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Abstract. Multivariate Laplace distribution is an important stochastic model that accounts for asymmetry and heavier than Gaussian tails often observed in practical data, while still ensuring the existence of the second moments. A Lévy process based on this multivariate infinitely divisible distribution is known as Laplace motion, and its marginal distributions are multivariate generalized Laplace laws. We review basic properties of the latter distributions and discuss a construction of a class of moving average vector processes driven by multivariate Laplace motion. These stochastic models extend to vector fields, which are multivariate both in the argument and the value and provide an attractive alternative to those based on Gaussianity, in presence of asymmetry and heavy tails in empirical data. An example from engineering shows modelling potential of this construction.

In memory of Professor Samuel Kotz

1. Introduction

The classical univariate Laplace distribution with mean zero and variance σ^2, introduced in [17], is a symmetric distribution given by the characteristic function (ChF)

$$\phi(t) = \frac{1}{1 + \frac{\sigma^2 t^2}{2}}$$

or, equivalently, by the probability density function (PDF)

$$f(x) = \frac{\sqrt{2}}{2\sigma} e^{-\sqrt{2}|x|/\sigma}, \quad x \in \mathbb{R}.$$

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While being neglected for many years, this distribution has been recently revived and extended to skew as well as multivariate settings, and is gaining popularity as an attractive alternative to Gaussianity (see [15] and references therein). While the term *multivariate Laplace law* is still a bit ambiguous, it applies most often to the class of symmetric, elliptically contoured distributions, given by the ChF

$$\phi(\mathbf{t}) = \frac{1}{1+\frac{1}{2}\mathbf{t}'\mathbf{\Sigma}\mathbf{t}}, \quad \mathbf{t}\in\mathbb{R}^d, \tag{1}$$

where $\mathbf{\Sigma}$ is a $d\times d$ non-negative definite matrix (which is the covariance matrix). An asymmetric generalization of this model, known as multivariate *asymmetric Laplace* (AL) distribution and denoted by $AL_d(\mathbf{\Sigma},\boldsymbol{\mu})$, has the ChF of the form (see [15])

$$\phi(\mathbf{t}) = \frac{1}{1+\frac{1}{2}\mathbf{t}'\mathbf{\Sigma}\mathbf{t} - i\boldsymbol{\mu}'\mathbf{t}}, \ \ t\in\mathbb{R}^d, \tag{2}$$

where $\boldsymbol{\mu}\in\mathbb{R}^d$ is the mean of the distribution and $\mathbf{\Sigma}$ is as before (although this time the covariance is $\mathbf{\Sigma}+\boldsymbol{\mu}\boldsymbol{\mu}'$). The significance of AL distributions is partially due to fact that these arise rather naturally as the only distributional limits for (appropriately normalized) random sums of independent and identically distributed (IID) random vectors with finite second moments,

$$\mathbf{X}^{(1)}+\cdots+\mathbf{X}^{(N_p)}, \tag{3}$$

where N_p has a geometric distribution with the mean $1/p$ (independent of the $\mathbf{X}^{(i)}$):

$$P(\nu_p = k) = p(1-p)^{k-1}, \ k=1,2,\ldots,$$

and p converges to zero. Since the sums such as (3) frequently appear in many applied problems in biology, economics, insurance mathematics, reliability, and other fields (see examples in [13] and references therein), AL distributions have a wide variety of applications (see [15]). The AL distributions play an analogous role among the heavy tailed geometric stable laws (which approximate the sums (3) without the restriction of finite second moment, see [14]) as Gaussian distributions do among the stable laws. Like Gaussian distributions, they have finite moments of all orders, and their theory is equally elegant and straightforward (see [15]). However, in spite of finiteness of moments, their tails are substantially longer than those of the Gaussian laws. This, coupled with the fact that they allow for asymmetry, renders them more flexible and attractive for modeling heavy tailed asymmetric data.

It should be noted that the AL distributions are infinitely divisible. This property allows for natural extension to more general random processes and fields. In this paper we discuss properties of one such construction in the multivariate setting, which was introduced in [22] (and, in one-dimensional case, was discussed in [1]). Namely, we consider random moving average fields driven by Laplace motion, which are multivariate in both the argument (n-dimensional) and the value (d-dimensional). The Laplace motion on the positive half-line is a Lévy process $\{\mathbf{\Lambda}(s),\ s \geq 0\}$ built upon the AL distribution (2). The increments of the process are independent and homogeneous, and the ChF of $\mathbf{\Lambda}(s)$ is the sth power of the AL ChF (2) so that the marginal distributions belong to multivariate *generalized asymmetric Laplace* (GAL) laws, or Bessel function distributions (the latter name relates to the fact that their PDFs involve Bessel special functions). Since these distributions play a crucial role in our construction of multivariate random fields, we shall first review their properties in Section 2. Beyond being a review this section contains also several new results. The main construction of random fields and their basic properties are presented in Section 3, which also includes some remarks on model fitting and estimation. Finally, an example of application for modelling two-dimensional process in time variable is presented in Section 4.

2. Generalized Laplace distributions

Here we review basic properties of the generalized asymmetric Laplace distributions, which play crucial role in constructing Laplace random fields. Although some of these results are known (and taken from [15]), others are new and presented here for the first time. We start with a formal definition of these laws.

Definition 1 (Multivariate Generalized Laplace Law). *A random vector in $\mathbb{R}^d$ is said to have a multivariate generalized asymmetric Laplace distribution (GAL) if its ChF is given by*

$$\phi(\mathbf{t}) = \left(\frac{1}{1 + \frac{1}{2}\mathbf{t}'\mathbf{\Sigma}\mathbf{t} - i\boldsymbol{\mu}'\mathbf{t}}\right)^s, \quad \mathbf{t} \in \mathbb{R}^d, \tag{4}$$

where $s > 0$, $\boldsymbol{\mu} \in \mathbb{R}^d$, and $\mathbf{\Sigma}$ is a $d \times d$ non-negative definite symmetric matrix. This distribution is denoted by $GAL_d(\mathbf{\Sigma}, \boldsymbol{\mu}, s)$.

Remark 1. If the distribution is one-dimensional ($d = 1$) with $\mathbf{\Sigma} = \sigma_{11}$ and $\boldsymbol{\mu} = \mu_1$, we obtain a univariate $GAL(\sigma, \mu, s)$ distribution studied in [15], given by the ChF

$$\phi(t) = \frac{1}{1 + \frac{\sigma^2 t^2}{2} - i\mu t}, \quad t \in \mathbb{R}, \tag{5}$$

where $\sigma = \sqrt{\sigma_{ii}}$ and $\mu = \mu_1$.

If the matrix $\mathbf{\Sigma}$ is positive-definite, the distribution is truly d-dimensional and has a PDF of the form (see [15])

$$p(\mathbf{x}) = \frac{2\exp(\boldsymbol{\mu}'\mathbf{\Sigma}^{-1}\mathbf{x})}{(2\pi)^{d/2}\Gamma(s)|\mathbf{\Sigma}|^{1/2}} \left(\frac{Q(\mathbf{x})}{C(\mathbf{\Sigma}, \boldsymbol{\mu})}\right)^{s-d/2} K_{s-d/2}(Q(\mathbf{x})C(\mathbf{\Sigma}, \boldsymbol{\mu})),$$

where $K_\lambda(\cdot)$ is the modified Bessel function of the third kind with index λ,

$$K_\lambda(u) = \frac{1}{2}\left(\frac{u}{2}\right)^\lambda \int_0^\infty t^{-\lambda-1} \exp\left(-t - \frac{u^2}{4t}\right) dt, \quad u > 0, \tag{6}$$

and

$$Q(\mathbf{x}) = \sqrt{\mathbf{x}'\mathbf{\Sigma}^{-1}\mathbf{x}}, \quad C(\mathbf{\Sigma}, \boldsymbol{\mu}) = \sqrt{2 + \boldsymbol{\mu}'\mathbf{\Sigma}^{-1}\boldsymbol{\mu}}. \tag{7}$$

This follows from the interpretation of a GAL random vector $\mathbf{Y} \sim GAL_d(\mathbf{\Sigma}, \boldsymbol{\mu}, s)$ as a subordinated Gaussian process,

$$\mathbf{Y} \stackrel{d}{=} \mathbf{X}(Z), \tag{8}$$

where Z has a standard gamma distribution with shape parameter s and the PDF

$$g(x) = \frac{x^{s-1}}{\Gamma(s)} e^{-x}, \quad x > 0, \tag{9}$$

while $\mathbf{X}$ is a d-dimensional Gaussian process with independent increments, $\mathbf{X}(0) = \mathbf{0}$, and $\mathbf{X}(1) \sim N_d(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ (d-dimensional normal distribution with mean vector $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$). The above representation, which follows by evaluating the ChF on the right hand side by conditioning on the gamma random variable Z, can also be expressed as

$$\mathbf{Y} \stackrel{d}{=} \boldsymbol{\mu} Z + Z^{1/2}\mathbf{X}, \tag{10}$$

where Z is as above and $\mathbf{X} \sim N_d(\mathbf{0}, \boldsymbol{\Sigma})$, showing that GAL distributions are location-scale mixtures of normal distributions. The stochastic representation (10) leads to many further properties of GAL random vectors, including moments, marginal and conditional distributions, and linear transformations.

