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Reconstruction of dielectrics from experimental data via a hybrid globally convergent/adaptive inverse algorithm

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Abstract

The validity of a synthesis of a globally convergent numerical method with the adaptive FEM technique for a coefficient inverse problem is verified on time resolved experimental data. Refractive indices, locations and shapes of dielectric abnormalities are accurately imaged.

Dedicated to the lasting memory of Academician Mikhail M. Lavrentiev (1932-2010), one of founders of the field of Inverse Problems.

1 Introduction

In [9] a globally convergent numerical method for a Coefficient Inverse Problems (CIP) for a hyperbolic PDE was developed. Next, a two-stage numerical procedure was proposed in [10, 11, 12]. In this procedure the technique of [9] is used as the first stage. Next, the Adaptive Finite Element method (adaptivity below) is used as the second stage for the refinement. In [24] the first stage was verified on *blind* experimental data. The goal of the current publication is to demonstrate that the two-stage numerical procedure applied to the same experimental data can significantly improve imaging results compared with the first stage only. Specifically, we now accurately reconstruct not only locations and refractive indices of dielectric abnormalities, as it was in [24], but their shapes as well.

The analytical part of this paper is focused on two recommendations for the mesh refinement in a posteriori error analysis for the adaptivity technique. While the first recommendation was derived in our previous publications [11, 12], the second one is *completely new*. To derive this recommendation, we extensively use results of [12], where the framework of

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Functional Analysis for the adaptivity for ill-posed problems was derived for the first time. We demonstrate numerically that this broader mesh refinement recommendation works well. The *main new element* in our a posteriori error analysis for the adaptivity is that we estimate now the accuracy of our approximation of the regularized coefficient on a certain mesh. The latter in turn leads to this new mesh refinement recommendation. Unlike this, previous works on the adaptivity for CIPs were concerned only with accuracy estimates of either the Tikhonov functional [11, 12] or of the Lagrangian [4]-[8],[20]. As a result, in publications [4] - [12] the mesh was refined in the regions where the modulus of the gradient of the Tikhonov functional has attained its maximal values. In the current paper we add a new element via refining the mesh in neighborhoods of those grid points where the reconstructed coefficient attains its maximal values (this coefficient is always positive). The first attempt to obtain *a posteriori* error estimate for the reconstructed coefficient rather than for the Lagrangian, i.e. to obtain an analog of (4.33), was undertaken in [5] via considering the second Fréchet derivative of the Lagrangian, which is somewhat close to the local strong convexity property of Theorem 4.2. However, some facts were not analytically established in [5]. Our derivation is completely different from one in this reference.

The main difficulty in applying the technique of [9]-[12] to our experimental data is caused by a *huge discrepancy* between these data and computationally simulated ones. This discrepancy can be seen via a visual comparison of Figures 2-a) and 2-b) (below). Because of this discrepancy, conventional data denoising techniques, like, e.g. Fourier transform, Hilbert transform, spline interpolation, etc. provide only an insignificant help in our case. Hence, it is necessary to apply a *radically new* data pre-processing procedure as a crucial preliminary step. The goal of this step is to obtain acceptable boundary conditions, which are used in our numerical method. This procedure is based on the intuition only. The single justification of it is the accuracy of reconstruction results.

Our data pre-processing procedure consists of three stages. First two stages were described in [24] (they were new at that time). Hence, they are presented only briefly in this paper for the convenience of the reader. The third stage is new, since it is designed solely for the adaptivity technique. Our two-stage algorithm does not assume neither a knowledge of the background medium nor a knowledge of the presence/absence of small "sharp" abnormalities of our interest in the medium. It uses only the knowledge of the target coefficient outside of the medium of interest. Applications are in the detection of explosives, since their refractive indices usually are much higher that those of regular materials, see http://www.clippercontrols.com. Other procedures of solving CIPs, which do not rely on locally convergent algorithms, can be found in [2, 14, 21, 27, 29, 30].

An excellent accuracy of the blind reconstruction of both locations and refractive indices of dielectric abnormalities in [24] has led to the statement there that the globally convergent method of [9, 10] "is completely validated now". The same is true for a new mathematical model, which was proposed in [10, 24] due to an approximation of this numerical technique. That approximation is caused by the truncation of the large value $\overline{s}$ of the so-called pseudo frequency $s > 0$, which is the parameter of the Laplace transform of the original hyperbolic PDE. Such an approximation is likely inevitable due to challenges of the development of

numerical methods for CIPs. Indeed, CIPs are both ill-posed and nonlinear. It is shown in [10, 24] that, from the analytical standpoint, the above truncation is neither better nor worse than the classical truncation of divergent asymptotic series in the Real Analysis.

To explain our need for the above two-stage procedure, we note that the number $1/\overline{s}$ cannot be made infinitely small in practical computations. At the same time, convergence estimates in global convergence theorems of [9, 10] depend on the small parameter $1/\overline{s}$. Hence, in practical terms, these theorems only guarantee that the solution obtained on the first stage is sufficiently close to the correct solution. However, they do not guarantee that the distance between computed and correct solutions can be made infinitely small. This opens the door for a refinement via a locally convergent technique. Indeed, the *key ingredient* for any such technique is a good first approximation for the solution. So, this approximation is provided on the globally convergent stage. It is shown below that, in the case of our experimental data, it is crucial that the good first guess for the solution taken from the globally convergent stage should be available for the adaptivity stage. On the other hand, it was demonstrated in section 8 of [24] that if a modified gradient method does not use the solution obtained on the first stage, then its performance is poor.

We have chosen the adaptivity because of our previous experience of [10]-[12]. It was shown in these references that the quasi-Newton method taken alone does not refine solution of the globally convergent stage. On the other hand, it was also demonstrated in these references that a significant refinement is achieved if adaptive meshes are used. The same observation is presented in this paper. The adaptivity for CIPs was first proposed in [4, 5] and was developed further in [6, 7, 8, 20]. It consists in minimizing either the Tikhonov functional [10]-[12] or the associated Lagrangian [4]-[8],[20] on a sequence of locally refined meshes in the FEM.

The new mesh is obtained from the previous one via a local mesh refinement in neighborhoods of those grid points, which provide the maximal input in the error of the calculation of that functional. Thus, the *main question* in the adaptivity is about the identification of those subdomains. This question is addressed via a posteriori error analysis. It is important that the error analysis does not use a knowledge of the exact solution. Instead, one should know an upper bound of that solution, and such a bound should be imposed a priori, in accordance with the Tikhonov principle [34]. It is worthy to mention that there is no rigorous guarantee that such local mesh refinements indeed improve the accuracy of the solution. Nevertheless, such improvements were constantly observed in computations of [4]-[8], [10]-[12], as well as in the current paper.

In section 2 we formulate both forward and inverse problems of our mathematical model. In section 3 we briefly outline the globally convergent stage of our two-stage numerical procedure. We refer to [9, 10] for details including formulations and proofs of global convergence theorems. The main new analytical part is in section 4, where we justify the new mesh refinement rule. In section 5 we describe the experimental setup. In section 6 the procedure of data simulation is described. This procedure is an integral part of our data pre-processing technique. In section 7 we describe how do we pre-process our experimental data. In section 8 we present our imaging results. Discussion is presented in section 9.

2 Statements of forward and inverse problems

As the forward problem, we consider the following Cauchy problem

$$\varepsilon_r(x)u_{tt} = \Delta u, \text{ in } \mathbb{R}^3 \times (0, \infty), \tag{2.1}$$

$$u(x, 0) = 0, u_t(x, 0) = \delta(x - x_0). \tag{2.2}$$

Here $\varepsilon_r(x)$ is the spatially variable dielectric constant (relative dielectric permittivity),

$$\varepsilon_r(x) = \frac{\varepsilon(x)}{\varepsilon_0}, \ \sqrt{\varepsilon_r(x)} = n(x) = \frac{c_0}{c(x)} \geq 1, \tag{2.3}$$

where ε_0 is the dielectric permittivity of the vacuum (which we assume to be the same as one in the air), $\varepsilon(x)$ is the spatially variable dielectric permittivity of the medium of interest, $n(x)$ is the refractive index of the medium of interest, $c(x)$ is the speed of the propagation of the EM field in this medium, and c_0 is the speed of light in the vacuum, which we assume to be the same as one in the air. We point out that it is the refractive index rather than the dielectric constant, which is measured in physics. The assumption $n(x) \geq 1$ means that the speed of the EM field propagation in the medium does not exceed the one in the air, which is reasonable.

Let $\Omega \subset \mathbb{R}^3$ be a convex bounded domain with the boundary $\partial\Omega \in C^3$. We assume that the coefficient $\varepsilon_r(x)$ of equation (2.1) is such that

$$\varepsilon_r(x) \in (1, d], \varepsilon_r(x) = 1 \text{ for } x \in \mathbb{R}^3 \backslash \Omega, \tag{2.4}$$

$$\varepsilon_r(x) \in C^2(\mathbb{R}^3). \tag{2.5}$$

The inequality $\varepsilon_r(x) \geq 1$ follows from (2.3). An upper estimate for the constant $d > 1$ is assumed to be known, although we do *not* assume that the number $d - 1$ is small.

Inverse Problem. *Suppose that the coefficient $\varepsilon_r(x)$ satisfies (2.4) and (2.5). Assume that the function $\varepsilon_r(x)$ is unknown in the domain Ω. Determine the function $\varepsilon_r(x)$ for $x \in \Omega$, assuming that the following function $g(x, t)$ is known for a single source position $x_0 \notin \overline{\Omega}$*

$$u(x, t) = g(x, t), \forall (x, t) \in \partial\Omega \times (0, \infty). \tag{2.6}$$

The assumption $\varepsilon_r(x) = 1$ for $x \in \mathbb{R}^3 \backslash \Omega$ means that one has air outside of the medium of interest Ω. The question of uniqueness of this Inverse Problem is a well known long standing open question. It is addressed positively only if the function $\delta(x - x_0)$ in (2.2) is replaced with a function $f(x)$ such that $f(x) \neq 0, \forall x \in \overline{\Omega}$. Corresponding uniqueness theorems were proven via the method of Carleman estimates [22, 23], also see a recent survey in [35]. Still, due to the applied aspect, it is worthy to develop numerical methods, assuming that the uniqueness question is addressed positively.

Remark 2.1. In section 9 we discuss some discrepancies between our mathematical model and the reality.

3 Brief outline of the globally convergent stage

Since the globally convergent stage was described in [9, 10], we outline it only briefly here referring for details to [9, 10]. In particular, these details include global convergence theorems. Consider the Laplace transform of the solution of the problem (2.1), (2.2),

$$w(x,s) = \int_0^\infty u(x,t)\,e^{-st}dt, s \geq \underline{s} = const. > 0. \tag{3.1}$$

Then $w(x,s) > 0$ for sufficiently large s. Consider the function $q(x,s) = \partial_s\left(s^{-2}\ln w(x,s)\right)$. Under certain conditions

$$D_x^\alpha D_s^k\left(\frac{\ln w(x,s)}{s^2}\right) = O\left(\frac{1}{s^{k+1}}\right), s \to \infty, k = 0,1; |\alpha| \leq 2. \tag{3.2}$$

We obtain a nonlinear integral differential equation for the function q for $x \in \Omega, s \in (\underline{s}, \infty)$ with Volterra integrals, in which the s-integration is carried out from an arbitrary $s \geq \underline{s}$ to ∞. One of the key features of this equation is that the unknown coefficient $\varepsilon_r(x)$ is not involved in it. The Dirichlet boundary condition for the function $q(x,s)$ at $\partial\Omega$ is generated by the function g in (2.6). If one would approximate the function q well, then one would also approximate the function $\varepsilon_r(x)$ well via backwards computations. The main difficulty then is to solve the resulting Dirichlet boundary value problem for q. To do this, we first truncate those Volterra integrals at a large value $s := \overline{s} > \underline{s}$. However, we complement that truncation by the so-called "tail function" $V(x,\overline{s}) \approx \overline{s}^{-2}\ln w(x,\overline{s})$. The tail function is unknown, although, it is small for large $\overline{s}$ because of (3.2). Hence, the resulting equation for q contains two unknown functions: q and V. The reason why we can approximate both of them is that we treat them separately: while we approximate q via inner iterations, we approximate V via outer iterations.

