}_h}\Big[\frac{\Delta t}{2}\int_{\partial K_-}[g]^2|G(\varphi)\cdot\mathbf{n}| - C_K\frac{h}{2}\|g\|_K^2\Big] + \frac{h+\Delta t}{2}\int_{\Gamma_+} g_-^2|G(\varphi)\cdot\mathbf{n}|\,d\nu.$$

Thus with a kick back argument and due to the presence of the small coefficients h_K and $C_K(<1)$, the contribution from the negative term can be hidden in the first term: $\|g\|_K$ in the triple-norm and we get, recalling (3.8), the desired result:

$$b(G(\varphi);g,g) \ge (1-h/2)|||g|||_\varphi^2,$$

and the proof is complete. □

4. Error estimates

Following the standard procedure, we let $\tilde f_h^n$ to be the interpolant of f with the interpolation error denoted by $\eta^n = f^n - \tilde f_h^n$ and set $\xi^n = f_h^n - \tilde f_h^n$, so that $e^n = f^n - f_h^n = \eta^n - \xi^n$. We shall use the following well-known results:

Proposition 4.1. *Assume that Ω is a sufficiently smooth domain and let $f\in C^1([0,T],W^{k,\infty}(\Omega)\cap W^{k+1,2}(\Omega))$. Then, we have the interpolation error estimates*

$$\|\eta\|_{L_2(\Omega,|G(\varphi_h^{n-1})\cdot\mathbf{n}|)} \le C_\Psi^v h^{k+1}\|f\|_{k+1}, \quad \max_{1\le n\le N}|||\eta^n|||_{\varphi_h^{n-1}} \le C_i h^{k+1/2}\|f\|_{k+1}, \tag{4.1}$$

where $C_\Psi^v = C_i|\Omega_v|\|\Psi\|_\infty$ and C_i is the interpolation constant.

Proposition 4.2. *[Trace theorem] Suppose that T is a Lipschitz domain. Then there is a constant $C_T = C|T|$ such that*

$$\|w\|_{L_2(\partial T)} \le C_T\|w\|_{L_2(T)}^{1/2}\|w\|_{H^1(T)}^{1/2}.$$

Proposition 4.1 can be proved as Theorem 4.4.3 in [10], see also [16]. For a proof of Proposition 4.2, see Brenner and Scott [5].

Lemma 4.1. *For each $n=1,2,\dots,N$, and with η^n and ξ_n defined as above, there are positive constants C and C' such that*

$$|b(G(\varphi_h^{n-1});\eta^n,\xi^n)| \le Ch|||\xi^n|||^2 + C'h^{-1}\|\eta^n\|_2^2 + |||\eta^n|||^2. \tag{4.2}$$

Proof. We use the definition of the triple norm and estimate the bilinear form as

$$\begin{aligned}
|b(G(\varphi_h^{n-1});\eta^n,\xi^n)| &= \big|\big(\eta^n+\Delta t G(\varphi_h^{n-1})\nabla\eta^n, \xi^n + hG(\varphi_h^{n-1})\nabla\xi^n\big)\\
&\quad + \Delta t\sum_{K\in\mathcal{C}_h}\int_{\partial K_G^-}[\eta^n]\xi_+^n|G(\varphi_h^{n-1})\cdot\mathbf{n}|\,d\nu\big|\\
&\le h^{-1}\|\eta^n\|_2^2 + \frac14 h\|\xi^n\|_2^2 + \Delta t\|G(\varphi_h^{n-1})\nabla\eta^n\|_2^2\\
&\quad + \frac{\Delta t}{4}h^2\|G(\varphi_h^{n-1})\nabla\xi^n\|_2^2 + \Delta t^{-1}\|\eta^n\|_2^2 + \frac{\Delta t}{4}h^2\|G(\varphi_h^{n-1})\nabla\xi^n\|_2^2\\
&\quad + \frac{\Delta t}{4}\|\xi^n\|_2^2 + \Delta t\|G(\varphi_h^{n-1})\nabla\eta^n\|_2^2 + \Delta t\Big|\sum_{K\in\mathcal{C}_h}\int_{\partial K_G^-}[\eta^n]\xi_+^n|G(\varphi_h^{n-1})\cdot\mathbf{n}|\,d\nu\Big|.
\end{aligned}$$