Remark 2. More general normal mixtures, where Z has a generalized inverse Gaussian distribution, were considered by Barndorff-Nielsen [2]. A generalized inverse Gaussian distribution with parameters (λ, χ, ψ) and denoted by $GIG(\lambda, \chi, \psi)$, has the PDF

$$p(x) = \frac{(\psi/\chi)^{\lambda/2}}{2K_\lambda(\sqrt{\chi\psi})} x^{\lambda-1} e^{-\frac{1}{2}(\chi/x+\psi x)}, \quad x > 0, \tag{11}$$

where K_λ is the modified Bessel function (6). The range of the parameters is as follows: $\chi \geq 0, \psi > 0, \lambda > 0$; $\chi > 0, \psi > 0, \lambda = 0$; $\chi > 0, \psi \geq 0, \lambda < 0$. Barndorff-Nielsen [2] considered mixtures of the form

$$\mathbf{Y} \stackrel{d}{=} \mathbf{m} + \boldsymbol{\mu} Z + Z^{1/2}\mathbf{X}, \tag{12}$$

where $\mathbf{X}$ is as before, $\boldsymbol{\mu} = \boldsymbol{\Sigma}\boldsymbol{\beta}$ with some d-dimensional vector $\boldsymbol{\beta}$, and $Z \sim GIG(\lambda, \chi, \psi)$. With the notation $\chi = \delta^2$, $\psi = \xi^2$, and $\alpha^2 = \xi^2 + \boldsymbol{\beta}'\boldsymbol{\Sigma}\boldsymbol{\beta}$, $\mathbf{Y}$ has a d-dimensional generalized hyperbolic distribution with index λ, denoted by $H_d(\lambda, \alpha, \boldsymbol{\beta}, \delta, \mathbf{m}, \boldsymbol{\Sigma})$ (a hyperbolic distribution is obtained for $\lambda = 1$, see, e.g., [4]).

By taking the *limiting case* $GIG(s,0,2)$ as the mixing distribution (which is a standard gamma distribution with shape parameter s) and setting $\boldsymbol{\Sigma\beta} = \boldsymbol{\mu}$ and $\mathbf{m} = \mathbf{0}$, so that $\delta^2 = 0$, $\xi^2 = 2$, and $\alpha = \sqrt{2 + \boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}}$, we then obtain the mixture $\boldsymbol{\mu}Z + Z^{1/2}\mathbf{X}$, where $\mathbf{X}$ is $N_d(\mathbf{0},\boldsymbol{\Sigma})$, independent of Z, which is multivariate GAL. This shows that the multivariate GAL laws can be obtained as a *limiting case* of the generalized hyperbolic distributions.

2.1. **Infinite divisibility.** As mentioned earlier, all AL distributions are infinitely divisible, and so are GAL distributions. Their Lévy measure presented below can be obtained either from that of AL laws given in [15] or from the representation (8) as a subordinated Brownian motion and Lemma 7, VI.2 of Bertoin [3].

Proposition 1. *Let* $\mathbf{Y}$ *have a truly d-dimensional* $GAL_d(\boldsymbol{\Sigma},\boldsymbol{\mu},s)$ *law. Then, the ChF of* $\mathbf{Y}$ *is of the form*

$$\Psi(\mathbf{t}) = \exp\left(\int_{R^n}\left(e^{i\mathbf{t}\cdot\mathbf{x}} - 1\right)\Lambda(d\mathbf{x})\right)$$

with

$$\frac{d\Lambda}{d\mathbf{x}}(\mathbf{x}) = \frac{2s\exp(\boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\mathbf{x})}{(2\pi)^{d/2}|\boldsymbol{\Sigma}|^{1/2}}\left(\frac{Q(\mathbf{x})}{C(\boldsymbol{\Sigma},\boldsymbol{\mu})}\right)^{-d/2} K_{d/2}(Q(\mathbf{x})C(\boldsymbol{\Sigma},\boldsymbol{\mu})),$$

where $Q(\mathbf{x})$ *and* $C(\boldsymbol{\Sigma},\boldsymbol{\mu})$ *are given by (7).*

2.2. **Mean vector and covariance matrix.** The relation between the mean vector $\mathbb{E}\mathbf{Y}$, the covariance matrix $\mathbb{E}[(\mathbf{Y}-\mathbb{E}\mathbf{Y})'(\mathbf{Y}-\mathbb{E}\mathbf{Y})]$ and the parameters $\boldsymbol{\mu} = (\mu_1,\ldots,\mu_d)$ and $\boldsymbol{\Sigma} = [\sigma_{ij}]_{i,j=1}^d$ of $\mathbf{Y} = (Y_1,\ldots,Y_d) \sim GAL_d(\boldsymbol{\Sigma},\boldsymbol{\mu},s)$ can be obtained from the representation (10). Indeed, we have $\mathbb{E}Y_i = \mu_i\mathbb{E}Z = \mu_i s$, so that

$$\mathbb{E}(\mathbf{Y}) = \mathbf{m}s.$$

Furthermore, since $\mathbb{E}(X_iX_j) = \sigma_{ij}$ and $\mathbb{E}Z^2 = s + s^2$, we have

$$\mathbb{E}(Y_iY_j) = \mu_i\mu_j\mathbb{E}Z^2 + \mathbb{E}(Z)\mathbb{E}(X_iX_j) = (s+s^2)\mu_i\mu_j + s\sigma_{ij},$$

so that

$$Cov(Y_i,Y_j) = \mathbb{E}(Y_iY_j) - \mathbb{E}(Y_i)\mathbb{E}(Y_j) = (s+s^2)\mu_i\mu_j + s\sigma_{ij} - \mu_i\mu_j s^2 = s(\mu_i\mu_j + \sigma_{ij}).$$

Thus, the covariance matrix of $\mathbf{Y}$ is

$$\mathbb{C}ov(\mathbf{Y}) = s(\mathbf{\Sigma} + \boldsymbol{\mu}\boldsymbol{\mu}').$$

More general cross-moments $\mathbb{E}[Y_1^{n_1} \cdots Y_d^{n_d}]$ can similarly be obtained from the representation (10) as well.

2.3. **Linear combinations and marginal distributions.** Here we discuss the properties of GAL vectors under linear transformations. The results presented below parallel those connected with AL distributions discussed in [15]. Our first result shows that all linear combinations of the components of $\mathbf{Y} \sim GAL_d(\mathbf{\Sigma}, \boldsymbol{\mu}, s)$ are jointly GAL.

Proposition 2. *Let $\mathbf{Y} = (Y_1, \ldots, Y_d) \sim GAL_d(\mathbf{\Sigma}, \boldsymbol{\mu}, s)$ and let $\mathbf{A}$ be an $l \times d$ real matrix. Then, the random vector $\mathbf{AY}$ is $GAL_l(\mathbf{\Sigma_A}, \boldsymbol{\mu}_\mathbf{A}, s)$, where $\boldsymbol{\mu}_\mathbf{A} = \mathbf{A}\boldsymbol{\mu}$ and $\mathbf{\Sigma_A} = \mathbf{A\Sigma A}'$.*

Proof. The assertion follows from the general relation

$$\phi_{\mathbf{AY}}(\mathbf{t}) = \mathbb{E}e^{i(\mathbf{AY})'\mathbf{t}} = \mathbb{E}e^{i\mathbf{Y}'\mathbf{A}'\mathbf{t}} = \phi_{\mathbf{Y}}(\mathbf{A}'\mathbf{t})$$

and the fact that the matrix $\mathbf{A\Sigma A}'$ is non-negative definite whenever $\mathbf{\Sigma}$ is. □

In particular, it follows that all univariate and multivariate marginals as well as linear combinations of the components of a multivariate GAL vector are GAL.