To solve the resulting problem, we divide the interval $[\underline{s}, \overline{s}]$ into N small subintervals. We assume that the function q is constant with respect to s on each of those subintervals. As a result, using the so-called Carleman Weight Function, which depends only on s, we obtain N elliptic Dirichlet boundary value problems for functions $q_n(x)$, where n is the number of the subinterval. Nonlinearities in these equations are mitigated due to the presence of the Carleman Weight Function. Hence, we can solve each of these Dirichlet boundary value problems iteratively via solving a linear problem on each step. Because originally we had Volterra integrals with respect to s, we can solve these problems sequentially starting from q_1. Let $q_{n,k}$ be the approximation for q_n obtained on the inner iteration number k and $V_{n,k}(x)$ be the corresponding approximation for the tail. Then we find the corresponding approximation $\varepsilon_r^{(n,k)}(x)$ for the function $\varepsilon_r(x)$, solve the problem (2.1), (2.2) with $\varepsilon_r := \varepsilon_r^{(n,k)}(x)$, calculate the Laplace transform $w_{n.k+1}(x,\overline{s})$ via (3.1) for it and find a new approximation $V_{n,k+1}(x) := \overline{s}^{-2}\ln w_{n,k+1}(x,\overline{s})$ for the tail. Convergence criteria for this algorithm are described in [9, 10, 11, 12, 24]. In particular, in our computations for experimental data we use the criterion described in subsection 7.1 of [24].

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4 The Adaptivity

4.1 Preliminaries

First, we need to make some remarks about the material of this section. It is well known that our CIP is a very complex problem with many yet unknown factors. It is natural therefore that some simplified assumptions should be made when developing the adaptivity theory for this CIP. We now list main such assumptions. First, in this section we make some assumptions about the smoothness of the data. They are certainly not true for the problem (2.1), (2.2), because of the $\delta-$ function in the initial condition. However, if one would replace the $\delta\left(x - x_0\right)$ with its approximation $\delta_\theta\left(x - x_0\right)$ (below), then smoothness would be restored, although it is outside of the scope of this paper to go further in this direction.

Also, in our computations we use the plane wave rather than the point source in (2.2). This is because we did not yet adapt our numerical code for the case of the point source. We have used the point source in (2.2) only to ensure the asymptotic behavior (3.2). In our numerical studies we verify this behavior computationally, see subsection 7.2 in [9]. We were unable to arrange the experimental signal to become a plane wave. Nevertheless, our data pre-processing procedure "enforces" it to become a plane wave, see subsection 7.1. In addition we impose an over-smoothness assumption of the function $f\left(x, t\right)$ when we require three rather than two t-derivatives of this function. In addition, since we extensively use results of the book [25], where only the Dirichlet boundary condition is considered, we work only with the Dirichlet boundary condition in subsection 4.3. It seems from §5 of Chapter 4 of [25] that the Neumann boundary condition can also be used, so as a lesser smoothness of $f\left(x, t\right).$ However, these assumptions would require a substantial and space consuming effort from us to work out results for forward hyperbolic problems, which would be similar with ones of Chapter 4 of [25]. Thus, for the sake of brevity, as well as because we are interested in inverse rather than forward problems, we have decided to use the Dirichlet boundary condition in subsection 4.3.

Next, we *assume* in subsection 4.4 that the result of subsection 4.3 is also valid for the Neumann boundary condition. Still, we point out that the resulting second mesh refinement recommendation works quite well numerically. Note that if we would replace in (4.5) $u \mid_{S_T} - g(x, t)$ with $\partial_n u \mid_{S_T} - p(x, t)$ (see (4.2) for notations), then we would work with Dirichlet boundary conditions in analogs of (4.6), (4.7), and our second mesh refinement recommendation would be completely rigorous then. Another discrepancy between our theory and and numercial studies is described in subsection 6.2. In this section we use sometimes the same notations for solutions of different PDEs. However, it is always clear from the context what is what here.

The function $\delta_\theta\left(x - x_0\right)$ is defined as

$$\delta_\theta\left(x - x_0\right) = \left\{ \begin{array}{c} C_\theta \exp\left(\frac{1}{|x-x_0|^2 - \theta^2}\right), \ |x - x_0| < \theta \\ 0, \ |x - x_0| \geq \theta \end{array} \right. , \int_{\mathbb{R}^3} \delta_\theta\left(x - x_0\right) dx = 1, \qquad (4.1)$$

where $\theta > 0$ is so small that $\delta_\theta\left(x - x_0\right) = 0$ for $x \in \Omega$ (recall that $x_0 \neq \overline{\Omega}$) and the constant

$C_\theta > 0$ is chosen to ensure the value of the integral in (4.1).

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with $\partial\Omega \in C^2$. We assume that there exists a function $a(x) \in C^\infty(\overline{\Omega})$ such that $a \mid_{\partial\Omega}= 0, \partial_n a \mid_{\partial\Omega}= 1$. An example of such a function was constructed in [11]. Let $T > 0$ be a number. Denote

$$Q_T = \Omega \times (0,T), S_T = \partial\Omega \times (0,T), \Omega_t = \{(x,\tau) : x \in \Omega, \tau = t\}, \forall t \in [0,T].$$

We can consider (2.1), (2.2), (2.6) as an initial boundary value problem for equation (2.1) in $(\mathbb{R}^3 \backslash \Omega) \times (0,T)$. Since by (2.4) $\varepsilon_r(x) = 1$ outside of Ω, this problem can be uniquely solved. Hence, the function $u(x,t)$ is known in $(\mathbb{R}^3 \backslash \Omega) \times (0,T)$. Hence, the following two functions g,p are known at S_T

$$u \mid_{S_T}= g(x,t), \partial_n u \mid_{S_T}= p(x,t). \tag{4.2}$$

Following [11,12], we assume everywhere in this section that there exist functions F, W such that

$$\begin{aligned}
F, W &\in& H^5(Q_T), \\
\partial_n F \mid_{S_T} &=& p(x,t), \partial_n W \mid_{S_T}= g(x,t), \\
\partial_t^i F(x,0) &=& \partial_t^i W(x,0) = 0, i = 1, ..., 4.
\end{aligned} \tag{4.3}$$

Fix a sufficiently small number $\omega \in (0,1)$. Keeping in mind that we need to work with piecewise linear functions, introduce the set Y of functions $c(x)$ satisfying the following conditions

$$Y = \left\{ \begin{array}{c} c \in C(\overline{\Omega}) \cap H^1(\Omega), \partial_{x_i} c \in L_\infty(\Omega), i = 1, 2, 3 \\ c(x) \in (1 - \omega, d + \omega) \text{ for } x \in \overline{\Omega}. \end{array} \right. \tag{4.4}$$

It is convenient to introduce the set of functions Z,

$$Z = \left\{ f : f \in C(\overline{\Omega}) \cap H^1(\Omega), \partial_{x_i} f \in L_\infty(\Omega) \right\}.$$

Hence, $Y \subset Z$. We turn Z into a Banah space via equipping it with the following norm

$$\|f\|_Z = \|f\|_{C(\overline{\Omega})} + \sum_{i=1}^3 \|\partial_{x_i} f\|_{L_\infty(\Omega)}.$$

4.2 The first mesh refinement recommendation

Let the function $z_\varrho(t) \in C^\infty[0,T], z_\varrho(t) = 0$ for $t \in [T - \varrho, T]$ and $z_\varrho(t) = 1$ for $t \in [0, T - 2\varrho]$, where $\varrho > 0$ is a sufficiently small number. We have introduced this function to ensure the compatibility condition for the solution of the adjoint problem (4.7) (below). Now we construct the Tikhonov regularization functional as

$$E(\varepsilon_r) = \frac{1}{2} \int_{S_T} (u \mid_{S_T} -g(x,t))^2 z_\varrho(t)\, dS_x dt + \frac{1}{2}\gamma \int_\Omega (\varepsilon_r - \varepsilon_r^{glob})^2 \; dx, \tag{4.5}$$

where ε_r^{glob} is the solution obtained on the globally convergent stage of our two-stage numerical procedure and γ is the regularization parameter. We assume in (4.5) that the function $\varepsilon_r \in Y$, where the set Y was defined in (4.4). Our goal now is to find a minimizer ε_r^{γ} of this functional, which is called *regularized solution* in the theory of ill-posed problems. In accordance with one of backbone principles of the regularization theory [34], we assume that there exists the unique exact solution ε_r^* of the original inverse problem satisfying conditions (2.4), (2.5). In particular, this means that $\varepsilon_r^* \in Y$. By the global convergence theorems of [9, 10], the function ε_r^{glob} provides a good approximation for ε_r^*. Hence, below in this section we work only in a small neighborhood, in terms of the norm in the space Z (subsection 4.1), of the exact solution ε_r^*. Thus, we assume that there exists unique minimizer ε_r^{γ} of the functional (2.2) in this neighborhood (also, see next subsection about existence and uniqueness). Consider the solution of the following problem, which we call *state problem*,

$$
\begin{aligned}
\varepsilon_r u_{tt} - \triangle u &= 0, \quad (x,t) \in Q_T, \\
u(x,0) = u_t(x,0) &= 0, \\
\partial_n u \mid_{S_T} &= p(x,t), \quad (x,t) \in S_T.
\end{aligned}
\tag{4.6}
$$

In addition, let the function $\lambda(x,t)$ be the solution of the following so-called *adjoint problem* where the time is reversed

$$
\begin{aligned}
\varepsilon_r \lambda_{tt} - \triangle \lambda &= 0, \quad (x,t) \in Q_T, \\
\lambda(x,T) = \lambda_t(x,T) &= 0, \\
\partial_n \lambda \mid_{S_T} &= z_\varrho(t)(g-u)(x,t), \quad (x,t) \in S_T.
\end{aligned}
\tag{4.7}
$$

Using results of books [19, 25] as well as (4.3) and above conditions imposed on the domain Ω, it was established in [11] that each of problems (4.6), (4.7) has unique weak solution and this solution belongs to $H^4(Q_T)$.

Consider a mesh in the domain Ω on which the functional $E(\varepsilon_r)$ is minimized. Let h be the maximal grid step size of this mesh. Suppose that there exists a unique minimizer $\varepsilon_{r,h}$ of the functional (4.5) on this mesh in that small neighborhood of ε_r^*. It was shown in [11] that the following approximate error estimate holds for the functional $E(\varepsilon_r)$

$$
|E(\varepsilon_r^{\gamma}) - E(\varepsilon_{r,h})| \le |E'(\varepsilon_{r,h})(\varepsilon_r^{\gamma} - \varepsilon_{r,h})|,
\tag{4.8}
$$

where E' is the Fréchet derivative of the functional E. The formula for $E'(\varepsilon_r)$ for $\varepsilon_r \in Y$ is [11, 12]

$$
E'(\varepsilon_r)(x) = \gamma(\varepsilon_r - \varepsilon_r^{glob}) - \int_0^T u_t \lambda_t dt, x \in \Omega.
\tag{4.9}
$$

Hence, we have to solve the following equation with respect to the function ε_r

$$
E'(\varepsilon_r)(x) = 0, x \in \Omega.
\tag{4.10}
$$

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By (4.8) and (4.9) the following approximate error estimate for the functional (4.5) is true

$$|E(\varepsilon_r^\gamma) - E(\varepsilon_{r,h})| \leq C \left\| E'(\varepsilon_{r,h}) \right\|_{L_\infty(\Omega)} \left\| \nabla \varepsilon_r^\gamma \right\|_{L_\infty(\Omega)} h, \tag{4.11}$$

where

$$|E'(\varepsilon_{r,h})(x)| = \left| \gamma \left(\varepsilon_{r,h} - \varepsilon_{r,h}^{glob} \right) - \int\limits_0^T (u_t \lambda_t)(x, t, \varepsilon_{r,h}) \, dt \right|, x \in \Omega, \tag{4.12}$$

where functions $u(x, t, \varepsilon_{r,h})$ and $\lambda(x, t, \varepsilon_{r,h})$ are weak solutions of problems (4.6) and (4.7) respectively for the case when the coefficient ε_r in (4.6) and (4.7) is replaced with $\varepsilon_{r,h}$. Here and below $C = C(\omega, \Omega) > 0$ denotes several constants depending only on the number ω in (4.4) and the domain Ω. Here $\varepsilon_{r,h}^{glob}$ is the linear interpolation of the function ε_r^{glob} on the above mesh. Hence, we should try to reduce the norm $\left\| E'(\varepsilon_{r,h}) \right\|_{L_\infty(\Omega)}$ via the next local mesh refinement, in order to reduce the error of computing the minimizer of the Tikhonov functional on the next refined mesh. To do this, we refine mesh in neighborhoods of those grid points where the function $|E'(\varepsilon_{r,h})(x)|$ attains its maximal values. As it was stated in Introduction, although there is no rigorous guarantee that such local mesh refinements improve the accuracy of the solution, we have always observed this numerically. These considerations led us in [11] to the following

First Mesh Refinement Recommendation. *Refine the mesh in neighborhoods of those grid points $x \in \Omega$ where the function $|E'(\varepsilon_{r,h})(x)|$ defined in (4.12) attains its maximal values. More precisely, refine the mesh in such subdomains of the domain Ω where*

$$|E'(\varepsilon_{r,h})(x)| \geq \varkappa \max_{\overline{\Omega}} |E'(\varepsilon_{r,h})(x)|, \tag{4.13}$$

where $\varkappa \in (0, 1)$ is the tolerance number.