We use Proposition 2.1, assumptions, and inverse inequality to bound

$$\begin{aligned}
\Delta t\|G(\varphi_h^{n-1})\nabla\eta^n\|_2^2 &\le C_v\Delta t\|\Psi^{n-1}-\Psi_h^{n-1}\|_\infty^2\|\nabla\eta^n\|_2^2 + C_v\Delta t\|\Psi^{n-1}\|_\infty^2\|\nabla\eta^n\|_2^2\\
&\le C_v(\Delta t)h^{-2}\|\eta^n\|_2^2.
\end{aligned}$$

Moreover, for the contribution from the boundary terms we use trace estimate as

$$\begin{aligned}
&\Delta t\Big|\sum_{K\in\mathcal{C}_h}\int_{\partial K_-(G)}[\eta^n]\xi^n_+|G(\varphi_h^{n-1})\cdot\mathbf{n}|\,d\nu\Big|\le\Delta t\sum_{K\in\mathcal{C}_h}\int_{\partial K_G^-}[\eta^n]^2|G(\varphi_h^{n-1})\cdot\mathbf{n}|\,d\nu\\
&\quad+\frac{\Delta t}{2}\sum_{K\in\mathcal{C}_h}\int_{\partial K_G^-}|\xi^n|^2|G(\varphi_h^{n-1})\cdot\mathbf{n}|\,d\nu\le|||\eta^n|||^2\\
&\quad+\frac{\Delta t\max C_K}{8}\Big(\sum_{K\in\mathcal{C}_h}\|\xi^n\|^2_{L_2(K,|G(\varphi_h^{n-1})\cdot\mathbf{n}|)}\Big)^{1/2}\Big(\sum_{K\in\mathcal{C}_h}\|\nabla\xi^n\|^2_{L_2(K,|G(\varphi_h^{n-1})\cdot\mathbf{n}|)}\Big)^{1/2}\\
&\quad\le|||\eta^n|||^2+C\frac{h}{16}\|\xi^n\|^2+C\frac{h}{16}(\Delta t)^2\|\nabla\xi^n\|^2\le|||\eta^n|||^2+C\frac{h}{8}\|\xi^n\|^2,
\end{aligned}$$

where in the last step we use inverse inequality. Summing up we have using $\Delta t\sim h$,

$$|b(G(\varphi_h^{n-1});\eta^n,\xi^n)|\le Ch|||\xi^n|||^2+C'h^{-1}\|\eta^n\|_2^2+|||\eta^n|||^2.$$

□

Our main result is the following error estimate.

Theorem 4.1. *Let $(f_h^n,\Psi^{n-1},\varphi^{n-1})\in\tilde V_h\times\tilde S_h\times\tilde W_h$ be the mixed discontinuous Galerkin finite element approximation of* (3.3)*, and (f,Ψ,φ) be the exact solution of the system* (1.2)-(1.3) *and* (2.1)*, such that $\|\nabla f^n\|_2+\|\nabla f^n\|_\infty\le C$, $\varphi^{n-1}\in W^{k+2,\infty}$ for $n=1,\dots,N$, and $f\in C^1([0,T],W^{k,\infty}\cap W^{k+1,2})$. Moreover assume that $h\approx\Delta t$. Then, there is a positive constants C, independent of h, φ and f, but may depend on the size of the velocity domain Ω_v, such that*

$$\max_{1\le n\le N}|||f^n-f_h^n|||_{\varphi_h^{n-1}}\le Ch^{k+1/2}.$$

Proof. By the definition of ξ^n and η^n the exact solution f^n at time $t=t_n$ satisfies

$$\begin{aligned}
b^e(G(\varphi^n);f^n,g):=\;&b(G(\varphi^n);f^n,g)-\big(\Delta t\Theta^n,g+hG(\varphi_h^{n-1})\nabla g\big)\\
=\;&\big(f^{n-1},g+hG(\varphi_h^{n-1})\nabla g\big),
\end{aligned}$$

where $\Theta^n=\frac{f^n-f^{n-1}}{\Delta t}-f_t^n$. Using Lemma 3.1, and (3.4) we may write