Corollary 1. *Let $\mathbf{Y} = (Y_1, \ldots, Y_d) \sim GAL_d(\mathbf{\Sigma}, \boldsymbol{\mu}, s)$, where $\mathbf{\Sigma} = [\sigma_{ij}]_{i,j=1}^d$. Then,*

(i) *For all $n \le d$, $(Y_1, \ldots, Y_n) \sim GAL_n(\tilde{\mathbf{\Sigma}}, \tilde{\boldsymbol{\mu}}, s)$, where $\tilde{\boldsymbol{\mu}} = (\mu_1, \ldots, \mu_n)$ and $\tilde{\mathbf{\Sigma}}$ is a $n \times n$ matrix with $\tilde{\sigma}_{ij} = \sigma_{ij}$ for $i, j = 1, \ldots, n$;*

(ii) *For any $\mathbf{b} = (b_1, \ldots b_d) \in \mathbb{R}^d$, the random variable $Y_\mathbf{b} = \sum_{k=1}^d b_k Y_k$, is univariate $GAL(\sigma, \mu, s)$ with $\sigma = \sqrt{\mathbf{b}'\mathbf{\Sigma b}}$ and $\mu = \boldsymbol{\mu}'\mathbf{b}$. Furthermore, if $\mathbf{Y}$ is symmetric (elliptically contoured) GAL, then $Y_\mathbf{b}$ is symmetric;*

(iii) *For all $i \le d$, $Y_i \sim GAL(\sigma, \mu, s)$ with $\sigma = \sqrt{\sigma_{ii}}$ and $\mu = \mu_i$.*

Proof. For part (i), apply Proposition 2 with $n \times d$ matrix $A = (a_{ij})$ such that $a_{ii} = 1$ and $a_{ij} = 0$ for $i \neq j$. For part (ii), apply Proposition 2 with $l = 1$ and compare the resulting ChF with (5). For part (iii) apply part (ii) with standard base vectors in $\mathbb{R}^d$. □

Remark 3. Corollary 1 part (ii) implies that the sum $\sum_{k=1}^{d} Y_k$ has a GAL distribution if all the $\{Y_k\}$ are components of a multivariate GAL random vector (and thus each Y_k is a univariate GAL). This is in contrast with a sum of IID GAL random variables, that generally does not have a GAL distribution.

Remark 4. If $\mathbf{Y}$ has a nonsingular GAL law (that is $\mathbf{\Sigma}$ is positive definite) and the matrix $\mathbf{A}$ is such that $\mathbf{A}\mathbf{A}'$ is positive-definite, then the vector $\mathbf{AY}$ has a nonsingular GAL law as well. In particular, this holds if $\mathbf{A}$ is a nonsingular square matrix.

Remark 5. We have shown in Corollary 1, part (ii), that if $\mathbf{Y}$ is multivariate GAL then all linear combinations of its components are univariate GAL. An interesting open question is whether the converse is true. Namely, if all linear combinations of a random vector $\mathbf{Y}$ in $\mathbb{R}^d$ are univariate GAL with the same shape parameter s, is $\mathbf{Y}$ multivariate GAL?

2.4. **Conditional distributions.** Below we present conditional distributions of GAL random vectors with a non-singular $\mathbf{\Sigma}$, taken from [15].

Theorem 1. *Let* $\mathbf{Y} \sim GAL_d(\mathbf{\Sigma}, \boldsymbol{\mu}, s)$ *have the ChF (4) with a non-singular* $\mathbf{\Sigma}$. *Let* $\mathbf{Y}' = (\mathbf{Y}_1', \mathbf{Y_2}')$ *be a partition of* $\mathbf{Y}$ *into* $r \times 1$ *and* $k \times 1$ *dimensional sub-vectors, respectively. Let* $(\boldsymbol{\mu}_1', \boldsymbol{\mu}_2')$ *and*

$$\mathbf{\Sigma} = \begin{pmatrix} \mathbf{\Sigma}_{11} & \mathbf{\Sigma}_{12} \\ \mathbf{\Sigma}_{21} & \mathbf{\Sigma}_{22} \end{pmatrix}$$

be the corresponding partitions of $\boldsymbol{\mu}$ *and* $\mathbf{\Sigma}$, *where* $\mathbf{\Sigma}_{11}$ *is an* $r \times r$ *matrix. Then,*
(i) If $s = 1$ *(so that* $\mathbf{Y}$ *is AL), then the conditional distribution of* $\mathbf{Y}_2$ *given* $\mathbf{Y}_1 = \mathbf{y}_1$ *is the generalized* k*-dimensional hyperbolic distribution* $H_k(\lambda, \alpha, \boldsymbol{\beta}, \delta, \boldsymbol{m}, \mathbf{\Delta})$ *having*

the density

$$p(\mathbf{y}_2|\mathbf{y}_1) = \frac{\xi^\lambda \exp(\boldsymbol{\beta}'(\mathbf{y}_2 - \boldsymbol{m}))K_{k/2-\lambda}(\alpha\sqrt{\delta^2 + (\mathbf{y}_2 - \boldsymbol{m})'\boldsymbol{\Delta}^{-1}(\mathbf{y}_2 - \boldsymbol{m})})}{(2\pi)^{k/2}|\boldsymbol{\Delta}|^{1/2}\delta^\lambda K_\lambda(\delta\xi)[\sqrt{\delta^2 + (\mathbf{y}_2 - \boldsymbol{m})'\boldsymbol{\Delta}^{-1}(\mathbf{y}_2 - \boldsymbol{m})}/\alpha]^{k/2-\lambda}}, \tag{13}$$

where $\lambda = 1 - r/2$, $\alpha = \sqrt{\xi^2 + \boldsymbol{\beta}'\boldsymbol{\Delta}\boldsymbol{\beta}}$, $\boldsymbol{\beta} = \boldsymbol{\Delta}^{-1}(\boldsymbol{\mu}_2 - \boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\mu}_1)$, $\delta = \sqrt{\mathbf{y}_1'\boldsymbol{\Sigma}_{11}^{-1}\mathbf{y}_1}$, $\boldsymbol{m} = \boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\mathbf{y}_1$, $\boldsymbol{\Delta} = \boldsymbol{\Sigma}_{22} - \boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}$, *and* $\xi = \sqrt{2 + \boldsymbol{\mu}_1'\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\mu}_1}$*;*
(ii) If $\boldsymbol{\mu}_1 = \mathbf{0}$*, then the conditional distribution of* $\mathbf{Y}_2$ *given* $\mathbf{Y}_1 = \mathbf{0}$ *is generalized Laplace* $GAL_k(\boldsymbol{\Sigma}_{2\cdot1}, \boldsymbol{\mu}_{2\cdot1}, s_{2\cdot1})$*, where*

$$s_{2\cdot1} = s - r/2, \;\; \boldsymbol{\Sigma}_{2\cdot1} = \boldsymbol{\Sigma}_{22} - \boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}, \;\; \boldsymbol{\mu}_{2\cdot1} = \boldsymbol{\mu}_2.$$

Our next result, taken from [15], provides expressions for the conditional mean vector and covariance matrix.

Proposition 3. *Let* $\mathbf{Y}$ *have a GAL law (4) with a non-singular* $\boldsymbol{\Sigma}$*. Let* $\mathbf{Y}$*,* $\boldsymbol{\mu}$*, and* $\boldsymbol{\Sigma}$ *be partitioned as in Theorem 1. Then,*

$$\mathbb{E}(\mathbf{Y}_2|\mathbf{Y}_1 = \mathbf{y}_1) = \boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\mathbf{y}_1 + (\boldsymbol{\mu}_2 - \boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\mu}_1)\frac{Q(\mathbf{y_1})}{C}R_{1-r/2}(CQ(\mathbf{y_1}))$$

and

$$\begin{aligned}\mathbb{V}ar(\mathbf{Y}_2|\mathbf{Y}_1 = \mathbf{y}_1) &= \frac{Q(\mathbf{y_1})}{C}(\boldsymbol{\Sigma}_{22} - \boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12})R_{1-r/2}(CQ(\mathbf{y_1})) \\ &+(\boldsymbol{\mu}_2 - \boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\mu}_1)(\boldsymbol{\mu}_2 - \boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\mu}_1)'\left(\frac{Q(\mathbf{y_1})}{C}\right)^2 G(\mathbf{y}_1),\end{aligned}$$

where $C = \sqrt{2 + \boldsymbol{\mu}_1'\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\mu}_1}$*,* $Q(\mathbf{y}_1) = \sqrt{\mathbf{y}_1'\boldsymbol{\Sigma}_{11}^{-1}\mathbf{y}_1}$*,* $R_s(x) = K_{s+1}(x)/K_s(x)$*, and*

$$G(\mathbf{y}_1) = (R_{1-r/2}(CQ(\mathbf{y_1}))R_{2-r/2}(CQ(\mathbf{y_1})) - R^2_{1-r/2}(CQ(\mathbf{y_1}))).$$

Remark 6. If $r = d-1$ and $\boldsymbol{\mu}_1'\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12} = \mu_d$, then we have linearity of the regression of Y_d on $Y_1, \ldots, Y_{d-1}$,

$$\mathbb{E}(Y_d|Y_1, \ldots, Y_{d-1}) = a_1Y_1 + \cdots + a_{d-1}Y_{d-1} \;\; (a.s.) \tag{14}$$

where $(a_1, \ldots, a_{d-1})' = \boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}$

2.5. **Polar representation.** All GAL distributions with mean zero ($\boldsymbol{\mu} = \mathbf{0}$) are elliptically contoured, as their ChF depends on $\mathbf{t}$ only through the quadratic form $\mathbf{t}'\boldsymbol{\Sigma}\mathbf{t}$. With a non-singular $\boldsymbol{\Sigma}$, they are also elliptically symmetric, and admit a polar representation given below, which generalizes similar representation of asymmetric Laplace distributions discussed in [15].