If we would take in (4.13) $\varkappa \approx 1$, then we would refine the mesh in too narrow regions. On the other hand, if we would take $\varkappa \approx 0$, then we would refine the mesh in almost the entire domain Ω, which is not efficient. Hence, the parameter $\varkappa$ should be chosen numerically. Below we take in (4.13) $\varkappa = 0.8$ for all computational meshes.

So, numerically we proceed as follows for both first and second mesh refinement recommendations. On each mesh we need to approximately solve equation (4.10). We start our computations on the same mesh on which the globally convergent method has worked. In our experience, this mesh does not provide an improvement of the image. For each newly refined mesh we first linearly interpolate the function $\varepsilon_r^{glob}(x)$ on it. Since this function is computed on the globally convergent stage as a linear combination of finite elements forming the initial mesh and since these finite elements are piecewise linear functions, then subsequent linear interpolations on finer meshes do not change this function. On each mesh we iteratively update approximations $\varepsilon_{r,h}^n$ of the function $\varepsilon_{r,h}$. To do so, we use the quasi-Newton method with the classic BFGS update formula with the limited storage [28]. Denote

$$g^n(x) = \alpha(\varepsilon_{r,h}^n - \varepsilon_r^{glob})(x) - \int_0^T (u_{ht} \lambda_{ht}) \left(x, t, \varepsilon_{r,h}^n \right) dt,$$

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where functions $u_h\left(x, t, \varepsilon_{r,h}^n\right), \lambda_h\left(x, t, \varepsilon_{r,h}^n\right)$ are FEM solutions of state and adjoint problems (4.6), (4.7) with $\varepsilon_r := \varepsilon_{r,h}^n$. We stop computing $\varepsilon_{r,h}^n$ if either $\|g^n\|_{L_2(\Omega)} \leq 10^{-5}$ or norms $\|g^n\|_{L_2(\Omega)}$ are stabilized. For a given mesh, let $\varepsilon_{r,h} := \varepsilon_{r,h}^n$ be the last computed function on which we have stopped. Next, we compute the function $|E'\left(\varepsilon_{r,h}\right)(x)|$ in (4.12) and consider all grid points in this mesh where (4.13) is fulfilled. Next, we refine the mesh in neighborhoods of all grid points satisfying (4.13). The stopping criterion for the mesh refinement process is described in sub-subsection 8.2.1.

4.3   Some estimates for a forward problem for a hyperbolic PDE

In this subsection we obtain some estimates, which are used in the next subsection for the derivation of the second mesh refinement recommendation. Let the function $f(x,t)$ be such that for $k = 0, 1, 2, 3$

$$\partial_t^k f \in L_2\left(\Omega_t\right), \forall t \in [0, T] \text{ and } \|f_k(t)\|_{L_\infty(0,T)} < \infty, \text{ where } f_k(t) = \left\|\partial_t^k f(x,t)\right\|_{L_2(\Omega_t)}. (4.14)$$

Let the function $c \in Y$, where the set Y was defined in (4.4). Consider the following initial boundary value problem

$$\begin{aligned}
c(x)\, u_{tt} &= \Delta u + f \quad \text{in } Q_T, \\
u(x,0) &= u_t(x,0) = 0, \\
u\left.\right|_{S_T} &= 0.
\end{aligned} \qquad (4.15)$$

Although upper estimates for the solution of this and more general hyperbolic problems are well known of course, see Chapter 4 of [25], constants in those estimates are not specified in the way we need them to be specified. More precisely, in order to derive our second mesh refinement recommendation, we need to specify the dependence that upper estimate from the function $c(x)$ for the case of the problem (4.15). To do this, we will naturally use Theorem 4.1 of Chapter 4 of [25]. The definition of the weak $H^1(Q_T)$−solution of the problem (4.15) can be found in §5 of Chapter 4 of [25] as well as in section 7.2 of [19].

   Theorem 4.1. *Let in (4.15) the coefficient $c(x) \in Y$, the function f satisfies (4.14) and $\partial_t^k f(x,0) = 0$ in Ω for $k = 0, 1, 2$. As above, assume that $\partial\Omega \in C^2$. Denote*

$$m = \max_{\overline{\Omega}} c(x),\, M = \|\nabla c\|_{L_\infty(\Omega)}.$$

Then there exists unique weak solution $u \in H^1(Q_T)$ of the problem (4.15). Furthermore,

$$\begin{aligned}
\partial_t^k u, \partial_t^j\left(\partial_{x_i} u\right) &\in L_2\left(\Omega_t\right), \forall t \in [0, T], k = 0, ..., 3; j = 0, 1, 2; i = 1, 2, 3, \qquad (4.16) \\
u, u_t &\in H^2\left(\Omega_t\right). \qquad (4.17)
\end{aligned}$$

In addition, functions $u, \partial_t u \in C\left(\overline{\Omega}_t\right), \forall t \in [0, T]$. Let $\overline{u}(t) = \|u_t(x,t)\|_{C\left(\overline{\Omega}_t\right)}$. Then the function $\overline{u}(t) \in L_\infty(0, T)$ and the following estimate holds

$$\|\overline{u}(t)\|_{L_\infty(0,T)} \leq Cm \exp(CMT)\left(\|f_{tt}\|_{L_2(Q_T)} + \|f_1(t)\|_{L_\infty(0,T)}\right). \qquad (4.18)$$

Proof. Denote $b(x) = 1/c(x)$. Then equation (4.15) can be rewritten as

$$u_{tt} = \nabla \cdot (b(x) \nabla u) - \nabla b \nabla u + \widetilde{f}(x, t), \tag{4.19}$$

where $\widetilde{f}(x, t) = b(x) f(x, t)$. Obviously,

$$\|\nabla b\|_{L_\infty(\Omega)} \leq \frac{M}{(1 - \omega)^2}, \left| \widetilde{f} \right| \leq \frac{|f|}{1 - \omega}. \tag{4.20}$$

Applying to equation (4.19) with initial and boundary conditions (4.15) Theorem 4.1 and formula (4.11) of Chapter 4 of [25], in combination with Theorems 3.1 and 3.2 from the same chapter, we obtain that there exists unique weak solution u of this problem and conditions (4.16), (4.17) are satisfied. Hence, transforming back equation (4.19) into equation (4.15), we obtain

$$\begin{aligned}
\Delta u_t &= c(x)(u_t)_{tt} - f_t(x, t) \text{ in } \Omega_t, \tag{4.21} \\
u_t \mid_{\partial\Omega} &= 0.
\end{aligned}$$

We can consider (4.21) as the Dirichlet boundary value problem for the Laplace equation in Ω_{1t}. Hence, using Theorem 4 of §2 of Chapter 4 of [26], we obtain that $u_t \in H^2(\Omega_t)$ and

$$\|u_t\|_{H^2(\Omega_t)} \leq C \left(m \left\| \partial_t^3 u \right\|_{L_2(\Omega_t)} + \|f_t\|_{L_2(\Omega_t)} \right). \tag{4.22}$$

Hence, we now need to estimate the norm $\left\| \partial_t^3 u \right\|_{L_2(\Omega_t)}$. Consider the solution of the following initial boundary value problem

$$\begin{aligned}
w_{tt} &= \nabla \cdot (b(x)\nabla w) - \nabla b \nabla w + f_{tt}(x, t), \\
w(x, 0) &= w_t(x, 0) = 0, \tag{4.23} \\
w \mid_{S_T} &= 0.
\end{aligned}$$

Then Theorem 4.1 of Chapter 4 of [25] in combination with Theorems 3.1 and 3.2 from the same chapter imply that there exists unique weak solution $w \in H^1(Q_T)$ of this problem and also

$$\partial_t^r w, \partial_t^p (\partial_{x_i} w) \in L_2(\Omega_t), \forall t \in [0, T], r = 0, 1, 2; p = 0, 1; i = 1, 2, 3. \tag{4.24}$$

It is easy to verify that actually $w = u_{tt}$. To do this in a simple way, one can first derive from (4.23) the initial boundary value problem for the function v,

$$v(x, t) = \int_0^t w(x, \tau) \, d\tau.$$

Next, using (4.24), one can establish that $w = v_t$. One can similarly establish that $v = u_t$.

We now estimate the function w using standard energy estimates. By (4.24) we can consider functions $\partial_t \partial_{x_i} w$. Multiply both sides of equation (4.23) by $2w_t$ and integrate the

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resulting equation over the domain $\Omega \times (0, t)$ for an arbitrary $t \in (0, T)$. Using integration by parts, we obtain

$$\int_{\Omega_t} w_t^2(x, t)\, dx + 2 \int_{\Omega} b(x)\, dx \int_0^t \partial_\tau (\nabla w(x, \tau))^2\, d\tau =$$

$$-2 \int_0^t \int_{\Omega} \nabla b(x) \nabla w(x, \tau) w_\tau(x, \tau)\, dx d\tau + 2 \int_0^t \int_{\Omega} f_{\tau\tau}(x, \tau) w_\tau(x, \tau)\, dx d\tau.$$

Using Cauchy-Schwarz inequality and (4.20), we obtain from here

$$\int_{\Omega_t} \left[w_t^2 + (\nabla w)^2 \right](x, t)\, dx \leq CM \int_0^t \int_{\Omega} \left[w_t^2 + (\nabla w)^2 \right](x, \tau)\, dx d\tau + C \int_0^t \int_{\Omega} f_{\tau\tau}^2(x, \tau)\, dx d\tau.$$

Therefore, Gronwall's inequality leads to

$$\int_{\Omega_t} \left[w_t^2 + (\nabla w)^2 \right](x, t)\, dx \leq C \exp(CMt) \|f_{tt}\|_{L_2(Q_T)}^2, \forall t \in (0, T). \qquad (4.25)$$

Since $w_t = \partial_t^3 u$, then (4.22) and (4.25) imply that

$$\|u_t\|_{H^2(\Omega_t)} \leq C \left(m \exp(CMt) \|f_{tt}\|_{L_2(Q_T)} + \|f_t\|_{L_2(\Omega_t)} \right). \qquad (4.26)$$

Now, by the Sobolev embedding theorem $H^2(\Omega_t) \subset C(\overline{\Omega})$ and $\|p\|_{C(\overline{\Omega}_t)} \leq C \|p\|_{H^2(\Omega_t)}, \forall p \in H^2(\Omega_t)$. Hence, using (4.26), we obtain

$$\|u_t\|_{C(\overline{\Omega}_t)} \leq C \left(m \exp(CMt) \|f_{tt}\|_{L_2(Q_T)} + \|f_t\|_{L_2(\Omega_t)} \right), \forall t \in (0, T).$$

Using this inequality and (4.14), we obtain (4.18). $\square$

4.4 The second mesh refinement recommendation

While in the subsection 4.3 we have estimated only the accuracy of the calculation of the Tikhonov functional on a mesh, now we want to estimate the distance between the minimizer of this functional on that mesh and the regularized solution. It is clear that this estimate is more valuable than the estimate (4.11). The price we pay for this is that we impose a more stringent condition on the regularized solution ε_r^γ, see below in this subsection. In this subsection we use results of [12] as well as of subsection 4.2.

Consider a triangulation Tr of the domain Ω_1 with a rather coarse mesh. We obtain a polygonal domain $\sigma \subseteq \Omega_1$. All tetrahedra forming finer meshes will be contained in σ.

Suppose that we have constructed an exceedingly fine mesh. Since corresponding finite elements are piecewise linear functions, consider all possible linear combinations of these functions. Then we obtain a finite dimensional space H and we equip H with the $L_2(\sigma)$ norm. Indeed, all norms in finite dimensional spaces are equivalent and it is quite convenient to work with the $L_2(\sigma)$ norm. So, the above mentioned more stringent condition we impose on the regularized coefficient ε_r^γ is that we assume now that the function $\varepsilon_r^\gamma \in H$ rather than being an element of an infinitely dimensional space. However, given that we work with problems of practical computations in which all functions are represented via finite elements, this condition is not an over-restrictive one.

Since it is not efficient to use an exceedingly large number of finite elements in computations, the idea of the adaptivity is to approximate ε_r^γ via minimizing of the functional (4.5) on a sequence of *locally* refined meshes. We assume below that finite elements forming each such mesh form a subspace of the space H. This assumption is reasonable, see arguments in section 2 of [12].

Let $\delta > 0$ be a small positive number characterizing the level of error in the data $\widetilde{g}$, see details in formula (7.2) of [12]. Since global convergence theorems of [9, 10] guarantee that the function $\varepsilon_r^{glob} \in Y$ provides a good approximation for the correct solution, then we assume that

$$
\begin{aligned}
\left\| \varepsilon_r^{glob} - \varepsilon_r^* \right\|_{L_2(\sigma)} &\leq \delta^{\mu_1}, \mu_1 = const. \in (0,1), \\
\gamma &= \delta^{\mu_2}, \mu_2 = const. \in (0, \min(\mu_1, 2(1-\mu_1))),
\end{aligned}
\tag{4.27}
$$

where μ_1 is a number of ones choice. Recall that the set of functions Y is defined in (4.4). In the regularization theory one usually studies the question about behavior of regularized solutions when $\delta \to 0$. So, as soon as the distance between a regularized and exact solutions is comparable with δ, the process is stopped [17, 34]. On the other hand, given a number $\eta \in (0,1)$, we obviously have that $\delta^\eta >> \delta$ for sufficiently small values of δ. This justifies the assumption (4.27).