$$\begin{aligned}
(1-h/2)|||\xi^n|||^2_{\varphi_h^{n-1}}&\le b(G(\varphi_h^{n-1});f_h^n-\tilde f_h^n,\xi^n)\\
&=b(G(\varphi_h^{n-1});f_h^n,\xi^n)-b(G(\varphi_h^{n-1});\tilde f_h^n,\xi^n)\\
&=\big(f_h^{n-1},\xi^n+hG(\varphi_h^{n-1})\nabla\xi^n\big)-b(G(\varphi_h^{n-1});\tilde f_h^n,\xi^n)\\
&=b^e(G(\varphi^n);f^n,\xi^n)-b(G(\varphi_h^{n-1});\tilde f_h^n,\xi^n)\\
&\quad+\big(\xi^{n-1}-\eta^{n-1},\xi^n+hG(\varphi_h^{n-1})\nabla\xi^n\big)\\
&=\big[b^e(G(\varphi^n);f^n,\xi^n)-b(G(\varphi_h^{n-1});f^n,\xi^n)\big]+b(G(\varphi_h^{n-1});\eta^n,\xi^n)\\
&\quad+\big(\xi^{n-1}-\eta^{n-1},\xi^n+hG(\varphi_h^{n-1})\nabla\xi^n\big):=J_1+J_2+J_3.
\end{aligned}$$

Here, Lemma 4.1 gives a bound for the J_2-term. J_1 and J_3 are combined error indicators for the mixed finite element (φ_h is computed, using DG approximated f_h), DG and BE approximations. Below, we estimate each J_1 and J_3, separately. As for the J_1-term, using the definition of b^e and (3.4),

$$\begin{aligned}
|J_1|=&\;|\big(f^n+\Delta tG(\varphi^n)\nabla f^n-\Delta t\Theta^n,\xi^n+hG(\varphi_h^{n-1})\nabla\xi^n\big)\\
&-\big(f^n+\Delta tG(\varphi^{n-1})\nabla f^n,\xi^n+hG(\varphi_h^{n-1})\nabla\xi^n\big)|\\
\le&\;|\big(\Delta t[G(\varphi^n)-G(\varphi_h^{n-1})]\nabla f^n,\xi^n+hG(\varphi_h^{n-1})\nabla\xi^n\big)|\\
&+|\big(\Delta t\Theta^n,\xi^n+hG(\varphi_h^{n-1})\nabla\xi^n\big)|:=J_{11}+J_{12}.
\end{aligned}$$

Evidently, we may write

$$\begin{aligned}
|J_{11}| &= \big| \big(\Delta t[G(\varphi^n) - G(\varphi^{n-1}) + G(\varphi^{n-1}) - G(\varphi_h^{n-1})]\nabla f^n, \xi^n + hG(\varphi_h^{n-1})\nabla\xi^n\big) \big| \\
&\leq \Delta t\big| \big([G(\varphi^n) - G(\varphi^{n-1})]\nabla f^n, \xi^n + hG(\varphi_h^{n-1})\nabla\xi^n\big) \big| \\
&\quad + \Delta t\big| \big([G(\varphi^{n-1}) - G(\varphi_h^{n-1})]\nabla f^n, \xi^n + hG(\varphi_h^{n-1})\nabla\xi^n\big) \big|.
\end{aligned}$$

Further, using Hölder and Young's inequalities, combined with the assumptions in the theorem, and the last estimate (2.30) of Proposition 2.1:

$$\begin{aligned}
|J_{11}| &\leq \Delta t\|\nabla_x(\varphi^n - \varphi^{n-1})\|_2\|\nabla f^n\|_\infty\|\xi^n + hG(\varphi_h^{n-1})\nabla\xi^n\|_2 \\
&\quad + \Delta t\|\nabla_x(\varphi^{n-1} - \varphi_h^{n-1})\|_\infty\|\nabla f^n\|_2\|\xi^n + hG(\varphi_h^{n-1})\nabla\xi^n\|_2 \\
&\leq C_v\Delta t\|f^n - f^{n-1}\|_2\|\nabla f^n\|_\infty\|\xi^n + hG(\varphi_h^{n-1})\nabla\xi^n\|_2 \\
&\quad + \Delta t\|\Psi^{n-1} - \Psi_h^{n-1}\|_\infty\|\nabla f^n\|_2\|\xi^n + hG(\varphi_h^{n-1})\nabla\xi^n\|_2 \\
&\leq C_v'\Delta t\|\xi^n + hG(\varphi_h^{n-1})\nabla\xi^n\|_2 + C(\Delta t)h^{k+1}|\log h|^{1/2} \\
&\qquad \cdot \Big(\|\varphi^{n-1}\|_{k+2,\infty} + |\log h|^{\delta_{k1}/2}\|\rho^{n-1}\|_{k,\infty}\Big)\|\xi^n + hG(\varphi_h^{n-1})\nabla\xi^n\|_2 \\
&\leq \Big(\frac{C_v'}{2} + \frac{1}{4C_{11}}\Big)\Delta t\left(\|\xi^n\|_2^2 + h^2\|G(\varphi_h^{n-1})\nabla\xi^n\|_2^2\right) \\
&\quad + C_{11}\Delta t h^{2k+2}|\log h|\left(\|\varphi^{n-1}\|_{k+2,\infty}^2 + |\log h|^{\delta_{k1}}\|\rho^{n-1}\|_{k,\infty}^2\right).
\end{aligned}$$