Proposition 4. *Let* $\mathbf{Y} \sim GAL_d(\boldsymbol{\Sigma}, \mathbf{0}, s)$*, where* $|\boldsymbol{\Sigma}| > 0$*. Then,* $\mathbf{Y}$ *admits the representation*

$$\mathbf{Y} \stackrel{d}{=} R\mathbf{H}\mathbf{U}^{(d)}, \tag{15}$$

where $\mathbf{H}$ *is a* $d \times d$ *matrix such that* $\mathbf{H}\mathbf{H}' = \boldsymbol{\Sigma}$*,* $\mathbf{U}^{(d)}$ *is a random vector uniformly distributed on the unit sphere* $\mathbb{S}_d$ *of* $\mathbb{R}^d$*, and* R *is a positive random variable, independent of* $\mathbf{U}^{(d)}$*, with the density*

$$f_R(x) = \frac{2x^{d/2+s-1}K_{d/2-s}(\sqrt{2}x)}{(\sqrt{2})^{d/2-2+s}\Gamma(s)\Gamma(d/2)}, \quad x > 0, \tag{16}$$

where K_v *is the modified Bessel function of the third kind.*

Proof. Write $\boldsymbol{\Sigma} = \mathbf{H}\mathbf{H}'$, where $\mathbf{H}$ is a $d \times d$ non-singular lower triangular matrix (see, e.g., [10], pp. 566, for a recipe of obtaining such matrix from a given non-singular $\boldsymbol{\Sigma}$). Then, the random vector $\mathbf{X} \sim N_d(\mathbf{0}, \boldsymbol{\Sigma})$ from (10) has the representation $\mathbf{X} = \mathbf{H}\mathbf{N}$, where $\mathbf{N} \sim N_d(\mathbf{0}, \mathbf{I})$. Further, $\mathbf{N}$, which is elliptically contoured, has the well known representation $\mathbf{N} \stackrel{d}{=} R_{\mathbf{N}}\mathbf{U}^{(d)}$. Here, $R_{\mathbf{N}}$ and $\mathbf{U}^{(d)}$ are independent, $\mathbf{U}^{(d)}$ is uniformly distributed on $\mathbb{S}_d$, and $R_{\mathbf{N}}$ is positive with the PDF

$$f_{R_{\mathbf{N}}}(x) = \frac{d \cdot x^{d-1}\exp(-x^2/2)}{2^{d/2}\Gamma(d/2+1)}, \quad x > 0. \tag{17}$$

Therefore, in view of the representation (10) with $\boldsymbol{\mu} = \mathbf{0}$, it is sufficient to show that $Z^{1/2}R_{\mathbf{N}}$ has density (16). To see this, apply standard conditioning argument and write the PDF of $Z^{1/2}R_{\mathbf{N}}$ as

$$f_{Z^{1/2}R_{\mathbf{N}}}(y) = dy\int_0^\infty \frac{x^{d/2-2}\exp(-\frac{1}{2}(x+2y^2/x))}{2^{d/2}\Gamma(s)\Gamma(d/2+1)}\left(\frac{y^2}{x}\right)^{s-1}dx. \tag{18}$$

Let $f_{\lambda,\chi,\psi}$ be the GIG density (11) with $\psi = 1$, $\chi = 2y^2$, and $\lambda = d/2 - s$. Then, the above relation becomes

$$f_{Z^{1/2}R_{\mathbf{N}}}(y) = \frac{2d \cdot y^{2s-1} K_\lambda(\sqrt{2}y)}{2^{d/2}\Gamma(s)\Gamma(d/2+1)(\chi)^{-\lambda/2}} \int_0^\infty f_{\lambda,\chi,\psi}(x)dx, \tag{19}$$

which, after some algebra, yields (16) since the function $f_{\lambda,\chi,\psi}$ integrates to one. □

2.6. **Limits of random sums.** Recall that multivariate asymmetric Laplace distributions are the only possible (weak) limiting distributions of (normalized) geometric random sums (3) as $p \to 0$ (and $N_p \xrightarrow{p} \infty$), see [15]. Similar result holds true for the GAL distributions under *negative binomial* (NB) random summation. Let $N_{p,s}$ be a NB random variable with parameters $p \in (0,1)$ and $s > 0$, so that

$$P(N_{p,s} = k) = \frac{\Gamma(s+k)}{\Gamma(s)k!} p^s (1-p)^k, \;\; k = 0,1,2,\ldots. \tag{20}$$

The following new result is an extension of Theorem 6.10.1 concerning AL distributions from [15] to the GAL case.

Theorem 2. *Let $\mathbf{X}^{(j)}$ be IID random vectors in $\mathbb{R}^d$ with mean vector $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$. For $p \in (0,1)$, let $N_{p,s}$ be a NB random variable (20), independent of the sequence $(\mathbf{X}^{(j)})$. Then, as $p \to 0$,*

$$a_p \sum_{j=1}^{N_{p,s}} (\mathbf{X}^{(j)} + \mathbf{b}_p) \xrightarrow{d} \mathbf{Y} \sim GAL_d(\boldsymbol{\Sigma}, \boldsymbol{\mu}, s), \tag{21}$$

where $a_p = p^{1/2}$ and $\mathbf{b}_p = \boldsymbol{\mu}(p^{1/2} - 1)$.

Proof. By the Cramér-Wald device, the convergence (21) is equivalent to

$$\mathbf{c}' a_p \sum_{j=1}^{N_{p,s}} (\mathbf{X}^{(j)} + \mathbf{b}_p) \xrightarrow{d} \mathbf{c}'\mathbf{Y}, \;\; \text{as } p \to 0,$$

for all vectors $\mathbf{c}$ in $\mathbb{R}^d$. Denoting $W_j = \mathbf{c}'(\mathbf{X}^{(j)} - \boldsymbol{\mu})$, $\mu = \mathbf{c}'\boldsymbol{\mu}$, $b_p = p^{1/2}\mu$, and $Y = \mathbf{c}'\mathbf{Y}$, we have

$$a_p \sum_{j=1}^{N_{p,s}} (W_j + b_p) \xrightarrow{d} Y \sim GAL(\sigma, \mu, s), \;\; \text{as } p \to 0. \tag{22}$$

Here, (W_j) are IID with mean zero and variance $\sigma^2 = \mathbf{c}'\mathbf{\Sigma}\mathbf{c}$, and Y is a univariate GAL random variable with the ChF ϕ given in (5). Writing (22) in terms of the ChFs we obtain

$$\left(\frac{p}{1-(1-p)e^{ip\mu t}\psi(p^{1/2}t)}\right)^s \to \phi(t), \tag{23}$$

where ψ is the ChF of the (W_j). Note that (23) is equivalent to the convergence

$$\frac{e^{-ip\mu t}-1}{p} + \frac{1-(1-p)\psi(p^{1/2}t)}{p} = I + II \to 1 + \sigma^2 t^2/2 - i\mu t.$$

It is easy to see that $I \to -i\mu t$ as $p \to 0$. To show the convergence

$$II = \frac{1-(1-p)\psi(p^{1/2}t)}{p} =\!\!\rightarrow 1 + \sigma^2 t^2/2 \tag{24}$$

we use Theorem 8.44 from [9]: since W_j has the first two moments, its ChF can be written as $\psi(u) = 1 + iu\mathbb{E}W_j + (iu)^2(\mathbb{E}X_j^2 + \delta(u))/2$, where δ denotes a bounded function of u such that $\lim_{u\to 0}\delta(u) = 0$. Since $\mathbb{E}W_j = 0$ and $\mathbb{E}W_j^2 = \sigma^2$, we apply the above with $u = p^{1/2}t$ to the lhs of (24) to obtain

$$\frac{t^2}{2}(\sigma^2 + \delta(p^{1/2}t)) + 1 - \frac{pt^2}{2}(\sigma^2 + \delta(p^{1/2}t)),$$

which converges to $1 + t^2\sigma^2/2$ as $p \to 0$. □

3. Moving average fields build upon generalized Laplace distributions

In this section we present several constructions of random moving average fields driven by Laplace motion. All these models are referred to as Laplace moving averages (LMA). This development is continuation of the ideas from [22] and of those that in one-dimensional case were introduced in [1]. Then we discuss properties of such fields and comment on available tools for model fitting statistical inference. The main component of the construction is a random independently scattered measure that has a generalized asymmetric multivariate Laplace distribution as marginals. These measures and their direct relation to multivariate Laplace motion are discussed first.