Let $\beta \in (0,1)$ be an arbitrary number. Denote

$$
\begin{aligned}
V_{\beta\delta^{\mu_2}}(\varepsilon_r^\gamma) &= \left\{ f \in H : \|f - \varepsilon_r^\gamma\|_{L_2(\sigma)} < \beta\delta^{\mu_2} \right\}, \\
V_{(1+\sqrt{2})\delta^{\mu_1}}(\varepsilon_r^*) &= \left\{ f \in H : \|f - \varepsilon_r^*\|_{L_2(\sigma)} < \left(1+\sqrt{2}\right)\delta^{\mu_1} \right\}.
\end{aligned}
$$

Let Y' be the set of restrictions of all functions $c \in Y$ on the polygonal domain σ. Combining Lemma 2.1, Theorem 7.2 and Theorem 7.3 of [12], we come up with the following

Theorem 4.2. *Assume that the domain Ω satisfies conditions formulated in subsection 4.1. In addition, assume that conditions (4.3) and (4.27) hold. Suppose that conditions (4.27) are satisfied and the function ε_r^* satisfies conditions (2.4), (2.5). Consider a locally refined mesh with the maximal grid step size h and let $M_h \subset H$ be the subspace of the space H generated by tetrahedral finite elements corresponding to this mesh. Then there exists a sufficiently small $\delta_0 \in (0,1)$ such that for all $\delta \in (0, \delta_0]$ the following assertions 1-5 are true:*

1. The functional (4.5) has unique minimizer $\varepsilon_r^\gamma \in V_{(1+\sqrt{2})\delta^{\mu_1}}(\varepsilon_r^*)$.

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2. $V_{\left(1+\sqrt{2}\right)\delta^{\mu_1}}\left(\varepsilon_r^*\right) \subset V_{\beta\delta^{\mu_2}}\left(\varepsilon_r^\gamma\right) \subset Y'.$

3. The functional (4.5) is strongly convex in $V_{\beta\delta^{\mu_2}}\left(\varepsilon_r^\gamma\right)$.

4. There exists a sufficiently small $h_0 = h_0\left(\delta_0, \|\nabla\varepsilon_r^\gamma\|_{L_\infty(\sigma)}\right)$ such that for all $h \in (0, h_0]$ the functional (4.5) has unique minimizer $\varepsilon_{r,h}$ on the set $V_{\beta\delta^{\mu_2}}\left(\varepsilon_r^\gamma\right) \cap M_h$.

5. The following a posteriori accuracy estimate of the reconstruction of the regularized coefficient holds

$$\|\varepsilon_r^\gamma - \varepsilon_{r,h}\|_{L_2(\sigma)} \le \frac{3}{\gamma}\left\|\widetilde{E}'\left(\varepsilon_{r,h}\right)\right\|_{L_2(\sigma)}, \tag{4.28}$$

where the function $\widetilde{E}'\left(\varepsilon_{r,h}\right)(x)$ has the form

$$\widetilde{E}'\left(\varepsilon_{r,h}\right)(x) = \gamma\left(\varepsilon_{r,h} - \varepsilon_r^{glob}\right) - \int\limits_0^T \left(u_t\lambda_t\right)(x, t, \varepsilon_{r,h})\, dt, x \in \sigma, \tag{4.29}$$

where functions $u(x, t, \varepsilon_{r,h})$ and $\lambda(x, t, \varepsilon_{r,h})$ are weak solutions of problems (4.6) and (4.7), respectively, for the case when the coefficient ε_r in (4.6) and (4.7) is replaced with $\varepsilon_{r,h}$ and, by the definition $\varepsilon_{r,h}(x) = 1$ for $x \in \Omega\backslash\sigma$.

Using Theorems 4.1 and 4.2, we now derive the second mesh refinement recommendation. As we have pointed out in subsection 4.1, we have considered the Dirichlet boundary condition in Theorem 4.1 instead of the Neumann boundary condition for convenience and brevity only. So, we assume now that Theorem 4.1 is true for the case of the Neumann condition in (4.15). Then functions $u_t(x, t, \varepsilon_{r,h}), \lambda_t(x, t, \varepsilon_{r,h}) \in C\left(\overline{\Omega}_{1t}\right), \forall t \in [0, T].$ Denote

$$\overline{u}_h(t) = \|u_t(x, t, \varepsilon_{r,h})\|_{C\left(\overline{\Omega}_{1t}\right)}, \overline{\lambda}_h(t) = \|\lambda_t(x, t, \varepsilon_{r,h})\|_{C\left(\overline{\Omega}_{1t}\right)}, \tag{4.30}$$

$$m_h = \max_{\overline{\Omega}}\varepsilon_{r,h}(x), M_h = \|\nabla\varepsilon_{r,h}\|_{L_\infty(\Omega)}. \tag{4.31}$$

Considering functions [11]

$$\widehat{u}(x, t, \varepsilon_{r,h}) = u(x, t, \varepsilon_{r,h}) - F, \widehat{\lambda} = \lambda(x, t, \varepsilon_{r,h}) - (W - a(x)u)z_\varrho(t)$$

and using Theorem 4.1, we obtain

$$\overline{u}_h(t) \le C_1 m_h \exp\left(CM_hT\right), \overline{\lambda}_h(t) \le C_1 m_h \exp\left(CM_hT\right). \tag{4.32}$$

Here and below $C_1 = C_1\left(\Omega, \omega, F, W, z_\varrho\right) > 0$ denotes different positive constants depending on parameters listed. Hence, (4.27)-(4.32) lead to

Theorem 4.3. *Assume that conditions of Theorem 4.2 are satisfied and the maximal grid step size h of the locally refined mesh under consideration is such that $h \in (0, h_0]$, where h_0 was defined in assertion number 4 of Theorem 4.2. Then there exists unique minimizer $\varepsilon_{r,h}$ of the functional (4.5) on the set $V_{\beta\delta^{\mu_2}}\left(\varepsilon_r^\gamma\right) \cap M_h$. In addition, assume that Theorem 4.1 is true for the case of the Neumann boundary condition in (4.15). Then the following a posteriori accuracy estimate of the reconstruction of the regularized coefficient holds*

$$\|\varepsilon_{r,h} - \varepsilon_r^\gamma\|_{L_2(\sigma)} \le \frac{C_1^2}{\delta^{\mu_2}}m_h^2 \exp\left(CM_hT\right), \tag{4.33}$$

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where the number μ_2 was defined in (4.27).

Recall that the idea of the adaptivity is to refine mesh locally at those regions which provide the biggest impact in *a posteriori* error estimate. Hence, it follows from (4.31) and (4.33) that we should refine mesh in neighborhoods of those grid points where: (a) the coefficient $\varepsilon_{r,h}$ attains its maximal its value m_h and also (b) where the modulus of the gradient of this coefficient attains its maximal value M_h. Thus, we obtain

The Second Mesh Refinement Recommendation. *Refine the mesh in such subdomains of the domain Ω where*

$$\varepsilon_{r,h}(x) \geq \alpha_1 \max_{\overline{\Omega}} \varepsilon_{r,h}(x) \ \text{and} \ |\nabla \varepsilon_{r,h}(x)| \geq \alpha_2 \max_{\overline{\Omega}} |\nabla \varepsilon_{r,h}(x)|, \qquad (4.34)$$

where $\alpha_1, \alpha_2 \in (0,1)$ are tolerance numbers.

Remarks 4.1.

1. It is important that (4.33) estimates the accuracy of the reconstruction of the regularized coefficient on that mesh via numbers m_h and M_h, which are directly related to the function $\varepsilon_{r,h}$, and this function is calculated already as the minimizer of the functional (4.5). In other words, numbers m_h and M_h are known, so as subdomains of the domain Ω where values of functions $\varepsilon_{r,h}(x), |\nabla \varepsilon_{r,h}|$ are close to m_h, M_h.

2. It is possible to obtain a sharper estimate than one in (4.33) via removing the large multiplier $\delta^{-\mu_2}$. This would require more accurate estimates of functions $u(x, t, \varepsilon_{r,h}), \lambda(x, t, \varepsilon_{r,h})$. However, we are not doing this here for brevity, since the main goal of the estimate (4.33) is to provide the second mesh refinement recommendation.

3. Since we want to image small "sharp" inclusions, within which $\varepsilon_r > 1$, then we can approximately assume that the maximal value of $|\nabla \varepsilon_{r,h}(x)|$ (in neighborhoods of those inclusions) is achieved at about the same points where the maximal value of the function $\varepsilon_{r,h}(x)$ is achieved. For this reason we use only the first formula (4.34) in our computations. Note that it is easier to verify this formula computationally than the second one.

5 The Experimental Setup

For brevity below x denotes both a vector $x = (x, y, z) \in \mathbb{R}^3$ and the first component of this vector, where z is the vertical coordinate. It is always clear from the context what is what there. Our source/detectors configuration is schematically depicted on Figure 1. The source has generated an electromagnetic (EM) wave. Only one component of the vector of the electric field was generated by our source. And only one component of this field was measured at the bottom side of the rectangular prism Ω depicted on Figure 1. We do not know which component was measured: we have only worked with the measured time dependent voltage in our computations. This prism is our computational domain Ω. It consisted of Styrofoam. Styrofoam is a material, whose relative permittivity $\varepsilon_r \approx 1$, i.e. the same as one in the air. The sizes of Ω were 240 mm×140 mm × 240 mm. Hence, sizes of front and back sides of the prism of Figure 1 are 240 mm× 240 mm and sizes of other four sides are 240 mm× 140 mm. The distance between the wave source and the top side of the domain Ω was 130 mm.
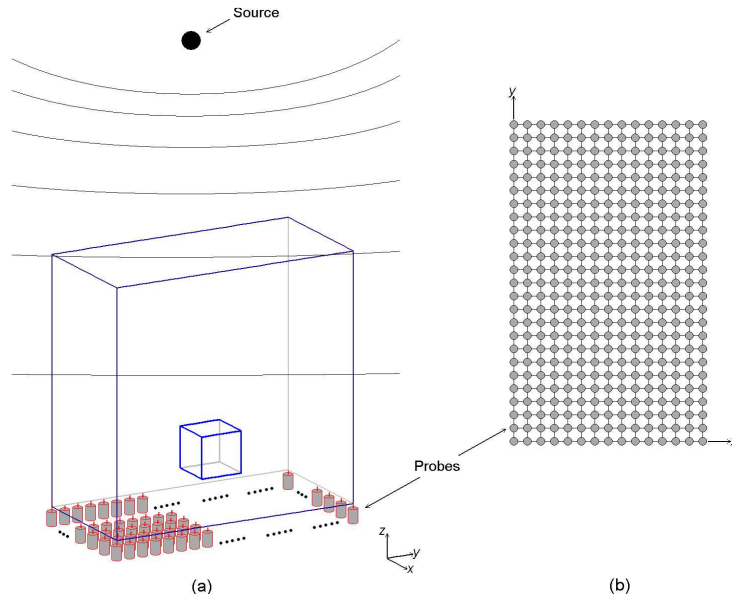
Figure 1: *Schematic diagram of the source/detectors configuration. a) The rectangular prism depicts our computational domain Ω . Only a single source location outside of this prism was used. Tomographic measurements of the scattered time resolved EM wave were conducted on the bottom side of this prism. b) Schematic diagram of locations of detectors on the bottom side of the prism Ω. The distance between neighboring detectors was 10 mm.*

The initializing pulse was 100 ps duration. Since the speed of the EM wave propagation in the air is 0.3 mm/ps, then it requires 433 ps $\approx$ 130/03 ps for this wave to travel from the source to the top boundary of Ω. Hence, it follows from (5.1) the wave did not yet reach the domain Ω during the 100 ps duration of this pulse. The initializing pulse was

$$f(t) = \begin{cases} \approx A \sin\left(\frac{\pi}{50}\tau\right), & \text{for } \tau \in (0, 100) \text{ ps}, \\ 0, & \text{for } \tau > 100 \text{ ps}, \end{cases} \tag{5.1}$$

where A is the amplitude. Our data processing procedure does not rely on a knowledge of A.