As for the J_{12}-term, using Taylor expansion $\|\Theta^n\|_2 \leq C\Delta t$, hence

$$\begin{aligned}
|J_{12}| &\leq \Delta t\|\Theta^n\|_2\|\xi^n + hG(\varphi_h^{n-1})\nabla\xi^n\|_2 \\
&\leq C_{12}(\Delta t)^2\left(\|\xi^n\|_2^2 + h^2\|G(\varphi_h^{n-1})\nabla\xi^n\|_2^2\right).
\end{aligned} \tag{4.3}$$

Next, for the term J_3 we have

$$\begin{aligned}
|J_3| &\leq \|\eta^{n-1} - \xi^{n-1}\|_2\|\xi^n + hG(\varphi_h^{n-1})\nabla\xi^n\|_2 \\
&\leq \frac{1}{2}\left(\|\eta^{n-1}\|_2^2 + \|\xi^{n-1}\|_2^2\right) + \frac{1}{2}\left(\|\xi^n\|_2^2 + h^2\|G(\varphi_h^{n-1})\nabla\xi^n\|_2^2\right).
\end{aligned} \tag{4.4}$$

Now adding the estimates for the terms J_{11}, J_{12}, J_2, J_3, using the mesh compatibility relation $\Delta t \sim h$ and hiding the terms involving $h|||\xi^n|||_2^2$,$h\|\xi^n\|_2^2$, and the first term on the right hand side of the estimate for $|J_{11}|$, the right hand side of (4.3), and $\frac{1}{2}\left(\|\xi^n\|_2^2 + h^2\|G(\varphi_h^{n-1})\nabla\xi^n\|_2^2\right)$ from the right hand side of (4.4), we end up with the bound

$$\begin{aligned}
\Big(\frac{1-Ch}{2}\Big)|||\xi^n|||^2 &\leq C_{11}\Delta t h^{2k+2}\left(\|\varphi^{n-1}\|_{k+2,\infty}^2 + |\log h|^{\delta_{k1}}\|\rho^{n-1}\|_{k,\infty}^2\right) \\
&\quad + C'h^{-1}\|\eta^n\|_2^2 + \frac{1}{2}\|\eta^{n-1}\|_2^2 + \frac{1}{2}\|\xi^{n-1}\|_2^2 \\
&\leq \tilde{C}h^{2k+1} + \frac{1}{2}\|\eta^{n-1}\|_2^2 + \frac{1}{2}\|\xi^{n-1}\|_2^2,
\end{aligned} \tag{4.5}$$

where in the last step we used the first interpolation error estimate in 4.1. Finally, iterating, the last terms in (4.5) in n, since $\xi^0 \equiv 0$, we have that

$$\frac{1}{2}\|\xi^{n-1}\|_2^2 \leq \frac{1}{2}|||\xi^{n-1}|||^2 \leq \hat{C}h^{2k+1} + \frac{1}{2(1-Ch)}\|\eta^{n-2}\|_2^2 + \ldots + \frac{1}{2(1-Ch)^{n-1}}\|\eta^0\|_2^2.$$

Hence, for each n the error bound

$$\begin{aligned}
|||\xi^n|||^2 &\leq Ch^{2k+1} + \frac{\|\eta^{n-1}\|_2^2}{(1-Ch)} + \frac{\|\eta^{n-2}\|_2^2}{(1-Ch)^2} + \frac{\|\eta^{n-3}\|_2^2}{(1-Ch)^3} + \ldots + \frac{\|\eta^0\|_2^2}{(1-Ch)^n} \\
&\leq Ch^{2k+1} + \frac{1}{1-Ch}\Big(\frac{1-(1-Ch)^n}{1-(1-Ch)}\Big)h^{2k+2} \leq Ch^{2k+1},
\end{aligned}$$

and consequently

$$\max_{1\leq n\leq N}|||\xi^n||| \leq Ch^{k+1/2}.$$

Now recalling that the second interpolation error estimate, cf 4.1, is also of order $h^{k+1/2}$, the proof is complete. □

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