3.1. **Multivariate Laplace motion and independent scattered measures.** For the generalized Laplace distribution it is convenient to build stochastic models using its infinite divisibility property. This property stands behind two important and related general concepts: an independently scattered random measure and Lévy motion. Let us recall the latter one.

Definition 2 (Laplace motion). *A vector valued Laplace motion* $\boldsymbol{\Lambda}(t)$ *in* $\mathbb{R}^d$ *with parameters* $(\boldsymbol{\Sigma}, \boldsymbol{\mu}, \nu)$ *defined on positive real line is a process with independent and homogeneous increments such that the increment over* t *and* $t + s$ *has the multivariate* $GAL_d(\boldsymbol{\Sigma}, \boldsymbol{\mu}, s/\nu)$ *distribution.*

Such a process can be conveniently represented as a multivariate Brownian motion subordinated to a Gamma process. Namely, if $\mathbf{A} = \sqrt{\boldsymbol{\Sigma}}$ (here a $d \times d$ matrix $\boldsymbol{\Sigma}$ is assumed to be positive definite), $\boldsymbol{\mu} \in \mathbb{R}^d$, $\mathbf{B}(t) = (B_1(t), \ldots, B_d(t))$, where the (B_i) are independent standard Brownian motions, and $\Gamma(t)$ is a standard gamma process (so that $\Gamma(1)$ has the standard exponential distribution), then the process

$$\boldsymbol{\Lambda}(t) = \mathbf{A}\mathbf{B}(\Gamma(t/\nu)) + \Gamma(t/\nu)\boldsymbol{\mu}$$

satisfies the properties defining the Laplace motion. The verification of this follows directly from (8) and (10).

The extension of Lévy motion to the case of multidimensional argument is through the concept of stochastic measure. In what follows, for a Borel measurable subset A of an Euclidean space, $m(A)$ stands for its Lebesgue measure.

Definition 3 (Stochastic Laplace Measure). *A stochastic Laplace measure* $\boldsymbol{\Lambda}$*, with parameters* $\nu > 0$*,* $\boldsymbol{\mu} \in \mathbb{R}^d$*, a positive definite* $d \times d$ *matrix* $\boldsymbol{\Sigma}$ *and controlled by a measure* m*, is a function that maps* $A \subseteq \mathbb{R}^n$*,* $m(A) < \infty$*, to a random variable*

$\mathbf{\Lambda}(A) \sim GAL_d(\mathbf{\Sigma}, \boldsymbol{\mu}, m(A)/\nu)$, *so that the ChF of* $\mathbf{\Lambda}(A)$ *is*

$$\phi_{\mathbf{\Lambda}(A)}(\boldsymbol{u}) = \left(1 - i\boldsymbol{\mu}'\mathbf{u} + \frac{1}{2}\mathbf{u}'\mathbf{\Sigma}\mathbf{u}\right)^{-m(A)/\nu}.$$

For disjoint sets A_i*, the variables* $\Lambda(A_i)$ *are independent, and with probability one*

$$\mathbf{\Lambda}\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathbf{\Lambda}(A_i).$$

The Laplace motion $\mathbf{\Lambda}$, which is defined on positive line, can be identified with a stochastic measure on the Borel sets of the half-line through the following definition of such a measure on a closed-open interval:

$$\mathbf{\Lambda}([a, b)) = \mathbf{\Lambda}(b) - \mathbf{\Lambda}(a).$$

The standard measure theoretical argument allows to extend such a measure to an arbitrary Borel set. The extension to the entire line can be obtained by considering an independent Laplace motion on the negative half-line.

3.2. **Multivariate Laplace moving averages.** Consider a square integrable real function f on $(-\infty, 0]$. The classical model that comes as a generalization to continuous time of the concept of time series is given by the following moving averages

$$X(t) = \int_{-\infty}^{t} f(s - t)\; dB(s),$$

where B represents Brownian motion extended over the entire real line. A natural generalization of this model is obtained by replacing B by an arbitrary Lévy motion. In particular, one can define

$$X(t) = \int_{-\infty}^{t} f(s - t)\; d\Lambda(s),$$

where Λ is Laplace motion with parameters σ, μ, ν and defined over the real line. As a result we obtain a strictly stationary process of the second order, which has been discussed in [1].

By using various kernel functions into a vector function $\mathbf{f} = (f_1, \dots, f_d)$, one can consider vector valued processes

$$\mathbf{X}(t) = \int_{-\infty}^{t} \mathbf{f}(s-t)\ \Lambda(ds) = \left(\int_{-\infty}^{t} f_1(s-t)\ d\Lambda(s), \dots, \int_{-\infty}^{t} f_d(s-t)\ \Lambda(ds) \right).$$

This model easily extends to the fields if we take Λ to be a Laplace measure on $\mathbb{R}^n$ controlled by the Lebesgue measure and consider

(25)

$$\mathbf{X}(\mathbf{p}) = \int_{\mathbb{R}^n} \mathbf{f}(\mathbf{s}-\mathbf{p})\ \Lambda(ds) = \left(\int_{\mathbb{R}^n} f_1(\mathbf{s}-\mathbf{p})\ d\Lambda(\mathbf{s}), \dots, \int_{\mathbb{R}^d} f_d(\mathbf{s}-\mathbf{p})\ \Lambda(d\mathbf{s}) \right).$$

From the distributional point of view such a random vector field is best described through ChFs. Consider the $d \times r$ matrix $\mathbf{F}(\mathbf{s}) = (f_{ij}) = (f_i(\mathbf{s}-\mathbf{p}_j))$ and an $r \times d$ matrix $\mathbf{u}$. We denote $\langle \mathbf{F}, \mathbf{u} \rangle = \sum_{i=1}^{d} \langle \mathbf{F}_{i\cdot}, \mathbf{u}_{\cdot i} \rangle$, i.e. the sum of inner products between the rows of $\mathbf{F}$ and the columns of $\mathbf{u}$. If the parameters of Λ are σ, μ and ν, then we can write the ChF of $(\mathbf{X}(\mathbf{p}_1), \dots, \mathbf{X}(\mathbf{p}_r))$ as follows

$$\phi(\mathbf{u}) = \exp\left(-\frac{1}{\nu} \int_{\mathbb{R}^n} \log\left(1 - i\mu \langle \mathbf{F}(\mathbf{s}), \mathbf{u} \rangle + \frac{\sigma^2}{2} \langle \mathbf{F}(\mathbf{s}), \mathbf{u} \rangle^2 \right) d\mathbf{s} \right). \tag{26}$$

The moments and cross-correlations for this stochastic vector field are readily available as discussed in [1]. In particular, for $\mathbf{X}(\mathbf{p}) = (X_1(\mathbf{p}), \dots, X_d(\mathbf{p})) = (X_1, \dots, X_d)$ we have the moments

$$\begin{aligned}
\mathbb{E}\, X_i &= \mu \cdot \int f_i, \\
\mathbb{E}\,(X_i - \mathbb{E}\, X_i)^2 &= \left(\mu^2 + \sigma^2\right) \cdot \int f_i^2, \\
\mathbb{E}\,(X_i - \mathbb{E}\, X_i)^3 &= \mu \left(2\mu^2 + 3\sigma^2\right) \cdot \int f_i^3, \\
\mathbb{E}\,(X_i - \mathbb{E}\, X_i)^4 &= 3\left(\mu^2 + \sigma^2\right)^2 \cdot \left(\int f_i^2 \right)^2 + \\
&\qquad + 3\left(2\mu^4 + 4\mu^2\sigma^2 + \sigma^4\right) \cdot \int f_i^4,
\end{aligned}$$

for $i = 1, \ldots, d$ and the cross-corelations

$$\mathbb{C}orr(X_i(\mathbf{t}), X_j(\mathbf{0})) = \frac{\int_{\mathbb{R}^n} f_i(\mathbf{s}-\mathbf{t}) f_j(\mathbf{s})\ d\mathbf{s}}{\sqrt{\int f_i^2 \int f_j^2}},\ i,j = 1,\ldots,d,. \tag{27}$$

where the integration, if not shown otherwise in the notation, is always understood over $\mathbb{R}^n$ and with respect the Lebesgue measure.