The time resolved signal was measured at some locations of the detector on the bottom side of the prism Ω, as indicated on Figure 1-b). On each detector location this signal was measured with the time interval of 20 picoseconds between two consecutive measurements for the total period of 12,300 picoseconds=12.3 nanoseconds. First, we were putting the detector at one location, sent the pulse and measured the time resolved scattering wave at this location. Next, we have moved the detector mechanically in a neighboring location and repeated the measurement, etc.. Hence, it is reasonable to assume in the mathematical

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model that the wave field was measured simultaneously at all those detectors. However, measurements was not carried out at other sides of this prism, see Figure 1. The reason why we have not performed measurements on other five sides of the rectangular prism Ω is that we have observed in our computational simulations that these sides were much less sensitive to the presence of dielectric abnormalities than the bottom side of Ω. Thus, we have prescribed to these sides the same values of the function $g(x,t)$ in (6) which we have obtained in our computational simulation of solving the problem (47) (subsection 6.1) for $\varepsilon_r(x) \equiv 1$, i.e. for the case when inclusion is not present.

We had two measurements at each detector location. First, we have measured the reference signal when the inclusion was not present. Second, we have measured the signal when the inclusion was present. In principle, our technique allows the measurement of the reference signal only at a few locations outside of the medium of interest: for the calibration purposes. The only reason why we have measured the reference signal for each location of the detector was that our current numerical implementation works only with the case when the initializing wave field is a plane wave. On the other hand, it was impossible to arrange a true plane wave in that experiment and so we had a spherical wave.

Our dimensionless computational domain Ω, the dimensionless distance $\widetilde{h}$ between two neighboring detectors and the dimensionless time t were [24]

$$\Omega = \{(x,y,z) \in [-2.4, 2.4] \times [-1.4, 1.4] \times [-2.4, 2.4]\}, \widetilde{h} = 0.2, \ t \in (0, 12). \qquad (5.2)$$

Let P be the bottom side of the domain Ω in (5.2),

$$P = \{(x,y,z) : (x,y) \in [-2.4, 2.4] \times [-1.4, 1.4], z = -2.4\}. \qquad (5.3)$$

6   Data Simulation

6.1   Data simulation

Since the computationally simulated data play an important role in our data pre-processing procedure, we outline here the solution of the forward problem for equation (2.1). Since it is practically impossible to solve the PDE (1) in the entire space $\mathbb{R}^3$, we have solved it in a larger rectangular prism $G = \{(x,y,z) \in [-3, 3] \times [-2, 2] \times [-5, 5]\}$. So, by (5.2) $\Omega \subset G$. Our initializing plane wave was $v(t)$,

$$v(t) = \begin{cases} \sin(\omega t), & \text{for } t \in \left(0, \frac{2\pi}{\omega}\right), \\ 0, & \text{for } t > \frac{2\pi}{\omega}, \omega = 7. \end{cases} \qquad (6.1)$$

Let ∂G_1 and ∂G_2 be respectively top and bottom sides of G and $\partial G_3 = \partial G \setminus (\partial G_1 \cup \partial G_2)$ be the rest of the boundary of G. We have numerically solved the following initial boundary

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value problem

$$\begin{aligned}
\varepsilon_r \left(x \right) u_{tt} &= \triangle u, \quad \text{in } G \times (0, T), T = 12, \\
u(x, 0) &= 0, \ u_t(x, 0) = 0, \text{ in } G, \\
\partial_n u \big|_{\partial G_1} &= v \left(t \right), \text{ on } \partial G_1 \times (0, 2\pi/\omega], \\
\partial_n u \big|_{\partial G_1} &= -\partial_t u, \text{ on } \partial G_1 \times (t_1, T), \\
\partial_n u \big|_{\partial G_2} &= -\partial_t u, \text{ on } \partial G_2 \times (0, T), \\
\partial_n u \big|_{\partial G_3} &= 0, \text{ on } \partial G_3 \times (0, T),
\end{aligned} \tag{6.2}$$

In the case when the data are simulated for the reference medium, we have in (6.2) $\varepsilon_r \left(x \right) \equiv 1$. We denote this solution as $u_1 \left(x, t \right)$. Thus, in (6.2) the plane wave is initialized at the top boundary ∂G_1 for times $t \in (0, 2\pi/\omega]$ and propagates into G . First order absorbing boundary conditions [18] were used on the top boundary for $t \in (2\pi/\omega, T)$ as well as on the bottom boundary ∂G_2 for $t \in (0, T)$. The zero Neumann boundary condition was used on the rest of the boundary of the prism G. The latter boundary condition is used because the "pure" plane wave with $\varepsilon_r \left(x \right) \equiv 1$ satisfies this condition. The problem (6.2) was solved by the hybrid FEM/FDM method described in [13]. In this method, FDM is used outside of the domain Ω, i.e. in $G \backslash \Omega$, and FEM, is used inside of Ω. The step size in the overlapping region was $\widetilde{h} = 0.2$ which is the same as the distance between any two neighboring detectors.

6.2   Solving problems (4.6) and (4.7) in the adaptivity

Although the above theory says that we should solve problems (4.6) and (4.7) in the domain Ω, we actually solve both of them in larger domains. Namely, in our computations the problem (4.6) is solved in the domain G with boundary conditions (6.2). And the problem (4.7) is solved in such a part of the domain G which is above the bottom side P of the prism Ω, i.e. in the subdomain $G' = G \cap \{z > -2.4\}$. Let $P_{obs} = \{z = -2.4\} \cap G$. Then by (45) the rectangle $P \subset P_{obs}$. When solving the problem (4.7), we use the boundary condition $\partial_n \lambda \,|_{P_{obs}} = (g - u) \,|_{P_{obs}}$, where g is our pre-processed experimental data. Hence, we actually need to know the function $g \left(x, t \right)$ not only on the rectangle P but also on a wider rectangle P_{obs}. In other words, we need to extend this function somehow from P to $P_{obs} \backslash P$. In general, this is a problem, which is similar with the very complicated problem of analytical continuation. However, using some features of our specific arrangement, we have found a different way of this extension via the so-called *third stage* of our data immersing procedure, which is described in subsection 7.4. We use the absorbing boundary condition $\partial_n \lambda = -\partial_t \lambda$ at $\{z = 5\} \cap \overline{G}'$ and we use the zero Neumann boundary condition at the rest of the boundary of the domain G'. We believe that the above theory of the adaptivity can be extended to this case, although we have not yet done this. So, the latter is another discrepancy, in addition to those described in subsection 4.1, between our theory of section 4 and numerical studies.

7   Data Pre-Processing

The main idea of this procedure is to *immerse* the experimental data in the computationally simulated ones. We have done this in three stages described in follow up sub-sections of this section. The third stage is new, whereas first two stages were described in [24]. The data pre-processing procedure provides us with the boundary data at $\partial\Omega$, which we use in our computations. Recall that we have not carried out measurements at $\partial\Omega \setminus P$ and have prescribed

$$u(x,t)\mid_{\partial\Omega \setminus P} := u_1(x,t)\mid_{\partial\Omega \setminus P},$$

where $u_1(x,t)$ is the solution of the problem (47) with $\varepsilon_r(x) \equiv 1$, see section 5. So, in this section we describe how we pre-process the data only at the bottom side P of the rectangular prism Ω.

a)   b)
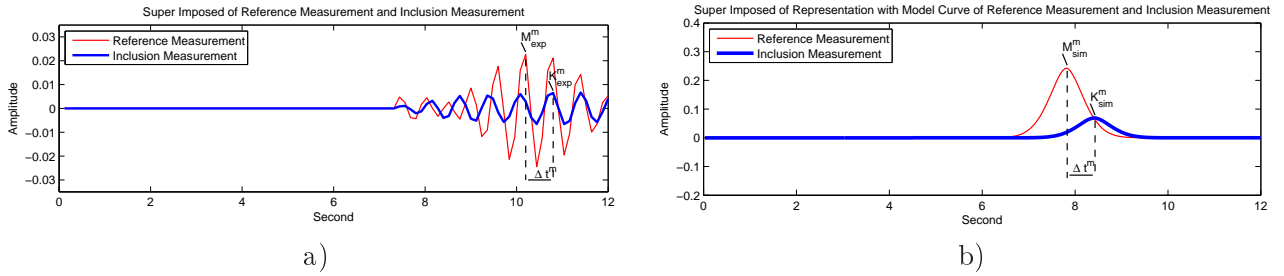
Figure 2:   *This figure explains the idea of the first stage of data immersing in the time domain. We have intentionally set to zero the small amplitude fluctuations before that first burst. a) Resulting superimposed experimental curves. The red curve (thin) is for the reference signal and the blue curve (thick) is for the signal with a dielectric inclusion present, both at the same location $x_m \in P$ of the detector number m. b) The red curve (thin) displays computationally simulated data $u_1(x_m, t)$. The blue curve (thick) $u_{incl}(x_m, t) = u_1(x_m, t - \Delta t^m) K_{\exp}^m / M_{\exp}^m$ represents a sample of the immersed experimental data in the time domain at the same detector location $x_m \in P$. It is only the blue curve (thick) with which we work further. The red curve (thin) is displayed for the illustration purpose only.*

7.1   First immersing in the time domain (the first immersing stage)

Let $x_m \in P$ be the detector number m at the bottom side P of the prism Ω, see (5.3) for P. Samples of unprocessed experimental data can be found on Figure 2 of [24]. As in [24], we work with the first burst only. Figure 2-a) displays a sample of the first burst after a partial denoising via the Fourier transform, see details in [24]. We have decided to "immerse" our experimental data in the computationally simulated data using the following two peaks for each detector x_m :

1. The largest peak in the red curve (thin line, reference medium) with the peak value of $M_{\exp}^m$.

a) $\varepsilon_r^{(2,2)} = 1.25, n^{(2,2)} = 1.12$ b) $\varepsilon_r^{(4,2)} = 2.49, n^{(4,2)} = 1.58$ c) $\varepsilon_r^{(5,2)} = 3.9, n^{(5,2)} = 1.97$
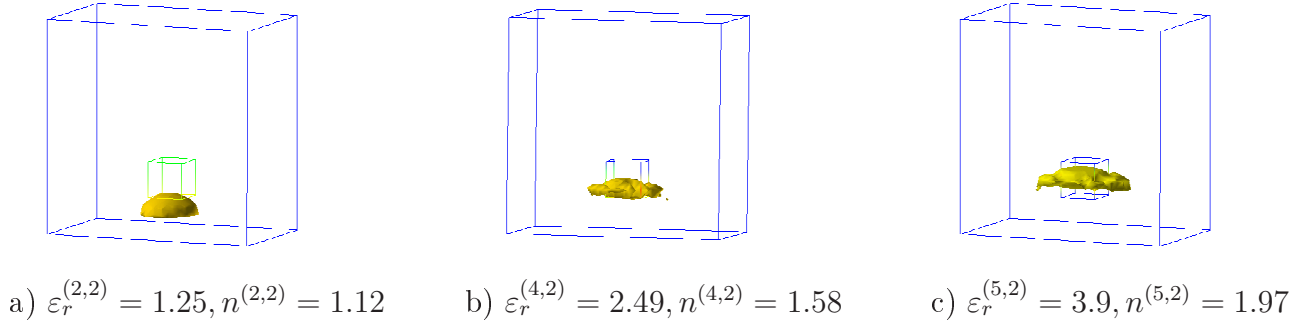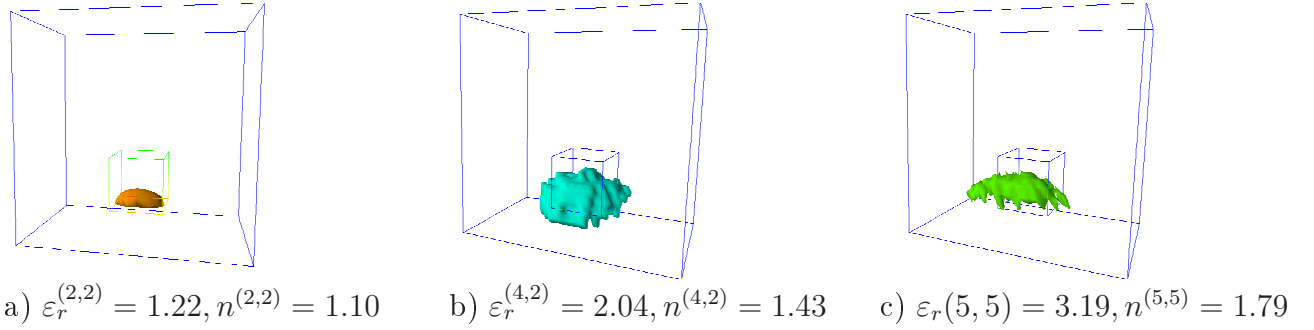
Figure 3: *Spatial distributions of iteratively computed dielectric constants $\varepsilon_r^{(n,k)}$ and refractive indexes $n^{(n,k)} = \sqrt{\varepsilon_r^{(n,k)}}$ for the Cube No. 1 (Table 1). The final image corresponds to $n^{(5,2)} := n_{glob} = 1.97$. See Table 2 for the reconstruction accuracy. Recall that refractive indices rather than dielectric constants are actually measured experimentally.*

2. The next peak after it in the blue curve (thick line, the medium with a dielectric inclusion present) with the peak value of $K_{\exp}^m$. This next peak was chosen because the presence of a dielectric inclusion results in a time delay of the EM wave, see (2.3).

Recall that the function $u_1(x, t)$ is the solution of the problem (6.2) with computationally simulated data for $\varepsilon_r \equiv 1$. Obviously $u_1\left(x^{(1)}, t\right) = u_1\left(x^{(2)}, t\right)$, $\forall x^{(1)}, x^{(2)} \in P$, $\forall t \in (0, T)$. Let $t := t_{ref}^{sim}$ be the time of the first arrival of the computationally simulated plane wave $u_1(x, t)$ at the plane P. In other words, for all $x \in P$ we have $u_1(x, t) = 0$ for $t < t_{ref}^{sim}$ and $u_1(x, t) > 0$ for such moments of time $t > t_{ref}^{sim}$ that are rather close to t_{ref}^{sim} with, see the reference curve on Fig. 2-b).

We point out that amplitudes of largest peaks of experimental curves for the reference medium were different for different detectors. Because it was impossible experimentally arrange the true plane wave for the reference medium, we actually had a spherical wave. Nevertheless, we have "forced" it to be a plane wave via applying the first stage of our data immersing procedure.

Let $y = y_m^{ref}(t)$ be the experimentally measured curve at the detector $\{x_m\}$ for the reference medium, i.e. when the dielectric inclusion was not present. Let the above chosen largest peak of this curve is achieved at $\left\{t = t_m^{ref}\right\}$ and its value is $y_m^{ref}\left(t_m^{ref}\right) = M_{\exp}^m$. Let $y = y_m^{incl}(t)$ be the experimentally measured curve at the detector $\{x_m\}$ for the case when inclusion is present. We choose such a local maximum of the function $y = y_m^{incl}(t)$ which is achieved at the first point $\left\{t = t_m^{incl}\right\}$ which follows after the point $\left\{t = t_m^{ref}\right\}$, see Fig. 2-a). Let $y_m^{incl}\left(t_m^{incl}\right) = K_{\exp}^m$. So, $K_{\exp}^m$ is the value of the latter peak, see Figure 2-a). On all detectors we have observed that $K_{\exp}^m \leq M_{\exp}^m$. This is because the presence of dielectrics decreases the amplitude of the EM wave. We enforce

$$K_{\exp}^m := M_{\exp}^m, \text{ if } \frac{K_{\exp}^m}{M_{\exp}^m} \geq \frac{2}{3}. \tag{7.1}$$

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a) $\varepsilon_r^{(2,2)} = 1.22, n^{(2,2)} = 1.10$ b) $\varepsilon_r^{(4,2)} = 2.04, n^{(4,2)} = 1.43$ c) $\varepsilon_r(5,5) = 3.19, n^{(5,5)} = 1.79$

Figure 4: *Spatial distributions of iteratively computed dielectric constants $\varepsilon_r^{(n,k)}$ and refractive indexes $n^{(n,k)} = \sqrt{\varepsilon_r^{(n,k)}}$ for the Cube No. 2 (Table 2). The final image corresponds to $n^{(5,5)} := n_{glob} = 1.79$, which is only 4.5% error compared with the experiment, see Table 2. Recall that refractive indices rather than dielectric constants are actually measured experimentally.*

Now we are ready to immerse our experimental data in the computationally simulated data. Let $\Delta t_m = t_m^{incl} - t_m^{ref}$ be the time delay between two above chosen peaks, see Figure 2-a). Then we set

$$u_{incl}(x_m, t) = \frac{K_{\exp}^m}{M_{\exp}^m} u_1(x_m, t - \Delta t_m). \tag{7.2}$$

So, (7.2) is our *first immersed data in the time domain* for the detector number $m.$ Figure 2-b) illustrates (7.2). By (7.1) and (7.2) if $K_{\exp}^m/M_{\exp}^m \geq 2/3$, then we set $u_{incl}(x_m, t) := u_1(x_m, t).$ After this data immersing, we use only the curve $u_{incl}(x_m, t)$ and do not use the curve for the reference medium anymore. We cannot rigorously justify our above decision to work with those peaks only. However, since our results of blind imaging in [24] were very accurate ones, then this justifies our purely intuitive choice.

7.2 The second stage of data immersing

Next, we apply the Laplace transform (3.1) to each function $u_{incl}(x_m, t)$ for nine values of $s = 3.5, 4, ..., 7.5.$ Denote $w_{incl}(x_m, s)$ the Laplace transform of the function $u_{incl}(x_m, t).$ Let $\widetilde{w}_{incl}(x_m, s) = -s^{-2} \ln w_{incl}(x_m, s)$ and $\overline{w}_{incl}(x, s)$ be the standard linear interpolation of the values $\{\widetilde{w}_{incl}(x_m, s)\}$ over the plane $P.$ We have observed that the function $\overline{w}_{incl}(x, s)$ is very noisy with respect to $x \in P.$ Hence, we have applied a smoothing procedure to the function $\overline{w}_{incl}(x_m, s)$ with respect to $(x, y) \in P$ for each of those nine values of $s.$ Specifically, we have used the *Lowess fitting procedure* in the 2D case, which we took from MATLABR 2009. We have obtained the function $\overline{w}_{smooth}(x, s).$ Let $w_1(x, s), x \in P$ be the Laplace transform of the function $u_1(x, t),$ i.e. for the case of the plane wave propagating in the air. Then we
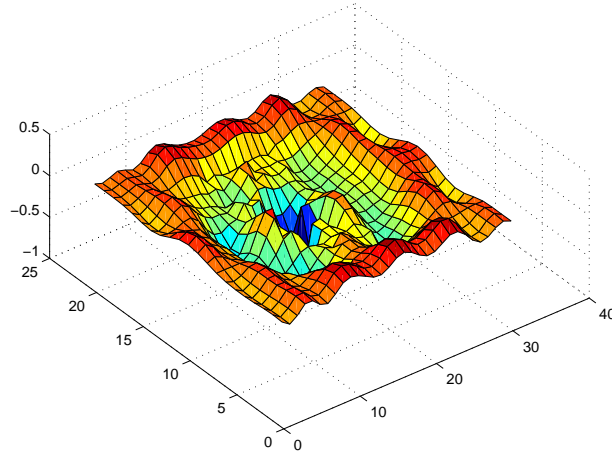
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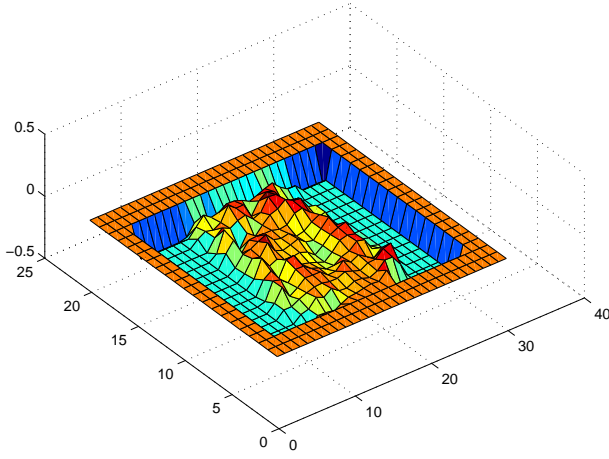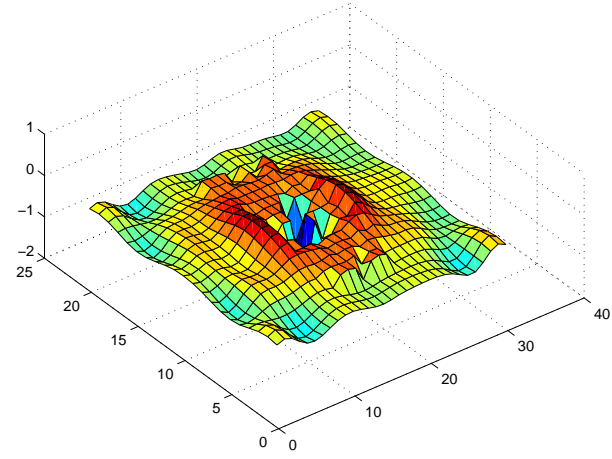
a) $t = 10.8$ b) $t = 10.8$
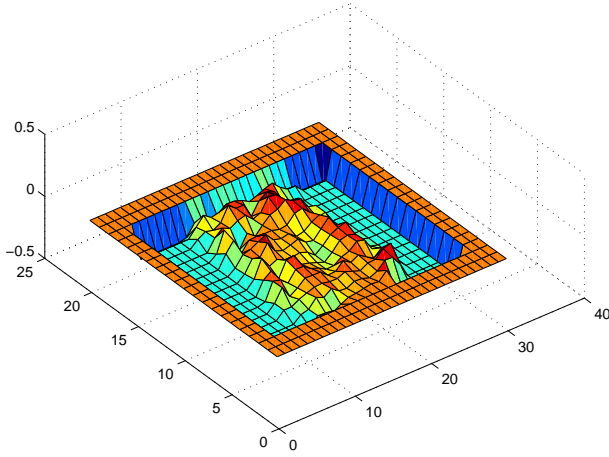
c) $t = 12.0$ d) $t = 12.0$

Figure 5: *a),c) The function $g(x,t)$, $x \in P$ for cube No.1 (Table 1). This is the function which is obtained via the first stage of data immersing procedure, i.e. this is the function $u_{incl}(x,t)$, $x \in P$. However, to solve the problem (15) in the adaptivity, we need to know this function at a wider rectangle $x \in P_{obs}$, see subsection 6.2. So, since $P \subset\subset P_{obs}$, we need to extend somehow the function $g(x,t)$ from P to P_{obs}. This extension is carried out via the third stage of our data immersing procedure described in subsection 7.4. b),d) present the resulting immersed data with $\beta = 0.1$.*

finally set for each of those nine values of s

$$\overline{w}_{immers}(x,s) = \begin{cases} \overline{w}_{smooth}(x,s), & \text{if } \overline{w}_{smooth}(x,s) \geq 0.985 \max_{\overline{P}} \overline{w}_{smooth}(x,s), \\ -s^{-2} \ln w_1(x,s), & \text{otherwise}, \end{cases}$$

see Figure 5 in [24]. So, we use the function $\overline{w}_{immers}(x,s)$ to obtain Dirichlet boundary conditions for above mentioned elliptic equations for functions q_n of the globally convergent method (section 2).

7.3   Reconstruction by the globally convergent method

It was shown in [24] that first and second immersing stages of two previous subsections are sufficient for the globally convergent algorithm. Our dielectric abnormalities to be imaged were two wooden cubes, see Table 1. Let CL be the center line, i.e. the straight line which is orthogonal to the plane P and which passes through the source of EM waves (Figure 1). Then $CL = \{(x, y, z) : x = y = 0\}$. We test our two-stage numerical procedure on two sets of experimental data. So, the center of our first cube was on CL, and the center of the second cube was off CL by 0.2 in dimensionless coordinates, which is equivalent with 10 mm. Images are presented on Figures 3 and 4. One can see from Table 2 that the error of the reconstruction of refractive indices is a few percent. The same conclusion was drawn in [24]. At the same time, it is clear from Figures 3 and 4 that it is desirable to improve images of shapes of these cubes. And this is why we use the adaptivity technique on the second stage.