In the above, the univariate Laplace motion has been used to introduce vector valued field. The vector of kernels $\mathbf{f}$ is almost entirely responsible for the multivariate distributional structure in the model. However, the modeling kernels can be a difficult task for non-Gaussian processes, as will be seen in Section 4. Therefore, next we shall discuss vector valued fields with a single kernel function, where the multivariate structure is introduced by a multivariate Laplace measure.

For a Laplace measure $\mathbf{\Lambda}$ in $\mathbb{R}^d$ and $f : \mathbb{R}^n \mapsto \mathbb{R}$, the following process will be called a moving average:

$$\mathbf{X}(\mathbf{p}) = \int_{\mathbb{R}^n} f(\mathbf{p}-\mathbf{s})\ d\mathbf{\Lambda}(\mathbf{s}). \tag{28}$$

Here, integration is defined through the extension from simple functions.

Theorem 3. *The moving average process $\mathbf{X}(\mathbf{p})$, $\mathbf{p} \in \mathbb{R}^n$, defined by (28), is a stationary vector valued stochastic field with the following mean and covariance function:*

$$\begin{aligned} \mathbb{E}\mathbf{X}(\mathbf{p}) &= \int_{\mathbb{R}^n} f(\mathbf{s})\ d\mathbf{s} \cdot \frac{\boldsymbol{\mu}}{\nu}, \\ \mathbb{C}ov\left(\mathbf{X}(\mathbf{p}), \mathbf{X}(\mathbf{0})'\right) &= f * \tilde{f}(\mathbf{p}) \cdot \frac{\mathbf{\Sigma} + \boldsymbol{\mu}\boldsymbol{\mu}'}{\nu}, \end{aligned} \tag{29}$$

where $\tilde{f}(\mathbf{s}) = f(-\mathbf{s})$.

Moreover, the marginal distributions of $\mathbf{X}(\mathbf{p})$ are given through the ChFs of $\mathbf{Y} = \sum_{i=1}^r a_i \mathbf{X}(\mathbf{p}_i)$ for each $m \in \mathbb{N}$, $a_i \in \mathbb{R}$, and $\mathbf{p}_i \in \mathbb{R}^n$,

$$\phi_{\mathbf{Y}}(\mathbf{u}) = \exp\left(-\frac{1}{\nu}\int_{\mathbb{R}^n} \log\left(1 - ig(\mathbf{p})\cdot\boldsymbol{\mu}'\mathbf{u} + g^2(\mathbf{p})\cdot\frac{\mathbf{u}'\mathbf{\Sigma}\mathbf{u}}{2}\right)\ d\mathbf{p}\right), \tag{30}$$

where $g(\mathbf{p}) = \sum_{i=1}^r a_i f(\mathbf{p}-\mathbf{p}_i)$.

Any vector process $\tilde{\mathbf{X}}(\mathbf{p})$ *obtained by subsetting from coordinates of* $\mathbf{X}(\mathbf{p})$ *is again a moving average process with the same kernel and with respect to a Laplace measure with parameters* ν, $\tilde{\boldsymbol{\mu}}$, *and* $\tilde{\boldsymbol{\Sigma}}$, *where the latter two are made of these entries in* $\boldsymbol{\mu}$ *and* $\boldsymbol{\Sigma}$, *indices of which were taken in* $\tilde{\mathbf{X}}(\mathbf{p})$.

Proof. By the standard extension argument it is enough to show the result for f being a simple function, $f(\mathbf{p}) = \sum_{j=1}^m c_j \mathbb{I}_{A_j}(\mathbf{p})$, where the (A_i) are disjoint subsets of $\mathbb{R}^n$ and $\mathbb{I}_{A_i}$ stands for the indicator function of the set A_i. By the definition of the stochastic integral, $\mathbf{Z} = \int_{\mathbb{R}^n} f(\mathbf{p}-\mathbf{s})\, d\boldsymbol{\Lambda}(\mathbf{s}) = \sum_{j=1}^m c_j \boldsymbol{\Lambda}(\mathbf{p} - A_j)$, which, for a fixed $\mathbf{p}$, is a linear combination of independent generalized Laplace vectors. The characteristic function of such a vector is given by

$$\begin{aligned}\phi_{\mathbf{Z}}(\mathbf{u}) &= \prod_{j=1}^m \left(1 - ic_j \cdot \boldsymbol{\mu}'\mathbf{u} + c_j^2 \cdot \frac{\mathbf{u}'\boldsymbol{\Sigma}\mathbf{u}}{2}\right)^{-m(A_j)/\nu} \\ &= \exp\left(-\frac{1}{\nu}\sum_{j=1}^m \log\left(1 - ic_j \cdot \boldsymbol{\mu}'\mathbf{u} + c_j^2 \cdot \frac{\mathbf{u}'\boldsymbol{\Sigma}\mathbf{u}}{2}\right) m(A_j)\right),\end{aligned}$$

which renders the characteristic function formula and in a consequence yields (strict) stationarity.

The mean and covariance of a multivariate GAL distribution as given in Definition 1 are $\boldsymbol{\mu}/\nu$ and $(\boldsymbol{\Sigma} + \boldsymbol{\mu}\boldsymbol{\mu}')/\nu$, respectively, see Subsection 2.2. The formula for the mean of a simple function f is an easy consequence of linearity of the expectation. The formula for the covariance follows easily from the independence of the increments for $\boldsymbol{\Lambda}$. The extensions to an arbitrary f are standard. The fact that $\tilde{\mathbf{Y}}_{\mathbf{p}}$ is again a Laplace moving average process with the indicated parameters is owed to the fact that a vector made of some coordinates of a multivariate GAL distribution is again GAL (see Subsection 2.3). □

We conclude our discussion by remarking on further generalization of Laplace driven models. Namely, the two presented models can be viewed as special cases of the following model

$$\mathbf{X}(\mathbf{p}) = \left(\int_{\mathbb{R}^n} f_1(\mathbf{s}-\mathbf{p})\Lambda_1(d\mathbf{s}), \dots, \int_{\mathbb{R}^n} f_d(\mathbf{s}-\mathbf{p})\Lambda_d(d\mathbf{s})\right), \tag{31}$$

where Λ_i are coordinates of some multivariate Laplace motion $\mathbf{\Lambda}$ in $\mathbb{R}^{d'}$, where $d' \leq d$ so it is not excluded that $\Lambda_i = \Lambda_j$ for some $i, j = 1, \dots d$. The obvious extension of the previous results on the form of ChFs and moments are not discussed here.

3.3. **Model fitting and statistical inference.** The foundations for estimation of the Laplace motion driven moving averages have been laid down in [1] and [19], where one dimensional Laplace distributions have been considered. The fitting of the models is based on the moments and these methods seem to work quite well even if compared to the likelihood based methods (only available for very special cases due to the lack of explicit formula for the densities in the general case), see [19]. In the previous section two vector fields models have been discussed. The coordinates of each of these models are one dimensional LMAs and the univariate estimation methods can be applied. There are two important aspects of statistical inference.

Firstly, the parameters σ (or the diagonal terms in $\mathbf{\Sigma}$ in the multivariate Laplace case), μ ($\boldsymbol{\mu}$) and ν of the Laplace motion can be obtained from the marginal distribution by the method of moments. Secondly, the estimation of the kernel function, that maybe parametric or not, has to be resolved. In contrast with the Gaussian case, here, the form of the kernel affects the properties of the model even if the resulting covariance is the same. In general, for models (25) and (31) the choice of the kernels must account for the cross-corelations in the data and for this crossco-relation formulas (27) would have to be used.

The model (28) is in this respect different. Since it uses a single kernel, the cross-corelations are essentially controlled by off-diagonal terms in $\mathbf{\Sigma}$, making this model a simpler one for fitting. For example, assume that some estimates $\hat{f}$, $\hat{\nu}$, $\hat{\mu}_i$ and σ_{ii}, $i = 1, \dots, d$, of the corresponding parameters are obtained based on univariate methods applied to the coordinates of multivariate data ($\hat{f}$ and $\hat{\nu}$ can be obtained by averaging the estimates obtained for each coordinate). Then using

(29) we can estimate σ_{ij}, for $i \neq j$ through

$$\hat{\sigma}_{ij} = \hat{\nu} \int_{\mathbb{R}^n} \frac{\widehat{\mathbb{C}ov}(X_i(\mathbf{p}), X_j(\mathbf{0}))}{\widehat{f} * \widetilde{\widehat{f}}(\mathbf{p})} w(\mathbf{p})\, d\mathbf{p} - \hat{\mu}_i \hat{\mu}_j, \tag{32}$$

where $\widehat{\mathbb{C}ov}(X_i(\mathbf{p}), X_j(\mathbf{0}))$ is the standard non-parametric estimate of the cross-covariance, w is a density function over R^n representing here an appropriate weight to account for different accuracy of this non-parametric estimate for various values of $\mathbf{p}$. For example, one choice of $w(\mathbf{p})$ is where the latter is inverse proportional to the standard deviation of $\widehat{\mathbb{C}ov}(X_i(\mathbf{p}), X_j(\mathbf{0}))$. While we do not discuss here kernel fitting in general situation, an example of fitting parameters to parametric kernels is given in Section 4, where an example of model given by (28) is presented.