Cube number	Original sizes, mm	Dimensionless sizes	Dimensionless coordinates of centers
1	$40 \times 40 \times 40$	$0.8 \times 0.8 \times 0.8$	$(0, 0, -1.2)$
2	$60 \times 60 \times 60$	$1.2 \times 1.2 \times 1.2$	$(0.2, 0, -1.2)$

Table 1: *Sizes and coordinates of centers of two wooden cubes used in experiments*

Cube number	Computed $n := n_{glob} = \sqrt{\varepsilon_r^{glob}}$	Measured n, error	Comput. error
1	1.97	2.07, 11%	4.8%
2	1.79	1.71, 3.5%	4.5%

Table 2: *Computed refractive indices $n := n_{glob} = \sqrt{\varepsilon_r^{glob}}$ and ones directly measured by the Waveguide Method*

7.4   The third stage of data immersing

This stage is new, since it was not a part of [24]. The function $u_{incl}(x_m, t)$ obtained in (7.2) is very noisy with respect to $x_m \in P$, see for example Figures 5-a), c) for this function. We know

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this function only at the bottom side P of the rectangular prism Ω. However, it was pointed out in subsection 6.2 that in order to solve the adjoint problem, we actually need to know this function on the bigger rectangle $P_{obs} = \{(x, y, z) : (x, y) \in [-3, 3] \times [-2, 2], z = -2.4\}$. So, since our experimental data were measured on a smaller rectangle P only, we need to complement them somehow on the set $P_{obs} \backslash P$. To do so, we have decided to solve the problem (6.2) with a certain coefficient ε_r: to complement the data on $P_{obs} \backslash P$. Let $\overline{u}(x, t)$ be this solution. Then we treat values of $\overline{u}(x, t)|_{P_{obs}}$ as a certain part of boundary values $g|_{P_{obs}}$ when solving the adjoint problem (4.7). In other words, this solution provides us with a new piece of data at the entire rectangle P_{obs}. Hence, the question now is: *How to choose the coefficient ε_r in equation (6.2) for this third immersing stage?*

We have decided to take in equation (6.2) the coefficient $\varepsilon_r^{glob}(x)$, which was obtained on the globally convergent stage of our two-stage numerical procedure, see Figures 3 and 4 as well as Table 2. Let $U_{ref}(x, t)$ be the solution of the problem (6.2) with $\varepsilon_r(x) := \varepsilon_r^{glob}(x)$. Thus, we define our second immersed data in the time domain as

$$u_{immers}(x, t)|_{P_{obs}} = \begin{cases} u_{incl}(x, t), & \text{if } u_{incl}(x, t) \geq \beta \max_{\overline{P}}(u_{incl}(x, t)) \text{ and } x \in P, \\ U_{ref}(x, t), & \text{otherwise,} \end{cases} \quad (7.3)$$

where the function $u_{incl}(x, t)$ is the standard linear interpolation of values $u_{incl}(x_m, t)$ in (7.2) over the rectangle P. In particular, (7.3) implies that $u_{immers}(x, t) = U_{ref}(x, t)$ for $x \in P_{obs} \setminus P$. In (7.3) the parameter $\beta \in (0, 1)$ should be chosen in numerical experiments. This parameter characterizes the amount of information which we take from the first immersed experimental data in (7.2). Comparison of Figures 5-a), c) with Figures 5-b), d) shows that the third stage of data immersing helps not only to obtain the data for $x \in P_{obs} \setminus P$ (rather than for $x \in P$ only), but also to significantly decrease the noisy component of the data resulting from the first immersing stage. We also show numerically below (Figure 10) that the change of the parameter β in the wide range $\beta \in (0.1, 0.985)$ does not significantly affect imaging results.

Thus, we now got a "double use" of the solution obtained on the globally convergent stage. First, for the data immersing via (7.3). Second, as the starting point for the adaptivity technique.

8   Reconstruction Results

While Figures 3-c) and 4-c) display reconstruction results via the globally convergent stage, we present in this section refined results which were obtained on the adaptivity stage. We use in this section both first and second mesh refinement recommendations of section 4.
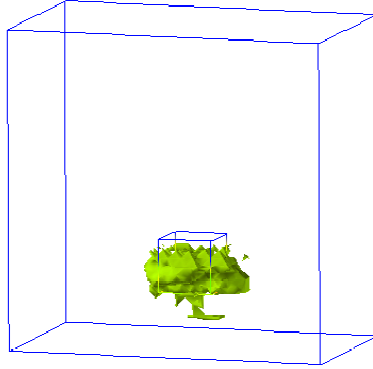
$$\varepsilon_{r,h} \approx 3.9, n_{glob} = \sqrt{\varepsilon_{r,h}} \approx 1.97$$

Figure 6: *The reconstruction result for the 1st stage of the adaptivity for the cube No. 1. Maximal values of the imaged coefficient are shown for the third refined mesh. The shape is not yet well reconstructed, although a comparison with Fig. 3-c) shows an improvement. The refractive index is reconstructed accurately (Table 2).*

8.1   Some details of the numerical implementation of the adaptivity

Recall that by the first mesh refinement recommendation we refine mesh locally in such a subdomain of the domain Ω where

$$\left| E' \left(\varepsilon_{r,h} \right) (x) \right| \geq \varkappa \max_{\overline{\Omega}} \left| E' \left(\varepsilon_{r,h} \right) (x) \right|, \tag{8.1}$$

where $\varkappa = const \in (0,1)$ is the tolerance number of our choice. The choice of $\varkappa$ depends on concrete values of $\left| E' \left(\varepsilon_{r,h} \right) (x) \right|$ and this should be done in numerical experiments. Below we take in (8.1) $\varkappa = 0.8$ for all computational meshes. Just as in [10, 11, 12], we have used a cut-off parameter B_{cut} on all refined meshes for the reconstructed coefficient $\varepsilon_{r,h}$. Namely, we took

$$\varepsilon_{r,h} (x) = \left\{ \begin{array}{ll} \varepsilon_{r,h} (x), & \text{if } |\varepsilon_{r,h} (x) - \varepsilon_{r,h}^{glob} (x) | \geq B_{cut}, \\ \varepsilon_{r,h}^{glob} (x), & \text{elsewhere.} \end{array} \right. \tag{8.2}$$

Our numerical experience of previous publications [10, 11, 12] has shown that it is important to use in the adaptive algorithm box constrains for the reconstructed coefficient via imposing that $1 \leq \varepsilon_{r,h} (x) \leq d$. While the inequality $\varepsilon_{r,h} (x) \geq 1$ follows from physics, see (2.3), we find a good estimate for the upper bound d on the basis of computed refractive indices from globally convergent part, i.e. on the basis of the second column of Table 2. Concrete values of B_{cut} and d can be found in subsections below.

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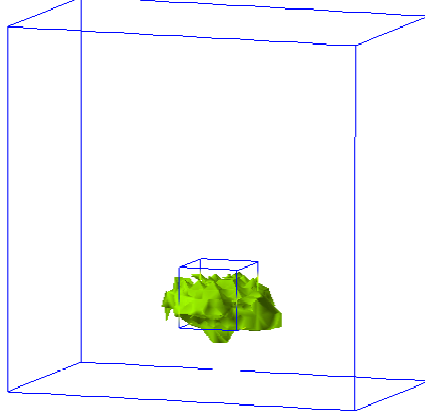
$$\varepsilon_{r,h} \approx 3.9, n_{glob} = \sqrt{\varepsilon_{r,h}} \approx 1.97$$

Figure 7: *The reconstruction result for the 2nd stage of the adaptivity for the cube No. 1. Thin lines (blue) indicate the correct cubical shape. Comparison with Fig. 6 shows an improvement of the image. The refractive index is reconstructed accurately (Table 2).*

8.2 Reconstruction results for the Cube No. 1

The function $\varepsilon_r^{glob}(x)$, which corresponds to Fig. 3-c), was taken as the starting point in adaptivity technique on all meshes, as well as the one generating the function U_{ref} in (7.3). We took in (8.2) $B_{cut} = 2$ for all refinements of the mesh. Since by Table 2 $\max \varepsilon_r^{glob}(x) = (1.97)^2 \approx 3.9$, we enforce that the coefficient $\varepsilon_r(x)$ belongs to the following set of admissible parameters $\varepsilon_r(x) \in C_M = \{1 \leq \varepsilon_r(x) \leq 4.4\}$.

8.2.1 The first stage of the mesh refinement

On this stage we refine mesh locally in the following two types of subdomains of the domain Ω :

1. The one where we follow the First Mesh Refinement Recommendation. That is, we refine mesh in all regions where (51) is fulfilled.

2. In addition, following the Second Mesh Refinement Recommendation and the third Remark 4.1, we refine the in such a subdomain where the coefficient imaged on the globally convergent stage attains values which are sufficiently close to its maximal value. In other words, we refine mesh in all regions where $\varepsilon_r^{glob}(x) \geq \alpha \max_{\overline{\Omega}} \varepsilon_r^{glob}(x)$, where the parameter $\alpha \in (0,1)$ is chosen in numerical experiments. In all calculations below we took

$$\alpha = 0.2, \beta = 0.985, \gamma = 0.001, \tag{8.3}$$

where γ is the regularization parameter of the Tikhonov functional (13).

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First, we use the same coarse mesh as the one on the globally convergent stage. Just as in [10]-[12], we have not observed any improvement of the image. Next, we use adaptively locally refined meshes. Let $\Gamma_T = P_{obs} \times (0, T)$. To figure out the stopping criterion with respect to the mesh refinements, we proceed similarly with Table 2 of [10] and Table 1 of [11]. Namely, let $\varepsilon_{r,h}(x)$ be the approximation for the true coefficient $\varepsilon_r(x)$, which is obtained on a certain mesh, and u be the corresponding solution of the forward problem (6.2). So, we analyze the behavior of computed L_2-norms of $\|u - u_{immers}\|_{L_2(\Gamma_T)}$. We have observed that these norms decrease with the number of mesh refinements. Next, this norm slightly increases on the 4th refinement. This is the same behavior as one in [10, 11]. Hence, we take the coefficient $\varepsilon_r(x)$ obtained after three mesh refinements as our final solution on the first stage of the adaptive algorithm. The resulting image is shown on Figure 6. Comparison of Figures 3-c) and 6 shows that the image of Fig. 6 is better than one of Fig. 3-c), whereas refractive indices are the same. However, the shape of the inclusion is not yet imaged well, although the size of the abnormality is computed rather well.

8.2.2 The second stage of the mesh refinement

One can derive from Fig. 6 that maximal values of the reconstructed coefficient are achieved in the subdomain Ω_2,

$$\Omega_2 = \{(x, y, z) : (x, y) \in [-0.5, 0.5] \times [-0.6, 0.6] \times [-1.4, -0.5]\}. \qquad (8.4)$$

Let $\overline{\varepsilon}_{r,h}(x)$ be the coefficient imaged on the first stage of the mesh refinement process. On the second mesh refinement stage we follow only the Second Mesh Refinement Recommendation and do not use the first one. Thus, we refine the mesh locally only in neighborhoods of those grid points where

$$x \in \left\{ \overline{\varepsilon}_r(x) \geq \alpha \max_{\overline{\Omega}} \overline{\varepsilon}_r(x) \right\} \cap \Omega_2, \qquad (8.5)$$

where α is taken from (8.3). The same stopping criterion for the number of mesh refinements as one above was used. Figure 7 displays the final image after four mesh refinements. Comparison of Figs. 7 and 6 shows an improvement of the image due to the second stage of the adaptivity.
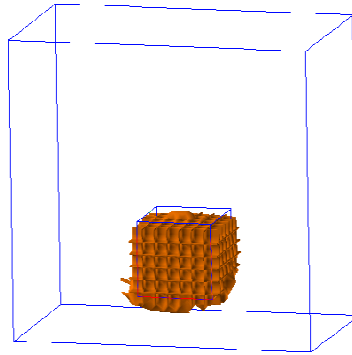
8.3 Reconstruction results for the Cube No. 2

We now apply the above adaptive two-stage technique to reconstruct the Cube No. 2 of Table 1. In doing so, we again get the first guess from the globally convergent method, see Fig. 4-c) and Table 2. Because of Table 2, we consider the following set of admissible parameters $\varepsilon_r(x) \in C_M = \{1 \leq \varepsilon_r(x) \leq 3.3\}$. Let j be the number of iterations in the quasi-Newton method on the 1st stage of the adaptivity. On all refined meshes we have chosen the cut-off parameters in (8.2) as: $B_{cut} = 0.91$ for $j = 1, 2$, for $j = 3$, $B_{cut} = 1.1$ and $B_{cut} = 2$ for $j > 3$.
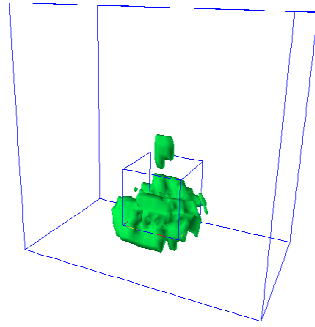
$$\varepsilon_{r,h} \approx 2.52, n_{glob} = \sqrt{\varepsilon_{r,h}} \approx 1.59$$

Figure 8: *The reconstruction result for the 1st stage of the adaptivity for the cube No. 2. Only maximal values of the imaged coefficient are shown for the 3rd refined mesh. The shape of the final imaged coefficient is better than one on Fig. 4-c). However, the imaged refractive index is lowered by about 19% compared with the imaged on the globally convergent stage.*

$$\varepsilon_{r,h} \approx 3.0, n_{glob} = \sqrt{\varepsilon_{r,h}} \approx 1.73$$

Figure 9: *The final reconstruction result for the cube No. 2. Only the final, 3rd mesh refinement, is shown. The imaged coefficient $\varepsilon_r(x) = 1$ outside of these images. All three components: shape, location and refractive index are imaged with a very good accuracy.*

8.3.1 The first stage of the mesh refinement

As in the case of cube No. 1, the use of the same mesh as one in the globally convergent method, did not lead to an improvement of the image. Again, just as in the case of the first cube, we refine the mesh in all regions where both first and second mesh refinement recommendations work, i.e. in all regions where the inequality (8.1) with $\varkappa = 0.8$ is valid, as well as in all regions where $\varepsilon_r^{glob}(x) \geq \alpha \max_{\overline{\Omega}} \varepsilon_r^{glob}(x)$. We have used parameters (8.3) as well as the same stopping criterion for the number of mesh refinements as one in sub-subsection 8.2.1. Because of this criterion, we have stopped on the 3th mesh refinement. The final image of the first stage of the adaptivity is displayed on Fig. 8. Comparing this image with one on Fig. 4-c) and with Table 2, we observe a slight improvement of the imaged shape while the value of the refractive index has decreased by about 19%. In addition, we observe that we have actually obtained two disconnected imaged inclusions.

8.3.2 The second stage of the mesh refinement

We use the same procedure as one for the first cube. First, we have to figure out an analog of the domain Ω_2 in (8.4). To define upper and lower boundaries for the vertical coordinate z of the subdomain of local mesh refinements, we have decided to use again the information obtained on the globally convergent stage. We see on Fig. 4-c) that, unlike Fig. 8, we have only a single rather than two inclusions. Hence, we have decided to refine mesh, in terms of the vertical coordinate z as follows:

1. The top boundary z_{top} should be slightly below the low boundary of the small imaged inclusion of Fig. 8.

2. The low boundary z_{low} should be slightly below the bottom of the larger imaged inclusion of Fig. 8.

3. Boundaries with respect to horizontal coordinates (x, y) of the mesh refinement subdomain were determined from the criterion $\overline{\varepsilon}_r(x) \geq \alpha \max_{\overline{\Omega}} \overline{\varepsilon}_r(x)$, for values of $\overline{\varepsilon}_r(x)$, which is similar with (8.5). Consider the subdomain $\Omega_3 \subset \Omega$, where

$$\Omega_3 = \{(x, y, z) : (x, y) \in [-0.6, 0.6] \times [-0.6, 0.6] \times [-1.8, -0.8]\}.$$

So, we refine the mesh, using the criterion (8.5), in which Ω_2 is replaced with Ω_3.

The same stopping criterion for the number of mesh refinements as one in sub-subsection 8.2.1 was used. Thus, we have concluded that the 3rd mesh refinement should be the final one. The resulting image is displayed on Fig. 9. A very accurate reconstruction of all three components of the Cube No. 2: shape, location and refractive index is evident.

8.4 Sensitivity to parameters β and γ

To investigate the sensitivity of our images to the choice of the regularization parameter γ in the Tikhonov functional (4.5), as well as to the parameter β in (7.3), we have performed further testing for cube No. 2 with different values of β and γ. Results are displayed on Figure 10. One can observe that the value of the regularization parameter γ does not impact

reconstruction results significantly. One can also see that images for $\beta = 0.985$ and $\beta = 0.5$ are almost the same, including imaged values of the refractive index. Surprisingly, images for the case $\beta = 0.1$ also look almost the same as ones for $\beta = 0.985$ and $\beta = 0.5$. However, values of the refractive index for $\beta = 0.1$ are lowered by about 10%. Hence, we conclude from Fig. 10 that our procedure is quite stable with respect to parameters β and γ.

8.5   Verification for the Cube No. 1

We have compared our imaging result for Cube No. 1 with computational simulations. To do so, we have computed the data for the forward problem for exactly the same cube as No. 1 in Table 1. We took $\epsilon_r = 4$ inside of this simulated cube. However, we have replaced $\omega = 7$ with $\omega = 14$ in (6.1) and (6.2), since this corresponds to the twice smaller dimensionless wavelength $2\pi/\omega$ in computational simulations. So, we have conjectured that having a two times lesser wavelength would result in a better image for the Cube No. 1. Indeed, with $\omega = 7$ the dimensionless wavelength in simulated data is 0.897, which is bigger than the dimensionless size 0.8 of the side of the first cube (Table 1). On the other hand, $\omega = 14$ gives us the dimensionless wavelength of $0.45 < 0.8$. Note that the dimensionless wavelength size of the side of the Cube No. 2 is $1.2 > 0.897$. We recall here the classical Rayleigh principle.

   We have applied the same procedure as above to the computationally simulated data. The resulting image is displayed on Fig. 11-a). One can observe a very good quality of this image from synthetic data. Next, we have applied the same procedure as above to the experimental data for the Cube No. 1 with the single difference that we now have used $\omega = 14$ instead of the previous $\omega = 7$. The resulting image is displayed on Fig. 11-b). One can observe a significant improvement compared with Figure 7. Hence, our conjecture about ω was partially materialized for the case of experimental data. Still, however, the image on Fig. 11-b) is not as good as the one for Cube No. 2 on Fig. 9.

9   Discussion

In this paper we have continued our work on experimental data of [24]. While only locations and refractive indices of dielectric abnormalities were accurately computed in [24], we now complement those by reconstructions of shapes. As a result, the shape of the Cube No. 2 is reconstructed with an excellent accuracy (Fig. 9). Furthermore, it was shown that our technique is quite stable with respect to some critical parameters which we choose in the reconstruction process (Fig. 10). The shape of the smaller Cube No. 1 was also reconstructed well (Fig. 11-b)), although the accuracy is not as good as the one of Cube No. 2. Just as in [24], refractive indices are imaged with a very good accuracy in both cases.

   The difference of qualities of images of our two cubes might likely be attributed to the classical Rayleigh principle. Indeed, the original wavelength λ_{em} of the EM wave in our experimental data was $\lambda_{\text{em}} = 3cm$. On the other hand, sizes of sides of Cubes No. 1 and No. 2 were respectively $4cm = 1.33\lambda_{\text{em}}$ and $6cm = 2\lambda_{\text{em}}$ (Table 1). Hence, we conjecture that the shape of the Cube No. 2 was imaged better than the one of Cube No. 1 because
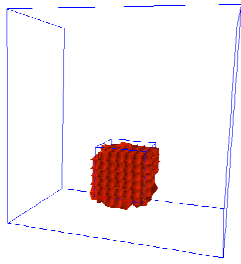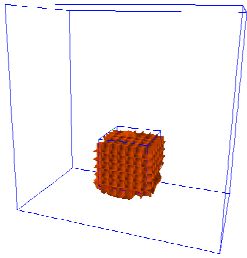
a) $\gamma = 0.001, \beta = 0.985$
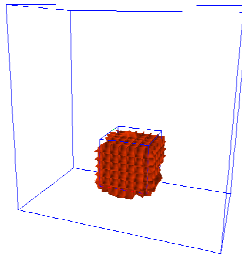$\varepsilon_{r,h} \approx 3.2, n_{glob} = 1.79$
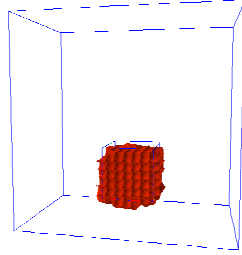
b) $\gamma = 0.01, \beta = 0.985$
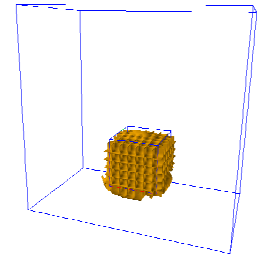$\varepsilon_{r,h} \approx 3.2, n_{glob} = 1.79$
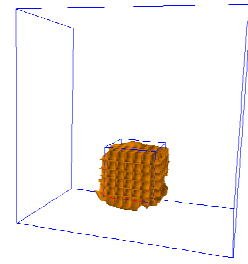
c) $\gamma = 0.1, \beta = 0.985$
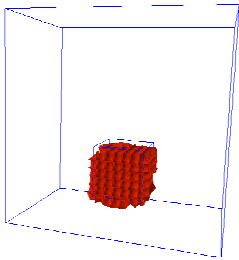$\varepsilon_{r,h} \approx 3.2, n_{glob} = 1.79$

d) $\gamma = 0.001, \beta = 0.5$
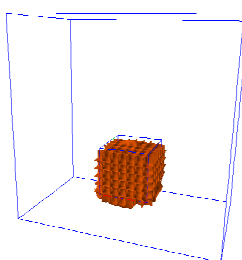$\varepsilon_{r,h} \approx 3.1, n_{glob} = 1.76$

e) $\gamma = 0.01, \beta = 0.5$
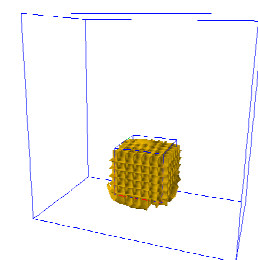$\varepsilon_{r,h} \approx 3.1, n_{glob} = 1.76$

f) $\gamma = 0.1, \beta = 0.5$
$\varepsilon_{r,h} \approx 3.0, n_{glob} = 1.73$

g) $\gamma = 0.001, \beta = 0.1$
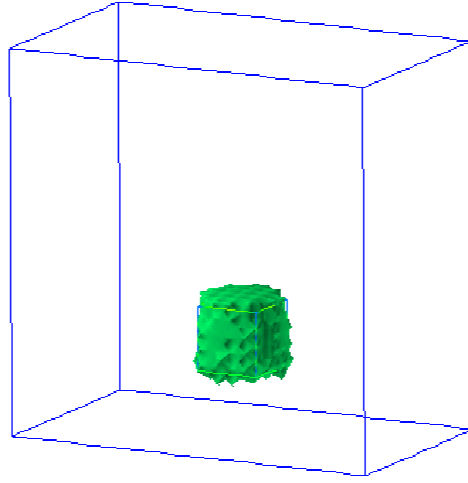$\varepsilon_{r,h} \approx 2.4, n_{glob} = 1.55$

h) $\gamma = 0.01, \beta = 0.1$
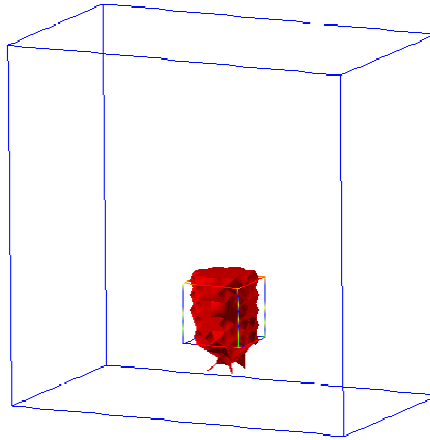$\varepsilon_{r,h} \approx 2.4, n_{glob} = 1.55$

i) $\gamma = 0.1, \beta = 0.1$
$\varepsilon_{r,h} \approx 2.4, n_{glob} = 1.55$

Figure 10: *Final reconstruction results for cube No. 2 with varying parameters β and γ. Lines (blue) indicate the correct cubical shape. Maximal values of the imaged coefficient are displayed. The computed value of the coefficient outside of imaged inclusions is 1.*

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a) $\varepsilon_{r,h} \approx 4.09, n_{glob} = \sqrt{\varepsilon_{r,h}} \approx 2.02$

b) $\varepsilon_{r,h} \approx 4.2, n_{glob} = \sqrt{\varepsilon_{r,h}} \approx 2.05$

Figure 11: *a) The image of computationally simulated cube No. 1 from computationally simulated data with $\omega = 14$ in (6.1) and (6.2). b) The image of cube No. 1 from experimental data with $\omega = 14$ in (6.1) and (6.2). The same imaging procedure as above was applied. Compared with Fig. 7, a significant improvement is observed. Still, however, the image of the shape is not as good as the one for Cube No. 2 on Fig. 9.*

our experimental data had $3cm$ wavelength "inscribed" in them. This question needs to be investigated further.

Compared with [24], the main new element here is that we have complemented the globally convergent method by the adaptivity technique. In addition, we have complemented the previously developed first rule of mesh refinements by the new second rule. The latter has required a significant analytical effort with an extensive use of results [11, 12, 25]. Two other important features of these studies are:

1. The use of the solution obtained on the globally convergent stage is *crucial* for obtaining above results via the adaptivity. At least for the case of our experimental data, the adaptivity does not work without the availability of this solution.

2. Our studies have consistently demonstrated that all analytical and numerical conclusions derived in our previous publications for computationally simulated data [9]-[12] were confirmed on experimental data.

Some discrepancies between our mathematical model and the reality are evident. It is well known that equation (2.1) cannot be derived from the Maxwell's system for the 3-D case if $\varepsilon_r \neq const$. In addition, we are not aware which of three components of the electric field was measured in experiments: we only knew that the time-resolved voltage was measured. Thus, we call (2.1) a *simplified* mathematical model of our physical process. A possible explanation why everything still works well is that the data immersing procedure "enforces" our data to be "good" for equation (2.1). A more complete investigation of this issue with the use of the full Maxwell's system seems to be worthy to pursue.

Another discrepancy is that in our globally convergent algorithm we need a certain asymptotic behavior of the Laplace transform of the function u, which can be derived from results of [32, 33], see (3.2) and Lemma 2.1 in [9]. In particular, that lemma requires at least the C^2-smoothness of the coefficient $\varepsilon_r(x)$. We verify that asymptotic behavior computationally, see subsection 7.2 [9]. However, the smoothness assumption of the function $\varepsilon_r(x)$ is obviously violated at the boundaries of our two cubes, which were used in experiments. More discrepancies can be derived from data immersing procedures described above. It might well take years to figure out how to handle all these discrepancies. In summary, it is rather surprising that, despite all these discrepancies, results of this publication as well as results of blind testing of [24] consistently demonstrate a very good reconstruction accuracy.

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