4. An illustration: parallel road tracks roughness

Responses of a vehicle traveling on road profiles modeled as stationary Gaussian processes have been extensively studied (see for example [21] and [18] for some recent studies). It is well-known in the vehicle engineering that a road surface can not be accurately represented by a Gaussian process, see [6]. The reason is that the actual roads contain short sections with above-average irregularity. As shown in [5], such irregularities cause most of the vehicle fatigue damage. A homogenous LMA model for a single track was proposed in [7], and appeared to represent the road roughness observed in real data quite well. Modeling only a single path along the road is a simplification, as a four-wheeled vehicle is subjected to excitations due to road roughness in the left and right wheel paths. Accounting for both paths is an important aspect of heavy vehicle fatigue assessment. Hence, it is natural to propose a bivariate stochastic model corresponding to parallel road tracks. Here, a bivariate version of the LMA is employed to parallel tracks modeling to provide a fairly accurate statistical description of road surface irregularities.

Let $Z_{\mathrm{R}}(x)$ and $Z_{\mathrm{L}}(x)$ denote the right and the left track elevations, respectively. We assume a homogenous road section and suppose that the right and the left tracks have the same stochastic distribution. The industry's standard is the so called MIRA spectrum (see [16]),

$$S_0(f) \;=\; \begin{cases} 10^{a_0}\left(\frac{f}{f_0}\right)^{-w_1}, & f \in [0.01, 0.20], \\ 10^{a_0}\left(\frac{f}{f_0}\right)^{-w_2}, & f \in [0.20, 10], \\ 0, & \text{otherwise}, \end{cases} , \tag{33}$$

where $f_0 = 0.2$. Here, as suggested in [11], 10^{a_0} is the basic roughness coefficient. The exponent w_1 describes energy distribution between components of wavelengths between 100 and 5 meters, while w_2, with wavelengths between 5 and 0.1 meters, describes the state of road deterioration.

In our example we shall slightly modify the spectrum as follows:

$$S_0(f) = \frac{10^{a_0} c^{-w_1}}{(1 + (f/cf_0)^2)^{w_1/2}} + \frac{10^{a_0} c^{-w_2}}{(1 + (f/cf_0)^2)^{w_2/2}}, \quad f \geq 0.01, \tag{34}$$

and zero otherwise. The road surface roughness will be modeled by means of LMA processes $Z_{\mathrm{R}}(x)$ and $Z_{\mathrm{L}}(x)$ having zero mean and skewness, the same kurtosis, and spectrum $S_0(f)$ defined in (34). For simplicity, only the spectrum is normalized so that the variances of $Z_{\mathrm{R}}(x)$ and $Z_{\mathrm{L}}(x)$ are one. The processes are correlated, with cross-covariance $r_{\mathrm{LR}}(\tau) = \mathbb{E}[Z_{\mathrm{L}}(x + \tau) \cdot Z_{\mathrm{R}}(x)]$. In the following it will be more convenient to use the cross spectrum

$$S_{\mathrm{LR}}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} r_{\mathrm{LR}}(\tau) e^{-i\omega\tau}\, d\tau.$$

The cross spectrum is real, and, as shown in [6], the correlation between the tracks in many measured signals is rather well described by the coherence function

$$S_{\mathrm{LR}}(\omega) = \exp(-\rho|\omega|). \tag{35}$$

Moreover, the values of ρ are often in the interval $[1, 7]$.

Knowledge of the spectrum is sufficient to define Gaussian road tracks. This can be done in several ways. Following [8], let us define three kernel functions $f_{\mathrm{R}}, f_{\mathrm{L}}$

and f_{LR} by the relations

$$\mathcal{F} f_{\mathrm{R}}(\omega) = \sqrt{2\pi\, S(\omega)} = F_{\mathrm{R}}(\omega), \tag{36}$$

$$\mathcal{F} f_{\mathrm{L}}(\omega) = F_{\mathrm{R}}(\omega)\sqrt{1-|S_{\mathrm{LR}}(\omega)|^2}, \tag{37}$$

$$\mathcal{F} f_{\mathrm{LR}}(\omega) = F_{\mathrm{R}}(\omega) S_{\mathrm{LR}}(\omega), \tag{38}$$

where $\mathcal{F}$ stands for the Fourier transform. Next, let $B_{\mathrm{R}}(x)$ and $B_{\mathrm{L}}(x)$ be independent Brownian motions. Then, the Gaussian moving average (GMA) model for the two tracks roughness is

$$Z_{\mathrm{R}}(x) = \int f_{\mathrm{R}}(x-u)\, dB_{\mathrm{R}}(u), \;\; Z_{\mathrm{L}}(x) = \int f_{\mathrm{LR}}(x-u)\, dB_{\mathrm{R}}(u) + \int f_{\mathrm{L}}(x-u)\, dB_{\mathrm{L}}(u). \tag{39}$$

The fact that the sum of Laplace distributed variables is no longer Laplace distributed makes the extension from vector valued GMA to vector valued LMA less obvious. Here, we shall employ uncorrelated but dependent Laplace motions $\Lambda_{\mathrm{R}}(x)$ and $\Lambda_{\mathrm{L}}(x)$. We are interested in symmetrical LMA processes and hence assume that $\Lambda(x)$ is a stochastic process with independent and stationary increments having distribution given by the ChF

$$\phi_{\Lambda(x)}(v) = \frac{1}{(1+\frac{\sigma^2 v^2}{2})^{\frac{x}{\nu}}}. \tag{40}$$

Given the kernel f and the excess kurtosis κ of LMA, the parameter ν is computed using the relation

$$\kappa = 3\nu \int f^4(x)\, dx.$$

The $\Lambda(x)$ process is sometimes represented as $\Lambda(x) = B(\Gamma(x))$ where $\Gamma(x)$ is the gamma motion. Now, introducing $\Lambda_{\mathrm{L}}(u) = B_{\mathrm{L}}(\Gamma(u))$ and $\Lambda_{\mathrm{R}}(u) = B_{\mathrm{R}}(\Gamma(u))$ we have

$$\begin{aligned} Z_{\mathrm{R}}(x) &= \int_{-\infty}^{+\infty} f_{\mathrm{R}}(x-u)\, d\,\Lambda_{\mathrm{R}}(u), \\ Z_{\mathrm{L}}(x) &= \int_{-\infty}^{+\infty} f_{\mathrm{LR}}(x-u)\, d\,\Lambda_{\mathrm{R}}(u) + \int_{-\infty}^{+\infty} f_L(x-u)\, d\,\Lambda_{\mathrm{L}}(u). \end{aligned} \tag{41}$$

The two spectral components in (34) have different physical origin. Hence, two LMA processes will be used here. First, the two kernels f_1, f_2 are defined by their Fourier transforms. In the example we shall use one asymmetrical kernel, and one symmetrical kernel having the following Fourier transforms:

$$\mathcal{F}f_1(\omega) = \frac{\sigma_1^{-1}}{(1+i\,\omega/\omega_0)^{(2p-1)\cdot w_1/2}(1+(\omega/\omega_0)^2)^{(1-p)\cdot w_1/2}}, \quad p \geq 1/2, \tag{42}$$

$$\mathcal{F}f_2(\omega) = \frac{\sigma_2^{-1}}{(1+(\omega/\omega_0)^2)^{w_2/4}}, \tag{43}$$

where σ_i and w_i are defined by the parameters in the MIRA spectrum. The asymmetry parameter p has to be estimated from the data. Finally, the kernels f_{L} and f_{LR} are defined by (42), (43), and equation (36-38).

We turn now to a numerical example. In Figure 1, left plot, one kilometer of measured (scaled to have variance one) parallel tracks are presented. Next, we present values of estimated parameters from the 5 km long measured signals. Estimates of skewness and kurtosis are 0.15, 5.02, respectively. The spectrum (34), with parameters $w_1 = 3.6$, $w_2 = 1.6$, is fitted to the data. In the figure, right plot, (34) spectrum (dashed line) is compared with nonparametric estimate (solid irregular line). The agreement between the two estimates is quite satisfactory. The parameter ρ in (35) is $\rho = 1.6$, while the asymmetry parameter $p = 0.6$ so the kernels appear to be close to the symmetric case ($p = 0.5$). The Laplace motion has parameter $\nu = 13.7$.

[Figure 1 about here.]

In Figure 2 (left plot) one km of simulated LMA tracks are presented. (In this type of applications synthetic and measured roads are equivalent if they induce the same amount of vehicle fatigue damage.) In Figure 2 (right plot) the accumulated damages in 20 simulated LMA and Gaussian road tracks are presented. The damages are normalized so that the value one assigned in the case when the simulated damage is equal to the observed in the measured signals. We can see that the Gaussian model is severely underestimating the damage, while the LMA model gives accurate predictions of the damage.

Let us comment on the issue why we do not use the LMA processes with symmetrical kernel to model the tracks. We have noted before that our choice $p = 0.6$ appears to be close to the symmetric case. However, looking only at the value of p is misleading. We repeated our numerical analysis with the symmetric kernel, and the results are presented in Figure 3. Note that the estimated damage for LMA with symmetric kernels is also closed to the observed one. However, high irregularity of the signals makes the symmetrical LMA unphysical as models of road surfaces, see Figure 2 (left plot).

[Figure 2 about here.]

Finally, for convenience of the reader, we give a definition of the damage. For a symmetrical zero mean response $Y(t)$, say, a simple damage accumulation model proposed in [12] (see also [20]) is that during a period of time length T the damage increment is

$$\Delta D_T = \kappa\beta 2^\beta \int_0^T (Y(t)^+)^{\beta-1} Y'(t)^+ \, dt, \quad x^+ = \max(0, x).$$

The failure is predicted when the damage exceeds threshold one. Here, κ and β are treated as deterministic material dependent constants. For vehicle components, β is usually in the range 3–8, making it most important to describe load cycles with large amplitude accurately.

[Figure 3 about here.]

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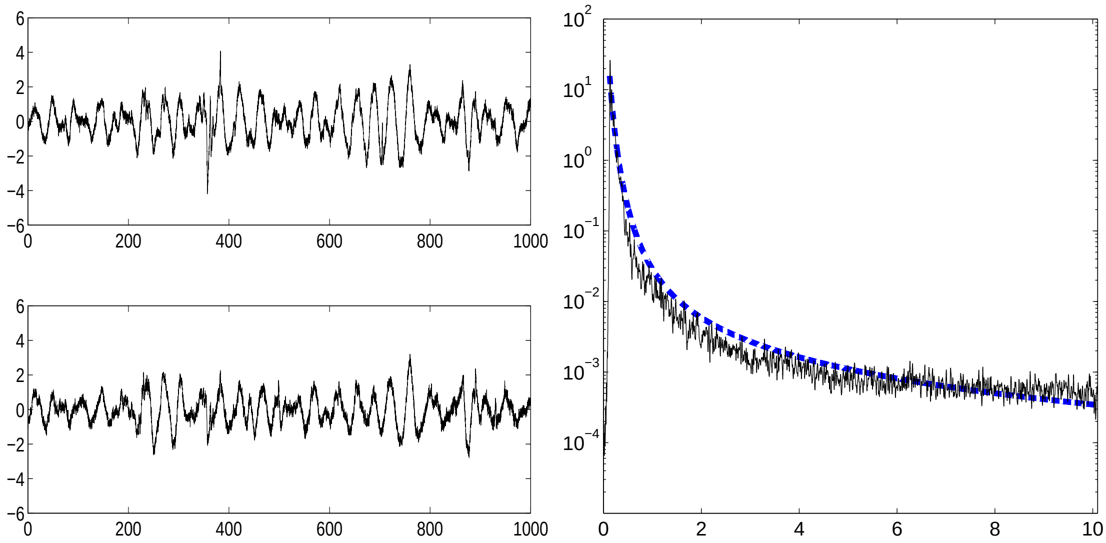

FIGURE 1. *(Left)* One kilometer measured parallel tracks. *(Right)* Estimated spectrum (34) – dashed line, nonparametric estimate – solid irregular line.

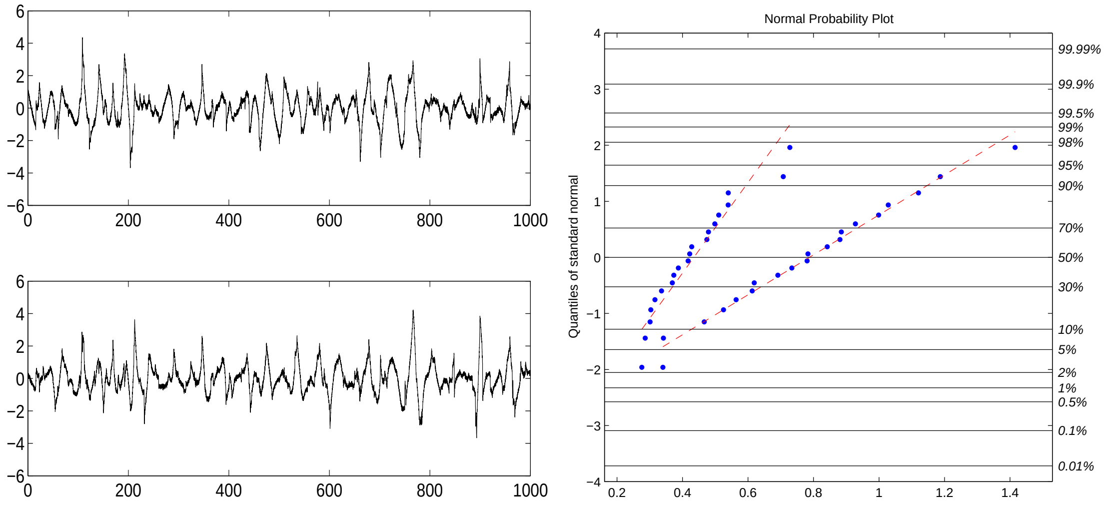

FIGURE 2. *(Left)* One kilometer simulated parallel tracks LMA with asymmetrical kernel. *(Right)* Scaled damages (value one corresponds to the observed damage) in 20 simulated LMA parallel tracks dots to the right and in the Gaussian tracks left dots.

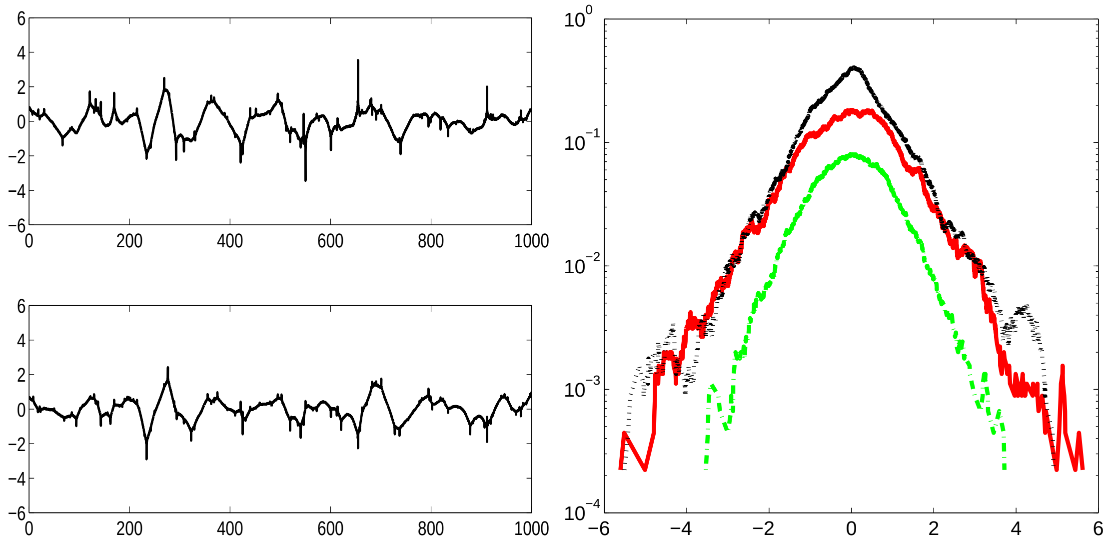

FIGURE 3. *(Left)* One kilometer simulated parallel tracks LMA with a symmetrical kernel. *(Right)* The observed number of upcrossing in 5 km measured road surface (dotted line), simulated number of upcrossings in LMA process (solid line) and in the Gaussian process (dashed dotted line).